The Positive Core of a Cooperative Game*

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Abstract

The positive core is a nonempty extension of the core of transferable utility games. If the core is nonempty, then it coincides with the core. It shares many properties with the core. Six well-known axioms which are employed in some axiomatizations of the core, the prenucleolus, or the positive prekernel, and one new intuitive axiom, characterize the positive core on any infinite universe of players. This new axiom requires that the solution of a game, whenever it is nonempty, contains an element which is invariant under any symmetry of the game.

1 Introduction

The positive core is a set-valued solution of cooperative transferable utility games. As far as we know the expression "positive core" was first mentioned in Orshan (1994). Its definition strongly relates to the definition of the prenucleolus. A preimputation belongs to the prenucleolus of a game, if it lexicographically minimizes the excesses of the coalitions. The definition of the positive core differs from the definition of the prenucleolus only inasmuch as the excesses are replaced by their **positive** parts. Hence, the prenucleolus is

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a subsolution of the positive core. On the other hand, if the core of a game is nonempty, then the positive parts of all excesses at elements of the core are zero, hence the core coincides with the positive core, whenever the core is nonempty.

Note that if an excess of a coalition S at a proposal x (see Section 2 for the formal definition) is positive, then it may be interpreted as the dissatisfaction of S when faced with x. If the excess is not positive, then the coalition is satisfied, that is, its dissatisfaction is 0. Hence, the positive core of a game consists of all preimputations that lexicographically minimize the dissatisfactions of the coalitions.

Though the core is regarded as one of the most intuitive solutions for cooperative games, there is (at least) one serious drawback: It specifies the empty set for many remarkable games. From a normative point of view, the elements of a solution of a game are interpreted as proposals of how to solve the game. If the solution applied to some game is empty, then this game cannot be solved. In this sense the core, when regarded a normative solution, should be applied to classes of balanced games (games wit a nonempty core) only. As soon as non-balanced games have to be considered the core seems to be a less suitable solution. It is then natural to replace the core by a solution which (a) contains the core as a subsolution and which (b) is nonempty when applied to any game under consideration. There are well-known nonempty core extensions. Indeed, the literature mentions various prebargaining sets (see, e.g., Aumann and Maschler (1964), Granot and Maschler (1997), or Sudhölter and Potters (2001)) which contain the core. However, even if the core is nonempty, they can be much larger than the core. They may contain counter intuitive proposals which are ruled out by the core. Recently Sudhölter and Peleg (2000) introduced and axiomatized a smaller nonempty core extension, the positive prekernel (see Section 2 for the precise definition). Though this solution is a subsolution of the prebargaining sets, the core of a balanced game may be a proper subset of the positive prekernel of this game. It is the aim of this paper to show that the positive core is characterized (see Theorem 4.8 and Corollary 4.11) by intuitive simple axioms. Only one property (see Definition 3.2) is employed, which is not used in some axiomatizations of the prekernel, the positive prekernel, the prenucleolus, or the core.

The paper is organized as follows: In Section 2 the notation and some definitions are presented. Moreover, some characterizations of solutions known from literature and relevant in the context of the positive core, are recalled.

The positive core satisfies nonemptiness, anonymity, covariance under strategic equivalence, and it is Pareto optimal and reasonable. The core satisfies three variants of reduced game properties: The reduced game property (RGP), the converse reduced game property (CRGP), and the reconfirmation property (RCP). The positive prekernel satisfies two of them, namely RGP and CRGP. In Section 3 it is shown that the positive core satisfies RGP and RCP. Moreover it shares a further property with the core, the (positive) prekernel, and the nucleolus. Indeed, these solutions allow for non-discrimination. This axiom requires from the solution of a game that it, whenever nonempty, contains a preimputation which does not discriminate (that is, which is invariant under all symmetries of the game).

The main results are contained in Section 4, which is divided into four subsections. We investigate the behavior and shape of solutions that are nonempty- and bounded-valued and satisfy covariance, anonymity, RGP, and RCP. In the first subsection it turns out (see Theorem 4.1) that the restrictions of such a solution to one- and two-person games is either the prenucleolus, or the relative interior of the positive core, or the positive core. In order to generalize this result to arbitrary games, it is shown (see Lemma 4.5) in Subsection 4.2 that the positive parts of the excesses at elements of the solution of a game have to coincide. For this and the preceding result the assumption of an infinite universe of potential players is crucial. In Subsection 4.3 Lemma 4.5 is used to prove that Theorem 4.1 can be generalized, if the solution is assumed to contain the prenucleolus.

Finally Subsection 4.4 shows that the unique solutions that satisfy the above properties **and** are convex-valued, are the prenucleolus, the relative interior of the positive core, or the positive core. Moreover, it is shown that anonymity and convex-valuedness can be replaced by ND (see Theorem 4.9). Hence, the positive core is the maximum nonempty and bounded-valued solution that satisfies covariance, anonymity, RGP, RCP, and allows for non-discrimination.

In Section 5 examples are presented which show that each of the axioms employed in Theorem 4.9 is logically independent of the remaining axioms. Moreover, some remarks are given.

2 Notation and Definitions

Let U be a set (the universe of players). Throughout this paper we shall assume for simplicity that $\{1,\ldots,k\} \subseteq U$ whenever $|U| \geq k$.

A *cooperative transferable utility game* – a *game* – is a pair (N,v) such that N is a finite nonempty subset of U (the player set) and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is a mapping (the coalition function). Here $2^N = \{S \subseteq N\}$ is the set of *coalitions*. The set of *feasible payoffs* of a game (N,v) is $X(N,v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$, where we use $x(S) = \sum_{i\in S} x_i$ $(x(\emptyset) = 0)$ for every $S \in 2^N$ and every $x \in \mathbb{R}^N$ as a convention. Additionally, x_S denotes the restriction of x to S, i.e. $x_S = (x_i)_{i\in S}$. For disjoint coalitions S, T let $(x_S, x_T) = x_{S\cup T}$. Moreover, $\mathcal{I}^*(N,v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ denotes the set of *preimputations*. A feasible payoff x is *individually rational*, if $x_i \geq v(\{i\})$ for all $i \in N$. An individually rational preimputation is called *imputation*.

A *solution* σ is a mapping that associates with every game (N,v) a set $\sigma(N,v) \subseteq X(N,v)$.

The following abbreviations are used to recall the definitions of several well-known solutions which are relevant in the sequel. Let (N,v) be a game, let $S \subseteq N$, let $k,\ell \in N$ be distinct, and let $x \in \mathbb{R}^N$. Then $e(S,x,v) = v(S) - x(S)$ denotes the *excess* of S, $\mu(x,v) = \max_{S\subseteq N} e(S,x,v)$ denotes the *maximal* excess and

$$s_{k\ell}(x,v) = \max\{e(S,x,v) \mid S \subseteq N, \ell \notin S \ni k\}$$

denotes the *maximal surplus* of k against ℓ at x with respect to (w.r.t.) (N,v).

The *core* of (N,v) is the set $\mathcal{C}(N,v) = \{x \in \mathcal{I}^*(N,v) \mid e(S,x,v) \leq 0 \ \forall S \subseteq N\}$ and the *prekernel* of (N,v) is the set

$$\mathcal{K}^*(N,v) = \{x \in \mathcal{I}^*(N,v) \mid s_{k\ell}(x,v) = s_{\ell k}(x,v) \ \forall k \in N \text{ and } \ell \in N \setminus \{k\}\}.$$

The *positive prekernel* of (N,v) is the set

$$\mathcal{K}^*_+(N,v) = \{x \in \mathcal{I}^*(N,v) \mid (s_{k\ell}(x,v))_+ = (s_{\ell k}(x,v))_+ \ \forall k \in N \text{ and } \ell \in N \setminus \{k\}\},$$

where $t_+ = \max\{t,0\}$ denotes the positive part of a real t. The *prenucleolus* of (N,v), denoted $\mathcal{V}(N,v)$, is the set of preimputations that lexicographically minimize the non-increasingly ordered vector of excesses of the coalitions. The set $\mathcal{V}(N,v)$ is a singleton $\{\nu(N,v)\}$.

Gillies (1959) introduced the core, for the prekernel and prenucleolus we refer to Maschler, Peleg, and Shapley (1972) and to Schmeidler (1969). The positive prekernel was introduced by Sudhölter and Peleg (2000). Now we are able to define the positive core.

Definition 2.1 *The* **positive core** *of a game* (N,v) *is the set* $\mathcal{C}_+(N,v)$ *defined by*

$$\mathcal{C}_+(N,v) = \{x \in \mathcal{I}^*(N,v) \mid (e(S,x,v))_+ = e(S,\nu(N,v),v)_+ \ \forall S \subseteq N\}.$$

The prenucleolus is the preimputation which minimizes the highest excess, then minimizes the number of coalitions attaining highest excess, then minimizes the second highest excess, and so on. Thus, the prenucleolus is yielded by iteratively solving a finite sequence of linear programs. The positive core can be computed by solving the same sequence of linear programs as long as the excesses are positive.

A member of the positive core lexicographically minimizes the non-increasingly ordered vector of **positive** excesses, whereas the prenucleolus lexicographically minimizes the non-increasingly ordered vector of **all** excesses. A member of the positive prekernel balances the maximal surplus of distinct players whenever it is **positive**, whereas a member of the prekernel balances the maximal surplus of **all** pairs of distinct players. In this sense the positive core of a game arises from the prenucleolus of the game in a similar way as the positive prekernel arises from the prekernel.

Clearly, $\mathcal{C}_+(N,v) = \mathcal{C}(N,v)$, if $\mathcal{C}(N,v) \neq \emptyset$, and $\nu(N,v) \in \mathcal{C}_+(N,v)$.

Some convenient and well-known properties of a solution σ are as follows: σ is *nonempty* (NE), if $\sigma(N,v) \neq \emptyset$ for every game (N,v). σ satisfies the *zero inessential two-person game property* (ZIG), if for every game (N,v) with $|N| = 2$ and $v = 0$, $\sigma(N,v) \neq \emptyset$. σ is *single-valued* (SIVA), if $|\sigma(N,v)| = 1$ for every game (N,v). σ is *covariant under strategic equivalence* (COV), if for games $(N,v),(N,w)$ with $w = \alpha v + \beta$ for some $\alpha > 0, \beta \in \mathbb{R}^N$ the equation $\sigma(N,w) = \alpha\sigma(N,v) + \beta$ holds. (Here we use the convention which identifies $\beta \in \mathbb{R}^N$ with the *additive* coalitional function, again denoted by β, on the player set N defined by $\beta(S) = \sum_{i \in S} \beta_i$ for all $S \in 2^N$.) If $w = \alpha v + \beta$ as above, then the games v and w are called *strategically equivalent.* σ satisfies *anonymity* (AN), if for any two games $(N,v),(M,w)$ the following condition holds: If $\pi : N \to M$ is a bijective mapping such that $\pi v = w$ $(\pi v(S) := v(\pi^{-1}(S)))$, then $\sigma(M,w) = \pi\sigma(N,v)$. (For $x \in \mathbb{R}^N$, $\pi x \in \mathbb{R}^{\pi(N)}$ is defined by $\pi x_j = x_{\pi^{-1}(j)}$ $\forall j \in \pi(N)$.) In this case (N,v) and (N,w) are *isomorphic* games. σ satisfies the *equal treatment property* (ETP), if for every game (N,v) and every $x \in \sigma(N,v)$ the following condition holds for every $i \in N$ and $j \in N \setminus \{i\}$: If $v(S \cup \{i\} = v(S \cup \{j\}$ for every $S \subseteq N \setminus \{i,j\}$ (i and j are *interchangeable* or *equals*), then $x_i = x_j$. σ satisfies *Pareto optimality* (PO), if $\sigma(N,v) \subseteq \mathcal{I}^*(N,v)$ for every game

(N, v). σ is *reasonable* (REAS), if, for every game (N, v), for every $x \in \sigma(N, v)$, and for every $i \in N$,

$$\min_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)) \leq x_i \leq \max_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)).$$

σ is *bounded-valued* (BOUND), if $\sigma(N, v)$ is bounded for every game (N, v). σ satisfies *weak unanimity for two-person games* (WUTPG), if, for every game (N, v) with $|N| = 2$, all imputations are members of $\sigma(N, v)$.

We now recall the definitions of some variants of the reduced game property. Let (N, v) be a game, $x \in X(N, v)$, and $\emptyset \neq S \subseteq N$. The *reduced game* $(S, v^{S,x})$ w.r.t. to S and x is defined by

$$v^{S,x}(T) = \begin{cases} 0 & , \text{ if } T = \emptyset, \\ v(N) - x(N \setminus S) & , \text{ if } T = S, \\ \max_{Q \subseteq N \setminus S} (v(T \cup Q) - x(Q)) & , \text{ otherwise}, \end{cases}$$

for every $T \subseteq S$. σ satisfies the *reduced game property* (RGP), if $x_S \in \sigma(S, v^{S,x})$ for every game (N, v), for every $\emptyset \neq S \subseteq N$, and every $x \in \sigma(N, v)$. σ satisfies the *converse reduced game property* (CRGP), if for every game (N, v) with $|N| \geq 2$ the following condition is satisfied for every $x \in \mathcal{I}^*(N, v)$: If, for every $S \subseteq N$ with $|S| = 2$, $x_S \in \sigma(S, v_{S,x})$, then $x \in \sigma(N, v)$. σ satisfies the *reconfirmation property* (RCP), if for every game (N, v), for every $\emptyset \neq S \subseteq N$, for every $x \in \sigma(N, v)$, and for every $y \in \sigma(S, v^{S,x})$, $(y, x_{N \setminus S}) \in \sigma(N, v)$.

With the help of these axioms the prekernel, the prenucleolus, the positive prekernel, and the core can be characterized. We recall some of the axiomatizations.

Theorem 2.2 (Peleg (1986)) *The unique solution that satisfies* NE*,* PO*,* COV*,* ETP*,* RGP*, and* CRGP *is the prekernel.*

Theorem 2.3 (Sobolev (1975)) *The unique solution that satisfies* SIVA*,* COV*,* AN*, and* RGP *is the prenucleolus, provided* $|U| = \infty$*.*

Theorem 2.4 (Sudhölter and Peleg (2000)) *The unique solution that satisfies* NE*,* AN*,* REAS*,* RGP*,* CRGP*, and* WUTPG *is the positive prekernel, provided* $|U| \geq 3$*.*

Theorem 2.5 (Hwang and Sudhölter (2001)) *The unique solution that satisfies* ZIG*,* COV*,* AN*,* RGP*,* RCP*,* CRGP*, and* BOUND *is the core, provided* $|U| \geq 5$*.*

Remark 2.6

(1) In Theorem 2.2, CRGP is a substitute for maximality, i.e., each solution that satisfies the remaining axioms is contained in the prekernel (see Peleg (1986)). Hence the prekernel is the **maximum** solution that satisfies the remaining five axioms.

(2) Orshan (1993) shows that AN can be replaced by ETP in Theorem 2.3.

(3) Note that the prenucleolus is also axiomatized by NE, COV, ETP, and RCP (see Orshan and Sudhölter (2003)).

Remark 2.7 The positive core of a game is the set of all preimputations that lexicographically minimize the non-increasingly ordered vector of the positive parts of the excesses. A proof of this alternative "definition" of the positive core is straightforward.

The following simple results are useful. Let Γ^2 be the set of games with at most two players.

Remark 2.8 Let (N, v) be a game, $\emptyset \neq S \subseteq N,\ k, \ell \in S,\ k \neq \ell$, and let $x \in \mathbb{R}^N$. Then

$$s_{k\ell}(x, v) = s_{k\ell}\left(x_S, v^{S,x}\right). \tag{2.1}$$

Remark 2.9 (Lemma 4.3 of Hwang and Sudhölter (2001)) If σ satisfies BOUND, COV, and RGP, then σ also satisfies PO.

Remark 2.10 (Lemma 7.1 of Sudhölter and Peleg (2000)) Let σ^1, σ^2 be solutions. If σ^1 satisfies PO and RGP, if σ^2 satisfies CRGP, and if $\sigma^1(N, v) \subseteq \sigma^2(N, v)$ for every game $(N, v) \in \Gamma^2$, then σ^1 is a subsolution of σ^2.

3 Properties and Examples

This section serves to show that the positive core has many properties in common with the core, the (positive) prekernel, and the nucleolus. Moreover, an algebraic characterization by balanced collections of coalitions is provided. We shall also use the *relative interior*, denoted rint $\mathcal{C}_+$ of the positive core. Let (N, v) be a game. Then

$$\text{rint } \mathcal{C}_+(N, v) = \{x \in \mathcal{C}_+(N, v) \mid \forall S \subseteq N : e(S, \nu(N, v), v) < 0 \Rightarrow e(S, x, v) < 0\}.$$

The positive core satisfies NE and, hence, ZIG, because it contains the prenucleolus. Clearly, it satisfies COV, AN, and PO. It is a subsolution of the positive prekernel which satisfies REAS (see Sudhölter and Peleg (2000)), thus it satisfies REAS as well. The relative interior of the positive core satisfies the same axioms. For two-person games with a non-empty core, $\mathcal{C}_+$ coincides with the core and for two-person games with an empty core it coincides with the prenucleolus. Therefore it satisfies, different from rint $\mathcal{C}_+$, WUTPG.

Remark 3.1 *It is well-known that the prekernel of any three-person game coincides with its prenucleolus. Hence, by Theorem 4.1 of Sudhölter and Peleg (2000), the positive core of any three-person game coincides with its positive prekernel.*

A further property of the positive core is interesting. Let (N, v) be a game. A *symmetry* of (N, v) is a permutation π of N such that $\pi v = v$. Let $\mathrm{SYM}(N, v)$ denote the group of symmetries of (N, v).

Definition 3.2 *A solution σ* **allows for non-discrimination** *(*ND*), if the following property is valid for every game (N, v): If $\sigma(N, v) \neq \emptyset$, then there exists $x \in \sigma(N, v)$ such that $\pi x = x$ for every $\pi \in \mathrm{SYM}(N, v)$.*

ND has the following simple interpretation: If the players of a game (N, v) are able to agree upon a proposal to solve the game (that is, if $\sigma(N, v) \neq \emptyset$), then ND requires that they should as well be able to agree upon a proposal, which is invariant under all symmetries of the game and which, hence, does not discriminate.

There is a "natural" strong version of ND. Indeed, a solution σ satisfies *strong non-discrimination* (SND), if **every** $x \in \sigma(N, v)$ is invariant under symmetries. Note that any solution σ that satisfies SIVA and AN also satisfies SND. Hence the prenucleolus satisfies SND and the (positive) prekernel, the core, the (relative interior of the) positive core, and many other well-known solutions satisfy ND. Note that there are also set-valued solutions that satisfy SND (e.g. the maximal satisfaction solution introduced by Sudhölter and Peleg (1998)).

The following approach results in a characterization of the positive core by balanced collections of coalitions and allows to deduce RGP and RCP.

For every game (N, v), for every $x \in \mathbb{R}^N$ and every $\alpha \in \mathbb{R}$ denote

$$\mathcal{D}(\alpha, x, v) = \{S \subseteq N \mid e(S, x, v) \geq \alpha\}.$$

Let $S \subseteq N$. The *characteristic vector* χ^S of S is the member of $\mathbb{R}^N$ which is given by

$$\chi_i^S = \begin{cases} 1, & \text{if } \ i \in S, \\ 0, & \text{if } \ i \in N \setminus S. \end{cases}$$

A collection $\mathcal{D} \subseteq 2^N$ of coalitions is *balanced* (over N), if there are coefficients $\gamma_S > 0,\ S \in \mathcal{D}$, such that $\sum_{S \in \mathcal{D}} \gamma_S \chi^S = \chi^N$. The collection $(\delta_S)_{S \in \mathcal{B}}$ is called a system of *balancing weights.*

We first recall a result of Kohlberg (1971) characterizing the prenucleolus.

Theorem 3.3 *Let (N, v) be a game and $x \in \mathcal{I}^*(N, v)$. Then the following assertions are equivalent: (a) $x = \nu(N, v)$. (b) For every $\alpha \in \mathbb{R}$ and every $y \in \mathbb{R}^N$ satisfying $y(N) = 0$ and $y(S) \geq 0$ for all $S \in \mathcal{D}(\alpha, x, v)$, $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v)$. (c) For every $\alpha \in \mathbb{R}$, $\mathcal{D}(\alpha, x, v)$ is balanced or empty.*

The equivalence of (a) and (b) may be proved directly and the equivalence of (b) and (c) is a direct consequence of the duality theorem of linear programming. In a completely analogous way the following result may be proved.

Theorem 3.4 *Let (N, v) be a game and $x \in \mathcal{I}^*(N, v)$. Then the following assertions are equivalent: (a) $x \in \mathcal{C}_+(N, v)$. (b) For every* **positive** *$\alpha \in \mathbb{R}$ and every $y \in \mathbb{R}^N$ satisfying $y(N) = 0$ and $y(S) \geq 0$ for all $S \in \mathcal{D}(\alpha, x, v)$, $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v)$. (c) For every* **positive** *$\alpha \in \mathbb{R}$, $\mathcal{D}(\alpha, x, v)$ is balanced or empty.*

Theorem 3.4 implies (see Theorem 6.3.14 in Peleg and Sudhölter (2003)) that the positive core satisfies RGP and RCP. Moreover, the relative interior of the positive core may be characterized by modifying Theorem 3.4 only inasmuch as $\mathcal{C}_+$ has to be replaced by rint $\mathcal{C}_+$ and **positive** has to be replaced twice by **nonnegative**. This modification implies that the relative interior of the positive core satisfies RGP and RCP as well. Hence, the following theorem is valid.

Theorem 3.5 *The positive core and its relative interior satisfy* RGP *and* RCP*.*

We shall now present an example which shows that the positive core does not satisfy CRGP. Sudhölter and Peleg (2002) use a similar example.

Example 3.6 Let $N = \{1,2,3,4\}$ and $v(S), S \subseteq N$, be defined by

$$v(S) = \begin{cases} 0 & , \text{ if } S \in \{\emptyset, N\}, \\ 2 & , \text{ if } S \in \{S^i \mid i = 1,2,3,4\}, \\ -2 & , \text{ otherwise,} \end{cases}$$

where $S^1 = \{1,2\}, S^2 = \{2,3\}, S^3 = \{3,4\}$, and $S^4 = \{1,4\}$. Note that $\mathrm{SYM}(N,v)$ is generated by the cyclic permutation, which maps 1 to 2, 2 to 3, 3 to 4, and 4 to 1; thus the game is *transitive*. (A game is called *transitive*, if its symmetry group is transitive.) As $\mathcal{V}$ satisfies ND and PO, $\nu(N,v) = 0$. Thus $e(S,\nu(N,v),v) = 2$ for $S \in \{S^i \mid i = 1,\ldots,4\}$ and $e(T,\nu(N,v),v) \leq 0$ for all other coalitions. Therefore,

$$\mathcal{C}_+(N,v) = \mathrm{convh}\{(1,-1,1,-1),(-1,1,-1,1)\},$$

where "convh" means "convex hull". Let $x \in \mathcal{C}_+(N,v), k \in N$, and $\ell \in N \setminus \{k\}$. Then $s_{k\ell}(x,v) = 2$ and $s_{k\ell}(x,v)$ is attained by some $S^i, i = 1, \ldots 4$. Thus

$$\mathrm{convh}(\{(2,-2,2,-2),(-2,2,-2,2)\} \subseteq \mathcal{K}^*(N,v).$$

The current example together with Theorem 2.4 shows that the positive core does not satisfy CRGP when $|U| \geq 4$.

Remark 3.7 *The positive core shares another property with the core: The positive core is convex-valued (CON). Indeed, Definition 2.1 directly implies that the positive core of a game is a compact convex polyhedral set.*

In the next section we shall need the following easy result.

Remark 3.8 (see, e.g. Remark 2.7 of Sudhölter (1997)) Let $\emptyset \neq N$ be a finite set and let $\mathcal{D}$ be a balanced collection over N. If $T \subseteq N$ satisfies $\chi_T \in \langle\{\chi_S \mid S \in \mathcal{D}\}\rangle$, then $\mathcal{D} \cup \{T\}$ is balanced. Here $\langle\cdots\rangle$ denotes the linear span. (We say that $\mathcal{D}$ **spans** $\mathcal{D} \cup \{T\}$.)

4 A Characterization of the Positive Core

In this section, which is divided into several subsections, we shall present a characterization of the positive core by intuitive simple axioms. We start with some investigations in the two-person case.

4.1 The Two-Person Case

Throughout this subsection we assume that $|U| \geq 3$ and that σ is a solution.

Theorem 4.1 *The solution σ satisfies* NE, BOUND, COV, AN, RGP, *and* RCP, *if and only if it coincides on Γ^2 with $\mathcal{V}$ or* rint $\mathcal{C}_+$ *or with $\mathcal{C}_+$.*

We postpone the proof of Theorem 4.1 and first prove two useful lemmata.

Lemma 4.2 *If σ satisfies* NE, BOUND, COV, AN, RGP, *and* RCP, *and if $(N,v) \in \Gamma^2$, then $\sigma(N,v) \subseteq \mathcal{C}_+(N,v)$.*

Proof: By Remark 2.9, σ satisfies PO and the assertion is true for any 1-person game. Assume, now, $N = \{1,2\}$ and let (N,v) be a game. Three cases may be distinguished.

(1) $|\mathcal{C}(N,v)| = 1$: By COV, $v(\{i\}) = 0, i \in N$, may be assumed. Hence, $v(N) = 0$. Let $x \in \sigma(N,v)$. Then $\alpha x \in \sigma(N,\alpha v) = \sigma(N,v)$ for every $\alpha > 0$ by COV, thus, by BOUND, the proof is finished.

(2) $\mathcal{C}(N,v) = \emptyset$: By COV we may assume that $v(\{1\}) = v(\{2\}) = 1$ and $v(N) = 0$. With $\widetilde{N} = \{1,2,3\}$ define $(\widetilde{N},w)$ by

$$w(S) = \begin{cases} 0 & , \text{ if } S = \emptyset, \widetilde{N}, \\ 1 & , \text{ otherwise.} \end{cases}$$

Then $\nu(\widetilde{N},w) = 0 \in \mathbb{R}^{\widetilde{N}}$ and the reduced game w.r.t. N and $\nu(\widetilde{N},w)$ coincides with (N,v). By RCP the proof is complete in this case as soon as it is shown that the unique member of $\sigma(\widetilde{N},w)$ is $\nu(\widetilde{N},w)$.

Assume, on the contrary, $\{\nu(\widetilde{N},w)\} \neq \sigma(\widetilde{N},w)$. By NE there exists $z \in \sigma(\widetilde{N},w)$ such that $z \neq 0$. By AN we may assume $z_1 \leq z_2 \leq z_3$. By PO,

$$z \in X := \{x \in \mathcal{I}^*(\widetilde{N},w) \mid x_1 \leq x_2 \leq x_3\}.$$

By BOUND it suffices to show that there exists a mapping $f : X \to X$ which has the following properties:

(a) For every $x \in X \setminus \{0\}$ the set $\{f^k(x) \mid k \in \mathbb{N}\}$ is not bounded. (f^k is the k-fold composition of f.)

(b) For every $x \in X \cap \sigma(\widetilde{N},w)$, $f(x) \in \sigma(\widetilde{N},w)$.

In order to construct the desired mapping f, define

$$X^{\leq} = \{x \in X \mid x_2 \leq 0\} \text{ and } X^{\geq} = \{x \in X \mid x_2 \geq 0\}.$$

We define $f^{\leq} : X^{\leq} \to \mathbb{R}^{\widetilde{N}}$ and $f^{\geq} : X^{\geq} \to \mathbb{R}^{\widetilde{N}}$ as follows. Let $x \in X^{\leq}, y \in X^{\geq}$, and let α, β be defined by

$$\alpha = 1 - \frac{(x_1+1)(x_3+2)}{2-x_2} \text{ and } \beta = 1 - y_1 + \frac{(y_1-1)(2-y_1)}{2+y_2}. \tag{4.1}$$

Note that the denominators are positive and that α and β may be expressed by

$$\alpha = -x_1 \frac{1+x_3}{2-x_2} \text{ and } \beta = -y_3 \frac{1-y_1}{2+y_2}, \tag{4.2}$$

because $x(\widetilde{N}) = y(\widetilde{N}) = 0$. Now, define

$$\begin{aligned} &f_2^{\leq}(x) = \min\{x_3, \alpha\}, \quad f_3^{\leq}(x) = \max\{x_3, \alpha\}, \quad f_1^{\leq}(x) = -f_2^{\leq}(x) - f_3^{\leq}(x), \\ &f_1^{\geq}(y) = \min\{y_1, \beta\}, \quad f_2^{\geq}(y) = \max\{y_1, \beta\}, \quad f_3^{\geq}(y) = -f_1^{\geq}(y) - f_2^{\geq}(y). \end{aligned}$$

By (4.2), $\alpha \geq 0 \geq \beta$. Hence

$$f^{\leq}(x) \in X^{\geq} \text{ and } f^{\geq}(y) \in X^{\leq}. \tag{4.3}$$

Define f by

$$f(x) = f^{\geq}(f^{\leq}(x)) \; \forall x \in X^{\leq} \text{ and } f(y) = f^{\geq}(y) \; \forall y \in X \setminus X^{\leq}.$$

We now show that f satisfies (4.1). Let $x \in X \setminus \{0\}$. We claim that

$$f_3(x) > x_3 \text{ and } f_1(x) \leq x_1 \tag{4.4}$$

If $x \in X \setminus X^{\leq}$, then $f(x) = f^{\geq}(x)$. By the definition of $f^{\geq}$ and (4.3), $f_1(x) \leq x_1$, $f_2(x) < 0$, and $f_3(x) > x_3$. If $x \in X^{\leq}$, we analogously obtain $f_1^{\leq}(x) < x_1$, $f_2^{\leq}(x) > 0$, and $f_3^{\leq}(x) \geq x_3$, hence $f_1(x) \leq f_1^{\leq}(x) < x_1$ and $f_3(x) > f_3^{\leq}(x) \geq x_3$. Thus, (4.4) is true. The continuity of f and (4.4) immediately imply (4.1).

In order to show (4.1) it suffices to prove the following assertions:

$$f^{\leq}(x) \in \sigma(\widetilde{N}, w) \; \forall x \in \sigma(\widetilde{N}, w) \cap X^{\leq} \tag{4.5}$$

$$f^{\geq}(y) \in \sigma(\widetilde{N}, w) \; \forall y \in \sigma(\widetilde{N}, w) \cap X^{\geq} \tag{4.6}$$

Let $x \in \sigma(\widetilde{N}, w) \cap X^{\leq}$ be arbitrarily. Then the coalition functions of the reduced games w.r.t. $\{1,3\}$ and w.r.t. $\{1,2\}$ satisfy

$$w^{\{1,3\},x}(\{1\}) = w^{\{1,3\},x}(\{3\}) = 1 - x_2, w^{\{1,3\},x}(\{1,3\}) = -x_2$$

and

$$w^{\{1,2\},x}(\{1\}) = w^{\{1,2\},x}(\{2\}) = 1, w^{\{1,2\},x}(\{1,2\}) = -x_3.$$

By the definition of $X^{\leq}$ the game $(\{1,2\}, w^{\{1,2\},x})$ is isomorphic (by interchanging players 2 and 3) to the game[1]

$$\left(\{1,3\}, \left(w^{\{1,3\},x} + (x_2 - 1, x_2 - 1)\right) \frac{x_3+2}{2-x_2} + (1,1)\right).$$

[1]If S is a nonempty finite subset of natural numbers, then we use for $x \in \mathbb{R}^S$ the notation $x = (x_{i_1}, \ldots, x_{i_{|S|}})$ with the convention $i_1 < \cdots < i_{|S|}$.

By AN and COV,

$$\hat{x} := ((x_1, x_3) + (x_2 - 1, x_2 - 1))\,\frac{x_3 + 2}{2 - x_2} + (1, 1) \in \sigma(\{1, 2\}, w^{\{1,2\},x}).$$

PO and

$$\hat{x}_2 = (x_3 + x_2 - 1)\frac{x_3 + 2}{2 - x_2} + 1 = (-x_1 - 1)\frac{x_3 + 2}{2 - x_2} + 1 = \alpha$$

yields $\hat{x} = (-x_3 - \alpha, \alpha)$. By RCP, $(\hat{x}, x_3) \in \sigma(\widetilde{N}, w)$. AN yields

$$f^{\leq}(x) = (\hat{x}_1, \min\{\hat{x}_2, x_3\}, \max\{\hat{x}_2, x_3\}) \in \sigma(\widetilde{N}, w).$$

Assertion (4.6) may be proved similarly by applying the fact that $(\{2, 3\}, w^{\{2,3\},y})$ is isomorphic to

$$\left(\{1, 3\}, \left(w^{\{1,3\},y} - (1, 1)\right)\frac{2 - y_1}{2 + y_2} + (1 - y_1, 1 - y_1)\right).$$

(3) $\mathcal{C}(N, v)$ has a nonempty interior: By COV we may assume that v is given by

$$v(S) = \begin{cases} 0 & , \text{ if } S = \emptyset, N, \\ -1 & , \text{ otherwise.} \end{cases}$$

Define $(\widetilde{N}, w)$ by adding the null-player 3 to (N, v), i.e., by $w(S) = v(S \setminus \{3\})$. By NE there is a member z of $\sigma(\widetilde{N}, w)$. We may assume $z_1 \leq z_2$ by AN.

Claim 1: $-2 < z_3 < 2$

Let $u = w^{N,z}$. If $z_3 \leq -2$ or $z_3 \geq 2$ respectively, then u is given by

$$u(\{i\}) = -1 - z_3, \text{ or } u(\{i\}) = -1 \; \forall i \in N \text{ and } u(N) = -z_3.$$

Hence (N, u) has a single valued core (in the case $z_3 = -2, +2$) or an empty core. In both cases (1) and (2) we have already shown that $\sigma(N, u)$ consists of the prenucleolus only. Hence, by RGP $z_1 = z_2$. Moreover, by PO, we obtain $z_1 = z_2 = -z_3/2$. Hence, $w^{\{1,3\},z}$ is given by

$$w^{\{1,3\},z}(\{1\}) = -1,\; w^{\{1,3\},z}(\{3\}) = 0,\; w^{\{1,3\},z}(\{1, 3\}) = \frac{z_3}{2} \text{ if } z_3 \leq -2$$

or by

$$w^{\{1,3\},z}(\{1\}) = \frac{z_3}{2},\; w^{\{1,3\},z}(\{3\}) = -1 + \frac{z_3}{2},\; w^{\{1,3\},z}(\{1, 3\}) = \frac{z_3}{2} \text{ if } z_3 \geq 2.$$

Hence, the core of $(\{1, 3\}, w^{\{1,3\},z})$ is a singleton (if $z_3 = 2$ or $z_3 = -2$) or empty. Therefore we know from cases (1) and (2) that (z_1, z_3) has to be the prenucleolus of the reduced game which is not the case. Hence Claim 1 is shown.

Claim 2: $(z_1, z_2) \in \mathcal{C}(N, w^{N,z})$

Assume the contrary. Two cases may be distinguished.

(i) $z_3 \geq 0$: The reduced game $(N, w^{N,z})$ is given by

$$w^{N,z}(\{1\}) = w^{N,z}(\{2\}) = -1,\ w^{N,z}(N) = -z_3. \tag{4.7}$$

By Claim 1 the core of this game has a nonempty interior, because $z_3 < 2$. By PO and the assumptions $z_1 \leq z_2$ and $(z_1, z_2) \notin \mathcal{C}(N, w^{N,z})$, we have $z_1 < -1$. Therefore,

$$w^{\{2,3\},z}(\{2\}) = -z_1,\ w^{\{2,3\},z}(\{3\}) = -1 - z_1,\ w^{\{2,3\},z}(\{2,3\}) = -z_1,$$

thus $\mathcal{C}(\{2,3\}, w^{\{2,3\},z}) = \emptyset$. Hence, by RGP and part (2),

$$z_2 = \frac{1 - z_1}{2} \text{ and } z_3 = -\frac{z_1 + 1}{2}.$$

Hence $z_2 > 1$. Therefore,

$$w^{\{1,3\},z}(\{1\}) = -1,\ w^{\{1,3\},z}(\{3\}) = 0,\ w^{\{1,3\},z}(\{1,3\}) = \frac{z_1 - 1}{2},$$

thus $\mathcal{C}(\{1,3\}, w^{\{1,3\},z}) = \emptyset$. Hence, by RGP and part (2), $z_3 = \frac{z_1+1}{4}$, which is negative.

(ii) $z_3 \leq 0$: The reduced game $(N, w^{N,z})$ is given by

$$w^{N,z}(\{1\}) = w^{N,z}(\{2\}) = -1 - z_3,\ w^{N,z}(N) = -z_3. \tag{4.8}$$

By Claim 1 the core of this game has a nonempty interior, because $z_3 > -2$. By PO and the assumptions $z_1 \leq z_2$ and $(z_1, z_2) \notin \mathcal{C}(N, w^{N,z}$, we have $z_1 < -1 - z_3$ and $z_2 > 1$. Therefore,

$$w^{\{1,3\},z}(\{1\}) = -1,\ w^{\{1,3\},z}(\{3\}) = 0,\ w^{\{2,3\},z}(\{1,3\}) = -z_2,$$

thus $\mathcal{C}(\{1,3\}, w^{\{1,3\},z}) = \emptyset$. Hence, by RGP and part (2),

$$z_1 = \frac{-1 - z_2}{2} \text{ and } z_3 = \frac{-z_2 + 1}{2}.$$

Hence $z_1 < -1$. Therefore,

$$w^{\{2,3\},z}(\{2\}) = \frac{z_2 + 1}{2},\ w^{\{2,3\},z}(\{3\}) = \frac{z_2 - 1}{2},\ w^{\{2,3\},z}(\{2,3\}) = \frac{z_2 + 1}{2},$$

thus $\mathcal{C}(\{2,3\}, w^{\{2,3\},z}) = \emptyset$. In case (2) we proved that the solution of a two-person game with an empty core must coincide with the prenucleolus, hence, by RGP, $z_3 = \frac{z_2-1}{4}$, which is positive.

In both cases, i.e., regardless of whether $z_3 \geq 0$ or $z_3 \leq 0$ we obtain a contradiction to our assumption that $z_N \notin \mathcal{C}(N, w^{N,z})$. Therefore Claim 2 is shown. By Claim 1 and (4.7) or (4.8), respectively, $w^{N,z}(\{1\}) + w^{N,z}(\{2\}) < w^{N,z}(N)$. Hence $(N, w^{N,z})$ is strategically equivalent to (N, v). Thus the proof is completed by applying RCP and COV. **q.e.d.**

Lemma 4.3 *If σ satisfies* NE, BOUND, COV, AN, RGP, *and* RCP, *then it is a subsolution of the positive prekernel.*

Proof: By Lemma 4.2, σ is a subsolution of $\mathcal{C}_+$ and, hence of $\mathcal{K}_+^*$, when restricted to Γ^2. By Theorem 2.2, $\mathcal{K}_+^*$ satisfies CRGP. Therefore Remark 2.10 applied to $\sigma^1 = \sigma$ and $\sigma^2 = \mathcal{K}_+^*$ completes the proof. **q.e.d.**

Proof of Theorem 4.1: Let (N, v) be a 2-person game and assume that $N = \{1, 2\}$. If the core of (N, v) is empty or single valued, then Lemma 4.2 finishes the proof. Hence, by COV, we may assume that (N, v) is given by $v(\{1\}) = v(\{2\}) = v(\emptyset) = 0$ and $v(N) = 1$. Assume that $\sigma(N, v)$ does not coincide with the prenucleolus. It remains to show that $\sigma(N, v) \supseteq \text{rint } \mathcal{C}_+(N, v)$. By AN and PO there is $x^0 \in \sigma(N, v)$ satisfying

$$0 \leq x_1^0 < x_2^0 = 1 - x_1^0 \leq 1 \tag{4.9}$$

It suffices to show that

(1) there is a convergent sequence $(\bar{x}^k)_{k \in \mathbb{N}}$ satisfying $\bar{x}^k \in \sigma(N, v)$ $\forall k \in \mathbb{N}$ with limit $(0, 1)$ and

(2) $\sigma(N, v)$ is convex.

Indeed, if $x \in \text{rint } \mathcal{C}(N, v)$, then choose $k \in \mathbb{N}$ such that $\bar{x}_1^k \leq \min\{x_1, x_2\}$ and notice that $(\bar{x}_2^k, \bar{x}_1^k) \in \sigma(N, v)$ is true by AN. Hence x is a convex combination of these two vectors.

ad 1: We proceed similarly as in the proof of Lemma 4.2 using $\widetilde{N} = \{1, 2, 3\}, X^{\leq}, X^{\geq}$ as defined in (2) but now defining the game $(\widetilde{N}, w)$ by

$$w(S) = \begin{cases} 0 & , \text{ if } S = \emptyset, \widetilde{N}, \\ -1 & , \text{ otherwise.} \end{cases}$$

Moreover, we put

$$\widetilde{X}^{\leq} = X^{\leq} \cap \mathcal{C}(\widetilde{N}, w), \widetilde{X}^{\geq} = X^{\geq} \cap \mathcal{C}(\widetilde{N}, w)$$

and define $g^{\leq} : \widetilde{X}^{\leq} \longrightarrow \mathbb{R}^{\widetilde{N}}$ by $g^{\leq}(x) = (-\alpha - x_3, \min\{\alpha, x_3\}, \max\{\alpha, x_3\})$, where

$$\alpha = \frac{(1 - x_1)(2 - x_3)}{2 + x_2} - 1 = x_1 \frac{x_3 - 1}{2 + x_2} \ \forall x \in \widetilde{X}^{\leq},$$

and $g^{\geq} : \widetilde{X}^{\geq} \longrightarrow \mathbb{R}^{\widetilde{N}}$ by $g^{\geq}(y) = (\min\{\beta, y_1\}, \max\{\beta, y_1\}, -\beta - y_1)$, where

$$\beta = \frac{(1 + y_1)(2 + y_1)}{2 - y_2} - 1 - y_1 = -y_3 \frac{1 + y_1}{2 - y_2} \ \forall y \in \widetilde{X}^{\geq}.$$

Note that the denominators are positive, because x and y are assumed to be elements of the core. (In fact, $\mathcal{C}(\widetilde{N}, w)$ is the convex hull of all permutations of $(-1, 0, 1)$.)

For any $x \in \widetilde{X}^{\leq}$ and for any $y \in \widetilde{X}^{\geq}$ we observe that $(\{1, 2\}, w^{\{1,2\},x})$ and $(\{2, 3\}, w^{\{2,3\},y})$, respectively, are isomorphic (by interchanging players 2 and 3 and by interchanging 1 and 2, respectively) to

$$\left(\{1, 3\}, \left(w^{\{1,3\},x} + (x_2 + 1, x_2 + 1)\right) \frac{2 - x_3}{2 + x_2} - (1, 1)\right)$$

and to

$$\left(\{1, 3\}, \left(w^{\{1,3\},y} + (1, 1)\right) \frac{2 + y_1}{2 - y_2} - (1 + y_1, 1 + y_1)\right),$$

respectively. Thus, by AN, RGP, RCP, and COV of the core, we conclude (see the proof of Lemma 4.2) that

$$g^{\leq}(x) \in \mathcal{C}(\widetilde{N}, w)\ \forall x \in \widetilde{X}^{\leq} \text{ and } g^{\geq}(y) \in \mathcal{C}(\widetilde{N}, w)\ \forall y \in \widetilde{X}^{\geq}.$$

The proof of the following properties is straightforward and skipped:

$$g^{\leq}(x) \in \widetilde{X}^{\geq},\ g_1^{\leq}(x) \leq x_1, \text{ and } g_3^{\leq}(x) \geq x_3\ \forall x \in \widetilde{X}^{\leq} \tag{4.10}$$

$$\forall x \in \widetilde{X}^{\leq} : g^{\leq}(x) = x \Leftrightarrow g_2^{\leq}(x) = 0 \Leftrightarrow x = 0 \text{ or } x = (-1, 0, 1) \tag{4.11}$$

$$g^{\geq}(y) \in \widetilde{X}^{\leq},\ g_1^{\geq}(y) \leq y_1, \text{ and } g_3^{\geq}(y) \geq y_3\ \forall y \in \widetilde{X}^{\geq} \tag{4.12}$$

$$\forall y \in \widetilde{X}^{\geq} : g^{\geq}(y) = y \Leftrightarrow g_2^{\geq}(y) = 0 \Leftrightarrow y = 0 \text{ or } y = (-1, 0, 1) \tag{4.13}$$

Therefore the composition $g = g^{\geq} \circ g^{\leq} : \widetilde{X}^{\leq} \to \widetilde{X}^{\leq}$ is well-defined and satisfies "monotonicity", i.e. it satisfies $g_1(x) \leq x_1$ and $g_3(x) \geq x_3$ for all $x \in X^{\leq}$. Hence for any $x \in \widetilde{X}^{\leq} \setminus \{0\}$ the sequence $(g^k(x))_{k \in \mathbb{N}}$ converges to $(-1, 0, 1)$.

In order to construct the sequence $(\bar{x}^k)_{k \in \mathbb{N}}$ with the desired properties we first show that $\widetilde{X}^{\leq}$ contains a nonzero member x^1 of $\sigma(\widetilde{N}, w)$. For this purpose choose $z \in \sigma(\widetilde{N}, w) \cap \left(\widetilde{X}^{\leq} \cup \widetilde{X}^{\geq}\right)$ which is possible by NE, AN, and Lemma 4.3. If $z \neq 0$, then put

$$x^1 = \begin{cases} z & , \text{ if } z \in \widetilde{X}^{\leq}, \\ g^{\geq}(z) & , \text{ otherwise,} \end{cases}$$

and observe that x^1 belongs to $\sigma(\widetilde{N}, w)$ by AN, RGP, RCP, and COV.

If $z = 0$, then observe that the reduced game $(N, w^{N,z})$ is strategically equivalent to (N, v), thus by NE, RGP, RCP, COV, and the assumption that $\sigma(N, v) \neq \{\nu(N, v)\}$, there is some $y \in \sigma(\widetilde{N}, w) \setminus \{0\}$. By AN and Lemma 4.2 we may assume that y belongs to $\widetilde{X}^{\leq} \cup \widetilde{X}^{\geq}$, thus we define $x^1 = y$ in the first and $x^1 = g^{\geq}(y)$ in the latter case.

Now the proof of this part may be finished by defining the sequence $(\bar{x}^k)_{k \in \mathbb{N}}$ as follows. Put $x^k = g^{k-1}(x^1)\ \forall k \in \mathbb{N}$ and define

$$\bar{x}^k = \left(\frac{x_1^k + 1}{2 - x_3^k}, \frac{x_2^k + 1}{2 - x_3^k}\right)\ \forall k \in \mathbb{N}.$$

Then the sequence $(\bar{x}^k)_{k \in \mathbb{N}}$ converges to $(0, 1)$, because $(x^k)_{k \in \mathbb{N}}$ converges to $(-1, 0, 1)$. It remains to show that $\bar{x}^k \in \sigma(N, v)$ for any natural number k. This assertion is implied by COV, RGP, and the observation that $v = \left(w^{N,x^k} + (1, 1)\right) \frac{1}{2 - x_3^k}$.

ad 2: In view of AN it suffices to show that $(\alpha, 1-\alpha)$ belongs to $\sigma(N, v)$ for any $0 < \alpha \leq \frac{1}{2}$. By (1), which was just proved, there exists $(a, 1 - a) \in \sigma(N, v)$ satisfying $0 \leq a \leq \alpha$. Additionally, we shall assume that $\frac{1-a}{2-a} \geq a$ (which is true, e.g., if $a \leq 1/4$). For any β such that $\frac{1-a}{2-a} \leq \beta \leq 1$ define $(\widetilde{N}, w_\beta)$ by

$$w_\beta(S) = \begin{cases} 1, & \text{if } S = \widetilde{N}, \\ \beta, & \text{if } S = N, \\ 0, & \text{otherwise,} \end{cases}$$

and choose $x^0 \in \sigma(\widetilde{N}, w_\beta)$. Then $x^0 \in \mathcal{K}^*_+(\widetilde{N}, w_\beta)$ by Lemma 4.3. It is well-known (see Sudhölter and Peleg (2000)) that the positive prekernel of a three-person game coincides with its positive core. Hence $x^0 \in \mathcal{C}(\widetilde{N}, w_\beta)$. We proceed by recursively applying RGP, RCP, and COV four times. In (2) and (4) AN is employed in addition. For a sketch of the four steps see Figure 4.1.

(1) Let (N, w^0) be the reduced game of $(\widetilde{N}, w_\beta)$ w.r.t. N and x^0. Then

$$w^0(\{1\}) = w^0(\{2\}) = 0,\ w^0(N) = 1 - x^0_3,$$

thus $x^1 = (a(1-x^0_3), (1-a)(1-x^0_3), x^0_3) \in \sigma(\widetilde{N}, w_\beta)$. (If $\beta = 1$, then $x^0_3 = 0$ and $w^0 = 0$.)

(2) Let $(\{2,3\}, w^1)$ be the reduced game of $(\widetilde{N}, w_\beta)$ w.r.t. $\{2,3\}$ and x^1. The inequalities $\beta \geq \frac{1-a}{2-a} \geq a \geq x^1_1$ imply $w^1(\{2\}) = \max\{0, \beta - x^1_1\} = \beta - x^1_1$. Also, $w^1(\{3\}) = 0$ and $w^1(\{2,3\}) = 1 - x^1_1$. Thus

$$x^2 = \big(a(1-x^0_3), (1-\beta)a + \beta - x^1_1, (1-\beta)(1-a)\big) \in \sigma(\widetilde{N}, w_\beta).$$

(3) Let (N, w^2) be the reduced game of $(\widetilde{N}, w_\beta)$ w.r.t. N and x^2. Then

$$w^2(\{1\}) = w^2(\{2\}) = 0,\ w^2(N) = 1 - x^2_3,$$

thus $x^3 = (a(1-x^2_3), (1-a)(1-x^2_3), x^2_3) \in \sigma(\widetilde{N}, w_\beta)$. Note that x^3 does **not** depend on x^0, because $x^2_3 = (1-\beta)(1-a)$.

(4) Let $(\{1,3\}, w^3)$ be the reduced game of $(\widetilde{N}, w_\beta)$ w.r.t. $\{1,3\}$ and x^3. The inequality $\beta \geq x^3_2$ is true, because $\beta \geq \frac{1-a}{2-a}$. Hence, $w^3(\{1\}) = \max\{0, \beta - x^3_2\} = \beta - x^3_2$. Also, $w^3(\{3\}) = 0$ and $w^3(\{1,3\}) = 1 - x^3_2$. Thus

$$x^4 = \big((1-a)(1-\beta) + \beta - x^3_2, x^3_2, a(1-\beta)\big) \in \sigma(\widetilde{N}, w_\beta).$$

Now the proof may be finished by considering the reduced game (N, w^4) of $(\widetilde{N}, w_\beta)$ w.r.t. x^4. Indeed, we have

$$w^4(\{1\}) = w^4(\{2\}) = 0,\ w^4(N) = 1 - x^4_3,$$

thus

$$y^\beta = \frac{1}{1 - x^4_3}(x^4_1, x^4_2) \in \sigma(N, v).$$

A careful inspection of the construction immediately shows that

$$y^\beta_1 = \begin{cases} a, & \text{if } \beta = 1, \\ \frac{1}{2}, & \text{if } \beta = (1-a)/(2-a). \end{cases}$$

The continuity of the mapping $\beta \mapsto y^\beta$ shows that there exist some $\widetilde{\beta}$, $\frac{1-a}{2-a} \leq \widetilde{\beta} \leq 1$, such that $y^{\widetilde{\beta}}_1 = \alpha$. **q.e.d.**

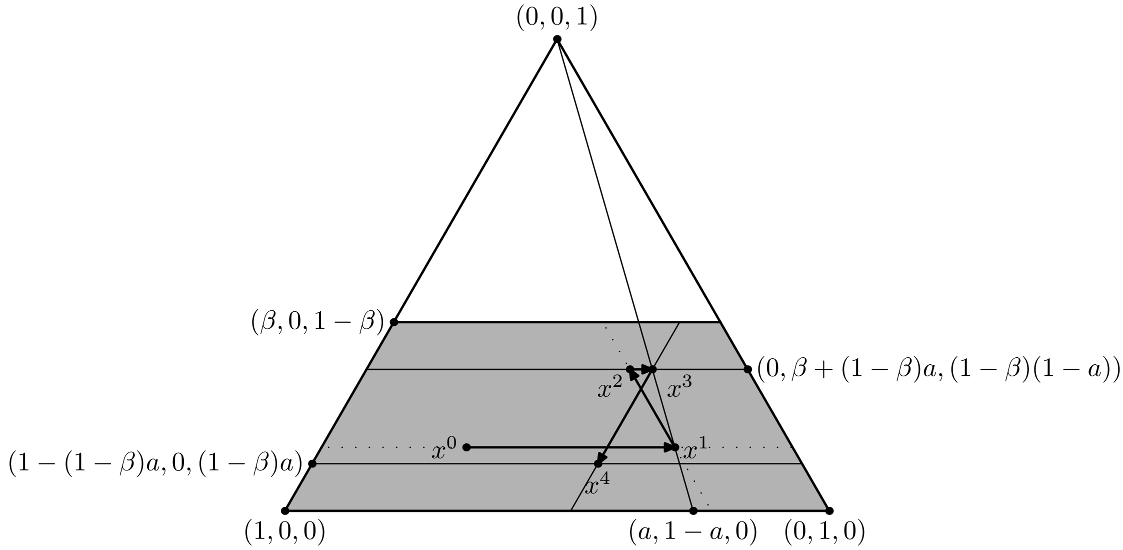

Figure 4.1: Sketch of the steps (1) - (4) in ad 2

4.2 Coincidence of Positive Excesses

Throughout this subsection we assume that the cardinality of U is infinite and that σ is a solution. If σ is a subsolution of the positive prekernel and if σ satisfies COV, RGP, and RCP, then it is shown that a coalition which has a positive excess at some member of the solution when applied to the game, has the same excess at **any** member of the solution of the game. The following simple lemma is useful.

Lemma 4.4 *Let (N, v) be a game, let $x \in \mathcal{K}_+^*(N, v)$, and let $k \in N$. If $\mu(x, v) > 0$ then there exist $S^k, S^{-k} \subseteq N$ such that*

$$e(S^k, x, v) = e(S^{-k}, x, v) = \mu(x, v),\ k \in S^k,\ \textit{and } k \notin S^{-k}.$$

Proof: It suffices to show that there exists $\ell \in N \setminus \{k\}$ such that

$$s_{k\ell}(x, v) = s_{\ell k}(x, v) = \mu(x, v). \tag{4.14}$$

Let $S \subseteq N$ be a coalition attaining $\mu(x, v)$. As $\mu(x, v) > 0$, $\emptyset \neq S \neq N$. If $k \in S$, then (4.14) holds for every $\ell \in N \setminus S$. If $k \notin S$, then (4.14) holds for every $\ell \in S$. **q.e.d.**

Lemma 4.5 *Let σ be a subsolution of the positive prekernel, let (N, v) be a game, let $x, y \in \sigma(N, v)$, and let $\bar{S} \subseteq N$. If σ satisfies* NE, COV, RGP, *and* RCP, *then*

$$e(\bar{S}, x, v) > 0 \Rightarrow e(\bar{S}, y, v) = e(\bar{S}, x, v).$$

Proof: Assume without loss of generality that $1 \in \bar{S}$ and that $N = \{1, \ldots, n\}$. Choose $* \in U \setminus N$, let us say $* = n + 1$, and put $\widetilde{N} = N \cup \{*\}$.

Choose $\alpha > (n + 1) \max_{P,Q \subseteq N}(v(P) - v(Q))$ and define the game $(\widetilde{N}, w)$ by

$$w(S)=\begin{cases} v(S\setminus\{*\}) & , \text{ if } S\subseteq N\setminus\{1\} \text{ or } 1,*\in S,\\ v(\bar S) & , \text{ if } S=\bar S,\\ v(N) & , \text{ if } S=N,\\ 0 & , \text{ if } S=\{*\},\\ v(S\setminus\{*\})-\alpha & , \text{ otherwise.}\end{cases}$$

By NE there exists $z\in\sigma(w)$. It is the aim to show the following assertions:

(1) $(N,w^{N,z})$ is strategically equivalent to (N,v).

(2) $s_{*1}(z,w)>0$ and $s_{*1}(z,w)$ is only attained by the coalition $\{*\}$.

(3) $s_{1*}(z,w)$ is only attained by $\bar S$.

Then we shall use RGP and RCP to "insert" x or y and immediately deduce the desired coincidence of the excesses. In order to prove (1) we shall show the following

Claim: $z_*\le 0$

Assume, on the contrary, $z_*>0$.

STEP 1: It is first proved that

$$\exists \bar k\in N\setminus\{1\}: z_{\bar k}<-\max_{P,Q\subseteq N}(v(P)-v(Q)). \tag{4.15}$$

Indeed, the assumption yields

$$s_{1*}(z,w)\ge e(N,z,w)=v(N)-z(N)=z_*>0.$$

Hence $s_{*1}(z,w)>0$ is implied, because $z\in\mathcal{K}^*_+(\widetilde N,w)$. Choose $T\subseteq N\setminus\{1\}$ such that $e(T\cup\{*\},z,w)>0$. Then $T\ne\emptyset$, thus

$$\begin{aligned} 0 &< v(T)-\alpha-z(T)-z_* &\le& \; v(T)-\alpha-z(T)\\ &= (v(T)-v(\emptyset))-\alpha-z(T) &<& \; -n\max_{P,Q\subseteq N}(v(P)-v(Q))-z(T).\end{aligned}$$

Hence there is $\bar k\in T\subseteq N\setminus\{1\}$ with $z_{\bar k}<-\max_{P,Q\subseteq N}(v(P)-v(Q))$.

STEP 2: Let $S\subseteq N\cup\{*\}$ attain $\mu(z,w)$. By Step 1 there exists $i\in N\setminus\{1\}$ such that $z_i<-\max_{P,Q\subseteq N}(v(P)-v(Q))$. The following assertions are shown:

(1) $\bar S$ attains $\mu(z,w)$.

(2) If $S\ne\bar S$, then $S\supseteq(N\cup\{*\})\setminus\bar S$.

(3) Player i does not belong to $\bar S$.

In order to show (1) the following claim is shown:

$$S\ne\bar S\Rightarrow i\in S. \tag{4.16}$$

Indeed, first note that $\{*\}$ does not attain $\mu(z,w)$ because $\mu(z,w)$ is positive. Let $T \subseteq (N \cup \{*\}) \setminus \{i\}$ such that $T \neq \bar{S}$ and $T \neq \{*\}$. If $\bar{S} \neq T \cup \{i\} \neq N$, then

$$e(T \cup \{i\}, z, w) - e(T, z, w) = v\left((T \cup \{i\}) \setminus \{*\}\right) - v(T \setminus \{*\}) - z_i > 0.$$

Also, if $T \cup \{i\} \in \{N, \bar{S}\}$, then

$$e(T \cup \{i\}, z, w) - e(T, z, w) = v(T \cup \{i\}) - v(T) - z_i + \alpha > 0.$$

Hence (4.16) is shown. By Lemma 4.4 there exists $S^{-i} \subseteq (N \cup \{*\}) \setminus \{i\}$ attaining $\mu(z,w)$. By (4.16), $S^{-i} = \bar{S}$, thus (1) is shown.

Now this step may be completed. In order to show (2) we assume, on the contrary, that there is some coalition S attaining maximal excess and violating (2). Then $i \in S$ by (4.16), because $S \neq \bar{S}$. Choose $j \in (N \cup \{*\}) \setminus (\bar{S} \cup S)$. Then

$$\mu(z,w) = e(S, z, w) = s_{ij}(z,w),$$

hence $s_{ji}(z,w) = \mu(z,w)$ which is impossible by (4.16). Moreover, (3) is valid because of the existence of S^{-i}.

STEP 3: By the preceding steps there is some coalition S attaining maximal excess such that $1 \notin S \ni *$. Moreover, $\{*\} \neq S$, hence

$$0 \geq e(S \cup \{1\}, z, w) - e(S, z, w) = v((S \cup \{1\}) \setminus \{*\}) - v(S \setminus \{*\}) + \alpha - z_1.$$

Thus $z_1 > n \max_{P,Q \subseteq N}(v(P) - v(Q))$. By (1) of the preceding step (the maximality of $e(\bar{S}, z, w)$) we obtain

$$0 < e(\bar{S}, z, w) = v(\bar{S}) - z_1 - z(\bar{S} \setminus \{1\}) < -((n-1) \max_{P,Q \subseteq N}(v(P) - v(Q))) - z(\bar{S} \setminus \{1\})$$

and, thus, there is $i \in \bar{S} \setminus \{1\}$ with $z_i < -\max_{P,Q \subseteq N}(v(P) - v(Q))$. The last observation contradicts (3) of the preceding step. Thus our claim is shown.

Now the proof may be finished. Let $u = w^{N,z}$ be the reduced game. By reasonableness of the positive prekernel, $z^* \geq -\alpha$. Hence, $z_* \leq 0$ implies that $u = v + (-z_*, \underbrace{0, \ldots, 0}_{n-1})$. By RCP and COV

$$\begin{aligned} z^1 &= (-z_*, 0, \ldots, 0, z_*) + (x, 0) \in \sigma(N \cup \{*\}, w) \quad \text{and} \\ z^2 &= (-z_*, 0, \ldots, 0, z_*) + (y, 0) \in \sigma(N \cup \{*\}, w). \end{aligned}$$

By REAS of the positive prekernel, $z_i^t \geq -\max_{P,Q \subseteq N}(v(P) - v(Q))$ for all $i \in N \setminus \{1\}$ and $t = 1, 2$. Therefore $s_{*1}(z^t, w)$ must be attained by $\{*\}$, thus this surplus is nonnegative. Moreover, $s_{1*}(z^t, w)$ can only be attained by N, if $z_* = 0$. Therefore $s_{1*}(z^1, w)$ cannot be attained by N, because $e(\bar{S}, x, v)$ is positive. Hence, $s_{1*}(z^t, w)$ must be attained by $\bar{S}$ for $t = 1, 2$.

Therefore $e(\bar{S}, z^1, w) = e(\bar{S}, x, v) + z_* = e(\{*\}, z^1, w) = -z_*$ and, similarly, $e(\bar{S}, z^2, w) = -z^*$. Thus,

$$z_* = -\frac{e(\bar{S}, x, v)}{2} \text{ and } z_* = -\frac{e(\bar{S}, y, v)}{2}.$$

Hence we conclude that $e(\bar{S}, x, v) = e(\bar{S}, y, v)$ is satisfied. **q.e.d.**

Corollary 4.6 *If the prenucleolus is a subsolution of* σ*, and* σ *satisfies* BOUND*,* COV*,* AN*,* RGP*, and* RCP*, then* σ *is a subsolution of the positive core.*

Proof: By Lemma 4.4 the solution σ is a subsolution of the positive prekernel, hence Lemma 4.5 can be applied. **q.e.d.**

4.3 The Prenucleolus and the General n-Person Case

During this subsection we assume that the cardinality of U is infinite and that σ is a solution. It is the aim to show that the prenucleolus, the relative interior of the positive core, and the positive core are the unique solutions that satisfy BOUND, COV, AN, RGP, RCP, and contain the prenucleolus as a subsolution. The main result of this section is the following theorem.

Theorem 4.7 *The prenucleolus is a subsolution of* σ *and* σ *satisfies* BOUND*,* COV*,* AN*,* RGP*, and* RCP*, if and only if* σ *coincides*

(1) with the prenucleolus **or**

(2) with the relative interior of the positive core **or**

(3) with the positive core.

Proof: The three solutions satisfy the desired properties. Let, now, σ be a solution that satisfies the axioms and contains the prenucleolus. By Theorem 4.1 the following three cases (1), (2), or (3) may occur.

(1) The solution σ coincides with the prenucleolus on the set of all 2-person games. By (2.1), σ must be a subsolution of the prekernel, thus σ satisfies ETP. By Theorem 4.7 of Orshan and Sudhölter (2003), σ is the prenucleolus.

(2) The solution σ coincides with the positive core on the set of all 2-person games. By Corollary 4.6, σ is a subsolution of the positive core. Let (N,v) be a game. It remains to show $\mathcal{C}_+(N,v) \subseteq \sigma(N,v)$. Without loss of generality we may assume that

$$N = \{1,\ldots,n\}, N^* := \{1^*,\ldots,n^*\} = \{n+1,\ldots,2n\} \subseteq U.$$

By COV we may assume that $\nu(N,v) = 0 \in \mathbb{R}^N$. If $\mathcal{C}_+(N,v) = \mathcal{V}(N,v)$, then NE completes the proof. Hence, let $x \in \mathcal{C}_+(N,v) \setminus \mathcal{V}(N,v)$ and put $\widetilde{N} = N \cup N^*$. It is no loss of generality to assume that

$$x_1 \geq \cdots \geq x_n. \tag{4.17}$$

We are going to define a game $(\widetilde{N},w)$ which allows to show that $\sigma(\widetilde{N},w)$ contains a member y such that

(a) the reduced game $(N, w^{N,y})$ coincides with (N,v) and

(b) the restriction y_N coincides with x.

Indeed the proof is finished by RGP as soon as $(\widetilde{N}, w)$ and y are constructed.

Let $k \in \mathbb{N}$ and $e_i \in \mathbb{R},\ i = 1, \ldots, k$, be defined by

$$\{e_i \mid i = 1, \ldots, k\} = \{v(S) \mid v(S) < 0\},\ e_1 > \cdots > e_k,$$

i.e., $e_1, \ldots, e_k$ are the different negative excesses of (N, v) at the prenucleolus in the strictly decreasing order. Moreover, define $\mathcal{S}^i,\ \mathcal{T}^i,\ \beta_i$ for all $i = 1, \ldots, k$ by the requirements

$$\mathcal{S}^i = \{S \subseteq N \mid v(S) = e_i\}, \tag{4.18}$$

$$\mathcal{T}^i = \{T \subseteq \bigcup_{j=1}^{i} \bigcup_{S \in \mathcal{S}^j} S\}, \tag{4.19}$$

$$\beta_i = \max_{S \in \mathcal{T}^i} x(S). \tag{4.20}$$

Note that $\mathcal{S}^i$ is the *i-th negative level set*. By construction

$$e_k < \cdots < e_1 < 0 = x(\emptyset) \leq \beta_1 \leq \cdots \leq \beta_k.$$

Put $\mathcal{T}^0 = \{\emptyset\}$ and $\beta_{k+1} = \max_{S \subseteq N} v(S) + \max_{T \subseteq N} x(T)$. We are now ready to define w. For $S, T \subseteq N$, let $w(S \cup T^*)$ be given by the following equation:

$$w(S \cup T^*) = \begin{cases} v(S) & , \text{ if } v(S) \geq 0 \text{ and } T = S, \\ e_i - \beta_i & , \text{ if } \left(\begin{array}{l} (S \in \mathcal{S}^i,\ T \in \mathcal{T}^i) \text{ or} \\ (S = \emptyset, T \in \mathcal{T}^i \setminus \mathcal{T}^{i-1}) \end{array} \right) \text{for some } 1 \leq i \leq k, \\ e_k - \beta_{k+1} & , \text{ otherwise.} \end{cases}$$

We are going to show that

$$\nu(\widetilde{N}, w) = 0 \in \mathbb{R}^{\widetilde{N}} \tag{4.21}$$

and

$$y^l := (\underbrace{x_1, \ldots, x_\ell}_{\ell}, \underbrace{0, \ldots, 0}_{n-\ell}, \underbrace{-x_1, \ldots, -x_\ell}_{\ell}, \underbrace{0, \ldots, 0}_{n-\ell}) \in \sigma(\widetilde{N}, w)\ \forall \ell = 1, \ldots, n. \tag{4.22}$$

(a) In order to show (4.21), it suffices to show in view of Theorem 3.3 that, for any $R \subseteq \widetilde{N}$,

$$\mathcal{D}(w(R), 0, w) = \{Q \subseteq \widetilde{N} \mid w(Q) \geq w(R)\}$$

is balanced (over $\widetilde{N}$). If $w(R) \geq 0$, i.e., $R = S \cup S^*$ for some S with $v(S) \geq 0$, then the required balancedness follows immediately from the fact that 0 is the prenucleolus of (N, v) and the observation that

$$\{Q \subseteq \widetilde{N} \mid w(Q) \geq w(R)\} = \{T \cup T^* \mid T \subseteq N,\ v(T) \geq v(S)\}.$$

Thus it remains to show that

$$\widehat{\mathcal{D}}^i := \{Q \subseteq \widetilde{N} \mid w(Q) \geq e_i - \beta_i\}$$

is balanced for any $i = 1, \dots, k$. By Remark 3.8 it suffices to show that $\widehat{\mathcal{D}}^i$ contains a balanced subset $\mathcal{D}^i$ that spans $\widehat{\mathcal{D}}^i$. In order to define $\mathcal{D}^i$ note that

$$S \cup \emptyset \in \widehat{\mathcal{D}}^i \ \forall S \in \bigcup_{j=1}^{i} \mathcal{S}^j \text{ and } \emptyset \cup \{t^*\} \in \widehat{\mathcal{D}}^i \ \forall \{t\} \in \mathcal{T}^i$$

for $i = 1, \dots k$. Hence we put

$$\mathcal{D}^i := \{S \cup S^* \mid v(S) \geq 0\} \cup \bigcup_{j=1}^{i} \mathcal{S}^j \cup \{\{t^*\} \mid \{t\} \in \mathcal{T}^i\} \ \forall i = 1, \dots, k.$$

By Theorem 3.3, $\{S \subseteq N \mid v(S) \geq 0\} \cup \bigcup_{j=1}^{i} \mathcal{S}^j$ is balanced over N, hence $\{S^* \subseteq N^* \mid v(S) \geq 0\} \cup \{S^* \subseteq N^* \mid S \in \bigcup_{j=1}^{i} \mathcal{S}^j\}$ is balanced over N^*. The observation that $\{\{t^*\} \mid \{t\} \in \mathcal{T}^i\} = \{\{t^*\} \mid t \in \bigcup_{j=1}^{i} \bigcup_{S \in \mathcal{S}^j} S\}$ shows that $\mathcal{D}^i$ is balanced. Also, it is straightforward to show that $\mathcal{D}^i$ spans $\widehat{\mathcal{D}}^i$ for every $i = 1, \dots, k$.

(b) Assertion (4.22) is shown recursively. Denote the prenucleolus $\nu(\widetilde{N}, w) = 0$ by y^0. By the definition of y^ℓ we conclude for every $T \subseteq N$ that $y^\ell(T^*) = -x(T \cap \{1, \dots, \ell\}) \geq -\beta_{k+1}$. Also, by (4.20), $y^\ell(T^*) \geq -\beta_i$ for every $T \in \mathcal{T}^i$ and every $i = 1, \dots, k$. Also, by (4.17), we have $y^\ell(S) \geq \min\{x(S), 0\}$ for all $S \subseteq N$. We conclude for $S, T \subseteq N$ that

$$e(S \cup T^*, y^\ell, w) \leq \max\{e(S, x, v), e_i\}, \text{ if } S \in \mathcal{S}^i \text{ for some } 1 \leq i \leq k. \tag{4.23}$$

Also,

$$e(S \cup T^*, y^\ell, w) < 0, \text{ if } w(S \cup T^*) = e_k - \beta_{k+1} \tag{4.24}$$

We conclude that

$$\left(e(S \cup T^*, y^\ell, w)\right)_+ = \left(w(S \cup T^*)\right)_+ .$$

Let $w^{\ell-1} := w^{\{\ell, \ell^*\}, y^{\ell-1}} \ \forall \ell = 1, \dots, n$. denote the coalition function of the "bilateral" reduced game w.r.t. $\{\ell, \ell^*\}$ and $y^{\ell-1}$. By (4.23) and (4.24),

$$(x^\ell, -x^\ell) \in \mathcal{C}(\{\ell, \ell^*\}, w^{\ell-1}) \ \forall \ell = 1, \dots, n. \tag{4.25}$$

The facts that the solution σ coincides with the core on 2-person games and that it contains the prenucleolus show that RCP, when applied recursively to $\ell = 1, \dots, n$, yields the following "chain" of arguments:

$$y^{\ell-1} \in \sigma(\widetilde{N}, w) \Rightarrow y^\ell \in \sigma(\widetilde{N}, w)$$

Now the proof of this case can be finished. Put $y = y^n$ and observe that $w^{N,y} = v$ by the definition of w. Thus RGP implies $y_N \in \sigma(N, v)$. Clearly $y_N = x$ is satisfied.

(3) The solution σ coincides with the relative interior of the positive core on the set of all 2-person games.

By Corollary 4.6, σ is a subsolution of the positive core. By copying the relevant part of the proof of the second case, it can be shown that the relative interior of the positive core is a subsolution of σ. Indeed, a careful inspection of the definition of the game $(\widetilde{N}, w)$, especially of (4.23) and (4.24), shows that (4.25) may be replaced by

$$(x^\ell, -x^\ell) \in \text{int } \mathcal{C}(\{\ell, \ell^*\}, w^{\ell-1}) \ \forall \ell = 1, \ldots, n,$$

whenever $x \in \text{rint } \mathcal{C}_+(N, v)$ is assumed. ("int $\mathcal{C}$" means the "interior of $\mathcal{C}$ relative to $\mathcal{I}^*$", i.e., for every game (M, u), int $\mathcal{C}(M, u) = \{x \in \mathcal{C}(M, u) \mid e(S, x, u) < 0 \ \forall \emptyset \neq S \subsetneqq M\}$.)

Hence it remains to show that $\sigma(N, v) \subseteq \text{rint } \mathcal{C}_+(N, v)$ for any game (N, v) satisfying $\mathcal{C}_+(N, v) \setminus \text{rint } \mathcal{C}_+(N, v) \neq \emptyset$. Take $x \in \mathcal{C}_+(N, v) \setminus \text{rint } \mathcal{C}_+(N, v)$ and let $\bar{S} \subseteq N$ be a coalition satisfying $0 = e(\bar{S}, x, v) > e(\bar{S}, \nu, v)$, where $\nu = \nu(N, v)$ denotes the prenucleolus. Without loss of generality we may assume $N = \{1, \ldots, n\}$, $1 \in \bar{S}$ and $* = n + 1 \in U$. Moreover, let $(N \cup \{*\}, w)$ defined by

$$w(S) = \begin{cases} v(S \setminus \{*\}) & , \text{ if } S \subseteq N \setminus \{1\} \text{ or } 1, * \in S, \\ v(\bar{S}) & , \text{ if } S = \bar{S}, \\ e(\bar{S}, \nu, v) & , \text{ if } S = \{*\}, \\ e(S \setminus \{*\}, \nu, v) - \alpha & , \text{ otherwise,} \end{cases}$$

where $\alpha := \mu(\nu, v) - \min_{S \subseteq N} e(S, \nu, v) - \min_{S \subseteq N} e(S, x, v)$.

Claim: $z := \nu(N \cup \{*\}, w) = (\nu, 0)$

For $\beta > e(\bar{S}, \nu, v)$,

$$\mathcal{D}(\beta, z, w) = \{S \in \mathcal{D}(\beta, \nu, v) \mid 1 \notin S\} \ \cup \{S \cup \{*\} \mid 1 \in S \in \mathcal{D}(\beta, \nu, v)\}.$$

For $-\alpha < \beta \leq e(\bar{S}, \nu, v)$,

$$\mathcal{D}(\beta, z, w) = \{S \in \mathcal{D}(\beta, \nu, v) \mid 1 \notin S\} \cup \{\bar{S}, \{*\}\} \cup \{S \cup \{*\} \mid 1 \in S \in \mathcal{D}(\beta, \nu, v)\}.$$

For $\beta \leq -\alpha$, $\mathcal{D}(\beta, z, w) = 2^{N \cup \{*\}}$. Hence our claim follows from Theorem 3.3.

The reduced game $(N, w^{N,z})$ coincides with (N, v). Assuming, on the contrary, that $x \in \sigma(N, v)$, yields $\widetilde{z} := (x, 0) \in \sigma(N \cup \{*\}, w)$ by RCP. Applying RGP yields

$$(x_1, 0) \in \sigma(\{1, *\}, w^{\{1,*\},\widetilde{z}}).$$

Moreover, this reduced game can easily be computed as

$$w^{\{1,*\},\widetilde{z}}(\{1\}) = e(\bar{S}, x, v) + x_1, w^{\{1,*\},\widetilde{z}}(\{*\}) = e(\bar{S}, \nu, v) < 0, w^{\{1,*\},\widetilde{z}}(\{1, *\}) = x_1.$$

Therefore the interior of the core of this reduced game is nonempty. The fact that $(x_1, 0)$ is not a member of this interior shows the desired contradiction. **q.e.d.**

4.4 Characterizations

As before we assume that $|U|$ is infinite and that σ is a solution. It is the aim to characterize the positive core by simple axioms using Theorems 4.7 and 4.1. Of course, we do not want to employ the property that the prenucleolus is a subsolution. Hence we need an additional axiom, which, together with the other properties, guarantee that the prenucleolus is a subsolution of σ. In what follows two proposals are presented.

Theorem 4.8 *The solution σ satisfies* NE, BOUND, COV, AN, RGP, RCP, *and* CON *(see Remark 3.7), if and only if σ coincides*

(1) with the prenucleolus **or**

(2) with the relative interior of the positive core **or**

(3) with the positive core.

Proof: The three mentioned solutions satisfy the required properties. Let σ satisfy the axioms. By Theorem 4.7 it suffices to prove that the prenucleolus is a subsolution of σ. Let (N, v) be a game. By COV we may assume without loss of generality that $\nu(N, v) = 0 \in \mathbb{R}^N$. According to Sobolev (1975) there exists a game $(\widetilde{N}, w)$ satisfying the following properties:

(1) $\nu(\widetilde{N}, w) = 0 \in \mathbb{R}^{\widetilde{N}}$.

(2) $N \subseteq \widetilde{N}$ and $w^{N,0} = v$.

(3) $(\widetilde{N}, w)$ is *transitive*[2].

By our infinity assumption on $|U|$ we may assume that $\widetilde{N} \subseteq U$. By NE, there exists $y \in \sigma(\widetilde{N}, w)$. By Remark 2.9, σ satisfies PO, thus $y(\widetilde{N}) = 0$. By AN, $\pi(y) \in \sigma(\widetilde{N}, w)$ for every $\pi \in \mathrm{SYM}(\widetilde{N}, w)$. Let

$$z = \frac{1}{|\mathrm{SYM}(\widetilde{N}, w)|} \sum_{\pi \in \mathrm{SYM}(\widetilde{N}, w)} \pi(y).$$

By CON, $z \in \sigma(\widetilde{N}, w)$. By transitivity, $z = 0 = \nu(\widetilde{N}, w)$. By RGP of $\boldsymbol{\nu}$, $z_N = \nu(N, v)$. By RGP of σ, $z_N \in \sigma(N, v)$. **q.e.d.**

Theorem 4.9 *The solution σ satisfies* NE, BOUND, COV, RGP, RCP, *and* ND, *if and only if σ coincides*

(1) with the prenucleolus **or**

(2) with the relative interior of the positive core **or**

(3) with the positive core.

Proof: The three mentioned solutions satisfy the required properties. Let σ satisfy the axioms. By Theorems 4.7 it suffices to prove (a) that the prenucleolus is a subsolution of σ and (b) that σ satisfies AN. Let (N, v) be a game. By COV we may assume without loss of generality that $\nu(N, v) = 0 \in \mathbb{R}^N$.

[2]A game is *transitive*, if its symmetry group is transitive, i.e., if for any two players $i, j \in N$ there is a symmetry $\pi \in \mathrm{SYM}(N, v)$ of the game which maps i to j.

We first prove (a). Let $(\widetilde{N}, w)$ be defined as in the proof of Theorem 4.8. By NE and ND and transitivity of $(\widetilde{N}, w)$ there exists $z \in \sigma(\widetilde{N}, w)$ such that $z_i = z_j$ for all $i, j \in \widetilde{N}$. By Remark 2.9, σ satisfies PO, thus $z = \nu(\widetilde{N}, w)$. By RGP of $\boldsymbol{\nu}$, $z_N = \nu(N, v)$. By RGP of σ, $z_N \in \sigma(N, v)$.

It remains to show (b). Let $\bar{N} \subseteq U$ be such that there is a bijection $\pi : N \to \bar{N}$ It has to be proved that $\pi(\sigma(N, v)) \subseteq \sigma(\bar{N}, \pi v)$. By the infinity assumption on $|U|$ there is a further set of players of the same cardinality in U disjoint from both, N and $\bar{N}$, thus we may assume that N and $\bar{N}$ are disjoint. Let $x \in \sigma(N, v)$. Then $\pi(x) \in \sigma(\bar{N}, \pi v)$ remains to be shown. We are going to define a “replicated” game $(N \cup \bar{N}, u)$ such that the reduced games w.r.t. N and $\bar{N}$ and the nucleolus coincide with the games (N, v) and $(\bar{N}, \pi v)$ respectively. Put $\alpha = \min_{R,S \subseteq N} \min\{e(S, \nu(N, v), v), e(S, x, v) + x(R)\}$ and define, for all $S \subseteq N,\ T \subseteq \bar{N}$,

$$u(S \cup T) = \begin{cases} v(S) & , \text{ if } T = \pi(S), \\ \alpha & , \text{ otherwise.} \end{cases}$$

Let $y = 0 \in \mathbb{R}^{N \cup \bar{N}}$ and observe that the equation

$$\mathcal{D}(\beta, y, u) = \begin{cases} \{S \cup \pi(S) \mid S \in \mathcal{D}(\beta, \nu(N, v), v)\} & , \text{ if } \beta > \alpha, \\ 2^{N \cup \bar{N}} & , \text{ otherwise} \end{cases}$$

is valid, thus $y = \nu(N \cup \bar{N}, u)$ by Theorem 3.3. By (a), $y \in \sigma(N \cup \bar{N}, u)$. Observe that $u^{N,y} = v$ by construction, thus $z := (x, y_{\bar{N}}) \in \sigma(N \cup \bar{N}, u)$ by RCP. Again the construction of u yields $u^{\bar{N},z} = \pi v - \pi(x)$, thus $\pi x \in \sigma(\bar{N}, \pi v)$ by COV and RGP. **q.e.d.**

Remark 4.10 In view of Theorem 4.8 it is possible to characterize the positive core as the maximum solution satisfying NE, BOUND, COV, AN, CON, RGP, and RCP. Alternatively, by Theorem 4.9, the positive core is the maximum solution that satisfies NE, BOUND, COV, ND, RGP, and RCP.

In order to avoid to characterize the positive core to be a **maximum** solution that satisfies several axioms, we may employ weak unanimity for two-person games (WUTPG).

Corollary 4.11 *There is a unique solution that satisfies* WUTP*,* NE*,* BOUND*,* COV*,* RGP*,* RCP*, and* ND*, and it is the positive core.*

5 On the Independence of the Axioms

Six examples are presented which show that each of the axioms (1) NE, (2) BOUND, (3) COV, (4) RGP, (5) RCP, and (6) ND is logically independent of the remaining axioms in Theorem 4.9.

Let U be a set, let $t \in \mathbb{R}$, and let (N, v) be a game. Let $\sigma^i,\ i = 1, \dots, 5$, be defined by

$$\begin{aligned}
\sigma^1(N, v) &= \mathcal{C}(N, v), \\
\sigma^2(N, v) &= \mathcal{I}^*(N, v), \\
\sigma^3(N, v) &= \{x \in \mathcal{I}^*(N, v) \mid \max\{e(S, x, v), t\} = \max\{e(S, \nu(N, v), v), t\}\ \forall S \subseteq N\}, \\
\sigma^4(N, v) &= \{x \in \mathcal{I}^*(N, v) \mid \mu(x, v) = \mu(\nu(N, v), v)\}, \text{ and} \\
\sigma^5(N, v) &= \mathcal{K}^*_+(N, v).
\end{aligned}$$

Let $\preceq$ be a total order of U. For every finite set $N \subseteq U$ let $\preceq_N$ be the restriction of $\preceq$ to N and let $\leq^N_{lex}$ be the induced lexicographical order on $\mathbb{R}^N$. Then σ^6 is defined by

$$\sigma^6(N, v) = \{x \in \mathcal{C}_+(N, v) \mid x \leq^N_{lex} y\ \forall y \in \mathcal{C}_+(N, v)\}.$$

It is straightforward to check that $\sigma^i,\ i = 1, \dots, 6$, satisfies all properties except the i-th one. If $|U| \geq 4$ and $t \neq 0$, then none of the solutions coincides with $\mathcal{C}_+$ or $\mathcal{V}$ or rint $\mathcal{C}_+$.

Remark 5.1 The infinity assumption on $|U|$ in Theorems 4.7, 4.8, 4.9, and in Corollary 4.11 is crucial. Indeed, if $|U| < \infty$, then define for any game (N, v),

$$\sigma(N, v) = \begin{cases} \mathcal{C}_+(N, v) & , \text{ if } N \subsetneqq U, \\ \{x \in \mathcal{I}^*(N, v) \mid x_S \in \mathcal{C}_+(S, v^{S,x})\ \forall \emptyset \neq S \subsetneqq N\} & , \text{ if } N = U. \end{cases}$$

Then σ satisfies all properties that were employed in the mentioned results. Also, there are examples which show that $\mathcal{C}_+$ is a proper subsolution of σ when $|U| \geq 4$.

Remark 5.2 We conjecture that CON is **not** logically independent of the remaining axioms in Theorem 4.8. Also, we do not know whether the characterizations given in Remark 4.10 and Corollary 4.11 are axiomatizations, because we do not have examples which show that AN, CON, or ND are independent of the remaining axioms.

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