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Abstract

We define *ternary voting games (TVGs)*, a generalization of *simple voting games (SVGs)*. In a play of an SVG each voter has just two options: voting 'yes' or 'no'. In a TVG a third option is added: abstention. Every SVG can be regarded as a (somewhat degenerate) TVG; but the converse is false. We define appropriate generalizations of the Shapley–Shubik and Banzhaf indices for TVGs. We define also the *responsiveness* (or *degree of democratic participation*) of a TVG and determine, for each n, the most responsive TVGs with n voters. We show that these maximally responsive TVGs are more responsive than the corresponding SVGs.

1. Introduction

The mathematical notion of *simple voting game (SVG)* has been widely used to model decision rules in legislatures, committees and other decision-making bodies (see, for example, Shapley, 1962; Felsenthal and Machover, 1995; Felsenthal, Machover and Zwicker, 1995, and the lists of references therein).

An SVG consists of a non-empty set W of non-empty subsets of a finite set N, such that W is monotone: whenever $S \subseteq T \subseteq N$ and $S \in W$, then $T \in W$ as well. The members of N are called *voters*, all subsets of N are called *coalitions*, and those belonging to W are called *winning* coalitions. This set-up is taken to model a real-life decision rule in the following sense. N represents an assembly of members, making decisions by division: a resolution or bill is passed iff the set of members voting for it is a winning coalition. (We use the term *division* in the sense in which it is used in the British House of Commons: the collective act whereby all members of the House declare their attitude towards a given question; we refer to each individual's declaration in a division as a *vote*.)

This, like all mathematical modelling, involves a certain amount of over-simplification.

But it seems to us that the most unrealistic aspect of the SVG set-up is that it allows a voter only two responses in any division: voting 'yes' or 'no', and does not recognize abstention as a distinct third option. To be sure, there are real-life decision rules which exclude abstention as a *tertium quid*. For example, in the Council of Ministers of the European Union, abstention usually counts as a 'no', except when unanimity is required — in which case abstention counts as a 'yes'. Similarly, in many legislatures passage of certain privileged bills (such as those involving basic constitutional issues) requires support of a special majority of *all* members; so that abstention amounts to a 'no' vote. Again, some committees operate under a rule whereby a resolution is passed unless a certain number of members oppose it; here abstention counts as a 'yes' vote. But these are exceptions; in most real-life decision rules, abstention is a distinct third option. This is true, in particular, of the rule most commonly used in most decision-making bodies: the *simple majority* rule, whereby a bill is passed iff more members vote for it than against it. In such cases there are configurations in which, *ceteris paribus*, an abstaining member could turn the failure of a bill into a success by voting 'yes'; and other configurations in which, *ceteris paribus*, the same member could turn the success of a bill into failure by changing abstention to 'no'.

We feel that squeezing the more usual decision rules into the SVG format involves serious distortion. We will therefore introduce in Section 2 a generalization of the SVG set-up, called a *ternary voting game (TVG)*, which recognizes abstention as an option alongside 'yes' and 'no' votes. A step in this direction was taken by Fishburn (1973, pp. 53–55); but he considers only self-dual weighted voting games. Our TVGs are in this respect considerably more general. On the other hand, unlike him, we do not allow ties.

The main purpose for which the SVG set-up was originally defined was the mathematical explication and analysis of the notion of *a priori* voting power of a voter in a given decision-making constitution. We shall show that the most commonly used voting-power measures, the Shapley–Shubik (S–S) and Banzhaf (Bz) indices, have extremely natural analogues for TVGs. (Fishburn (1973) defines an analogue of the Bz index, but not of the S-S index, for his weighted ternary games.)

In Section 3 we extend to TVGs the notion of *responsiveness*, first introduced for SVGs by Dubey and Shapley (1979) following a suggestion by Rae (1969). Roughly speaking, the responsiveness of a voting game is the degree to which the average voter can influence the outcome of a division. Dubey and Shapley (1979, Theorem 2) proved that of all SVGs with a given number of voters, greatest responsiveness is achieved by the absolute majority rule, according to which the winning coalitions are those containing more than half the voters. We prove a similar result for TVGs: of all TVGs with a given number of voters, the most responsive is the simple majority rule, according to which a resolution is passed iff more voters vote for it than against it. Finally, we prove that the simple majority rule with a given number of voters is more responsive than the absolute majority rule with the same number of voters.

2. The formal framework

We assume that the reader is familiar with the definitions of SVG, the S–S index and the Bz index. (These definitions can be found, inter alia, in Shapley and Shubik's (1954) original paper, in Banzhaf's (1965) original paper, as well as in the papers by Dubey and Shapley (1979) and by Felsenthal and Machover (1995).) However, we shall reformulate these definitions here in equivalent forms, which will make them easier to generalize.

Throughout, we let N be an arbitrary non-empty finite set. We refer to N as the *assembly*, to any of its members as a *voter* and to any subset of N as a *coalition*. We put $n = |N|$.

We shall be interested in integer-valued functions defined on N. These functions can be added and multiplied by (integer) coefficients, as n-dimensional vectors. For each $a \in N$, we define χ_a as the function on N such that $\chi_a(a) = 1$ and $\chi_a(x) = 0$ for all other $x \in N$.

We define a partial order $\leq$ among all integer-valued functions on N by putting

$$f \leq g \iff fx \leq gx \text{ for all } x \in N.$$

Definition 2.1 By a *binary division* [of N] we mean a map B from N to the set $\{-1, 1\}$. We denote by B^- and B^+ the inverse images of $\{-1\}$ and $\{1\}$ respectively under the binary division B:

$$B^- = \{x \in N : Bx = -1\}, \qquad B^+ = \{x \in N : Bx = 1\}.$$

By an *SVG* [on N] we mean a mapping $\mathcal{V}$ from the set $\{-1, 1\}^N$ of all binary divisions of N to $\{-1, 1\}$, satisfying the following three conditions:

(1) If B is the binary division such that $B^- = N$, then $\mathcal{V}(B) = -1$.

(2) If B is the binary division such that $B^+ = N$, then $\mathcal{V}(B) = 1$.

(3) If B_1 and B_2 are binary divisions such that $B_1 \leq B_2$, then $\mathcal{V}(B_1) \leq \mathcal{V}(B_2)$.

The *dual* of $\mathcal{V}$, denoted by '$\mathcal{V}^*$' is determined by the identity

$$\mathcal{V}^*(B) = -\mathcal{V}(-B) \text{ for every binary division } B.$$

Comments. A binary division B is interpreted as a voting division, not allowing abstentions, on a bill tabled in the assembly N. B^- and B^+ are interpreted as the set of 'no' voters and the set of 'yes' voters, respectively, in the division B. The value Bx is interpreted as the degree of voter x's support for the bill in question. Thus $B_1 \leq B_2$ means that every voter supports the bill in B_2 at least as much as in B_1.

An SVG $\mathcal{V}$ is interpreted as a decision rule that assigns an outcome $\mathcal{V}(B)$ to each binary division B: a negative outcome -1 represents defeat and a positive outcome 1 represents passage of the bill in question. Thus, condition (1) of the definition means that a bill opposed unanimously by all the voters must be defeated. Similarly, condition (2) means that a bill supported unanimously by all the voters must pass.

Condition (3) says that an SVG must be *monotone*: if each voter's degree of support for the bill in B_2 is at least as great as in B_1, and the outcome of B_1 is positive, then the outcome of B_2 must be positive as well.

Our definition of SVG translates into the more conventional definition as follows. To each binary division B there corresponds the coalition B^+ of B's 'yes' voters; this correspondence is one-one, because $B^- = N - B^+$. The set $\mathcal{W}$ of *winning* coalitions in the SVG $\mathcal{V}$ is given by

$$\mathcal{W} = \{B^+ : B \text{ is a binary partition such that } \mathcal{V}(B) = 1\}.$$

Finally, note that the winning coalitions of $\mathcal{V}^*$ are precisely the complements of the losing (non-winning) coalitions of $\mathcal{V}$. This agrees with the conventional definition of duality (see, for example, Dubey and Shapley, 1979, p. 109).

Definition 2.2 By a *tripartition* or *ternary division* [of N] we mean a map T from N to the set $\{-1, 0, 1\}$. We denote by T^-, T^0 and T^+ the inverse images of $\{-1\}$, $\{0\}$ and $\{1\}$ respectively under T:

$$T^- = \{x \in N : Tx = -1\},$$

$$T^0 = \{x \in N : Tx = 0\},$$

$$T^+ = \{x \in N : Tx = 1\}.$$

By a *TVG* [on N] we mean a mapping $\mathcal{U}$ from the set $\{-1, 0, 1\}^N$ of all tripartitions of N to $\{-1, 1\}$, satisfying the following three conditions:

(1) If T is the tripartition such that $T^- = N$, then $\mathcal{U}(T) = -1$.

(2) If T is the tripartition such that $T^+ = N$, then $\mathcal{U}(T) = 1$.

(3) If T_1 and T_2 are tripartitions such that $T_1 \leq T_2$, then $\mathcal{U}(T_1) \leq \mathcal{U}(T_2)$.

The *dual* of $\mathcal{U}$, denoted by '$\mathcal{U}^*$' is determined by the identity

$$\mathcal{U}^*(T) = -\mathcal{U}(-T) \quad \text{for every tripartition T.}$$

Comments. Definition 2.2 is evidently modelled on Definition 2.1. The only difference is that a tripartition classifies the voters into three sets instead of two: in addition to the sets T^- and T^+ of 'no' and 'yes' voters, respectively, there is also a possibly non-empty set T^0 of abstainers. Tx, the degree of voter x's support for the bill in T, may be -1, 0 or 1.

A TVG assigns a negative or positive outcome (-1 or 1, respectively) to each tripartition. As before, a negative outcome, -1, is interpreted as defeat and a positive outcome, 1, as passage of the bill in question.

Conditions (1), (2) and (3) of Definition 2.2 mean the same, *mutatis mutandis*, as their counterparts in Definition 2.1.

Every SVG can be regarded, in a sense, as a TVG; for example, as a somewhat degenerate TVG that conflates abstention and 'no'. Thus, with each tripartition T we may associate the binary division B_T such that $B_T{}^+ = T^+$; then, with any given SVG $\mathcal{V}$ we may associate the TVG $\mathcal{V}'$ defined by

$$\mathcal{V}'(T) = \mathcal{V}(B_T) \quad \text{for every tripartition } T. \tag{2.3}$$

Note, however, that this way of associating a TVG with a given SVG $\mathcal{V}$ is not unique: we could equally have associated with $\mathcal{V}$ a TVG that conflates abstention with 'yes'. Moreover, the association between the SVG $\mathcal{V}$ and the corresponding $\mathcal{V}'$ is not quite natural in that it fails to respect some properties of SVGs and relations among SVGs. For example, the dual of $\mathcal{V}'$ (in the sense of Definition 2.2) does not in general correspond to the dual of $\mathcal{V}$ (in the sense of Definition 2.1).

It is important to point out that there are decision rules that can be represented as TVGs but not as SVGs. We shall see an example of this later on.

Our next task is to extend the definition of the S–S index ϕ to TVGs. From the conventional definition of ϕ for SVGs it is not at all obvious how this might be done in a natural way. However, we shall give here an alternative characterization of ϕ — whose equivalence to the conventional definition is proved in Felsenthal and Machover (1996) — which lends itself readily to our task.

We shall denote by 's' and 'd' (short for *sinister* and *dexter*) the canonical left-hand and right-hand projections from a cartesian product of two sets.

Recall that $n = |N|$. We denote by Γ the set of all bijections from N to $\{1, \ldots, n\}$. Thus, each $\pi \in \Gamma$ induces a total order on N. We therefore refer to π as an *ordering* of the voters, and we say, for example, that voter a *precedes* voter b in π if $\pi a < \pi b$.

Definition 2.4 By a *binary roll-call* [over N] we mean an ordered pair R, such that $\mathrm{s}R \in \Gamma$ and $\mathrm{d}R$ is a binary division of N. We denote by '$\mathcal{B}_N$' the set of all binary roll-calls over N: the cartesian product $\Gamma \times \{-1, 1\}^N$.

We say that two binary roll-calls, R_1 and R_2, *agree up to* voter a, if $\mathrm{s}R_1 = \mathrm{s}R_2$ and $\mathrm{d}R_1(x) = \mathrm{d}R_2(x)$ for all $x \in N$ such that $\mathrm{s}R_1(x) \leq \mathrm{s}R_1(a)$.

If $\mathcal{V}$ is an SVG and R is a binary roll-call, we say that voter a is the *pivot of R under $\mathcal{V}$* — briefly, $a = \mathrm{Piv}(\mathcal{V}, R)$ — if a is the first voter in the ordering $\mathrm{s}R$ satisfying the condition:

(*) $\mathcal{V}(\mathrm{d}S) = \mathcal{V}(\mathrm{d}R)$ for every binary roll-call S that agrees with R up to a.

Comments. It is helpful to visualize a binary roll-call R as follows. The voters are called one by one, in the order specified by $\mathrm{s}R$. As voter x is called, he or she declares 'yes' or 'no', according as $\mathrm{d}R(x)$ is 1 or -1.

Thus, to say that two binary roll-calls, R_1 and R_2, agree up to voter a means that the voters are called in the same order in both roll-calls, and the declarations of all voters up to and including a are the same in both roll-calls.

Next, let $\mathcal{V}$ be an SVG, which we hold fixed for the purpose of this discussion. If R is a roll-call, then dR is a binary division; so $\mathcal{V}$ assigns to dR an outcome $\mathcal{V}(\mathrm{d}R)$, which may be -1 or 1. We can regard $\mathcal{V}(\mathrm{d}R)$ also as the outcome of the roll-call R under $\mathcal{V}$.

Thus, condition (*) means that the outcome of R under $\mathcal{V}$ would not have altered if the voters who are called after a were to change their declarations in an arbitrary way. Every roll-call has at least one such voter — trivially, the last voter to be called satisfies (*) — but there may be several such voters. The *first* voter in R satisfying (*) is the pivot Piv($\mathcal{V}$, R).

Thus Piv($\mathcal{V}$, R) can be characterized as the last voter in R whose declaration makes any difference to the outcome $\mathcal{V}(\mathrm{d}R)$. If this voter would have changed his or her declaration, the outcome might have been different; but if the declarations of subsequent voters were to be changed in an arbitrary way, the outcome would have been unaffected.

Now, it is proved in Felsenthal and Machover (1996) that if $\mathcal{V}$ is an SVG on N and $a \in N$, then $\phi_a(\mathcal{V})$, the value of the S–S index for a in $\mathcal{V}$, is the number of binary roll-calls R such that a = Piv($\mathcal{V}$, R) divided by the cardinality of the set $\mathcal{B}_N$ of all binary roll-calls. Since the latter cardinality is clearly $2^n n!$, we have:

$$\phi_a(\mathcal{V}) = |\{R \in \mathcal{B}_N : a = \mathrm{Piv}(\mathcal{V}, R)\}|/2^n n!. \tag{2.5}$$

(If $\mathcal{B}_N$ is regarded as a probability space, with all binary roll-calls equally probable, then $\phi_a(\mathcal{V})$ is the probability that a is the pivot under $\mathcal{V}$ of a randomly chosen binary roll-call.)

We shall use this formula to define, by analogy, an S–S index for TVGs.

Definition 2.6 By a *ternary roll-call* [over N] we mean an ordered pair R, such that s$R \in \Gamma$ and dR is a tripartition of N. We denote by '$\mathcal{T}_N$' the set of all ternary roll-calls over N: the cartesian product $\Gamma\times\{-1, 0, 1\}^N$.

We say that two ternary roll-calls, R_1 and R_2, *agree up to* voter a, if $sR_1 = sR_2$ and $dR_1(x) = dR_2(x)$ for all $x \in N$ such that $sR_1(x) \leq sR_1(a)$.

If $\mathcal{U}$ is a TVG and R is a ternary roll-call, we say that voter a is the *pivot of R under* $\mathcal{U}$ — briefly, $a = \text{Piv}(\mathcal{U}, R)$ — if a is the first voter in the ordering sR satisfying the condition:

(*) $\mathcal{U}(dS) = \mathcal{U}(dR)$ for every ternary roll-call S that agrees with R up to a.

Next, we define, for any TVG $\mathcal{U}$ on N and any $a \in N$,

$$\phi_a(\mathcal{U}) = |\{R \in \mathcal{T}_N : a = \text{Piv}(\mathcal{U}, R)\}|/3^n n!. \qquad (2.7)$$

Comments. Definition 2.6 is evidently modelled on Definition 2.4, and the explanations we have provided for the latter apply to the former, *mutatis mutandis.* The analogy between formulas (2.5) and (2.7) is also obvious. In fact, many of the properties of the ordinary S–S index (for SVGs) have analogues that apply to the new index we have defined here. For example, it is easy to see that the latter, like the former, is self-dual: $\phi_a(\mathcal{U}^*) = \phi_a(\mathcal{U})$ for any TVG $\mathcal{U}$ and any voter a.

We would like to point out that if $\mathcal{V}$ is an SVG and $\mathcal{V}'$ is the TVG we have associated with it, defined by (2.3), then $\phi_a(\mathcal{V}') = \phi_a(\mathcal{V})$. This can be proved by an argument very similar to that used in proving the main result of Felsenthal and Machover (1996). We omit the proof here.

We can now present an example of a TVG that cannot be faithfully represented as an SVG. Let $N = \{a, b, c\}$ and let the decision rule be that a resolution is carried if a supports it and at least one of the two other voters does not oppose it. An easy calculation shows that in this TVG the S-S index of a is 22/27. But in an SVG with three voters the S-S index of each voter must be an integral multiple of 1/6. Hence our TVG cannot be represented faithfully as an SVG.

Our final task in this section is to define [an analogue of] the Bz index to TVGs. This turns out to be quite easy. We start by introducing some new terminology.

Let B be a binary division of N and let $a \in N$. We note that if $Ba = 1$, then $B - 2\chi_a$ is the binary division that has the value -1 at a, but coincides with B for all $x \neq a$. Similarly, if $Ba = -1$, then $B + 2\chi_a$ is the binary division that has the value 1 at a, but coincides with B for all $x \neq a$.

Now let $\mathcal{V}$ be an SVG. If $Ba = 1$, $\mathcal{V}(B) = 1$ and $\mathcal{V}(B - 2\chi_a) = -1$, we shall say that a is *positively critical for B in* $\mathcal{V}$. Similarly, if $Ba = -1$, $\mathcal{V}(B) = -1$ and $\mathcal{V}(B + 2\chi_a) = 1$, we shall say that a is *negatively critical for B in* $\mathcal{V}$.

Clearly, the number of binary divisions for which a is positively critical in $\mathcal{V}$ is equal to the number of binary divisions for which a is negatively critical in $\mathcal{V}$. It is easy to see that this number is equal to $\eta_a(\mathcal{V})$, the Bz score (also known as the Bz *count* or *number of swings*) of a in $\mathcal{V}$ (cf. Dubey and Shapley, 1979 p. 102; Felsenthal and Machover, 1995).

Recall that β and β' — respectively the relative and absolute Bz index for SVGs — are given by

$$\beta_a(\mathcal{V}) = \eta_a(\mathcal{V})/\Sigma\{\eta_x(\mathcal{V}) \mid x \in N\}, \tag{2.8}$$

$$\beta'_a(\mathcal{V}) = \eta_a(\mathcal{V})/2^{n-1}. \tag{2.9}$$

The absolute Bz index has the following probabilistic interpretation. Suppose that, in ignorance of the voters' real intentions and predispositions, we make the Laplacian assumption that the voters will vote independently of each other, each saying 'yes' or 'no' with equal probability of 1/2. Then $\beta'_a(\mathcal{V})$ is the probability of obtaining a division for which a is positively or negatively critical, and so could change the outcome by changing his or her vote from 'yes' to 'no' or vice versa. (See Dubey and Shapley, 1979, pp. 102–103; For a detailed discussion of the significance of this Laplacian probabilistic model, see Felsenthal, Machover and Zwicker, 1995, Section 6.)

It is now clear how to proceed in the case of TVGs. Let $\mathcal{U}$ be a TVG and let T be a tripartition. If $Ta \geq 0$, $\mathcal{U}(T) = 1$ and $\mathcal{U}(T - \chi_a) = -1$, we shall say that a is *positively critical for T in* $\mathcal{U}$. Similarly, if $Ta \leq 0$, $\mathcal{U}(T) = -1$ and $\mathcal{U}(T + \chi_a) = 1$, we shall say that a is *negatively critical for B in* $\mathcal{U}$. Next, we lay down the following

Definition 2.10 For any TVG $\mathcal{U}$ and voter a, the *Bz score of a in* $\mathcal{U}$, denoted by '$\eta_a(\mathcal{U})$', is the number of tripartitions for which a is positively critical in $\mathcal{U}$.

Further, the *relative Bz index* and *absolute Bz index* respectively are defined for TVGs as follows:

$$\beta_a(\mathcal{U}) = \eta_a(\mathcal{U})/\Sigma\{\eta_x(\mathcal{U}) \mid x \in N\}, \tag{2.11}$$

$$\beta'_a(\mathcal{U}) = \eta_a(\mathcal{U})/3^{n-1}. \tag{2.12}$$

Comments. Clearly, $\eta_a(\mathcal{U})$ is also equal to the number of tripartitions for which a is negatively critical in $\mathcal{U}$.

Here too the absolute Bz index has a probabilistic interpretation. Suppose that, in ignorance of the voters' real intentions and predispositions, we make the Laplacian assumption that the voters will vote independently of each other, each saying 'yes', 'no' or abstaining with equal probability of 1/3. Then $\beta'_a(\mathcal{U})$ is the probability of obtaining a tripartition for which a is positively or negatively critical, and so could change the outcome by reducing or increasing his or her degree of support for the bill in question.

Finally, we must point out that if $\mathcal{V}$ is an SVG and $\mathcal{V}'$ is the TVG we have associated with it, defined by (2.3), then $\beta_a(\mathcal{V}')$ is in general unequal to $\beta_a(\mathcal{V})$. For example, let $N = \{a, b, c\}$ and let $\mathcal{V}$ be the SVG such that $\mathcal{V}(B) = 1$ iff B^+ contains a and at least one of the other two voters. Then $\beta_a(\mathcal{V}) = 3/5$ but $\beta_a(\mathcal{V}') = 5/9$.

3. Responsiveness

Let us put, for any integer-valued function f defined on N,

$$\text{Maj}f = \Sigma\{fx \mid x \in N\}.$$

If f is a binary division or a tripartition, we call Maj$f$ the *majority in f*, because in these cases Maj$f$ equals the number of 'yes' voters minus the number of 'no' voters.

Further, let us put, for any SVG $\mathcal{V}$ and any TVG $\mathcal{U}$,

$$H(\mathcal{V}) = \Sigma\{\eta_x(\mathcal{V}) \mid x \in N\}, \qquad H(\mathcal{U}) = \Sigma\{\eta_x(\mathcal{U}) \mid x \in N\}.$$

Dubey and Shapley (1979, pp. 106 and 124) argue that $H(\mathcal{V})$ is 'a kind of democratic participation index', which measures the sensitivity of the SVG $\mathcal{V}$ to the wishes of the 'average voter'. In our view, it is $H(\mathcal{V})/2^{n-1}n$ rather than $H(\mathcal{V})$ which is suitable to serve as a democratic participation index. Of course, if one compares SVGs with the same number of voters, then the difference between the two measures is unimportant, merely a matter of scaling. But in comparing SVGs with different numbers of voters, $H(\mathcal{V})/2^{n-1}n$ ought to be preferred, for the following reasons.

First, note that if $\mathcal{V}'$ is obtained from $\mathcal{V}$ by introducing a new voter who is a dummy (that is, whose vote never affects the outcome of a division), then $H(\mathcal{V}') = 2H(\mathcal{V})$. It is intuitively unacceptable that the introduction of a dummy should make a voting rule twice as democratic.

Second, $H(\mathcal{V})/2^{n-1}n$ is the average of β' over all the voters of $\mathcal{V}$; so — according to the probabilistic interpretation of β' — we may regard it as the degree to which the 'average voter' can affect the outcome of a division.

An analogous argument shows that $H(\mathcal{U})/3^{n-1}n$ may be regarded as the degree to which the 'average voter' in a TVG $\mathcal{U}$ can affect the outcome of a tripartition. These considerations justify the following definition.

Definition 3.1 The *responsiveness* R($\mathcal{V}$) of an SVG $\mathcal{V}$ on N is $H(\mathcal{V})/2^{n-1}n$. The *responsiveness* R($\mathcal{U}$) of a TVG $\mathcal{U}$ on N is $H(\mathcal{U})/3^{n-1}n$. We denote by '$\delta(n)$' the greatest value of R($\mathcal{V}$) as $\mathcal{V}$ ranges over all SVGs with n voters; and by '$\tau(n)$' the greatest value of R($\mathcal{U}$) as $\mathcal{U}$ ranges over all TVGs with n voters.

Using our terminology and notation, a result proved by Dubey and Shapley (1979, Theorem 2) can be re-stated as follows.

Theorem 3.2 The value $\delta(n)$ is given by the following recursion:

$$\delta(1) = 1, \qquad \delta(2) = 1/2, \qquad \delta(2m+1) = \delta(2m),$$

$$\delta(2m+2) = \{(2m+1)/(2m+2)\}\delta(2m).$$

Moreover, this greatest responsiveness $\delta(n)$ is achieved by an SVG $\mathcal{V}$ with n voters iff $\mathcal{V}$ satisfies the following two conditions:

(i) $\mathcal{V}(B) = 1$ *for every binary division B such that* $\mathrm{Maj}(B) > 0$,

(ii) $\mathcal{V}(B) = -1$ *for every binary division B such that* $\mathrm{Maj}(B) < 0$.

An SVG $\mathcal{V}$ satisfying the conditions of Theorem 3.2 may be said to embody the *absolute majority* rule: binary divisions in which over half of the voters vote 'yes' have a positive outcome, and those in which over half of the voters vote 'no' have a negative outcome.

For odd n, there is exactly one such SVG (up to isomorphism). But for even n there are several SVGs satisfying the absolute majority rule: for some Bs such that $\mathrm{Maj}(B) = 0$ the outcome $\mathcal{V}(B)$ may be 1, while for other such Bs it may be -1. The absolute majority rule does not make any particular prescription in these cases.

Note that δ is a descending step-function of n. (It is quite easy to show that $\delta(n)$ tends to 0 as n increases, but we shall not need this.)

In the following theorem we determine the TVGs that achieve greatest responsiveness. (The proof is an adaptation of the argument used by Dubey and Shapley (1979).)

Theorem 3.3 Among all TVGs $\mathcal{U}$ with n voters, the greatest value of H, and hence greatest responsivenes, is achieved by a TVG $\mathcal{U}$ iff $\mathcal{U}$ satisfies the following two conditions:

(i) $\mathcal{U}(T) = 1$ *for every tripartition T such that* $\mathrm{Maj}(T) > 0$,

(ii) $\mathcal{U}(T) = -1$ *for every tripartition T such that* $\mathrm{Maj}(T) < 0$.

Proof: First we prove that conditions (i) and (ii) are necessary. Let $\mathcal{U}$ be a TVG with n voters, and suppose that there is a tripartition T such that $\mathrm{Maj}(T) > 0$ but $\mathcal{U}(T) = -1$. We shall show that $\mathcal{U}$ does not achieve the greatest value of H.

Without loss of generality we may assume that T is a maximal member (in the partial ordering $\leq$) of the set of tripartitions S for which $\mathcal{U}(S) = -1$. Indeed, if there exists a tripartition S such that $T < S$ and $\mathcal{U}(S) = -1$, then $\mathrm{Maj}(S) > 0$, since Maj is clearly monotone; so we could use S instead of T.

Define a mapping $\mathcal{U}'$ from the set of all tripartitions to $\{-1, 1\}$ by putting $\mathcal{U}'(T) = 1$, and $\mathcal{U}'(S) = \mathcal{U}(S)$ for all tripartitions $S \neq T$. It is easy to verify that $\mathcal{U}'$ is in fact a TVG.

Let us compare $H(\mathcal{U})$ with $H(\mathcal{U}')$. The only differences in the Bz scores between $\mathcal{U}$ and $\mathcal{U}'$ are those relating to T. By the maximality of T, all the members of T^- and of T^0 are negatively critical for T in $\mathcal{U}$; but they are not negatively critical for T in $\mathcal{U}'$, because $\mathcal{U}'(T) = 1$. On the other hand, by the monotonicity of $\mathcal{U}$, all the members of T^0 and of T^+ are positively critical for T in $\mathcal{U}'$; but they are not positively critical for T in $\mathcal{U}$, because $\mathcal{U}(T) = -1$. Hence, in going from $\mathcal{U}$ to $\mathcal{U}'$, H loses $|T^-| + |T^0|$ units and gains $|T^0| + |T^+|$ — a net gain of

$|T^+| - |T^-|$ units. Since Maj(T) > 0, this net gain is positive, so $H(\mathcal{U}') > H(\mathcal{U})$.

By an entirely symmetric argument, if there is a tripartition T such that Maj(T) < 0 but $\mathcal{U}(T) = 1$, $\mathcal{U}$ does not achieve the greatest value of H. This shows that conditions (i) and (ii) are necessary.

To prove sufficiency, let $\mathcal{U}$ be any TVG with n voters that satisfies these conditions. Let T be any tripartition such that Maj(T) = 0. Define $\mathcal{U}'$ by putting $\mathcal{U}'(T) = -\mathcal{U}(T)$, and $\mathcal{U}'(S) = \mathcal{U}(S)$ for all tripartitions $S \neq T$. It is easy to verify that $\mathcal{U}'$ is in fact a TVG. Moreover, an argument like the one used in the first half of this proof shows that $H(\mathcal{U}') = H(\mathcal{U})$. It follows that all TVGs satisfying (i) and (ii) have the same value of H. This must therefore be the greatest value of H. ■

TVGs satisfying the conditions of Theorem 3.3 may be said to embody the *simple majority* rule: tripartitions in which more voters vote 'yes' than 'no' have a positive outcome, and those in which the reverse is true have a negative outcome. (No particular prescription is made for tripartitions in which the number of 'yes' voters is equal to that of 'no' voters.)

Our next task is to calculate the value $\tau(n)$, the greatest responsiveness achievable by a TVG with n voters. To this end we define for each positive n, an SVG $\mathcal{V}_n$ and a TVG $\mathcal{U}_n$, both having n voters, as follows:

$$\mathcal{V}_n(B) = 1 \iff B \text{ is a binary division such that } \mathrm{Maj}(B) > 0,$$

$$\mathcal{U}_n(T) = 1 \iff T \text{ is a tripartition such that } \mathrm{Maj}(T) > 0.$$

$\mathcal{V}_n$ satisfies the conditions of Theorem 3.2. Moreover, by symmetry all voters in $\mathcal{V}_n$ have the same value of β', which must therefore equal $\delta(n)$. Similarly, by Theorem 3.3 and symmetry, all voters in $\mathcal{U}_n$ have the same value of β', which must equal $\tau(n)$.

Using the probabilistic interpretations of $\beta'(\mathcal{V}_n)$ and $\beta'(\mathcal{U}_n)$ presented in Section 2, we can give $\delta(n)$ and $\tau(n)$ the following probabilistic interpretations. If all n voters vote, independently of each other, 'yes' or 'no' with equal probability of 1/2, then the probability that a given voter a will be positively or negatively critical in $\mathcal{V}_n$ for a randomly chosen binary division is $\delta(n)$. Similarly, if all n voters, independently of each other, vote 'yes' or 'no' or abstain with equal probability of 1/3, then the probability that a given voter a will be positively or negatively critical in $\mathcal{U}_n$ for a randomly chosen tripartition is $\tau(n)$. Using these interpretations, we shall prove the following result.

Theorem 3.4 $\tau(1) = 1$. *For any positive integer* n, $\tau(n + 1)$ *is given by the formula*

$$\tau(n + 1) = \Sigma\{{}^n\mathrm{C}_k\, 2^k 3^{-n} \delta(k + 1) \mid k = 0, 1, \ldots, n\},$$

where ${}^n\mathrm{C}_k$ *is the binomial coefficient 'n over k'*.

Proof: That $\tau(1) = 1$ is obvious. To prove the formula for $\tau(n + 1)$, let us consider a particular voter a in the TVG $\mathcal{U}_{n+1}$. $\tau(n + 1)$ is the prior probability that in this TVG a will be critical in a randomly chosen tripartition.

Now let k be a natural number, $0 \le k \le n$. Suppose we know that, among the voters other than a, k particular voters, say $b_1, \ldots, b_k$, do not abstain, while the remaining $n - k$ voters do abstain. Given this information, a and the k non-abstainers $b_1, \ldots, b_k$ are essentially in the situation of playing the SVG $\mathcal{V}_{k+1}$; hence the conditional probability that a is critical is equal to $\delta(k + 1)$.

Moreover, the prior probability that $b_1, \ldots, b_k$ do not abstain and the remaining $n - k$ voters do abstain is $(2/3)^k(1/3)^{n-k}$; and there are ${}^n\mathrm{C}_k$ different ways of selecting the non-abstainers $b_1, \ldots, b_k$. This gives us the term with $\delta(k + 1)$ in the formula for $\tau(n + 1)$ and concludes the proof. ■

Corollary 3.5 For every positive integer n, $\tau(n + 1) > \delta(n + 1)$.

Proof: The coefficients of the $\delta(k + 1)$ in the expression for $\tau(n + 1)$ are precisely the terms in the binomial expansion of $(2/3 + 1/3)^n$; so these coefficients add up to 1. Thus $\tau(n + 1)$ is a weighted average of $\delta(1), \ldots, \delta(n + 1)$. By Theorem 3.2, $\delta(n + 1)$ is the smallest of these $n + 1$ quantities, and is actually smaller than some of them. ■

Thus we have shown that, for all $n > 1$, $\mathcal{U}_n$ is more responsive than $\mathcal{V}_n$. This means that the simple majority rule is more responsive than the absolute majority rule.

4. Discussion

The fact that, for all $n > 1$, $\mathcal{U}_n$ is more responsive than $\mathcal{V}_n$ implies that if one is concerned with devising a decision rule such that the 'average voter' will have maximal prior probability of affecting the outcome of a division, then:

(1) Division should be allowed to take place without requirement for a quorum;

(2) Each of the n members participating in the division should be allowed to abstain, vote in favor, or vote against;

(3) A resolution should be deemed to pass if a simple majority of the members participating in the division support it;

(4) A resolution should be deemed to fail if a simple majority of the members participating in the division oppose it.

However, there still remains the problem of how to treat ties. From the point of view of responsiveness it makes no difference if ties are disallowed (i.e., a resolution is either deemed to pass or it is deemed not to pass once a tie occurs), or if a tie is broken randomly. Nevertheless, if one is also concerned with averting the *quorum* paradox (cf. Felsenthal, 1991), then one should break ties randomly. Given that resolutions are passed by simple majority of the participating voters, the quorum paradox consists in the fact that the probability of 'correct decision' may not be a monotonic function of the number of members participating in the division if ties are not broken randomly. In this context, a 'correct decision' is the outcome that would result if *all* eligible n members of a voting body participate in the division. Although random tie-breaking is a common practice in public elections, it is quite uncommon in smaller decision-making bodies such as committees and parliaments; nevertheless, there exist some real-life examples where parliaments, too, resorted to this practice (e.g., in the Swedish Riksdag during the period 1973-1976).

As far as the S-S and Bz indices are concerned, we have shown that if $\mathcal{V}$ is an SVG and $\mathcal{V}'$ is the TVG associated with it, defined by (2.3), then $\phi_a(\mathcal{V})$ always equals $\phi_a(\mathcal{V}')$, but $\beta_a(\mathcal{V})$ is in general unequal to $\beta_a(\mathcal{V}')$. This implies that both the weak and strong forms of the *bicameral* paradox afflicting the S-S index in SVGs (cf. Felsenthal, Machover and Zwicker, 1995) must also afflict this index in TVGs. Moreover, although $\beta_a(\mathcal{V})$ is in general unequal to $\beta_a(\mathcal{V}')$, it can nevertheless be shown, by using the same examples as in Felsenthal and Machover (1995), that the *donation*, *bloc* and *transfer* paradoxes, from which the Bz index suffers in SVGs, apply also in TVGs.

Finally, it must be pointed out that several real-life examples analyzed in the literature as though they were SVGs should properly be regarded as TVGs. In this respect they are like the simple example presented in the comments following Definition 2.7. This applies, in particular, to the UN Security Council, as well as to the US legislative system consisting of the Senate, the House of Representatives and the veto-wielding President.

During the period 1945-1965 the UN Security Council consisted of 11 members — five permanent members and six non-permanent members. The (original) Article 27 of the UN Charter stated:

(1) Each member of the Security Council shall have one vote.

(2) Decisions of the Security Council on procedural matters shall be made by an affirmative vote of seven members.

(3) Decisions of the Security Council on all other matters shall be made by an affirmative vote of seven members including the concurring votes of the permanent members;...

In 1966 the number of non-permanent members was increased from six to 10, and the word 'seven' in clauses (2) and (3) of Article 27 was replaced by 'nine'. Ostensibly, the wording of Article 27(3) of the UN Charter implies that in non-procedural matters an explicit 'yes' vote by all permanent members is needed to pass a resolution. However, in practice, as of 1946 abstention by a permanent member is not interpreted as a veto; and as of 1947 and 1950 the same applies to non-participation in the vote and absence, respectively, of a permanent member. (For the interpretation in practice of Article 27(3) of the UN Charter with respect to abstention, non-participation or absence of a permanent member, see Simma, 1994, pp. 447–454 and references cited therein.) This means that on non-procedural matters a resolution is carried in the UN Security Council if it is supported by at least nine (or, before 1966, seven) members and not explicitly opposed by any permanent member. This rule is essentially a TVG, and cannot be faithfully represented as an SVG. However, Shapley (1962, p. 65), Rapoport (1970, pp. 218-219), Brams (1975, pp. 182-191), Lucas, (1982, p. 196), Riker (1982, p. 52), Coleman (1986, p. 198), Lambert (1988, p. 230), Brams, Affuso and Kilgour (1989, p. 58), as well as several others, have ignored the actual practice and have stuck to the literal reading of Article 27(3), which they have interpreted as an SVG.

Using the wrong model can have a very significant effect on the numerical results. For example, using the unsuitable SVG model for the UN Security Council, Straffin (1982, pp. 314-315) finds that $\phi = 0.19627$ and $\beta = 0.16693$ for each of the five permanent members and that $\phi = 0.00186$ and $\beta = 0.01654$ for each of the 10 non-permanent members. But if one calculates the S-S and relative Bz indices while viewing the UN Security Council (appropriately) as a TVG, one obtains that $\phi = 0.1636$ and $\beta = 0.1009$ for each of the five permanent members and that $\phi = 0.0182$ and $\beta = 0.0495$ for each of the 10 non-permanent members. Thus the more realistic (TVG) model ascribes to each non-permanent member of the UN Security Council — according to both the S-S and relative Bz indices — a much greater relative *a priori* voting power than does the SVG model. (It could be argued that since abstention by a non-permanent member is always counted as a 'no' vote, these members have in effect two voting options — 'yes' or 'no' — whereas only for the permanent members abstention is a distinct *tertium quid*. The results obtained for the UN Security Council according to this "mixed" SVG/TVG model are as follows: $\phi = 0.1762$ and $\beta = 0.1038$ for each of the five permanent members; and $\phi = 0.0119$ and $\beta = 0.0482$ for each of the 10 non-permanent members.)

In the US Congress, business in each of the two Houses of Congress can only take place if a (simple) majority of its members are present (cf. Article 1, Section 5(1) of the US Constitution), and this ruling has been interpreted to mean that if the necessary quorum is attained then an ordinary bill (as distinct from overriding a presidential veto) is deemed to pass in each House if it is supported by a simple majority of the members *participating in the division*. (Both the Speaker of the House of Representatives and the President of the Senate have a casting vote which they can use in order to break ties). In case the President vetoes a bill passed by the two Houses of Congress, Article 1, Section 7(2) of the US Constitution states that the President "shall return it, with his objections, to that House in which it shall have originated, who shall enter the objections at large on their journal, and proceed to reconsider it. If after reconsideration two-thirds of that House shall agree to pass the bill, it shall be sent, together with the objections, to the other House, by which it shall likewise be reconsidered, and if approved by two-thirds of that House, it shall become law...". The US Supreme Court has ruled on January 7, 1919 (cf. Missouri Pacific Railway Co. vs State of Kansas, 248 U.S. 276) that the words 'two-thirds of that House' in this Article should be interpreted to mean two-thirds of the members *participating in the division* rather than two-thirds of all the members in that House. So in this legislative system too, the decision rule treats abstention as different from both 'yea' and 'nay' — and hence it is essentially a TVG. However, Shapley and Shubik (1954, p. 789), Shapley (1962, p. 60), Brams (1975, p. 192), Lucas (1982, p. 212), Lambert (1988, p. 235), Brams, Affuso and Kilgour (1989, p. 62), and several others, have wrongly stated that in order for an ordinary bill to pass in each House it must be supported by a simple majority of *all* its members, and in order to override a presidential veto two-thirds of *all* its members must agree, thus making it an SVG.

The mis-representation of the US legislature as an SVG by Shapley and Shubik (1954) is particularly tantalizing. For, in discussing the chairman's tie-breaking function (p. 788) they are perfectly aware that an absence of a member of the Senate during a division counts neither as a 'yes' nor as a 'no'; and they expressly state that 'in the passage of ordinary legislation, ... perfect attendance [in the Senate] is unlikely even for important issues ...'. But in the very next paragraph (p. 789), when applying their index to the US legislature, they revert to the mis-statement of the decision rule: 'It takes majorities of Senate and House, with the President, or two-thirds majorities of Senate and House without the President, to enact a bill. We take *all* [our emphasis] the members of the three bodies and consider them voting... .'

It seems that we are confronted here with a clear-cut case of theory-laden (or theory-biased) observation. Scientists, equipped with a ready-made theoretical conception, 'observe' in reality phenomena that fit that conception. And where the phenomena do not quite fit the theory, they are mis-perceived and tweaked into the theoretical mould.

We are not suggesting that any of that mis-representation of reality is deliberate or even conscious. Speaking for ourselves, we can attest that so long as we worked within the SVG paradigm those mis-representations did not evoke in us more than a vague feeling of malaise. It is only after we had invented, partly by chance, the alternative theoretical model of TVGs presented in this paper, which does admit abstentions, that we became acutely aware of that widespread distortion.

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