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Tree enterprises and bankruptcy ventures A game theoretic similarity due to a graph theoretic proof ☆

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Abstract

In a tree enterprise, users reside at the nodes of the tree and their aim is to connect themselves, directly or indirectly, to the root of the tree. The construction costs of arcs of the tree are given by means of the arc-cost-function associated with the tree. Further the bankruptcy venture is described in terms of the estate of the bankrupt firm and the claims of the various creditors. The first objective of the paper is to provide conditions (on the claims and the surplus of the claims in the bankruptcy venture) which are sufficient and necessary for the bankruptcy venture to agree with some tree enterprise. It is established that the bankruptcy venture agrees with some tree enterprise if and only if the surplus of claims in the bankruptcy venture is at most the size of the second smallest claim (in the weak sense). For that purpose, both the tree enterprise as well as the bankruptcy venture are modelled as a cooperative game with transferable utility. Within the framework of cooperative game theory, the proof of the equivalence theorem concerning the tree enterprise game and the bankruptcy game, under the given circumstances, is based on graph-theoretic tools in a tree structure. As an adjunct to the proof of the equivalence theorem, the solution concept of the nucleolus for specific tree enterprises is determined.

Keywords: Tree; Cooperative game; Tree game; Bankruptcy game

1. The standard tree enterprise in comparison with the bankruptcy venture: game-theoretic models

Let (V,E) be a directed graph with node set V and arc set E, which is provided with a (nonnegative) arc-cost-function $a:E \to \mathbb{R}_+$. The graph can be regarded as the

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mathematical model of a cable-television network in which one central supplier (the central station broadcasting cable-television signals) is located at a distinguished node (denoted by 0) and the users in the remaining nodes so that every user resides at exactly one node. The aim of the users is to connect themselves, directly or indirectly, to the central supplier by constructing connecting-links (i.e., arcs of the graph) through which the cable-television signals are transmitted. The construction costs of connecting-links are given by means of the arc-cost-function associated with the graph. Throughout this paper it is supposed that the users in the network are connected to the central supplier through a standard tree enterprise. We say the directed graph is a *standard tree enterprise* if, for each vertex, there is a unique path from the distinguished node 0, called the *root* of the tree, to that vertex. All arcs are directed away from the root of the tree and the construction of arcs not in the tree is regarded infeasible (or too costly).

In the game-theoretic literature the former standard tree enterprise has been modelled as a cooperative *cost game* (N,c). The *player set* N represents the set of users in the cable-television network (i.e., nodes of the standard tree enterprise different from the root) and the *cost function* $c:2^N \to \mathbb{R}$ is defined so that, for each coalition $S \subset N$,

$$\text{the cost } c(S) \text{ equals the least cost to connect all users in } S \text{ to the central supplier 0 via arcs of the standard tree enterprise.} \tag{1.1}$$

The so-called *standard tree enterprise game* (N,c) of (1.1) was discussed in [9–11]. The game-theoretic analysis of the standard tree enterprise deals with the cost allocation problem of setting charges to the users in order to allocate the overall cost $c(N)$. How to allocate the least total construction costs of connecting-links to the users in the network?

In addition to the cost allocation problem in a standard tree enterprise, we address the division problem how to divide the estate of a bankrupt firm among various creditors. The problem is that the claims of the creditors are mutually inconsistent in that the estate is insufficient to meet all of the claims. In the game-theoretic literature the latter bankruptcy venture has been modelled as a cooperative *savings game* (N,u). The *player set* N represents the set of creditors and the *savings function* $u:2^N \to \mathbb{R}$ is defined so that, for each coalition $S \subset N$, the worth $u(S)$ equals either zero or what is left of the estate after each creditor outside S is paid his claim. That is,

$$u(S) := \max\left[0,\ E - \sum_{j \in N \setminus S} d_j\right] \qquad \text{for all } S \subset N, \tag{1.2}$$

where E denotes the estate of the bankrupt firm and d_i the claim of creditor $i, i \in N$. It is always supposed that the estate E satisfies $0 < E \leqslant \sum_{j \in N} d_j$ (otherwise, the bankruptcy problem would not exist). The so-called *bankruptcy game* (N,u) of (1.2) was introduced in [12] and is studied in [2–8].

Notice that the game model of the standard tree enterprise is formulated in terms of cost figures, whereas the game model of the bankruptcy venture is in terms of savings. In order to be able to compare both game models, there is associated with the given cost function $c: 2^N \to \mathbb{R}$ of (1.1) the *cost savings function* $w: 2^N \to \mathbb{R}$ defined by $w(S) := \sum_{j \in S} c(\{j\}) - c(S)$ for all $S \subset N$. Here the worth $w(S)$ represents the cost savings that would result from cooperation between the members of S instead of acting alone. Clearly, individuals earn no cost savings, i.e., $w(\{i\}) = 0$ for all $i \in N$. Generally speaking, the bankruptcy game (N, u) of (1.2) fails to be a zero-normalized game, i.e., $u(\{i\}) = 0$ for all $i \in N$. Throughout the remainder of the paper, we replace the bankruptcy game (N, u) by its zero-normalized version (N, v) defined by

$$v(S) := u(S) - \sum_{j \in S} u(\{j\}) \quad \text{for all } S \subset N,$$

or equivalently,

$$v(S) = \sum_{j \in S} \min[d_j,\ \Delta] - \min\left[\sum_{j \in S} d_j,\ \Delta\right] \quad \text{for all } S \subset N. \tag{1.3}$$

Here the nonnegative *surplus of claims* defined by $\Delta := \sum_{j \in N} d_j - E$ represents the part of the claims that cannot be met by the estate.

The (first) main goal of the paper is to establish that, under certain circumstances, the bankruptcy division problem can be treated as an equivalent of the cost allocation problem in some standard tree enterprise. In terminology of the corresponding game models, it will be proved in Section 2 that, under certain conditions on the estate and the claims, the zero-normalized bankruptcy game agrees with the cost savings game arising from some standard tree enterprise game. Thus, given the game (N, v) of (1.3), we look for a standard tree enterprise game (N, c) of (1.1) satisfying $v(S) := \sum_{j \in S} c(\{j\}) - c(S)$ for all $S \subset N$. The main results of Section 2 are stated in Theorems 2.1 and 2.3.

The solution part of cooperative game theory deals with the study of all kinds of solution concepts, such as the core, stable sets, bargaining set, (pre)kernel, nucleolus, Shapley value and τ-value (cf. [5]). In the context of the bankruptcy game (N, u) of (1.2), almost all solution concepts are well studied and determined in an appropriate manner (cf. [2, 3, 5, 12]), whereas, in the setting of the standard tree enterprise game (N, c) of (1.1), one has to go to a lot of trouble to determine the above solution concepts, especially the nucleolus (cf. [10]).

As an adjunct to the forthcoming proof of Theorem 2.3, Section 3 deals with the concept of the nucleolus for the specific standard tree enterprises which we shall encounter within the proof of Theorem 2.3. The (second) main goal of the paper is to exploit the equivalence between bankruptcy ventures and standard tree enterprises in the sense that we first discuss the elegant description of the nucleolus for those specific bankruptcy ventures (taken from the development of the nucleolus for general bankruptcy ventures) and next we transfer the obtained result for the nucleolus from the bankruptcy venture to the underlying standard tree enterprise by means of a simple relationship between cost and savings allocations.

Remark 1.1. Every zero-normalized *2-person* bankruptcy game (N,v) of (1.3) agrees with the cost savings game arising from the tree enterprise game of which the arc-cost-function a of the underlying chain enterprise on node set $\{0,1,2\}$ is given by $a((0,1)) := v(\{1,2\})$ and $a((1,2))$ arbitrarily chosen from $\mathbb{R}_+$.

Remark 1.2. If there is no surplus in the bankruptcy problem (i.e., $\Delta = 0$), then the zero-normalized *n-person* bankruptcy game (N,v) of (1.3) is the trivial null game (i.e., $v(S) = 0$ for all $S \subset N$). The trivial null game, on its turn, agrees with the cost savings game arising from the standard tree enterprise game of which the underlying standard tree enterprise on node set $N \cup \{0\}$ is composed of the arcs $(0,i)$, $i \in N$ (with an arbitrarily chosen arc-cost-function).

Remark 1.3. It is well-known (cf. [3, 5]) that the bankruptcy game (N,u) of (1.2) is a *convex game*, i.e., $u(S \cup \{i\}) - u(S) \leqslant u(T \cup \{i\}) - u(T)$ for all $i \in N$ and all S, $T \subset N$ such that $S \subset T \subset N \backslash \{i\}$. Consequently, the zero-normalized bankruptcy game (N,v) of (1.3) is convex as well. Further, it is known (cf. [10]) that the standard tree enterprise game (N,c) of (1.1) is a *concave game*, i.e., $c(S \cup \{i\}) - c(S) \geqslant c(T \cup \{i\}) - c(T)$ for all $i \in N$ and all S, $T \subset N$ such that $S \subset T \subset N \backslash \{i\}$. Thus, the cost savings game associated with a standard tree enterprise game is a convex game. From the viewpoint of the convexity property for games, both types of a game (standard tree enterprise game versus bankruptcy game) match.

Example 1.4. Involving the standard tree enterprises on node set $\{0,1,2,3\}$, there are four possibilities for the structure of the underlying tree. We list the arc set of each possible form of the standard tree enterprise Γ and describe the cost savings game (N,w) arising from the corresponding standard tree enterprise game (N,c). Write $\{1,2,3\} = \{i_1, i_2, i_3\}$.

- $\Gamma = \{(0,1), (0,2), (0,3)\}$ and (N,w) agrees with the trivial null game, i.e., $w(S) = 0$ for all $S \subset N$ (see Remark 1.2).
- $\Gamma = \{(0,i_1), (0,i_2), (i_1,i_3)\}$ and (N,w) is given by $w(\{i_1,i_3\}) = w(\{i_1,i_2,i_3\}) = a((0,i_1))$ and $w(S) = 0$ otherwise. Notice that in this situation the users i_1 and i_3 are substitutes, that is interchanging the roles of the users i_1 and i_3 does not change the cost savings game (N,w).
- $\Gamma = \{(0,i_1), (i_1,i_2), (i_1,i_3)\}$ and (N,w) is given by $w(S) = (|S| - 1)a((0,i_1))$ for all $S \subset N$, $S \neq \emptyset$. Here $|S|$ denotes the number of members of coalition S. Notice that this game is symmetric, that is the cost savings merely depend on the number of users in coalition S and not the users themselves.
- $\Gamma = \{(0,i_1), (i_1,i_2), (i_2,i_3)\}$ and (N,w) is given by $w(\{i_1,i_2\}) = w(\{i_1,i_3\}) = a((0,i_1))$, $w(\{i_2,i_3\}) = a((0,i_1)) + a((i_1,i_2))$ and $w(\{i_1,i_2,i_3\}) = 2a((0,i_1)) + a((i_1,i_2))$. Notice that in this situation the users i_2 and i_3 are substitutes.

Example 1.5. Look at the bankruptcy venture with the three different claims $d_1 = 100$, $d_2 = 150$, $d_3 = 175$, and the estates $E = 300$ and $E = 250$, respectively.

(i) Suppose $E=300$. Then the zero-normalized three-person bankruptcy game (N,v) of (1.3) is given by $v(\{i\})=0$ for all $i\in N$, $v(\{1,2\})=v(\{1,3\})=100$, $v(\{2,3\})=125$ and $v(N)=225$. It appears that this bankruptcy venture can be associated with the chain enterprise on node set $\{0,1,2,3\}$ of which the arc-cost-function a is given by $a((0,1))=100$, $a((1,2))=25$, $a((2,3))=0$. The standard tree enterprise game (N,c) corresponding to this chain enterprise is given by $c(\{1\})=100$ and $c(S)=125$ for all $S\subset N, S\neq\emptyset,\{1\}$. Obviously, the cost savings game associated with this standard tree enterprise game (N,c) agrees with the zero-normalized bankruptcy game (N,v).

(ii) Suppose $E=250$. Then the zero-normalized three-person bankruptcy game (N,v) of (1.3) is given by $v(\{i\})=0$ for all $i\in N$, $v(\{1,2\})=75$, $v(\{1,3\})=100$, $v(\{2,3\})=150$ and $v(N)=250$. Since this zero-normalized three-person game does not agree with one of the four possible types of a cost savings game listed in Example 1.4, we conclude that this bankruptcy venture cannot be associated with any standard tree enterprise on node set $\{0,1,2,3\}$.

2. The representation of bankruptcy ventures as standard tree enterprises

In this section we are concerned with the possible representation of the zero-normalized bankruptcy game as the cost savings game arising from a standard tree enterprise game. That is, we aim to provide conditions (on the claims and the surplus of claims in the bankruptcy venture) which are sufficient and necessary for the cost savings relationship between zero-normalized bankruptcy games and standard tree enterprise games.

Let $N=\{1,2,\dots,n\}$ be the set of creditors. Given the claims d_i, $i\in N$, of the creditors, we may order these claims, without loss of generality, so that $0<d_1\leqslant d_2\leqslant\cdots\leqslant d_n$. The number of members of coalition S is denoted by $|S|$. In view of Remark 1.1, we assume throughout this section that there are at least *three creditors*, so $n\geqslant 3$.

Theorem 2.1. *Let $n\geqslant 3$. If the surplus of claims in the bankruptcy venture is at most the size of the second smallest claim (in the weak sense, i.e., $0\leqslant\Delta\leqslant d_2$ where $\Delta:=\sum_{j\in N}d_j-E$), then the zero-normalized n-person bankruptcy game (N,v) of (1.3) can be represented as the cost savings game arising from a standard tree enterprise game with user set N.*

Proof. Suppose that $0\leqslant\Delta\leqslant d_2$. From $d_j\geqslant d_2\geqslant\Delta$ for all $j\in N\backslash\{1\}$ and formula (1.3), we deduce that the zero-normalized n-person bankruptcy game (N,v) is given by

$$v(S)=\begin{cases}(|S|-1)\Delta & \text{for all } S\subset N\backslash\{1\},\ S\neq\emptyset,\\ \min[d_1,\ \Delta]+(|S|-2)\Delta & \text{for all } S\subset N \text{ with } 1\in S,\ S\neq\{1\}.\end{cases}$$

Consider the chain enterprise on node set $N \cup \{0\}$ of which the arc-cost-function a is given by

$$a((0,1)) := \min[d_1,\ \Delta], \qquad a((1,2)) := \max[0,\ \Delta - d_1]$$

$$a((i-1,i)) := 0 \quad \text{for all } i \in N \backslash \{1,2\}.$$

Let (N,c) be the standard tree enterprise game corresponding to this chain enterprise. Then $c(\{1\}) = \min[d_1,\ \Delta]$ and $c(S) = \min[d_1,\ \Delta] + \max[0,\ \Delta - d_1] = \Delta$ for all $S \in 2^N \backslash \{\emptyset, \{1\}\}$. Particularly, $c(\{j\}) = \Delta$ for all $j \in N \backslash \{1\}$. Now it follows immediately that $\sum_{j \in S} c(\{j\}) - c(S) = v(S)$ for all $S \subset N$. Hence, the game (N,v) is the cost savings game arising from the standard tree enterprise game (N,c) associated with the given chain enterprise (provided that $0 \leqslant \Delta \leqslant d_2$). □

Remark 2.2. The arc-cost-function a listed in the proof of Theorem 2.1 is not unique since the cost of the arc $(n-1,n)$ may be arbitrarily chosen from $\mathbb{R}_+$ instead of zero. Although the cost $c(S)$ of any coalition S containing user n (i.e., the unique leaf of the chain) will be increased by the size of $a((n-1,n))$, the corresponding cost savings game does not change.

Theorem 2.3. *Let $n \geqslant 3$ and suppose that the surplus of claims in the bankruptcy venture is positive (that is, $\Delta := \sum_{j \in N} d_j - E > 0$). If the zero-normalized n-person bankruptcy game (N,v) of (1.3) can be represented as the cost savings game arising from a standard tree enterprise game with user set N, then the surplus of claims is at most the size of the second smallest claim.*

Proof. Suppose that the zero-normalized bankruptcy game (N,v) of (1.3) agrees with the cost savings game arising from some standard tree enterprise game (N,c) with user set N (given that $n \geqslant 3$ and $\Delta > 0$). That is, the cost savings relationship $\sum_{j \in S} c(\{j\}) - c(S) = v(S)$ holds for all $S \subset N$ and moreover, formula (1.3) involving the worth $v(S)$, $S \subset N$, holds as well.

Let $p \geqslant 1$ be the degree of the root 0 in the underlying standard tree enterprise Γ. We think the tree enterprise Γ on node set $N \cup \{0\}$ to be composed of various subtree enterprises $\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(p)}$, rooted at node 0, on node sets $N^{(1)} \cup \{0\}, N^{(2)} \cup \{0\}, \ldots, N^{(p)} \cup \{0\}$, respectively (so, the various coalitions $N^{(\ell)}$, $1 \leqslant \ell \leqslant p$, do not contain the root 0 and form a partition of the player set N). Clearly, by definition, the standard tree enterprise game (N,c) satisfies $c(N) = \sum_{\ell=1}^{p} c(N^{(\ell)})$. Therefore, the overall cost savings for the grand coalition N in Γ are attainable as the sum of the overall cost savings in the various subtrees $\Gamma^{(\ell)}$ on node sets $N^{(\ell)} \cup \{0\}$, $1 \leqslant \ell \leqslant p$. Our objective is to study the possible decomposition of Γ into various subtrees and next, to determine the overall cost savings in each subtree of Γ.

For the moment let us concentrate on one subtree $\Gamma^{(1)}$, rooted at node 0, of Γ. Let node $i_1 \in N^{(1)}$ be the unique follower of node 0 in $\Gamma^{(1)}$, that is $\Gamma^{(1)}$ contains the arc $(0, i_1)$. In case node i_1 has no followers in $\Gamma^{(1)}$ (that is, $\Gamma^{(1)}$ contains exactly one arc), then the subtree $\Gamma^{(1)}$ generates no cost savings. Without loss of generality we may suppose that node i_1 has at least one follower, say node $i_2 \in N^{(1)}$, in $\Gamma^{(1)}$, so the arc (i_1, i_2) belongs to $\Gamma^{(1)}$. The cost savings for the coalition $\{i_1, i_2\}$ in $\Gamma^{(1)}$ can be determined in two ways: on the one hand, $c(\{i_1\})+c(\{i_2\})-c(\{i_1,i_2\})=a((0,i_1))$ and on the other, by using (1.3), $v(\{i_1,i_2\})=\min[d_{i_1},\Delta]+\min[d_{i_2},\Delta]-\min[d_{i_1}+d_{i_2},\Delta]$. From this we deduce that

$$a((0,i_1))=\min[d_{i_1},\Delta]+\min[d_{i_2},\Delta]-\min[d_{i_1}+d_{i_2},\Delta]. \tag{2.1}$$

Without loss of generality we may suppose that $a((0,i_1))>0$ in $\Gamma^{(1)}$ (otherwise, if $a((0,i_1))=0$, then the arc $(0,i_1)$ is costless and we may reconsider node i_1 as the root instead of node 0, and so on). From (2.1), together with $a((0,i_1))>0$, it follows that $d_{i_1}+d_{i_2}>\Delta$ and hence, (2.1) reduces to

$$a((0,i_1))=\min[d_{i_1},\Delta]+\min[d_{i_2},\Delta]-\Delta \quad \text{in } \Gamma^{(1)}. \tag{2.2}$$

Consider an arbitrary third node $i_3 \in N^{(1)}$ in $\Gamma^{(1)}$ (which node is always connected, directly or indirectly, with node i_1). The previous reasoning applies once again (replace i_2 by i_3) to conclude that

$$a((0,i_1))=\min[d_{i_1},\Delta]+\min[d_{i_3},\Delta]-\Delta \quad \text{in } \Gamma^{(1)}. \tag{2.3}$$

As a consequence of (2.2) and (2.3), we obtain that

$$\min[d_{i_2},\Delta]=\min[d_{i_3},\Delta] \quad \text{for all } i_3 \in N^{(1)}\backslash\{i_1\}. \tag{2.4}$$

Since $d_{i_1}+d_{i_2}>\Delta$, we get that $\sum_{\ell\in N^{(1)}} d_\ell>\Delta$ and together with (1.3), this implies that the overall cost savings $v(N^{(1)})$ in the nontrivial subtree $\Gamma^{(1)}$ are the size of $\sum_{\ell\in N^{(1)}}\min[d_\ell,\Delta]-\Delta$. If it would happen that these overall cost savings in $\Gamma^{(1)}$ do not cover the overall cost savings $v(N)$ in Γ, which are the size of $\sum_{\ell\in N}\min[d_\ell,\Delta]-\Delta$, then the nonzero remaining cost savings are the size of $\sum_{\ell\in N\backslash N^{(1)}}\min[d_\ell,\Delta]$. The latter remaining amount, however, cannot be met by the cost savings generated by any number of subtrees different from $\Gamma^{(1)}$, because each nontrivial subtree $\Gamma^{(k)}$ on node set $N^{(k)}\cup\{0\}$ generates cost savings of which the size is strictly less than $\sum_{\ell\in N^{(k)}}\min[d_\ell,\Delta]$ (due to the fact that $\Delta>0$). We conclude that the underlying standard tree enterprise Γ contains exactly one nontrivial subtree rooted at node 0 (so, we ignore the trivial subtrees, rooted at node 0, of Γ consisting of one arc). Let node $i_1 \in N$ be the unique follower of node 0 in Γ and let node $i_2 \in N$ be some follower of node i_1 in Γ. So, both arcs $(0,i_1)$ and (i_1,i_2) belong to Γ. Furthermore, $d_{i_1}+d_{i_2}>\Delta$ and (2.2) holds. By (2.4), the essential result is as follows:

$$\min[d_j,\Delta]=\min[d_{i_2},\Delta] \quad \text{for all } j\in N\backslash\{i_1\}. \tag{2.5}$$

In order to establish that $\Delta \leqslant d_2$, we distinguish two cases regarding the number of followers of node i_1 in Γ.

Case 1: Suppose that node i_1 has at least two followers, say nodes i_2, $i_3 \in N$, in Γ, so both arcs (i_1, i_2) and (i_1, i_3) belong to Γ. The cost savings for the coalition $\{i_1, i_2, i_3\}$

in Γ can be determined in two ways: on the one hand, $c(\{i_1\}) + c(\{i_2\}) + c(\{i_3\}) - c(\{i_1, i_2, i_3\}) = 2a((0, i_1))$ and on the other, by using (1.3), $v(\{i_1, i_2, i_3\}) = \sum_{k=1}^{3} \min[d_{i_k}, \Delta] - \Delta$. From the induced equality $2a((0, i_1)) = \sum_{k=1}^{3} \min[d_{i_k}, \Delta] - \Delta$ and (2.2), (2.3), it follows that $\min[d_{i_1}, \Delta] = \Delta$. In summary, we obtain that $d_{i_1} \geqslant \Delta$ and $a((0, i_1)) = \min[d_{i_2}, \Delta] = \min[d_{i_3}, \Delta]$.

Now we are in a position to prove that $\Delta \leqslant d_2$. Assume, on the contrary, that $\Delta > d_2$. From $\Delta > d_2 \geqslant d_1$ and (2.5), we derive that $\Delta > d_j$ for all $j \in N \setminus \{i_1\}$. Together with $d_{i_1} \geqslant \Delta$, this yields that $i_1 = n$ and $d_1 = d_2 = \cdots = d_{n-1}$. Further, $a((0, n)) = \min[d_{i_2}, \Delta] = d_1$. We distinguish two subcases.

Subcase 1: Suppose that each node $j \in N \setminus \{n\}$ is a follower of node n, so all arcs of the form (n, j), $j \in N \setminus \{n\}$, belong to Γ. The cost savings for the coalition $N \setminus \{n\}$ in Γ can be determined in two ways: on the one hand, $\sum_{j \in N \setminus \{n\}} c(\{j\}) - c(N \setminus \{n\}) = (n-2)a((0, n)) = (n-2)d_1 > 0$ and on the other, by using (1.3), $v(N \setminus \{n\}) = (n-1)d_1 - \min[(n-1)d_1, \Delta]$. Since the cost savings are strictly positive, the induced equality $(n-2)d_1 = (n-1)d_1 - \min[(n-1)d_1, \Delta]$ reduces to $(n-2)d_1 = (n-1)d_1 - \Delta$, that is $d_1 = \Delta$.

Subcase 2: Suppose that not all nodes $j \in N \setminus \{n\}$ are followers of node n. Then there exists a node, say node $i_4 \in N \setminus \{n, i_2, i_3\}$, which is connected with some follower of node n, say with node i_2 (without loss of generality). So the arc (i_2, i_4) belongs to Γ. The cost savings for the coalition $\{n, i_2, i_4\}$ in Γ can be determined in two ways: on the one hand, $c(\{n\}) + c(\{i_2\}) + c(\{i_4\}) - c(\{n, i_2, i_4\}) = 2a((0, n)) + a((n, i_2)) = 2d_1 + a((n, i_2))$ and on the other, by using (1.3), $v(\{n, i_2, i_4\}) = 2\min[d_{i_2}, \Delta] = 2d_1$. The induced equality reduces to $a((n, i_2)) = 0$. Next the cost savings for the coalition $\{i_2, i_3, i_4\}$ in Γ can be determined in two ways: on the one hand, $c(\{i_2\}) + c(\{i_3\}) + c(\{i_4\}) - c(\{i_2, i_3, i_4\}) = 2a((0, n)) + a((n, i_2)) = 2d_1 > 0$ and on the other, by using (1.3), $v(\{i_2, i_3, i_4\}) = 3d_1 - \min[3d_1, \Delta]$. Since the cost savings are strictly positive, the induced equality $2d_1 = 3d_1 - \min[3d_1, \Delta]$ reduces to $2d_1 = 3d_1 - \Delta$, that is $d_1 = \Delta$.

In both subcases we arrive at $d_1 = \Delta$ which result contradicts the assumption that $\Delta > d_2 \geqslant d_1$. We conclude that $\Delta \leqslant d_2$. This completes the first case.

Case 2: Suppose that node i_1 has exactly one follower, namely node $i_2 \in N$, in Γ. Consider some follower $i_3 \in N$ of node i_2 in Γ, so the arc (i_2, i_3) also belongs to Γ. The cost savings for the coalition $\{i_1, i_2, i_3\}$ in Γ can be determined in two ways: on the one hand, $c(\{i_1\}) + c(\{i_2\}) + c(\{i_3\}) - c(\{i_1, i_2, i_3\}) = 2a((0, i_1)) + a((i_1, i_2))$ and on the other, by using (1.3), $v(\{i_1, i_2, i_3\}) = \sum_{k=1}^{3} \min[d_{i_k}, \Delta] - \Delta$. From the induced equality $2a((0, i_1)) + a((i_1, i_2)) = \sum_{k=1}^{3} \min[d_{i_k}, \Delta] - \Delta$ and (2.2), (2.3), it follows that $a((0, i_1)) + a((i_1, i_2)) = \min[d_{i_2}, \Delta]$. Now we are in a position to prove that $\Delta \leqslant d_2$. We distinguish two subcases.

Subcase 1: Suppose that node i_2 has at least two followers, say nodes $i_3, i_4 \in N$, in Γ, so both arcs (i_2, i_3) and (i_2, i_4) belong to Γ. The cost savings for the coalition $\{i_2, i_3, i_4\}$ in Γ can be determined in two ways: on the one hand, $c(\{i_2\}) + c(\{i_3\}) + c(\{i_4\}) - c(\{i_2, i_3, i_4\}) = 2a((0, i_1)) + 2a((i_1, i_2)) = 2\min[d_{i_2}, \Delta]$ and on the other, by using (1.3), $v(\{i_2, i_3, i_4\}) = 3\min[d_{i_2}, \Delta] - \min[\sum_{k=2}^{4} d_{i_k}, \Delta]$. The induced equality reduces to $\min[\sum_{k=2}^{4} d_{i_k}, \Delta] = \min[d_{i_2}, \Delta]$ and therefore, $d_{i_2} \geqslant \Delta$.

Subcase 2: Suppose that node i_2 has exactly one follower, say node $i_3 \in N$, in Γ. In case node i_3 has at least two followers in Γ, it can be shown that the arc (i_2, i_3) is costless as well as $d_{i_2} \geqslant \Delta$. In a more general setting, consider a chain enterprise consisting of the arcs (i_{k-1}, i_k), $1 \leqslant k \leqslant t$, in Γ, where $i_0 := 0$, so that node i_t has at least two followers, say nodes i_{t+1}, $i_{t+2} \in N$, and the arcs (i_{k-1}, i_k), $3 \leqslant k \leqslant t-1$, are costless. The cost savings for the coalition $\{i_k \mid 1 \leqslant k \leqslant t+1\}$ in Γ can be determined in two ways: on the one hand, $a((0, i_1)) + (t-1)\min[d_{i_2}, \Delta] + a((i_{t-1}, i_t))$ and on the other, by using (1.3), $\min[d_{i_1}, \Delta] + t\min[d_{i_2}, \Delta] - \Delta$. From the induced equality and (2.2), it follows that $a((i_{t-1}, i_t)) = 0$.

Next the cost savings for the coalition $\{i_t, i_{t+1}, i_{t+2}\}$ in Γ can be determined in two ways: on the one hand, $2\min[d_{i_2}, \Delta]$ and on the other, by using (1.3), $3\min[d_{i_2}, \Delta] - \min[\sum_{k=t}^{t+2} d_{i_k}, \Delta]$. The induced equality reduces to $\min[\sum_{k=t}^{t+2} d_{i_k}, \Delta] = \min[d_{i_2}, \Delta]$ and therefore, by (2.5), $d_{i_2} \geqslant \Delta$.

Eventually, we end up with a chain enterprise consisting of the arcs (i_{k-1}, i_k), $1 \leqslant k \leqslant n$, where $i_0 := 0$, so that the arcs (i_{k-1}, i_k), $3 \leqslant k \leqslant n-2$, are costless. Notice that this chain has a unique leaf, namely node i_n. The cost savings for the grand coalition N in Γ can be determined in two ways: on the one hand, $a((0, i_1)) + (n-2)\min[d_{i_2}, \Delta] + a((i_{n-2}, i_{n-1}))$ and on the other, by using (1.3), $\min[d_{i_1}, \Delta] + (n-1)\min[d_{i_2}, \Delta] - \Delta$. From the induced equality and (2.2), it follows that $a((i_{n-2}, i_{n-1})) = 0$. Next the cost savings for the coalition $N \backslash \{i_1\}$ in Γ can be determined in two ways: on the one hand, $(n-2)\min[d_{i_2}, \Delta] + a((i_{n-2}, i_{n-1})) = (n-2)\min[d_{i_2}, \Delta]$ and on the other, by using (1.3), $(n-1)\min[d_{i_2}, \Delta] - \min[\sum_{k=2}^{n} d_{i_k}, \Delta]$. The induced equality reduces to $\min[\sum_{k=2}^{n} d_{i_k}, \Delta] = \min[d_{i_2}, \Delta]$ and therefore, $d_{i_2} \geqslant \Delta$.

In both subcases we arrive at $d_{i_2} \geqslant \Delta$. From $d_{i_2} \geqslant \Delta$ and (2.5), we derive that $d_j \geqslant \Delta$ for all $j \in N \backslash \{i_1\}$ and hence, $d_2 \geqslant \Delta$. This completes the second case as well as the proof of the theorem. □

Corollary 2.4. *Provided that $n \geqslant 3$, the zero-normalized n-person bankruptcy game (N, v) of* (1.3) *can be represented as the cost savings game arising from a standard tree enterprise game with user set N if and only if the surplus of claims in the bankruptcy venture is at most the size of the second smallest claim (in the weak sense, i.e., $0 \leqslant \Delta \leqslant d_2$).*

Remark 2.5. The final stage of the proof of Theorem 2.3 can be shortened whenever there are at least three different sizes of claims among all claims d_i, $i \in N$. Given the latter circumstances, it follows immediately from (2.5) that $d_j \geqslant \Delta$ for all $j \in N \backslash \{i_1\}$ and hence, $d_2 \geqslant \Delta$.

Involving the bankruptcy venture with the three different claims $d_1 = 100$, $d_2 = 150$, $d_3 = 175$ (see Example 1.5), the surplus of claims Δ equals $425 - E$ and thus, $d_2 \geqslant \Delta$ iff $E \geqslant 275$. Particularly, by Corollary 2.4, the bankruptcy venture corresponding to the estate $E = 250$ cannot be associated with any standard tree enterprise on node set $\{0, 1, 2, 3\}$ (which result was already shown by means of Example 1.4).

Remark 2.6. Let us compare the current results with the results obtained by Driessen [6] who studied the interrelationship between bankruptcy games and minimum cost spanning tree games.

In the context of the spanning tree enterprise, the *player set* N represents the set of users in the cable-television network (i.e., nodes of the complete graph which is provided with a nonnegative arc-cost-function) and the *cost function* $c: 2^N \to \mathbb{R}$ is defined so that, for each coalition $S \subset N$, the cost figure $c(S)$ equals the least cost to connect all users in S to the central supplier 0 via arcs of a spanning tree on node set $S \cup \{0\}$. This type of a game is well-known as a *minimum cost spanning tree game*.

Driessen [6] established that the zero-normalized n-person bankruptcy game (N, v) of (1.3) can be represented as the cost savings game arising from a minimum cost spanning tree game with user set N if and only if one of the following two conditions is satisfied:

- The surplus of claims Δ in the bankruptcy venture is at most the size of the third smallest claim (in the weak sense, i.e., $0 \leqslant \Delta \leqslant d_3$)
- There exists a unique natural number $k \in \{3, 4, \ldots, n-1\}$ such that $d_k < \Delta \leqslant d_{k+1}$ and $\sum_{\ell=1}^{k} d_\ell \leqslant \Delta$.

Consequently, bankruptcy ventures that can be associated with some standard tree enterprise, can also be associated with some spanning tree enterprise (since $\Delta \leqslant d_2$ implies $\Delta \leqslant d_3$). However, the bankruptcy venture with the three different claims $d_1 = 100$, $d_2 = 150$, $d_3 = 175$ (see Example 1.5) and the variable estate E satisfying $250 \leqslant E < 275$, cannot be associated with any standard tree enterprise, but it can still be associated with some spanning tree enterprise on node set $\{0, 1, 2, 3\}$. For instance, consider the zero-normalized three-person bankruptcy game (N, v) in case $E = 250$ (see Example 1.5). It appears that the game (N, v) can be associated with the spanning tree enterprise on node set $\{0, 1, 2, 3\}$ of which the arc-cost-function a is given by $a((0,1)) = a((0,2)) = a((0,3)) = 350$, $a((1,2)) = 275$, $a((1,3)) = 250$, $a((2,3)) = 200$. Indeed, the cost savings game arising from the corresponding minimum cost spanning tree game equals the game (N, v). For further properties in the context of bankruptcy games and minimum cost spanning tree games, see [1, 6].

3. An application: the case of the nucleolus for a standard tree enterprise

In order to exploit the equivalence between bankruptcy ventures and standard tree enterprises, let us determine the nucleolus for the specific standard tree enterprises which we encountered within the proof of Theorem 2.3.

Let node $1 \in N$ be the unique follower of node 0 in the standard tree enterprise Γ and let node $2 \in N$ be some follower of node 1 in Γ (assuming that $n \geqslant 3$ and Γ has no trivial subtrees, rooted at node 0, consisting of one arc). So, both arcs $(0, 1)$ and $(1, 2)$ belong to Γ, where $a((0, 1)) > 0$. We distinguish two possibilities for the structure of the standard tree enterprise.

Structure 1: Suppose that node 1 has at least two followers in Γ (cf. case one in the proof of Theorem 2.3). It turns out that any arc in Γ, which is not incident with a leaf or node 0, is costless. Hence, the cost savings game (N,w) arising from the corresponding standard tree enterprise game (N,c) is given by $w(S)=(|S|-1)a((0,1))$ for all $S\subset N$, $S\neq\emptyset$. Notice that this game is symmetric, that is the cost savings merely depend on the number of users in coalition S and not the users themselves. Because of the symmetry property of the nucleolus, the total cost savings $w(N)$ are shared equally among the users.

Structure 2: Suppose that node 1 has exactly one follower, namely node $2\in N$, in Γ (cf. case two in the proof of Theorem 2.3). It turns out that any arc in Γ, which is not incident with a leaf or node 1, is costless. Hence, the cost savings game (N,w) arising from the corresponding standard tree enterprise game (N,c) is given by

$$w(S)=\begin{cases}(|S|-1)[a((0,1))+a((1,2))] & \text{for all } S\subset N\backslash\{1\},\ S\neq\emptyset, \\ & \quad \text{or } S=\{1\}, \\ a((0,1))+(|S|-2)[a((0,1))+a((1,2))] & \text{for all } S\subset N \text{ with } 1\in S, \\ & \quad S\neq\{1\}.\end{cases}$$

With the standard tree enterprise Γ, we associate the bankruptcy venture of which the claims d_i, $i\in N$, of the creditors and the estate E of the bankrupt concern are given by $d_1:=a((0,1))$, $d_i:=a((0,1))+a((1,2))$ for all $i\in N\backslash\{1\}$, and $E:=(n-1)a((0,1))+(n-2)a((1,2))$. Clearly, the surplus of claims in this bankruptcy venture satisfies $\Delta=a((0,1))+a((1,2))$, so $\Delta=d_i$ for all $i\in N\backslash\{1\}$ and $\Delta\geqslant d_1$. Now it follows from (1.3) that the corresponding zero-normalized n-person bankruptcy game (N,v) agrees with the cost savings game (N,w). Consequently, the cost allocation problem of allocating the least total costs $c(N)$ to the users in the standard tree enterprise is equivalent to the division problem of dividing the estate E among the creditors. In fact, any cost allocation $y\in\mathbb{R}^n$ in the standard tree enterprise corresponds to a savings allocation $x\in\mathbb{R}^n$ in the bankruptcy venture by means of

$$y_i=c(\{i\})+\max[0,d_i-\Delta]-x_i \quad \text{for all } i\in N$$

(based on the relationship that $c(S)=\sum_{j\in S}c(\{j\})+\sum_{j\in S}u(\{j\})-u(S)$ for all $S\subset N$, where (N,u) is the corresponding n-person bankruptcy game).

Let us apply the procedure involving the determination of the nucleolus for general bankruptcy ventures (cf. [2, 5]) to this specific bankruptcy venture. Because of $E\geqslant\frac{1}{2}\sum_{j\in N}d_j$, each creditor gets at least half of his claim. Each pair of creditors, not containing creditor 1, are substitutes (that is interchanging the roles of both creditors does not change the zero-normalized bankruptcy game (N,v)) and therefore, all creditors different from creditor 1 earn the same amount according to the nucleolus concept. We distinguish two cases.

Case 1: Suppose that the estate is significant, i.e., $E\geqslant\sum_{j\in N}d_j-\frac{1}{2}nd_1$, or equivalently, $(n-2)a((0,1))\geqslant 2a((1,2))$. Each creditor $i\in N$ receives the amount the size of $x_i=d_i-\alpha$ where $0\leqslant\alpha\leqslant\frac{1}{2}d_1$ is determined by the efficiency condition $\sum_{j\in N}d_j-n\alpha=E$. Thus, $\alpha=n^{-1}\Delta$ and we conclude that the cost allocation $y\in\mathbb{R}^n$, which represents the nucleolus for this specific standard tree enterprise, is given by $y_i = c(\{i\}) - x_i = c(\{i\}) - d_i + n^{-1}\Delta = n^{-1}[a((0,1)) + a((1,2))] + \delta_i$ for all $i \in N$, where δ_i represents the cost of the arc incident with user i, provided that user i is a leaf in Γ (otherwise, $\delta_i = 0$).

That is, according to the nucleolus concept, the least total costs $c(N)$ is allocated in such a way that each leaf in the standard tree enterprise is charged its separable cost (i.e., the cost of the arc incident with the leaf) and the remaining nonseparable costs the size of $a((0,1)) + a((1,2))$ is equally charged to all users in the standard tree enterprise. This completes the first case.

Case 2: Suppose that the estate is large, i.e., $E \leqslant \sum_{j \in N} d_j - \frac{1}{2}nd_1$, or equivalently, $(n-2)a((0,1)) \leqslant 2a((1,2))$. Creditor 1 receives $x_1 = \frac{1}{2}d_1$, while each other creditor $i \in N \setminus \{1\}$ receives the amount the size of $x_i = \alpha$ where $\frac{1}{2}d_2 \leqslant \alpha \leqslant d_2 - \frac{1}{2}d_1$ is determined by the efficiency condition $\frac{1}{2}d_1 + (n-1)\alpha = E$. Thus, $\alpha = (n-1)^{-1}[(n-\frac{3}{2})a((0,1)) + (n-2)a((1,2))]$ and we conclude that the cost allocation $y \in \mathbb{R}^n$, which represents the nucleolus for this specific tree enterprise, is given by $y_1 = c(\{1\}) - x_1 = \frac{1}{2}a((0,1))$ and $y_i = c(\{i\}) - x_i = a((0,1)) + a((1,2)) + \delta_i - (n-1)^{-1}[(n-\frac{3}{2})a((0,1)) + (n-2)a((1,2))] = (n-1)^{-1}[\frac{1}{2}a((0,1)) + a((1,2))] + \delta_i$ for all $i \in N \setminus \{1\}$.

That is, according to the nucleolus concept, the least total costs $c(N)$ is allocated in such a way that each leaf in the standard tree enterprise is charged its separable cost, node 1 is charged half of the cost of its connecting arc to the supplier, whereas the remaining costs the size of $\frac{1}{2}a((0,1)) + a((1,2))$ is equally charged to all users in the standard tree enterprise, except for user 1. This completes the second case.

For the general treatment of the nucleolus for standard tree enterprises, we refer to [10]. Let us conclude the paper with a remark concerning the possible representation of the cost savings game corresponding to a standard tree enterprise game as the zero-normalized version of some bankruptcy game. It is still an open problem to provide, if possible, necessary and sufficient conditions on the data of the standard tree enterprise in order to establish this (reversed) representation of standard tree enterprises versus bankruptcy ventures.

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