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An Incomplete Cooperation Structure for a Voting Game Can Be Strategically Stable

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Aumann and Myerson (1988) defined a noncooperative linking game leading to the formation of cooperation structures. They asked whether it is possible for a simple game to have a stable structure in which no coalition forms, i.e., in which the cooperation graph is not internally complete but is connected and stable. We answer this question affirmatively; specifically, we present a simple proper weighted majority game with a connected incomplete structure, and we prove it to be stable under *any* protocol for the strategic formation of new links. This result implies that strategically refused communication can be a robust stable phenomena. *Journal of Economic Literature* Classification Numbers: C71, C72.

1. INTRODUCTION

Consider the formation of a government in a multi-party parliament where parties are assumed to be disciplined, i.e., each party acts as one agent. Assume that parties can make bilateral cooperation agreements—open a line of communication. A coalition of parties is a winning coalition with respect to a given cooperation structure (a graph of bilateral links), if it contains a *connected* winning coalition, i.e., a coalition of parties that agree directly or indirectly to cooperate and possesses the required number of seats in parliament to govern.

Intuitively, if two parties *do not* form a link, it is as if they *commit to refuse* direct cooperation, but they do not rule out the possibility of cooperation via a third party. Thus, a right wing party and a left wing party might refuse to form a government comprising only their two parties, even

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though they both may have enough seats in parliament; but they both may agree to sit in the government providing that a center party is included. This kind of behavior might seem strange from a strategic point of view. If we measure the power of a party in a given cooperation structure by its Myerson value [the Shapley value of the game induced by the cooperation structure—Myerson (1977)], then it seems even more unreasonable when we observe that in this game adding a single link only benefits the two linking parties. However, it turns out that strategically refusing new links may be the best thing to do, since these links may motivate other links leading eventually to a decrease in power.

Aumann and Myerson (1988) suggested a model for the endogenous formation of cooperation structures. By allowing players to strategically form or to refuse links they wished to study the nature of the cooperation structures that will emerge under the assumption that payoffs are determined by the Myerson value. A natural question asked by Aumann and Myerson was whether a coalition will form? Or, stated in terms of the situation described above, will all communication channels be open in the formed government? Aumann and Myerson were able to prove that for a small class of voting games the answer to this question is positive. They showed that for general cooperative games the answer is negative. However, they left the question for simple (i.e., win−lose) games open.

In this paper we show that there is a simple game (actually it is even a proper weighted majority game) with a cooperation structure that is stable (no new links will be formed) but is not ‘‘internally complete’’ (no coalition has emerged). Moreover, this structure is robust in the sense that new links will not be formed no matter what protocol for link formation is used.

The stable structure we present has the following three properties: It is connected, i.e., every pair of parties is connected via a chain of links. It is incomplete—not all links are present. It is stable under every protocol, i.e., no new links will be added strategically to the cooperation structure under the restriction to subgame perfect equilibria in a linking game with an arbitrary protocol. The third property implies that when a new link is offered to two parties, at least one of them will refuse it—the reason being that the refusing party foresees a decrease in its strength if that link is added to the cooperation structure due to the triggering of the addition of other links.

Our example is of a parliament with 19 seats and eight parties having $5, 1, 2, 2, 2, 2, 4, 1$ seats, respectively, where a government is required to have at least 12 seats to rule. The cooperation structure depicted in Fig. 1 satisfies the three properties above. For the most part this paper is devoted to proving that this structure indeed satisfies stability.

Basically, under this structure, players $1, 2, 6, 7,$ and 8 stand to lose, eventually, if new links are added to the cooperation structure. What we

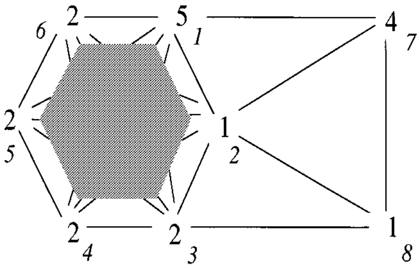

FIG. 1. We denote this graph by g; numbers at the vertices denote the "weight" (number of seats) of each party, and the small numbers in italic are used to index these eight parties. The gray area indicates that each of the players 1 through 6 is connected to each other one.

observe is a tangled web of credible threats which are made by these players and which lend this structure its stability.

Various approaches were used in the study of coalition formation and communication structures. Hart and Kurz (1983) incorporated a strategic approach to the formation of coalition structures by allowing agents to choose coalitions. Bloch (1996) analyzed the sequential nature of coalition structure formation. These studies and others [cf. the references in Bloch (1996)] considered coalition structures—the choice of belonging to a coalition—as opposed to the approach taken by Aumann and Myerson (1988) where communication or bilateral cooperation act as primitives and coalitions may not form at all. In a sense the former analyze *which* coalitions can be formed rather than *whether* coalitions will be formed at all. Approaching coalition formation from the communication perspective, Kirman *et al.* (1986) analyzed an economy where randomly formed (non-strategic) communication structures determine admissible coalitions. Random communication and coalition formation was also studied by Rosenthal (1992). In Borm, van den Nouweland and Tijs (1994) and the references there one can find the most related work on communication and coalition formation. In van den Nouweland (1995) one can find, among other things, an investigation and various extensions of the Aumann and Myerson model.[1]

This paper is constructed as follows: In Section 2 we present the linking game as defined in Aumann and Myerson (1988). Section 3 contains two lemmas used in the proof of stability, which is given in Section 4. The last section contains some remarks.

[1] It should be noted that recently a counterexample to the formation of coalitions in convex games under the Aumann and Myerson model was found by R. Holzman (private communication), this answers a question raised by van den Nouweland (1995, Conjecture 6.1.8).

2. DEFINITIONS

Let N be a finite non-empty set of players. Let g be a non-directed graph whose set of vertices is N, i.e., $g \subset \{\{i, j\} \mid i, j \in N \text{ and } i \neq j\}$. Let v be a coalitional form game on the player set N, i.e., let $v\colon 2^N \to \mathbb{R}$ and let $v(\emptyset) = 0$. Given (N, v), the *linking game given the graph* g [Aumann and Myerson (1988)] is defined as follows: The game starts with players linked according to the graph g. At each stage a pair of players is offered to form a link. If both players agree then the link is formed and the graph is updated (the link is added to the current graph); once formed, a link cannot be destroyed. The links are offered according to some definite list of links; this list is always assumed to be finite and includes all possible links. The list of links and the history of offers and responses at each stage are assumed to be common knowledge among players. Once the list is exhausted the offers start again from the top of the list. The game stops once no new links are formed in a consecutive run of the whole list. If h is the graph reached when the game ends then the payoff to each player is the Myerson value for the player in the game v with the cooperation structure given by the graph h [Myerson (1977)].

We denote by $\varphi_i^h(v)$ the Myerson value for player $i \in N$ in the game v with cooperation structure given by h, we will omit the notation for the game v when there is no risk of confusion, e.g., we will write φ_i^h. We denote by φ the Shapley value of the game {recall that $\varphi = \varphi^{c^N}$ [Myerson (1977)] where C^N is the complete graph on the set N}, and we define the sets of players $N_h^+ = \{i \in N \mid \varphi_i^h > \varphi_i\}$, $N_h^0 = \{i \in N \mid \varphi_i^h = \varphi_i\}$, and $N_h^- = \{i \in N \mid \varphi_i^h < \varphi_i\}$. Let $PEG(g)$—perfect equilibrium graphs of g—denote the set of all graphs that are reached by a sub-game perfect equilibrium in the linking game given g, with any finite list (v is fixed). The Myerson value operator is naturally defined on $PEG(g)$ using $\varphi(PEG(g)) = \{\varphi^h \mid h \in PEG(g)\}$. A graph g is called *stable* if for every possible list of links, all sub-game perfect equilibria of the linking game given g dictate that no additional links are formed, i.e., the game stops with the graph g. Using the notation above, g is stable if and only if $PEG(g) = \{g\}$. The restriction of a graph g to a subset of vertices S is defined as the graph $g/S := \{\{i, j\} \in g \mid i, j \in S\}$. Two vertices i, j are said to be connected in a graph g if there exist $n > 1$ and $i_1, \dots, i_n$ such that $i_1 = i$, $i_n = j$, and for all $k = 1, \dots, n-1$ we have $\{i_k, i_{k+1}\} \in g$. A non-empty set of vertices S is said to be connected in the graph g if each two vertices in S are connected in g. A connected set S is called a connected component of a graph g if it is connected in g and none of the vertices in S are connected to vertices not in S. Denote by $c^S = \{\{i, j\} \mid i, j \in S \text{ and } i \neq j\}$ the complete graph on the set of vertices S. A graph g is called *internally complete* if its restriction to any of its connected components is a complete graph.

The partition of a set of vertices S into connected components induced by a graph g is denoted by $S/g = \{T \subset S \mid T$ is a connected component of $g/S\}$. Thus a graph g on a player set N is internally complete if for all $S \in N/g$ one has $g/S = c^S$. The game restricted to (induced by) the graph g is denoted v/g and defined [Myerson (1977)] by $v/g(S) := \sum_{T \in S/g} v(T)$. Recall that $\varphi(v/g) = \varphi^g(v)$.

3. TWO LEMMAS

LEMMA 1. *Let g be a graph which satisfies $c^{N_g^- \cup N_g^0} \subset g$ and $N_g^+ \neq \emptyset$. If $\varphi^h = \varphi$ for all $h \in PEG(g) \setminus \{g\}$ then $PEG(g) = \{g\}$.*

Proof. Consider an arbitrary list of links L (finite and including all possible links). Adding a new link to g requires that at least one of the players in N_g^+ agrees to that link. Consider the last of the links in the list which includes a member of N_g^+, and denote the link by l. Now consider the list L' obtained by the cyclic permutation of L starting with the link l. Assume that none of the links prior to l were made under the list L. Then that link will be made only if there is a sub-game perfect equilibrium under the list L' at which the link l is made. But under our assumption, had there been such an equilibrium, it would have yielded the payoff φ, which means that at least one of the players in the link l will be worse off if he agrees to the link under the list L (recall that since no prior links where made, if l is not added to the graph, the linking game results with the original graph g). Going to the last link prior to l which includes a member of N_g^+ and assuming no prior links were made, we know that if this link is not added than l is not added. Using the same argument and the fact that we are considering only sub-game perfect equilibria, that link will not be added either. Using backward induction we see that none of the links of L which include a member of N_g^+ will be formed, hence no links will be formed under L. We have actually shown that g is stable under L, and since L was arbitrary g is stable. ■

LEMMA 2. *Let g be a maximal (inclusionwise) graph which satisfies $\varphi^g \neq \varphi$ and $c^{N_g^-} \subset g$. If $c^{N_g^- \cup N_g^0} \subset g$ then g is stable.*

Proof. Let h be a maximal graph strictly containing g which satisfies $\varphi^h \neq \varphi$, hence $c^{N_h^-} \not\subset h$. Since h is maximal all graphs strictly containing it yield the Shapley value. Thus under any list the players in N_h^- will always agree to form links with one another. So $h \notin PEG(h)$. Using backward induction, all graphs h strictly containing g which satisfy $\varphi^h \neq \varphi$ (and hence $c^{N_h^-} \not\subset h$ by g's maximality) must satisfy $h \notin PEG(h)$. This follows from the fact that the players in N_h^- know that if they link so will the players getting less than the Shapley value in the resulting graph, and so eventually we reach a graph where all the players receiving less than the Shapley value are connected. This graph must yield the Shapley value since g is maximal. But if $h \in PEG(g)$ then $h \in PEG(h)$, this is seen by taking the list leading from g to h in sub-game equilibrium. By the definition of the linking game, applying this list starting with the graph h with no new links added is supported by a sub-game perfect equilibrium. We conclude that any graph h strictly containing g which satisfies $\varphi^h \neq \varphi$ is not in $PEG(g)$, in other words for all $h \in PEG(g) \setminus \{g\}$ we have $\varphi^h = \varphi$ which implies according to Lemma 1 that g is stable. ■

4. PROOF OF STABILITY

Recall that we are considering the weighted majority game [12; 5, 1, 2, 2, 2, 2, 4, 1] and the cooperation structure given by the graph g in Fig. 1. The Shapley value of this game is

$$\varphi = \left(\frac{122}{420}, \frac{22}{420}, \frac{41}{420}, \frac{41}{420}, \frac{41}{420}, \frac{41}{420}, \frac{90}{420}, \frac{22}{420} \right)$$

and the Myerson value of the game with the cooperation structure given by g is[2]

$$\varphi^g = \left(\frac{123}{420}, \frac{27}{420}, \frac{42}{420}, \frac{38}{420}, \frac{38}{4210}, \frac{38}{420}, \frac{91}{420}, \frac{23}{420} \right).$$

Note that $N_g^- = \{4, 5, 6\}$, $c^{N_g^-} \subset g$, and $N_g^0 = \emptyset$. By Lemma 2 it suffices to show that for all graphs h such that $h \supset g$ and $h \neq g$, one gets either $\varphi^h = \varphi$ or $c^{N_h^-} \not\subset h$. Let h be a graph strictly containing g, we divide the proof according to the following four cases.

Case 1. $\{1, 8\} \in h$. In this case the Myerson value of players 1, 2, and 8 is at least their Shapley value. This results from the fact that for every $i \in \{1, 2, 8\}$ and for every coalition S such that $i \notin S$, $v(S) = 0$, and $v(S \cup \{i\}) = 1$ we also get that the coalition $S \cup \{i\}$ is connected in h, i.e., $v/h(S \cup \{i\}) = 1$ and $v/h(S) = 0$. Furthermore, if in the graph h player 7 is linked to any of the players 3, 4, 5, or 6, then $\varphi^h = \varphi$ since all winning coalitions in v are winning in v/h. If player 7 is not linked to the set of players $\{3, 4, 5, 6\}$ then we have $N_h^- = \{3, 4, 5, 6, 7\}$. To see this observe that

[2] The Shapley value is approximately $\varphi \cong (0.290, 0.052, 0.098, 0.098, 0.098, 0.098, 0.214, 0.052)$ and the Myerson value is approximately $\varphi^g \cong (0.293, 0.064, 0.100, 0.090, 0.090, 0.090, 0.217, 0.055)$.

for every $i \in \{3,4,5,6,7\}$ and every coalition S, $v/h(S \cup \{i\}) - v/h(S) = 1$ implies $v(S \cup \{i\}) - v(S) = 1$, but $v/h(\{3,4,5,6,7\}) - v/h(\{3,4,5,6,7\} \setminus \{i\}) = 0$ while $v(\{3,4,5,6,7\}) - v(\{3,4,5,6,7\} \setminus \{i\}) = 1$, hence players $3,4,5,6$, and 7 get strictly less than their Shapley value.

Case 2. $h \subset g \cup \{\{3,7\}, \{4,7\}, \{5,7\}, \{6,7\}\}$. Here for every $i \in \{1,8\}$ and for every coalition S, if $v/h(S \cup \{i\}) - v/h(S) = 1$ then we have $v(S \cup \{i\}) - v(S) = 1$. Yet there exists a coalition to which players 1 or 8 contribute in v but not in v/h, namely, $v/h(\{1,4,5,6,8\}) - v/h(\{1,4,5,6,8\} \setminus \{i\}) = 0$ while $v(\{1,4,5,6,8\}) - v(\{1,4,5,6,8\} \setminus \{i\}) = 1$. Thus, the Myerson value under h for players 1 and 8 is strictly less than their Shapley value.

Case 3. $\{i,7\} \in h$ and $\{j,8\} \in h$ for some $i \in \{3,4,5,6\}$, $j \in \{4,5,6\}$. In this case the winning coalitions in the game v/h are exactly those of the game h, and we have $\varphi^h = \varphi$.

Case 4. $h \subset g \cup \{\{4,8\}, \{5,8\}, \{6,8\}\}$. First we note that this case exhausts all of the remaining graphs. Graphs in which player 7 links with the set of players $\{3,4,5,6\}$ fall into one of the previous cases. Case 1 includes such links when 1 and 8 also link, Case 2 includes all graphs in which only such links are made, and Case 3 covers the rest of the graphs in which player 7 links with players $\{3,4,5,6\}$. Since Case 1 covers all graphs which include the link $\{1,8\}$, we are left only with graphs in which player 7 and the set $\{3,4,5,6\}$ are not connected and the link $\{1,8\}$ does not appear. This leaves us with graphs in which the only links added are the links of player 8 to the set of players $\{4,5,6\}$. For all graphs h of this case we have

$$\varphi^h = \left(\frac{249}{840}, \frac{49}{840}, \frac{79}{840}, \frac{79}{840}, \frac{79}{840}, \frac{79}{840}, \frac{177}{840}, \frac{49}{840}\right)$$

while

$$\varphi = \left(\frac{122}{420}, \frac{22}{420}, \frac{41}{420}, \frac{41}{420}, \frac{41}{420}, \frac{41}{420}, \frac{90}{420}, \frac{22}{420}\right)$$

thus $N_h^- = \{3,4,5,6,7\}$ and this set is not completely connected in h. ■

5. REMARKS

The example we gave has the following property: If you start with the cooperation structure given in Fig. 1, then subgame perfect equilibrium implies you will remain with that cooperation structure. However, it is not determined whether this (or any incomplete cooperation structure for a voting game) can be both stable and a result of a subgame perfect equilibrium when starting out with no links at all. We note that one stable structure may be contained in another stable structure (both for the same game); this is indeed the case in the following game:

Let $N = \{1, \dots, 8\}$ be the set of players and let v be defined as follows,

$$v(S) = \begin{cases} 1 & |S \cap \{1,2,3,4,5,6\}| \geq 5 \\ 1 & 1,6 \in S \text{ and } |S \cap \{2,3,4,5\}| = 2 \\ 1 & \{1,7,8\} \subset S \\ 0 & \text{otherwise} \end{cases}.$$

Here the graph $g = c^{\{1,2,3,4,5\}} \cup \{\{1,6\}, \{7,8\}\}$ and the graph $h = g \cup \{\{2,7\}, \{3,7\}, \{6,8\}\}$ are both stable. Furthermore, the graph g has two connected components. The proof of stability for these graphs can be done by observing all graphs strictly containing these graphs and using Lemma 2 as before. However, it is quite tedious and mechanical and thus is omitted.[3]

These examples demonstrate that even in the most simple cooperative situations, communication unfolds a complicated strategic structure that allows for sophisticated intrigue and carefully balanced (yet robust) non-trivial equilibria.

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[3]The length of the proof in this case stems from the fact that this simple game is not proper, and it is not a weighted majority game.