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GAMES OF THREATS

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GAMES OF THREATS

ELON KOHLBERG* AND ABRAHAM NEYMAN⁺

ABSTRACT. A game of threats on a finite set of players, N , is a function d that assigns a real number to any coalition, $S \subseteq N$, such that $d(S) = -d(N \setminus S)$. A game of threats is not necessarily a coalitional game as it may fail to satisfy the condition $d(\emptyset) = 0$. We show that analogs of the classic Shapley axioms for coalitional games determine a unique value for games of threats. This value assigns to each player an average of the threat powers, $d(S)$, of the coalitions that include the player.

1. INTRODUCTION

The Shapley value is the most widely studied solution concept of cooperative game theory. It is defined on coalitional games, which are the standard objects of the theory.

A *coalitional game* on a finite set of players, N , is a function v that assigns a real number to any subset (“coalition”), $S \subseteq N$, such that $v(\emptyset) = 0$. The amount $v(S)$ may be interpreted as the *worth* of S , i.e., what the players belonging to S can jointly get by coordinating their efforts.

A *value* is a function that assigns to each coalitional game a vector of payoffs, one for each player, that reflects the a priori evaluation of each player’s position in the game. In his celebrated paper [2] Shapley proposed four desirable properties (“axioms”) and proved the remarkable result that there exists a unique function satisfying these properties. This function, the Shapley value, can be described as follows. The value of player i is an average of the marginal contributions, $v(S \cup i) - v(S)$, of player i , where the average is taken over all random orders of N , with S denoting the subset of players that precede i in the random order.

A *game of threats* is a function d that assigns a real number to any coalition, $S \subseteq N$, such that $d(S) = -d(N \setminus S)$. The amount $d(S)$ may be interpreted as the threat power of the coalition S , i.e., the maximal difference between the total amounts that the players belonging to S and the players belonging to $N \setminus S$ receive, when the players in S coordinate their efforts to maximize this difference and the players in $N \setminus S$ coordinate their efforts to minimize it. Games of threats arise naturally in value theory for strategic games [1]. There, the condition $d(S) = -d(N \setminus S)$ is a consequence of the minmax theorem; see the Appendix.

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Now, games of threats need not be coalitional games, as they may fail to satisfy the condition $d(\emptyset) = 0$. Thus, if we wish to obtain an a priori evaluation for games of threats then we must develop a value theory for such games. This paper does that. We show that there is a unique function, from games of threats to n -dimensional payoff vectors, that satisfies the analogs of Shapley's four axioms; and that this function can be described as follows. The value of a player is the average of the threat powers, $d(S)$, of the coalitions that include the player. Specifically, if $d_{i,k}$ denotes the average of $d(S)$ over all k -player coalitions that include i , then the value of player i is the average of $d_{i,k}$ over $k = 1, 2, \dots, n$.

We take three approaches: a derivation of the results from the Shapley value theory for coalitional games, a direct derivation based on the formula, and a direct derivation based on the random-order approach. This last derivation establishes a formula for the Shapley value of games of threats that is analogous to the formula for coalitional games: the value of player i is the average of $d(S \cup i)$, where the average is taken over all random orders of N , with S denoting the subset of players that precede i in the random order.

We end this introduction by noting that the existence and uniqueness of the Shapley value for games of threats is an essential component in the proof of existence and uniqueness of a value for strategic games [1].

2. GAMES OF THREATS - DEFINITION

A *coalitional game of threats* is a pair (N, d) , where

- $N = \{1, \dots, n\}$ is a finite set of players.
- $d: 2^N \rightarrow \mathbb{R}$ is a function such that $d(S) = -d(N \setminus S)$ for all $S \subseteq N$.

Example:

$N = \{1, 2, 3\}$.

$d(\emptyset) = -1$, $d(1) = d(2) = -1$, $d(3) = 0$, $d(1, 2) = 0$, $d(1, 3) = d(2, 3) = 1$, $d(1, 2, 3) = 1$.

Denote by $\mathbb{D}(N)$ the set of all coalitional games of threats. By choosing, for every $S \subseteq N$, either S or $N \setminus S$, we can describe any $d \in \mathbb{D}(N)$ by means of 2^{n-1} numbers, thereby identifying $\mathbb{D}(N)$ with $\mathbb{R}^{2^{n-1}}$. One convenient choice is $(d(S))_{S \ni 1}$.

With this choice, the above example is described as follows.

$$d(1) = -1, \quad d(1, 2) = 0, \quad d(1, 3) = 1, \quad d(1, 2, 3) = 1.$$

3. THE SHAPLEY VALUE

Let $\psi: \mathbb{D}(N) \rightarrow \mathbb{R}^n$ be a map that associates with each game of threats an allocation of payoffs to the players. Following Shapley [2], we consider the following axioms.

For all games of threats $(N, d), (N, e)$, for all players i, j , and for all real numbers α, β (and using the notation $\psi_i d$ for $\psi(d)(i)$), the following properties hold:

- **Efficiency** $\sum_{i \in N} \psi_i d = d(N)$.

- **Linearity** $\psi(\alpha d + \beta e) = \alpha\psi d + \beta\psi e$.
- **Symmetry** $\psi_i(d) = \psi_j(d)$ if i and j are substitutes in d (i.e., if $d(S \cup i) = d(S \cup j)$ $\forall S \subseteq N \setminus \{i, j\}$).
- **Null player** $\psi_i d = 0$ if i is a null player in d (i.e., if $d(S \cup i) = d(S)$ $\forall S \subseteq N$).

Definition 1. A map $\psi: \mathbb{D}(N) \rightarrow \mathbb{R}^n$ satisfying the above axioms is called a value.

Theorem 1. There exists a unique value for games of threats. It may be described as follows:

$$(1) \quad \psi_i d = \frac{1}{n} \sum_{k=1}^n d_{i,k},$$

where $d_{i,k}$ denotes the average of the $d(S)$ over all k -player coalitions that include i .

In the example, $\psi_1 = \psi_2 = \frac{1}{3} \times (-1) + \frac{1}{3} \times \frac{1+0}{2} + \frac{1}{3} \times 1 = \frac{1}{6}$, $\psi_3 = \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times 1 = \frac{2}{3}$.

Note: Formula (1) allocates to each player a weighted average of the $d(S)$ over the coalitions S that include that player. The weight is the same for all coalitions of the same size but different for coalitions of a different size. Specifically, for each $k = 1, \dots, n$, the total weight of $\frac{1}{n}$ is divided among the $\binom{n-1}{k-1}$ coalitions of size k that include i :

$$(2) \quad \psi_i d = \frac{1}{n} \sum_{k=1}^n \frac{1}{\binom{n-1}{k-1}} \sum_{\substack{S: i \in S \\ |S|=k}} d(S) = \frac{1}{n} \sum_{S: i \in S} \frac{1}{\binom{n-1}{|S|-1}} d(S).$$

Note: Henceforth we shall refer to the map of equation 1 as the *Shapley value* for games of threats.

4. DERIVATION FROM CLASSIC SHAPLEY VALUE THEORY

Let $\mathbb{V}(N) := \{v: 2^N \rightarrow \mathbb{R}, v(\emptyset) = 0\}$ be the set of standard coalitional games on N . It can be identified with a subspace of \mathbb{R}^{2^n} of dimension $2^n - 1$.

Let $K: \mathbb{V}(N) \rightarrow \mathbb{D}(N)$ and $L: \mathbb{D}(N) \rightarrow \mathbb{V}(N)$ be defined by

$$(Kv)(S) = v(S) - v(N \setminus S)$$

and

$$(Ld)(S) = \frac{1}{2}d(S) + \frac{1}{2}d(N).$$

Note that the map L is linear, efficient (i.e., $L(d)(N) = d(N)$), symmetric (i.e., if i and j are substitutes in $d \in \mathbb{D}(N)$ then i and j are substitutes in $Ld \in \mathbb{V}(N)$), and preserves null players (i.e., if i is a null player in $d \in \mathbb{D}(N)$ then i is a null player in $Ld \in \mathbb{V}(N)$).

Thus, if φ is a value on $\mathbb{V}(N)$, then $\varphi \circ L$ is a value on $\mathbb{D}(N)$.

Note that $K \circ L$ is the identity on $\mathbb{D}(N)$; therefore – in particular – K is surjective. It follows that, if ψ_1 and ψ_2 are two different values on $\mathbb{D}(N)$, then $\psi_1 \circ K$ and $\psi_2 \circ K$ are two different values on $\mathbb{V}(N)$.

Thus if the value on $\mathbb{V}(N)$ is unique then the value on $\mathbb{D}(N)$ is unique. As the Shapley value φ is the unique value on $\mathbb{V}(N)$, $\varphi \circ L$ is the unique value on $\mathbb{D}(N)$.

This completes the proof of the existence and uniqueness of a value on $\mathbb{D}(N)$.

In order to show that the map ψ of Theorem 1 is, in fact, the Shapley value on $\mathbb{D}(N)$, it suffices to show that $\psi = \varphi \circ L$.

Recall that the Shapley value φ on $\mathbb{V}(N)$ is defined as follows. If $v \in \mathbb{V}(N)$ then

$$(3) \quad \varphi_i v := \frac{1}{n!} \sum_{\mathcal{R}} (v(P_i^{\mathcal{R}} \cup i) - v(P_i^{\mathcal{R}})),$$

where the summation is over the $n!$ possible orderings of the set N and where $P_i^{\mathcal{R}}$ denotes the subset of those $j \in N$ that precede i in the ordering \mathcal{R} .

Denote $S = P_i^{\mathcal{R}} \cup i$. In the ordering \mathcal{R} , the marginal contribution of player i is $v(S) - v(S - i)$. In the reverse ordering the marginal contribution is $(v(N \setminus S \cup i) - v(N \setminus S))$. Since the set of reverse orderings is the same as the set of orderings, we may replace each summand in the r.h.s. of (3) by the average marginal contributions in the ordering and its reverse. Thus

$$\varphi_i v = \frac{1}{n!} \sum_{S \ni i} \sum_{\mathcal{R}: P_i^{\mathcal{R}} = S - i} \left(\frac{1}{2} ((v(S) - v(N \setminus S)) + (v(N \setminus S \cup i) - v(S - i))) \right).$$

But as S ranges over the subsets of N that include i , so does $N \setminus S \cup i$. Thus we have

$$\varphi_i v = \frac{1}{n!} \sum_{S \ni i} (s - 1)!(n - s)! (v(S) - v(N \setminus S)).$$

Recall that $v(S) = (Ld)(S) = \frac{1}{2}d(S) + \frac{1}{2}d(N)$, and so $v(S) - v(N \setminus S) = d(S)$. Therefore the above equation can be rewritten as

$$\varphi_i v = \frac{1}{n} \sum_{k=1}^n \frac{(k-1)!(n-k)!}{(n-1)!} \sum_{S \ni i, |S|=k} d(S) = \frac{1}{n} \sum_{k=1}^n d_{i,k},$$

which is formula (1). Thus $(\varphi \circ L)d = \varphi(Ld) = \psi d$.

Remark: The map K is linear, efficient, symmetric, and preserves null players. Thus, if ψ is a value on $\mathbb{D}(N)$ then $\psi \circ K$ is a value on $\mathbb{V}(N)$. In particular, the existence of a value on $\mathbb{D}(N)$ implies the existence of a value on $\mathbb{V}(N)$.

Young [3] showed that the existence and uniqueness theorem for the Shapley value in $\mathbb{V}(N)$ remains valid when the axioms of linearity and null player are replaced by an axiom of *marginality*, which requires that the value of a player i in a game v depend only on the player's marginal contributions, $v(S \cup i) - v(S)$. Now the map L preserves marginality (i.e.,

if the marginal contributions of player i are the same in two games $d_1, d_2 \in \mathbb{D}(N)$ then the marginal contributions are the same in $Ld_1, Ld_2 \in \mathbb{V}(N)$; therefore the same argument as above implies the following.

Corollary 1. *The Shapley value is the unique mapping $\psi: \mathbb{D}(N) \rightarrow \mathbb{R}^n$ satisfying the axioms of efficiency, symmetry, and marginality.*

Remark: The map K also satisfies marginality. Thus the existence of a value satisfying Young's axioms on $\mathbb{D}(N)$ implies the existence of such a value on $\mathbb{V}(N)$.

We end this section by noting that the results for games of threats are also valid for constant-sum games.

Let $\mathbb{C}(N) := \{v: 2^N \rightarrow R, v(S) + v(N \setminus S) = v(N) \text{ for all } S \subseteq N\}$ be the set of *constant-sum* coalitional games on N . It can be viewed as a subspace of \mathbb{R}^{2^n} of dimension 2^{n-1} .

Let $K': \mathbb{V}(N) \rightarrow \mathbb{C}(N)$ and $L': \mathbb{C}(N) \rightarrow \mathbb{V}(N)$ be defined by

$$(K'v)(S) = \frac{1}{2}(v(S) - v(N \setminus S) + v(N))$$

and

$$(L'c)(S) = c(S).$$

Then both L' and K' are linear, efficient, symmetric, preserve null players, and preserve marginality, and $K' \circ L'$ is the identity on $\mathbb{C}(N)$. Therefore,

Corollary 2. *The Shapley value is the unique mapping $\psi: \mathbb{C}(N) \rightarrow \mathbb{R}^n$ satisfying the axioms of linearity, efficiency, symmetry, and null player, as well as the unique such mapping satisfying the axioms of efficiency, symmetry, and marginality.*

5. DIRECT DERIVATION FROM THE FORMULA OF THEOREM 1

Definition 2. *Let $T \subseteq N, T \neq \emptyset$. The unanimity game, $u_T \in \mathbb{D}(N)$, is defined by*

$$u_T(S) = \begin{cases} |T| & \text{if } S \supseteq T, \\ -|T| & \text{if } S \subseteq N \setminus T, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2. *The unanimity games span $\mathbb{D}(N)$.*

Proof. It is sufficient to show that the 2^{n-1} games $(u_T)_{T \ni 1}$ are linearly independent. Suppose, then, that $\sum a_j u_{T_j} = 0$, where $T_i \neq T_j$ for $i \neq j$ and not all the a_j are zero.

Since, for $i \neq j$, $T_i \neq T_j$ and $T_i \cap T_j \supseteq \{1\} \neq \emptyset$, neither set is contained in the other's complement and therefore

$$(4) \quad u_{T_j}(T_i) = \begin{cases} 1 & \text{if } T_i \supseteq T_j, \\ 0 & \text{otherwise.} \end{cases}$$

Among the T_j for which the coefficient a_j is non-zero choose one, say T_1 , with a minimum number of players. Then for any $j > 1$, $T_1 \not\supseteq T_j$ and therefore by (4), $u_{T_j}(T_1) = 0$. \square

Proof of Theorem 1.

In the unanimity game u_T , all $i \notin T$ are null players and all $i \in T$ are substitutes. It follows that any map that satisfies the efficiency, symmetry, and null player axioms, is uniquely determined on u_T :

$$(5) \quad \psi_i u_T = \begin{cases} \frac{1}{|T|} & \text{for } i \in T, \\ 0 & \text{for } i \notin T. \end{cases}$$

If the map also satisfies linearity then – by Proposition 2 – it is determined on all of $\mathbb{D}(N)$. This establishes uniqueness.

It is easy to verify that the map ψ defined in (1) satisfies linearity and symmetry. To verify that it satisfies the null player axiom, proceed as follows. Consider the basic condition $d(S) = -d(N \setminus S)$. As S ranges over all sets of size k that include i , $N \setminus S$ ranges over all sets of size $n - k$ that do not include i . Averaging over all these sets, we have

$$(6) \quad d_{i,k} = -\underline{d}_{i,n-k},$$

where $\underline{d}_{i,k}$ denotes the average of $d(S)$ over all k -player sets that do not include i .

Now, if i is a null player then $d(S) = d(S \cup i)$ for any set S that does not include i . Taking the average over all such sets S of size $n - k$, we have $\underline{d}_{i,n-k} = d_{i,n-k+1}$, which – combined with (6) – yields

$$d_{i,k} = -d_{i,(n-k+1)}.$$

Summing over $k = 1, \dots, n$ and dividing by n yields $\psi_i d = -\psi_i d$. Thus $\psi_i d = 0$.

It remains to prove efficiency, namely, that

$$(7) \quad \sum_{i=1}^n \psi_i d = d(N).$$

Let D_k denote the average of the $d(S)$ over all k -player coalitions. Since $d_{i,k}$ is the average of all the $d(S)$ with $i \in S$ and $|S| = k$, it follows that by symmetry the average of $d_{i,k}$ over $i = 1, \dots, n$ is D_k . Thus

$$\sum_{i=1}^n \psi_i d = \sum_{i=1}^n \frac{1}{n} \sum_{k=1}^n d_{i,k} = \sum_{k=1}^n \frac{1}{n} \sum_{i=1}^n d_{i,k} = \sum_{k=1}^n D_k.$$

Note that there is just one n -player coalition, namely, N , so $d_N = d(N)$. Thus, to prove (7), we must show that

$$\sum_{k=1}^{n-1} D_k = 0.$$

But this follows from the fact that

$$D_k = -D_{n-k} \text{ for } k = 1, \dots, n-1,$$

which in turn follows from the basic condition $d(S) = -d(N \setminus S)$, as seen by noting that as we take an average of the left side over all the sets of size k , the right side is averaged over all the sets of size $n - k$. \square

Note: The theorem remains valid when the axiom of linearity is replaced by the weaker axiom of additivity. The proof is the same, with the additional observation that efficiency, symmetry, and the null player axiom determine the value not only on the unanimity games but also on any multiple, cu_T , of such games.

6. DIRECT DERIVATION BY THE RANDOM ORDER APPROACH

In this section we provide an alternative definition of the Shapley value for games of threats that is analogous to the classic definition:

Proposition 3.

$$(8) \quad \psi_i d = \frac{1}{n!} \sum_{\mathcal{R}} d(P_i^{\mathcal{R}} \cup i),$$

where the summation is over the $n!$ possible orderings of the set N and where $P_i^{\mathcal{R}}$ denotes the subset of those $j \in N$ that precede i in the ordering \mathcal{R} .

Proof. The equation says that $\psi_i d$ is the average of the $d(S)$ over the sets S that include i , weighted according to the frequency of those random orderings where the players in S appears first, with i last among them.

The number of such orderings is $(|S| - 1)!(n - |S|)!$. Thus

$$(9) \quad \begin{aligned} \psi_i d &= \frac{1}{n!} \sum_{S: i \in S} (|S| - 1)!(n - |S|)! d(S) \\ &= \frac{1}{n!} \sum_{k=1}^n \sum_{\substack{S: i \in S \\ |S|=k}} (k - 1)!(n - k)! d(S), \end{aligned}$$

which is the same as (2). \square

Note that equation (8) may be rewritten as follows:

$$(10) \quad \psi_i d = -1/n! \sum_{R^*} d(P_i^{\mathcal{R}^*})$$

$$(11) \quad = -1/n! \sum_R d(P_i^{\mathcal{R}}),$$

where \mathcal{R}^* denotes the reverse ordering of \mathcal{R} .

To see (10), observe that $P_i^{\mathcal{R}^*}$ is the complement of $P_i^{\mathcal{R}} \cup i$, hence $d(P_i^{\mathcal{R}^*}) = -d(P_i^{\mathcal{R}} \cup i)$.

To see (11), observe that the map, $R \mapsto R^*$, from an ordering to its reverse, is a surjective and injective map on the space of orderings.

Note further that equation (9) can be rewritten as follows:

$$(12) \quad \begin{aligned} \psi_i d &= -\frac{1}{n!} \sum_{S: i \notin S} |S|!(n - |S| - 1)! d(S) \\ &= -\frac{1}{n!} \sum_{k=0}^{n-1} \sum_{\substack{S: i \notin S \\ |S|=k}} k!(n - k - 1)! d(S). \end{aligned}$$

This follows from equation (10), which says that $\psi_i d$ is minus the average of the $d(S)$ over the sets S that exclude i , weighted according to the frequency of those random orderings where the players in S appear first, with i immediately following them.

Next, we provide an alternative proof of Theorem 1 that's based on the random order approach.

Let ψ be the map defined by (8).

It is easy to verify that ψ satisfies linearity and symmetry. To prove efficiency, proceed as follows.

If two games satisfy the efficiency condition, (7), then so does any linear combination of them. By Proposition 2, then, it is sufficient to prove efficiency on the unanimity games.

Consider the unanimity game u_T and let $i \in T$. Then $u_T(P_i^{\mathcal{R}} \cup i) = 1$ if $(P_i^{\mathcal{R}} \cup i) \supseteq T$; i.e., if i is the last to appear among the members of T , and zero otherwise. Clearly, the probability that – in a random ordering – i is last among $|T|$ players is $\frac{1}{|T|}$. Thus $\psi_i u_T = \frac{1}{|T|}$.

Similarly, if $i \notin T$, then $u_T(P_i^{\mathcal{R}} \cup i) = 1$ if all members of T appear before i , -1 if all members of T appear after i , and zero otherwise. Clearly, the number of orderings in which i appears after all the members of T is the same as the number of orderings in which i appears before all the members of T (think of reversing the orderings), and hence $\psi_i u_T = 0$.

This establishes (5), which implies that $\sum_{i=1}^n \psi_i d = |T| \frac{1}{|T|} = 1$.

It remains to verify the null player axiom.

If i is a null player then $d(P_i^{\mathcal{R}} \cup i) = -d(P_i^{\mathcal{R}^*}) = -d(P_i^{\mathcal{R}^*} \cup i)$. As \mathcal{R} ranges over all the random orderings, so does \mathcal{R}^* . Therefore,

$$\psi_i d = \frac{1}{n!} \sum_{\mathcal{R}} d(P_i^{\mathcal{R}} \cup i) = -\frac{1}{n!} \sum_{\mathcal{R}^*} d(P_i^{\mathcal{R}^*} \cup i) = -\psi_i d,$$

which implies that $\psi_i d = 0$.

APPENDIX A. THE VALUE OF STRATEGIC GAMES

A strategic game is a triple $G = (N, A, g)$, where

- $N = \{1, \dots, n\}$ is a finite set of players,
- A^i is the finite set of player i 's pure strategies,
- $g = (g^i)_{i \in N}$, where $g^i: \prod_{i \in S} A^i \rightarrow \mathbb{R}$ is player i 's payoff function.

Denote by X^S the probability distributions on $\prod_{i \in S} A^i$; these are the correlated strategies of the players in S .

Let $G \in \mathbb{G}(N)$. Define the *threat power* of coalition S as follows:¹

$$(\delta G)(S) := \max_{x \in X^S} \min_{y \in X^{N \setminus S}} \left(\sum_{i \in S} g^i(x, y) - \sum_{i \notin S} g^i(x, y) \right).$$

By the minmax theorem, $\delta G(S) = -\delta G(N \setminus S)$ for any $S \subseteq N$. Thus δG is a game of threats.

We define the *value*, γG , of the strategic game G by taking the Shapley value of the game of threats δG .

In [1] it is proved that γ is the unique map that satisfies five properties that are desirable in a map providing an a priori evaluation of the position of each player in a strategic game. Four of these properties are analogs of the Shapley axioms for the value of cooperative games.

Formula (1) then implies the following: the value of a player in an n -person strategic game G is an average of the threat powers, $(\delta G)(S)$, of the subsets of which the player is a member. Specifically, if $\delta_{i,k}$ denotes the average of $(\delta G)(S)$ over all k -player coalitions that include i , then the value of player i is the average of $\delta_{i,k}$ over $k = 1, 2, \dots, n$.

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¹Expressions of the form max or min over the empty set are ignored.