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# SEQUENTIAL AGGREGATION OF JUDGMENTS

By

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מרכז פדרמן לחקר הרציונליות

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# Sequential aggregation of judgments

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**Abstract** We consider a standard model of judgment aggregation as presented, for example, in Dietrich (2015). For this model we introduce a *sequential aggregation procedure* (SAP) which uses the majority rule as much as possible. The ordering of the issues is assumed to be exogenous. The exact definition of SAP is given in Section 3. In Section 4 we construct an intuitive *relevance relation* for our model, closely related to conditional entailment. Unlike Dietrich (2015), where the relevance relation is given exogenously as part of the model, we require that the relevance relation be derived from the agenda. We prove that SAP has the property of *independence of irrelevant issues* (III) with respect to (the transitive closure of) our relevance relation. As III is weaker than the property of proposition-wise independence (PI) we do not run into impossibility results as does List (2004) who incorporates PI in some parts of his analysis. We proceed to characterize SAP by anonymity, restricted monotonicity, local neutrality, restricted agenda property, and independence of past deliberations (see Section 5 for the precise details). Also, we use this occasion to show that Roberts's (1991) characterization of *choice by plurality voting* can be adapted to our model.

*Keywords:* Judgment aggregation; Sequential procedure; Axiomatization; Relevance; Independence of Irrelevant Propositions (IIP).

*JEL Classification:* D70, D71.

## Introduction

We have two goals in this paper. The first is to argue that, practically, rules for judgment aggregation are sequential. The second is to describe and axiomatize a special sequential judgment aggregation rule. We start with the first objective. Let us consider the Doctrinal Paradox (see Example 4 on page 4). The three judges must first decide whether $p$ (= the contract is valid) is true. They might first ask whether $q$ (= the contract has been violated) is true but this will only lead to a permutation of their two decisions). Finally, they apply the law that "$p \wedge q$ if and only if $g$ (= the defendant is guilty)" to decide whether or not $g$ is true. One may now argue in general that a group of people cannot decide simultaneously on two (non-trivial) binary choices by majority rule. This is because majority decision takes time for communication, discussion, and persuasion,

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and we require throughout the entire paper that binary choices be resolved by majority rule if necessary (and not, for example, by forming subcommittees). Moreover, in certain situations the majority rule cannot handle more than two alternatives, as is evident from the Condorcet Paradox.

We are not, of course, the first to consider sequential aggregation. The first to do so, as far as we know, is List (2004). Conceptually, we use the same ideas: the first proposition is determined by majority rule. We proceed by induction: if propositions $p_1,\dots,p_k$ were chosen, $k \geq 1$, then we check whether $p_1 \wedge, \dots \wedge p_k \models q$ for some $q$ in the issue $I_{k+1}$. If the answer is positive, then we choose $q$. Otherwise, we choose the $(k+1)$-th proposition by majority. We immediately obtain anonymity, rationality, and unanimity. The main differences between our approach and List's are the following. (1) List incorporates in his algorithm *proposition-wise independence* (PI) (also called *independence of irrelevant alternatives*) except towards the end of his paper. As a result of this assumption, his conclusions are mainly negative. We use the weaker assumption of *independence of irrelevant propositions* due to Dietrich (2015). Thus we are able to obtain positive results. (2) List is also interested in the path independence of his algorithm, that is, independence of the collective judgment of the ordering of the issues (which might be arbitrary to some extent). We have in mind a parliament or a cabinet (or, more generally, a committee) that has to resolve a stream of issues that arrive one after the other. Thus, the issues in our model are conceived to be temporally ordered. This is a useful model but not, perhaps, the most general one.

We now describe briefly the contents of our paper. We start with the basic definitions that are relevant to the standard model of judgment aggregation. In Section 2 we adapt to the standard model a result of Roberts (1991) that yields an axiomatization of choice by plurality voting (CPV). His work relies on prior works of Young (1975) and Richelson (1978). The axioms for CPV are *anonymity, neutrality, unanimity*, and *reinforcement*. Section 3 presents our judgment aggregation rule as described in the second paragraph and illustrates it with the Doctrinal Paradox. We proceed with a modification of Dietrich's concept of *relevance relation*. In Dietrich (2015) PI is weakened to independence of irrelevant propositions (IIP), which is derived from an arbitrary relevance relation $R(P)$ on the agenda. To eliminate arbitrariness we require that the relevance relation be derived from the agenda. First we restrict ourselves only to entailments (implications); however, this works only for two issues. Then we devise a (rather sophisticated) intuitive relevance relation $R$ that is closely related to *conditional entailment*. Our *sequential aggregation procedure* (SAP) satisfies IIP with respect to the transitive closure $R^*$ of $R$. We conclude in Section 5 with a characterization of SAP. First, naturally SAP is rational and has full domain. It also satisfies *anonymity, restricted monotonicity, local neutrality*, and the *reduced agenda property* (i.e., sequentiality). The last property of the characterization is *independence of past deliberations* (IPD). It means that society's choice on an issue I depends only on society's choices on the previous issues and the choices of the individuals on I. We would like to add that intuitively SAP is the sequential judgment aggregator that uses the majority rule in its decision most extensively. One might argue that using the relationship between the propositions of the agenda more strongly might lead to a more efficient aggregator. However, this remains to be seen.

## 1 The model

There is a finite group of decision makers (or players) $N = \{1, 2, \ldots, n\},\ \ n \geq 2$ . They are examining a set of propositions $X = \{p_1, \ldots, p_k, ...\}$ that may be finite or infinite. With each proposition $p \in X$ the negation of $p$, $\neg p$ is also in $X$. An agenda $A_k = \{p_1, \neg p_1, \ldots, p_k, \neg p_k\}$ is a finite subset of $X$ that contains with each proposition $q \in A_k$ its negation $\neg q$. An *issue* is a pair of propositions $I = (p, \neg p)$. Thus, the agenda is partitioned into a finite set of issues: $A_k = \{I_1, \ldots, I_k\}$. A judgment $J$ is a subset of $A$ with the property that whenever $q \in J$, then $\neg q$ is not in $J$. A judgment $J$ is *complete* if for each $p$ not in $J$ we have $\neg p \in J$. A certain nonempty set $\mathscr{J}$ of complete judgments is known to all players as the set of *rational judgments*. A judgment $J$ is *consistent* if it is contained in a rational judgment. A set of propositions $S \subset A$ *entails* a proposition $p \in A$, denoted by $S \models p$, if whenever $S$ is contained in a rational judgment $J$, then $p \in J$. By this definition, the relation of entailment satisfies the following properties: for any propositions $p \in A$ and $q \in A$ and sets of propositions $S \subset A$ and $T \subset A$,

*Monotonicity:* If $S \models p$ and $T \supseteq S$ then $T \models p$.

*Transitivity:* If $S \models p$ and $S \cup \{p\} \models q$ then $S \models q$.

These two properties imply the following weaker version of transitivity:

*Weak Transitivity:* If $S \models p$ and $p \models q$ then $S \models q$.

To obtain significant results, the set of rational judgments must satisfy some minimal properties. To that end we make the following assumption (see Dietrich 2016).

**Assumption** The set $\mathscr{J}$ of rational judgments has *no tautologies*; that is, there is no proposition $p \in A$ such that $p \in J$ for all $J \in \mathscr{J}$.

This assumption also guarantees that the set $\mathscr{J}$ of rational judgments is "rich" enough in the sense that for each $p \in J$ there is $J \in \mathscr{J}$ such that $p \in J$.

**Definition 1** A *judgment aggregation problem* (JAP) is a 4-tuple $g = (N, A_k, \neg, \mathscr{J})$, where $N$ is the set of players (decision makers, judges, etc.), $A_k$ is the agenda, and $\mathscr{J}$ is the set of rational judgments.

**Definition 2** An *aggregation function* (AF) for a JAP is a function $F : \mathscr{J}^n \to \mathscr{J}$.

*Example 1 Propositional Calculus*. Let $\mathscr{L}$ be a propositional language on a given (countable) set of atoms, endowed with the following functions: for $p \in \mathscr{L}$, $\neg p$ (not $p$) (with $\neg p\ \neq p$ and $\neg\neg p = p$), $p_1 \wedge p_2$ (both

$p_1$ and $p_2$ are true), $p_1 \vee p_2$ ($p_1$ or $p_2$ is true), and $p_1 \Rightarrow p_2$ ($p_1$ implies $p_2$). The set of *rational judgments* is the set of judgments with no logical contradictions.

*Example 2 The semantic model* (see, e.g., Dietrich 2014, Section 2). In this model, the propositions are subsets of a finite set $\Omega = \{a_1, a_2, \ldots, a_m\}$ and the negation of a proposition $p \subset \Omega$ is its complement w.r.t. $\Omega$; $\neg p = \Omega \setminus P$. The *entailment* $\models$ is represented by set *inclusion* $\subset$, the *conjunction* $\wedge$ is represented by *intersection* $\cap$, and the *disjunction* $\vee$ is represented by set *union* $\cup$.

*Example 3 Preference aggregation*. Given a set $S = \{a, b, \ldots\}$ of social alternatives, the propositions are of the form $a \succ b$ (or $a \succeq b$). A judgment of a player is his (complete or incomplete, weak or strict) preference order on the set of social alternatives, and consistency is imposed by the acyclicity of the (strict) preferences.

*Example 4 The Doctrinal Paradox.* In the situation described in the Doctrinal Paradox, our AF provides a complete and consistent aggregation and the "paradox" is just a manifestation of the fact that the resulting aggregated judgment depends on the order in which the issues are decided. In the propositional calculus setting the paradox is presented as follows. Consider three judges deliberating on the following issues:

- $p$ – The contract is legally valid.
- $q$ – The defendant has broken the contract.
- $g$ – The defendant is liable.
- By law, $g \Leftrightarrow p \wedge q$.

Assume that the judgments of the three judges are those given in the following table (where 1 indicates that the proposition is true and 0 indicates that it is false):

Issues

| | $p$ | $\neg p$ | $q$ | $\neg q$ | $g$ | $\neg g$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 3 | 0 | 1 | 1 | 0 | 0 | 1 |

Aggregation of propositions by simple majority voting yields:

Issues

| | $p$ | $\neg p$ | $q$ | $\neg q$ | $g$ | $\neg g$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 3 | 0 | 1 | 1 | 0 | 0 | 1 |
| | **1** | **0** | **1** | **0** | **0** | **1** |

This aggregated judgment is **inconsistent** as $p$ and $q$ are accepted and yet $\neg g$ is also accepted. In other words, the "paradox" is that

– By the *premise-based rule*:

  $p$ and $q$ are accepted and hence the verdict is $g$ (*guilty*).

– By the *conclusion-based rule*:

  $\neg g$ is accepted by majority rule and the verdict is *not guilty*.

## 2 Choice by plurality voting (CPV)

**Definition 3** Let $g = (N, A_k, \neg, \mathscr{J})$ be a JAP. A *judgment aggregation correspondence* (JAC) is a function $F : \mathscr{J}^N \to 2^{\mathscr{J}}$, assigning a set of judgments to each judgment profile.

**Definition 4** *Choice by plurality voting (CPV)* is the aggregation correspondence $F$ defined by:

$$F(J^N) = \{J^i, i \in N : J^i \in J^N \text{ and } \mid j' : J^{j'} = J^j \mid \leq \mid j' : J^{j'} = J^i \mid, \forall j \in N\}$$

In words, given a judgment profile, the AC chooses those judgments in the profile that are shared by the largest number of judges. This aggregation correspondence shares the following properties:

– *Anonymity*: For all profiles $J^N \in \mathscr{J}^N$ and for all permutations $\pi$ of $N = \{1, 2, \dots, n\}$,

$$F(J^{\pi(1)}, \dots, J^{\pi(n)}) = F(J^1, \dots, J^n).$$

– *Neutrality*: For all permutations $\sigma$ of $\mathscr{J}$ and for all profiles $J^N \in \mathscr{J}^N$,

$$F(\sigma(J^1), \dots, \sigma(J^n)) = \sigma(F(J^1, \dots, J^n)).$$

– *Unanimity*: For all judgments $J \in \mathscr{J}$,

$$F(J, \dots, J) = \{J\}.$$

– *Reinforcement*: Let $(N, A_k, \neg, \mathscr{J})$ and $(M, A_k, \neg, \mathscr{J})$ be two judgment aggregation problems with the same agenda and disjoint sets of judges, $N$ and $M$; $N \cap M = \emptyset$.

  If $F(J^N) \cap F(J^M) \neq \emptyset$, then (in JAP $(N \cup M, A_k, \neg, \mathscr{J})$),

$$F(J^N, J^M) = F(J^N) \cap F(J^M).$$

**Theorem 1** *The choice by plurality voting is the only judgment aggregation correspondence that satisfies anonymity, neutrality, unanimity, and reinforcement.*

*Proof* This follows readily from Roberts (1991) who, following Young (1975) and Richelson (1978), considered a choice function (or correspondence) from an abstract set $X$ of alternatives and any number of voters: $f : \bigcup_{n=1}^{\infty} X^n \to P_0(X)$, where $P_0(X)$ is the set of nonempty subsets of $X$. Roberts provided several sets of axioms characterizing the CPV correspondence in his abstract aggregated choice model. Our characterization theorem is a special case of Roberts's results for $X = \mathscr{J}$ that states that our stated properties, *anonymity, neutrality, unanimity*, and *reinforcement*, characterize the CPV correspondence, (Theorem 3 (case 4) in Roberts 1991). ∎

*Example 5 (The Doctrinal Paradox revisited.)* For the classical example of the Doctrinal Paradox,

Issues

| | $p$ | $\neg p$ | $q$ | $\neg q$ | $g$ | $\neg g$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 3 | 0 | 1 | 1 | 0 | 0 | 1 |

we have $F(pqg, p\neg q\neg g, \neg pq\neg g) = \{pqg, p\neg q\neg g, \neg pq\neg g\}$.

In other words, the judgment of each of the judges can be chosen.

Consider now the following variant of the situation with five judges:

Issues

| | $p$ | $\neg p$ | $q$ | $\neg q$ | $g$ | $\neg g$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 3 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 4 | 0 | 1 | 1 | 0 | 0 | 1 |
| Judge 5 | 0 | 1 | 0 | 1 | 0 | 1 |

We note that the same "paradox" persists, but now $F(J^N) = \{pqg\}$. In particular, the verdict is *Guilty*.

Consider now the following variant of the situation with five judges:

Issues

| | $p$ | $\neg p$ | $q$ | $\neg q$ | $g$ | $\neg g$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 3 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 4 | 0 | 1 | 1 | 0 | 0 | 1 |
| Judge 5 | 0 | 1 | 1 | 0 | 0 | 1 |

Again, the same "paradox" persists, but now $F(J^N) = \{p\neg q\neg g, \neg pq\neg g\}$. In particular, the verdict is *'Not guilty'*.

## 3 Sequential aggregation procedure (SAP)

Given an agenda with $k$ issues $A_k = \{I_1, \ldots, I_k\}$, when the issues are ordered (for example, temporally), we write a judgment as an ordered array $J = (q_1, \ldots, q_k)$ where $q_\ell \in I_\ell;\ \ell = 1, \ldots, k$, and we denote:

- $J_\ell = q_\ell$, the judgment for the $\ell$-th issue $I_\ell$.
- $J_{|\ell} = (q_1, \ldots, q_\ell)$, the judgment for the first $\ell$ issues $\{I_1, \ldots, I_\ell\}$.

For any profile $J^N \in \mathscr{J}^N$ we denote:

- $J^N_\ell = (J^1_\ell, \ldots, J^n_\ell)$, the profile of judgments for the issue $I_\ell$.
- $J^N_{|\ell} = (J^1_{|\ell}, \ldots, J^n_{|\ell})$, the profile of judgments for the first $\ell$ issues $\{I_1, \ldots, I_\ell\}$.

Let $g = (N, A_k, \neg, \mathscr{J})$ be a JAP and let $S$ be a union of issues in $g$; then $S$ defines the *sub-problem* $g(S) = (N, S, \neg, \mathscr{J} \cap S)$ where $\mathscr{J} \cap S = \{J \cap S | J \in \mathscr{J}\}$. When the issues are ordered we define a sequential aggregation function as follows.

**Definition 5** Let $A_k = (I_1, \ldots, I_k)$ and $S_\ell = (I_1 \cup, \ldots, \cup I_\ell),\ \ell = 1, \ldots, k$. A *sequential aggregation function* for $g$ is a sequence of AF's, $(F_1, \ldots, F_k)$, where $F_\ell$ is an aggregation function of $g(S_\ell)$ for $\ell = 1, \ldots, k$, such that for every profile $J^N = (J^1, \ldots, J^n)$ and every $\ell = 1, \ldots, k-1$,

$$F_\ell(J^1 \cap S_\ell, \ldots, J^n \cap S_\ell) = F_{\ell+1}(J^1 \cap S_{\ell+1}, \ldots, J^n \cap S_{\ell+1}) \cap S_\ell.$$

**Definition 6** Let $(N, A_k, \neg, \mathscr{J})$ be a JAP with an agenda consisting of $k$ issues (that is, $\#A_k = 2k$). The *sequential aggregation procedure (SAP)* is the sequential aggregation function defined inductively on $k$ as follows.

- For $k = 1$, i.e., $A_1 = \{p, \neg p\}$, choose between $p$ and $\neg p$ by majority rule (with anonymous tie-breaking).
- Assume that SAP has been defined for $k \geq 1$ and consider an (ordered) agenda with $k+1$ issues: $A_{k+1} = (\{p_1, \neg p_1\}, \ldots, \{p_k, \neg p_k\}, \{p_{k+1}, \neg p_{k+1}\})$. For a given profile $J^N \in \mathscr{J}^N$, let $SAP(J^N_{|k}) = (q_1, ..., q_k)$. Then,
  1. If $\{q_1, ..., q_k\} \models p_{k+1}$, then SAP chooses $p_{k+1}$ for the $(k+1)$-th issue.
  2. If $\{q_1, ..., q_k\} \models \neg p_{k+1}$, then SAP chooses $\neg p_{k+1}$ for the $(k+1)$-th issue.
  3. Otherwise, we call $\{p_{k+1}, \neg p_{k+1}\}$ a *free issue*, and SAP chooses from $\{p_{k+1}, \neg p_{k+1}\}$ by majority rule with *anonymous tie-breaking* rule.

*Remark 1* Note that the above-defined SAP is indeed a sequential aggregation function according to Definition 5 and that $F_k(J^N)$ is consistent for all $J^N \in \mathscr{J}^N$.

*Remark 2* We emphasize that the foregoing SAP depends on the order of introducing the members of $A$ that we have chosen. Different orderings yield different aggregators, as is the case in the well-known Doctrinal Paradox.

*Remark 3* The above-defined procedure is actually a family of procedures since the *anonymous tie-breaking* rule need not be the same for all free issues. Different tie-breaking rules may lead to different aggregated judgments. However, this is relevant only when the number of judges $n$ is even since when $n = 2k+1$ is odd, no tie can occur and SAP determines the aggregated judgment uniquely.

*Example 6 (The Doctrinal Paradox revisited)* . The classical example of the Doctrinal Paradox is

Issues

| | $p$ | $\neg p$ | $q$ | $\neg q$ | $g$ | $\neg g$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 3 | 0 | 1 | 1 | 0 | 0 | 1 |

If we apply our SAP with the order of issues $(p,q,r)$ we obtain:

Issues

| | $p$ | $\neg p$ | $q$ | $\neg q$ | $g$ | $\neg g$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 3 | 0 | 1 | 1 | 0 | 0 | 1 |
| $SAP(J)$ | **1** | **0** | **1** | **0** | **1** | **0** |

That is, the aggregate judgment is $pqg$ (in particular, the defendant is liable).

If the order of issues is $(p,g,q)$ we obtain

Issues

| | $p$ | $\neg p$ | $g$ | $\neg g$ | $q$ | $\neg q$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 1 | 0 | 0 | 1 |
| Judge 3 | 0 | 1 | 0 | 1 | 1 | 0 |
| $SAP(J)$ | **1** | **0** | **0** | **1** | **0** | **1** |

That is, the aggregate judgment is $p\neg g\neg q$ (in particular, the defendant is *not* liable). The same aggregated judgment is obtained for the order $(g,p,q)$, while the orders $(q,g,p)$ and $(g,q,p)$ yield $q\neg g\neg p$.

We shall argue that in each aggregation problem there is a natural order in which the issues are deliberated. In this example $p$ and then $q$ seem to be the natural temporal order. However, even when the order is given, the aggregation procedure is vulnerable to manipulation. For example, in the above-described situation judge 3, who thinks that the contract is invalid ($\neg p$) and therefore thinks that the defendant is not liable, may dishonestly vote for $\neg q$ in order to reach the verdict "not liable" ($\neg g$).

The following is readily verified:

**Proposition 1** *$SAP = \{F_1, \ldots, F_k\}$ shares the following properties:*

1. *Rationality: The aggregated $F_k(J^N)$ is consistent and complete.*
2. *Anonymity.*
3. *Unanimity.*
4. *Restricted Agenda: If $\ell \leq k$ then: $F_\ell(J^N_{|\ell}) = (F_k(J^N))_{|\ell}, \ \forall J^N \in \mathscr{J}^N$.*

## 4 Relevance Relations: From IIA to III

The most crucial axiom in Arrow's impossibility theorem is IIA – *independence of irrelevant alternatives*. The analogue axiom for judgment aggregation would be PI – *proposition-wise independence*. It turns out that this axiom is too strong and, together with a few mild assumptions, it readily yields impossibility results (see, e.g., List 2012). Any attempt to obtain positive results must go through weakening this axiom. Such a weakening was suggested by Dietrich (2015) who replaced PI by IIP – *independence of irrelevant propositions*, with respect to an abstract given relevance relation. We adopt this idea but attempt to derive the *relevance relation* from the agenda: we will derive a "natural" *relevance relation* between propositions in the agenda and show that our proposed aggregation function satisfies IIP. We first recall that Dietrich assumed that the (abstract) relevance relation $R$ between propositions satisfies two conditions (we adopt Dietrich's notation and write $\{\pm p\}$ for $\{p, \neg p\}$):

- *Negation-invariance* (Dietrich 2015 Equation (1), p. 470):

$$qRp \Leftrightarrow q'Rp' \text{ if } q' \in \{\pm q\} \text{ and } p' \in \{\pm p\}.$$

- *Non-underdetermination* (Dietrich 2015 p. 470): every proposition is settled by the judgments on the relevant propositions, i.e., for every $p \in X$ and every consistent set $S$ of the form $S = \{q | q' \in R(p) \text{ where } q' \in \{\pm q\}\}$, one of the following conditions holds:
  - either $S$ entails $p$ ($S$ is then called an *R-explanation* of $p$),
  - or $S$ entails $\neg p$ ($S$ is then called an *R-refutation* of $p$).

We notice that a relation $R$ satisfying negation invariance is actually a relation between *issues*; therefore, we will adopt this terminology and define a relevance relation between the issues of the agenda $A = \{I_1, \ldots, I_k\}$.

**Definition 7** A *relevance relation $R$* is a reflexive and acyclic binary relation between the issues of the agenda $A$. "$I_j$ is relevant to $I_h$" is denoted by $I_jRI_h$ and for $I_h \in A$, the set $R(I_h) = \{I_j | I_jRI_h\}$ is the set of issues relevant to issue $I_h$. For convenience, when no confusion may arise, we use the same notation for the set of propositions in these issues, i.e.,

$$R(I_h) = \cup\{p_j, \neg p_j\} : I_j = \{p_j, \neg p_j\} R I_h.$$

The analogue of the IIA axiom is the III axiom (*independence of irrelevant issues*) defined as follows.

**Definition 8 (Independence of irrelevant issues (III))** . Given a JAP, $g = (N, A_k, \neg, \mathscr{J})$, a judgment aggregation function $F : \mathscr{J}^N \to \mathscr{J}$ satisfies *independence of irrelevant issues* (III) w.r.t. the relevance relation $R$, if for all $J_1^N, J_2^N \in \mathscr{J}^N$, and for all $I_h \in A$,

$$\left[J_1^i \cap R(I_h) = J_2^i \cap R(I_h),\ \forall i\ \in N,\ \text{and}\ p^* \in I_h\right] \Longrightarrow \left[p^* \in F(J_1^N) \Leftrightarrow p^* \in F(J_2^N)\right].$$

**Example:** If $R(I_h) = \{I_h\}$ for all $I_h \in A$, then for $p^* \in I_h$,

$$\left[J_1^i \cap R(I_h) = J_2^i \cap R(I_h),\ \forall i\ \in N\right] \Leftrightarrow \left[p^* \in J_1^i \Leftrightarrow p^* \in J_2^i;\ \forall i\ \in N; \forall p^* \in I_h\right],$$

and III is equivalent in this case to proposition-wise independence (PI).

The first natural attempt to derive a relevance relation from the agenda is

**Definition 9 (Relevance by direct entailment)** . Given an agenda $A_k$ of $k$ issues and a fixed order $A_k = \{I_1, \ldots, I_k\}$, the relevance relation $EM$ (*entailment*) is a correspondence $EM : A_k \to 2^{A_k}$ defined by,

$$I_j \in EM(I_h)\ \text{ if }\ j \le h\ \text{ and }\ [\exists\, q^* \in I_j \text{ and } \exists p^* \in I_h \text{ such that } q^* \models p^*].$$

When $p \in I_h$ we also write $EM(p)$ for $EM(I_h)$.

*Remark 4* We note that

1. This relevance relation is *reflexive* $(I_h \in EM(I_h))\, ; \forall I_h \in A$, but it is *not transitive*.
2. This relevance relation is *not symmetric*; that is, $I_j R I_h$ does not imply $I_h R I_j$. Furthermore, for $j \neq h$, if $I_j R I_h$ then $I_h R I_j$ cannot hold even if $p^* \models q^*$ for some $q^* \in I_j$ and $p^* \in I_h$ since $j \le h$ excludes $h \le j$ for $j \neq h$. In other words, the issue $I_h$ is irrelevant to the issue $I_j$ even if there is a logical implication since it is decided *after* $I_j$.

Nevertheless, for the case of two issues we have:

**Proposition 2** *For $k = 1, 2$, the aggregation function F, given in Definition 3, satisfies independence of irrelevant issues (III) w.r.t. the relevance relation EM defined by Definition 9.*

*Proof* We have to prove that for each $j \le k,\ p \in \{p_j, \neg p_j\}$, and all $J_1^N, J_2^N \in \mathscr{J}^N$,

$$J_1^i \cap EM(p) = J_2^i \cap EM(p),\ \forall i\ \in N \Longrightarrow \left[p \in F(J_1^N) \Leftrightarrow p \in F(J_2^N)\right].$$

1. For $k = 1,\ A_1 = \{p, \neg p\}$ and $EM(p) = \{p\}$. By our assumption $p \in J_1^i$ if and only if $p \in J_2^i$ for all $i \in N$. As $p$ is admitted to the collective choice set by majority rule, $p \in J_1$ if and only if $p \in J_2$ where $J_t = F(J_t^N),\ t = 1, 2.$

2. For $k = 2$, $A_2 = (\{p_1, \neg p_1\}, \{p_2, \neg p_2\})$. By part 1, we have only to consider the second issue. Let $p \in \{p_2, \neg p_2\}$. We distinguish the following cases:
   2.1. $EM(p) = \{p\}$ (and thus $EM(\neg p) = \{\neg p\}$). Then $F(J_1^N)$ and $F(J_2^N)$ are determined by majority rule. As $EM(p) = \{p\}$ and $p \in J_1^i$ if and only if $p \in J_2^i$ for all $i \in N$, it follows that $p \in J_1$ if and only if $p \in J_2$.
   2.2. There is $q \in \{p_1, \neg p_1\}$ such that $q \models p$. By our assumptions $q \in J_1^i$ if and only if $q \in J_2^i$. Thus, the first element in our choice (i.e., the first issue) is determined uniquely (by part 1.). Hence the second element is also determined uniquely (by our assumptions, as it is implied by the first).
   2.3. There is $q \in \{p_1, \neg p_1\}$ such that $q \models \neg p$; then $q \in EM(p)$. By our assumptions we have the same choice for both profiles in the first issue and therefore the same selection for the second issue in both profiles . This completes the proof. ∎

Unfortunately, Proposition 2 cannot be extended to $k > 2$. Furthermore, the following example shows that for $k > 2$, our aggregation function SAP cannot satisfy III w.r.t. any relevance relation between two propositions based only on binary implications between the propositions or their negations.

*Example 7* Consider the following the agenda with three issues $A_3 = \{I_1, I_2, I_3\}$ corresponding to the following three propositions and their negations (put in the semantic setting[2]):

$$p_1 = \{a_1, a_2, a_5, a_6\} \quad \neg p_1 = \{a_3, a_4, a_7, a_8\}$$
$$p_2 = \{a_1, a_3, a_7, a_8\} \quad \neg p_2 = \{a_2, a_4, a_5, a_6\}$$
$$p_3 = \{a_1, a_4, a_7, a_8\} \quad \neg p_3 = \{a_2, a_3, a_5, a_6\}$$

First, observe that there is no entailment relation between any two of the propositions and their negations; that is, $EM(I_j) = \{I_j\}$ for $j = 1, 2, 3$. Next we see that $p_1 \wedge p_2 \models p_3$ , $\neg p_1 \wedge \neg p_2 \models p_3$, and $p_1 \wedge \neg p_2 \models \neg p_3$.

For the order of issues $(I_1, I_2, I_3)$ our aggregation function yields

$$F((p_1, p_2, p_3), (p_1, \neg p_2, \neg p_3), (\neg p_1, p_2, p_3)) = (p_1, p_2, p_3),$$

as $p_1$ and $p_2$ are decided by majority rule and $I_3$ is determined by $p_1 \wedge p_2 \models p_3$.

Changing $p_2$ in the judgment of the third voter to $\neg p_2$ yields

$$F((p_1, p_2, p_3), (p_1, \neg p_2, \neg p_3), (\neg p_1, \neg p_2, p_3)) = (p_1, \neg p_2, \neg p_3),$$

since $p_1$ and $\neg p_2$ are decided by majority rule and then $I_3$ is determined since $p_1 \wedge \neg p_2 \models \neg p_3$. This contradicts III since $I_2$ is irrelevant to $I_3$.

[2] In all our examples using a finite semantic logic, we take $\mathscr{J}$ to be the set of all complete and consistent (i.e., with nonempty intersection) judgments.

In view of our last example, if our objective is to have our aggregation function $F$ satisfy III, we must introduce a relevance relation of a wider range than that of simple implication.

**Definition 10** Let $j \leq h,\ h > 1$. The issue $I_j$ is relevant to the issue $I_h$ (notation $I_j R I_h$) if there exist $p \in I_h,\ \ q \in I_j$, and a set of issues $(I_\ell)_{\ell \in L}$, where $L \subset \{1,\ldots,h-1\}$ (which may be empty), and $q_\ell \in I_\ell,\ \ell \in L$ such that the set $S = \{q_\ell | \ell \in L\}$ satisfies the following requirements:

$$S \cup q \quad \text{is consistent} \tag{1}$$

$$S \cup q \models p \tag{2}$$

$$S \not\models p \tag{3}$$

*Interpretation* Denoting by $\mathscr{J}_h$ the set of all rational judgments of the issues $(I_1,\ldots,I_h)$, for distinct issues $(j < h)$, the intuition formalized in this definition is that the issue $\{\pm q\}$ is relevant to proposition $\{\pm p\}$ if the following conditions hold:

1. The issue $\{\pm q\}$ is decided (appears in our given order) before the issue $\{\pm p\}$.
2. All $J \in \mathscr{J}_h$ satisfy $S \cup q \subset J \Rightarrow p \in J$. ($S \cup q \models p$.)
3. $\exists J^* \in \mathscr{J}_h$ such that $S \cup \neg q \cup \neg p \subset J^*$. ($S \not\models p$ while $S \cup q \models p$.)

*Remark 5* Note that $R$ is reflexive: $p \in R(p)$ (by $p \models p$). Also, for $L = \emptyset$ (hence $S = \emptyset$), the conditions (1),(2),(3) reduce to straight entailment $q \models p$, and hence the relevance relation $R$ is an extension of the implication relation; that is, $EM(p) \subset R(p)$ for all propositions $p \in A$.

*Remark 6* This relevance relation is very closely related to the notion of *conditional entailment* introduced first by Nehring and Puppe and then defined again by Dietrich and List: “ $q$ conditionally entails $p$ (denoted by $q \models^* p$) if there is $S \subseteq A$ that is consistent both with $q$ and with $\neg p$ such that $S \cup \{q\} \models p$” (see Dietrich and List 2008, p. 21.) The relation to the relevance relation $R$ in Definition 10 is: the issue $I_j$ is relevant to the issue $I_h$ $(j < h)$ if there exist $p \in I_h$, and $q \in I_j$ such that $q$ conditionally entails $p$ (i.e., $q \models^* p$).

The relevance relation in Definition 10 is *not transitive* as is demonstrated by the following example presented in the semantic setting.

*Example 8* Let $W = \{a,b,c,d,e,f,g,h,m\}$ and consider the following issues $(I_1,I_2,I_3,I_4)$, where $I_j = \{q_j, \neg q_j\}$,
$j = 1,2,3,4$, with the propositions:

$$\begin{array}{ll} q_1 = \{a,b\} & \neg q_1 = \{c,d,e,f,g,h,m\} \\ q_2 = \{c,d,e\} & \neg q_2 = \{a,b,f,g,h,m\} \\ q_3 = \{a,b,c,f,g\} & \neg q_3 = \{d,e,h,m\} \\ q_4 = \{a,c,g,h\} & \neg q_4 = \{b,d,e,f,m\} \end{array}$$

With respect to our relevance relation (Definition 10), we have:

- $q_1 \models q_3$ (and $q_1 \models \neg q_2$), and hence $q_1 \in R(q_3)$ (and $q_1 \in R(q_2)$).
- $q_2 \wedge q_3 \models q_4$, $q_2 \not\models q_4$ and $q_3 \not\models q_4$, and hence $q_3 \in R(q_4)$ (and $q_2 \in R(q_4)$).

We claim that $q_1$ is not relevant to $q_4$. Indeed:

- $q_1 \wedge \neg q_2 = q_1 \not\models q_4$ (or$\neg q_4$) ($q_1 \not\models I_4$ for short).
- $q_1 \wedge q_3 = q_1 \not\models I_4$ and $\neg q_1 \wedge q_2 = q_2 \not\models I_4$.
- $\neg q_1 \wedge \neg q_2 = \{f,g,h,m\} \not\models I_4$, and $\neg q_1 \wedge q_3 = \{c,f,g\} \not\models I_4$.
- Finally, $\neg q_1 \wedge \neg q_3 = \{d,e,h,m\} \not\models I_4$, completing the check of all pairs of propositions including $q_1$.

We proceed checking all triples of propositions including $q_1$:

- $q_1 \wedge q_2 = \emptyset$, eliminating the two triples $q_1 \wedge q_2 \wedge q_3$ and $q_1 \wedge q_2 \wedge \neg q_3$.
- $q_1 \wedge \neg q_2 = q_1$, eliminating the two triples $q_1 \wedge \neg q_2 \wedge q_3$ and $q_1 \wedge \neg q_2 \wedge \neg q_3$, by our results for pairs.
- $\neg q_1 \wedge q_2 \wedge q_3 \models q_4$ and $\neg q_1 \wedge q_2 \wedge \neg q_3 \models \neg q_4$; however, in both cases $\neg q_1$ is redundant for the entailment and therefore it does not satisfy the conditions for relevance to $q_4$ or $\neg q_4$.

The remaining two triples to check are:

- $\neg q_1 \wedge \neg q_2 \wedge q_3 = \{f,g\} \not\models I_4$.
- Finally, $\neg q_1 \wedge \neg q_2 \wedge \neg q_3 = \{h,m\} \not\models I_4$.

This completes the proof that $I_1 \notin R(I_4)$, and hence this relevance relation is *not transitive*.

The following proposition will be used in our proofs in the sequel.

**Proposition 3** *For any $p \in I_h$ and any restricted consistent judgment $J_{|h-1}$, the following holds:*

$$J_{|h-1} \models p(or\ \neg p) \ \ \textit{if and only if} \ \ J_{|h-1} \cap R(p) \models p(or\ \neg p).$$

*Proof* The “if” part follows since $J_{|h-1} \cap R(p) \subset J_{|h-1}$ (by the monotonicity of the entailment) .

For the “only if” part assume that $J_{|h-1} \models p(\text{or } \neg p)$ and $J_{|h-1} \cap R(p) \not\models p(\text{or } \neg p)$. If the propositions in $J_{|h-1} \setminus R(p)$ are removed one by one from $J_{|h-1}$, there must be a first case in which, when $\tilde{q} \notin R(p)$ is removed, the entailment $\models p$ (or $\models \neg p$) no longer holds. Taking in Definition 10 the set $S \subseteq J_{|h-1}$ to be the set of propositions removed up to that stage (before removing $\tilde{q}$), we have that $\tilde{q} \in R(p)$ in contradiction to $\tilde{q} \in J_{|h-1} \setminus R(p)$. ∎

Although the transitivity of our relevance relation is not required for the previous proposition, it seems to be necessary for the III property of SAP as is demonstrated by the following example (built on Example 8) in which III is violated.

*Example 9 (violation of III).*
Let $W = \{a,b,c,d,e,f,g,h,m\}$, $W' = \{a',b',c',d',e',f',g',h',m'\}$, and $\Omega = W \cup W'$. Let $q_1,q_2,q_3,q_4$ be the following subsets of $W$ (and their complements), defined in Example 8:

$$q_1 = \{a,b\} \qquad q_1^c = \{c,d,e,f,g,h,m\}$$
$$q_2 = \{c,d,e\} \qquad q_2^c = \{a,b,f,g,h,m\}$$
$$q_3 = \{a,b,c,f,g\} \quad q_3^c = \{d,e,h,m\}$$
$$q_4 = \{a,c,g,h\} \qquad q_4^c = \{b,d,e,f,m\}$$

For $k = 1,\dots,4$, let $q'_k$ be the subset of $W'$ defined by $q'_k = \{w' \in W' | w \in q_k\}$ and consider the following five propositions (subsets) in $\Omega$:

$$q_{10} = q_1 \cup W',\ q_{01} = W \cup q'_1,\ q_{kk} = q_k \cup q'_k,\ \ k = 2,3,4,$$

and the corresponding five issues:

$$I_{10} = \{q_{10}, \neg q_{10}\},\ I_{01} = \{q_{01}, \neg q_{01}\},\ \ I_{kk} = \{q_{kk}, \neg q_{kk}\},\ \ k = 2,3,4.$$

Considering the agenda of five (ordered) issues, $A = (I_{10}, I_{01}, I_{22}, I_{33}, I_{44})$, we have:

- $I_{10} \wedge I_{01} \models q_{33}$, $I_{01} \not\models q_{33}$ (and $I_{10} \not\models q_{33}$), hence $q_{10} \in R(I_{33})$ (and $q_{01} \in R(I_{33})$).
- $I_{22} \wedge I_{33} \models q_{44}$, $I_{22} \not\models q_{44}$ (and $I_{33} \not\models q_{44}$), hence $q_{33} \in R(I_{44})$ (and $q_{22} \in R(I_{44})$).

*Claim 1* The issue $I_{10}$ is not relevant to the issue $I_{44}$, that is, $I_{10} \notin R(I_{44})$ (non-transitivity).

*Proof* See Appendix.

Assume that three judges debating the five issues presented above, have the following profile of judgments $J_1^N$:

| | Issues | $I_{10}$ | $I_{01}$ | $I_{22}$ | $I_{33}$ | $I_{44}$ |
|---|---|---|---|---|---|---|
| $J_1^N$ | Judge 1 | $q_{10}$ | $q_{01}$ | $\neg q_{22}$ | $q_{33}$ | $\neg q_{44}$ |
| | Judge 2 | $\neg q_{10}$ | $q_{01}$ | $q_{22}$ | $\neg q_{33}$ | $\neg q_{44}$ |
| | Judge 3 | $q_{10}$ | $\neg q_{01}$ | $q_{22}$ | $q_{33}$ | $q_{44}$ |

Aggregation of these judgments according to SAP yields:

| | Issues | $I_{10}$ | $I_{01}$ | $I_{22}$ | $I_{33}$ | $I_{44}$ |
|---|---|---|---|---|---|---|
| $J_1^N$ | Judge 1 | $q_{10}$ | $q_{01}$ | $\neg q_{22}$ | $q_{33}$ | $\neg q_{44}$ |
| | Judge 2 | $\neg q_{10}$ | $q_{01}$ | $q_{22}$ | $\neg q_{33}$ | $\neg q_{44}$ |
| | Judge 3 | $q_{10}$ | $\neg q_{01}$ | $q_{22}$ | $q_{33}$ | $q_{44}$ |
| | **SAP** | $\mathbf{q_{10}}$ | $\mathbf{q_{01}}$ | $\neg\mathbf{q_{22}}$ | $\mathbf{q_{33}}$ | $\neg\mathbf{q_{44}}$ |

($q_{10}, q_{01}$ are obtained by majority voting, $q_{10} \wedge q_{01} \models \neg q_{22}$, $q_{10} \wedge q_{01} \models q_{33}$, and $\neg q_{44}$ is obtained by majority voting).

Consider now the following profile of judgments $J_2^N$, which differs from the profile $J_1^N$ only by Judge 1 switching opinion on issue $I_{10}$ (which is irrelevant to $I_{44}$), from $q_{10}$ to $\neg q_{10}$ :

| Issues | $I_{10}$ | $I_{01}$ | $I_{22}$ | $I_{33}$ | $I_{44}$ |
|---|---|---|---|---|---|
| $J_2^N$ Judge 1 | $\neg q_{10}$ | $q_{01}$ | $\neg q_{22}$ | $q_{33}$ | $\neg q_{44}$ |
| Judge 2 | $\neg q_{10}$ | $q_{01}$ | $q_{22}$ | $\neg q_{33}$ | $\neg q_{44}$ |
| Judge 3 | $q_{10}$ | $\neg q_{01}$ | $q_{22}$ | $q_{33}$ | $q_{44}$ |
| **SAP** | $\neg \mathbf{q_{10}}$ | $\mathbf{q_{01}}$ | $\mathbf{q_{22}}$ | $\mathbf{q_{33}}$ | $\mathbf{q_{44}}$ |

($\neg q_{10}, q_{01}, q_{22}, q_{33}$ are obtained by majority voting and then $\neg q_{10} \wedge q_{01} \wedge q_{22} \wedge q_{33} \models q_{44}$).

As $I_{10}$ is irrelevant to $I_{44}$, this is in contradiction to the III property.

In view of Example 9 we take the transitive closure of our relevance relation.

**Definition 11** The relevance relation $R^*$ is the *transitive closure* of the relevance relation $R$ given in Definition 10.

Since $R^*(p) \supseteq R(p)$ for all propositions $p$, Proposition 3 clearly holds also for the relevance relation $R^*$ and we have:

**Corollary 1** *For any $p \in I_h$ and any restricted consistent judgement $J_{|h-1}$ the following holds:*

$$J_{|h-1} \models p(or\ \neg p) \ \textit{ if and only if } \ J_{|h-1} \cap R^*(p) \models p(or\ \neg p)$$

**Proposition 4** *Our aggregation function F (SAP), given in Definition 3, satisfies III w.r.t. the relevance relation $R^*$ given in Definition 11.*

*Proof* Let $J_1^N, J_2^N \in \mathscr{J}^N$, and let $p \in I_h$. We have to prove that if $J_1^i \cap R^*(p) = J_2^i \cap R^*(p)$ for all $i \in N$; then $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$. Actually we will prove a stronger result. Namely, under the same conditions $F(J_1^N) \cap R^*(p) = F(J_2^N) \cap R^*(p)$; that is, not only does $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$ but also $q \in F(J_1^N)$ if and only if $q \in F(J_2^N)$ for all $q \in R^*(p)$. In other words, if $J_1^i \cap R^*(p) = J_2^i \cap R^*(p)$ for all $i \in N$, then not only the appearance of $p$ is the same in both $F(J_1^N)$ and $F(J_2^N)$ but this is true for all propositions relevant to $p$.

The proof is by induction on $h$. The case $h = 1$ follows from our assumptions, the reflexivity of $R^*(\cdot)$, and the definition of $F$. Let $h > 1$ and assume by induction that the claim is true for $j = 1, \ldots, h-1$.

Note first that from the transitivity of $R^*$ we have $q \in R^*(p) \Rightarrow R^*(q) \subset R^*(p)$ and therefore from

$$J_1^i \cap R^*(p) = J_2^i \cap R^*(p), \ \forall i \in N$$

we also have (by intersecting both sides with $R^*(q)$),

$$J_1^i \cap R^*(q) = J_2^i \cap R^*(q), \ \forall i \in N, \ \forall q \in R^*(p)$$

and therefore by the induction hypothesis,

$$F(J_1^N) \cap R^*(q) = F(J_2^N) \cap R^*(q), \ \ \forall q \in I_j, \ j < h, \ q \in R^*(p),$$

and hence

$$(F(J_1^N))_{|h-1} \cap R^*(p) = (F(J_2^N))_{|h-1} \cap R^*(p). \qquad (4)$$

We distinguish two cases.

1. If $(F(J_1^N))_{|h-1} \models p$. In this case, it must also be that $(F(J_2^N))_{|h-1} \models p$.
   Indeed, by Corollary 1 we have $(F(J_1^N))_{|h-1} \cap R^*(p) \models p$ and, by Equation (4), $(F(J_2^N))_{|h-1} \cap R^*(p) \models p$. Applying Corollary 1 again we have $(F(J_2^N))_{|h-1} \models p$.
   Similarly, if $(F(J_1^N))_{|h-1} \models \neg p$ then also $(F(J_2^N))_{|h-1} \models \neg p$.
   It follows that in this case SAP chooses $p$ (or $\neg p$) in both $J_1^N$ and $J_2^N$. Combining this with Equation (4), we get $F(J_1^N) \cap R^*(p) = F(J_2^N) \cap R^*(p)$.
2. If $(F(J_1^N))_{|h-1} \not\models p$ and $(F(J_1^N))_{|h-1} \not\models \neg p$, then again by Corollary 1 and Equation (4) (by the same argument as in part 1.) we also have $(F(J_2^N))_{|h-1} \not\models p$ and $(F(J_2^N))_{|h-1} \not\models \neg p$. Hence the issue $\{p, \neg p\}$ is decided by simple majority voting in both profiles. Since for all $i \in N$, $p \in J_1^i$ if and only if $p \in J_2^i$, we get $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$. Combining this with Equation (4) we get $F(J_1^N) \cap R^*(p) = F(J_2^N) \cap R^*(p)$, completing the proof. ∎

## 5 Characterization of SAP

SAP is a sequential aggregation function $(F_1, \ldots, F_k)$ for a JAP $g = (N, A_k, \neg, \mathscr{J})$, where $F_\ell : \mathscr{J}_\ell^N \to \mathscr{J}_\ell$ and for $\ell = 1, \ldots, k.$, $\mathscr{J}_\ell = \{J \cap (I_1 \cup, \ldots, \cup I_\ell) | J \in \mathscr{J}\}$ (see Definition 5). Thus, *full domain* and *rationality* are guaranteed by definition.

Other properties of SAP established so far are:

(AN) *Anonymity.*
$F_k$ is anonymous: $F_k(J^{\pi(1)}, \ldots, J^{\pi(n)}) = F_k(J^1, \ldots, J^n)$ for any permutation $\pi$ of $N = \{1, 2, \ldots, n\}$, and any profile $J^N \in \mathscr{J}^N$.

(U) *Unanimity.*
$F_k$ is unanimous: $F_k(J, \ldots, J) = J$ for all $J \in \mathscr{J}$.

(REIN) *Reinforcement.*

(RA) *Restricted Agenda*: $F_\ell(J^N_{|\ell}) = F_k(J^N) \cap (I_1 \cup, \ldots, \cup I_\ell)$ for all $J^N \in \mathscr{J}^N$ and all $1 \leq \ell \leq k$, which follows from the fact that SAP is a sequential aggregation function (Definition 5).

(III) *Independence of irrelevant issues* with respect to the relevance relation $R^*$ given in Definition 11.

For our characterization of SAP we introduce the following three properties:

(RM) *Restricted Monotonicity.*
$F$ satisfies restricted monotonicity if for any $i \in N$, $1 \leq \ell \leq k$, and for any $J^N \in \mathscr{J}^N$ and $\tilde{J}^N \in \mathscr{J}^N$ such

that $q^i_\ell = \neg p_\ell,\ \tilde{q}^i_\ell = p_\ell$ and $\tilde{q}^{i'}_{\ell'} = q^{i'}_{\ell'}$ for all $i' \neq i$ or $\ell' \neq \ell$,

$$\text{if } \ (F(J^N))_\ell = p_\ell \ \text{ then } \ (F(\tilde{J}^N))_\ell = p_\ell.$$

(IPD) *Independence of Past Deliberations.*

$F$ satisfies independence of past deliberations if for all $1 \leq \ell < k$ and for any profiles $J^N$ and $\tilde{J}^N$,

$$\text{if } \ F_\ell(J^N_{|\ell}) = F_\ell(\tilde{J}^N_{|\ell}) \ \text{ and } \ J^N_{\ell+1} = \tilde{J}^N_{\ell+1} \ \text{ then } \ (F_k(J^N))_{\ell+1} = (F_k(\tilde{J}^N))_{\ell+1}.$$

For an $\ell$-judgment $J_{|\ell}$, denote by $(J_{|\ell})^N = (J_{|\ell}, \ldots, J_{|\ell})$ the $\ell$-profile in which all judges have the same $\ell$-judgment $J_{|\ell}$.

Given $J^N_\ell = (q^1_\ell, \ldots, q^n_\ell)$, a profile of judgments on issue $I_\ell$, denote by $\overrightarrow{J^N_\ell}$ this profile ordered with all $p_\ell$ first and then $\neg p_\ell$, that is, $\overrightarrow{J^N_\ell} = (p_\ell, \ldots, p_\ell, \neg p_\ell, \ldots, \neg p_\ell)$. We are now ready to state our last axiom:

(LN) *Local Neutrality.*

$F$ is locally neutral if for all $1 \leq \ell < k$ and all $J^N \in \mathscr{J}^N$, if both $(F_\ell(J^N_{|\ell}), p_{\ell+1})$ and $(F_\ell(J^N_{|\ell}), \neg p_{\ell+1})$ are consistent, then

$$(F_{\ell+1}((F_\ell(J^N_{|\ell})^N, \neg J^N_{\ell+1}))_{\ell+1} = \begin{cases} p_{\ell+1} & \text{if } \quad \overrightarrow{\neg J^N_{\ell+1}} = \overrightarrow{J^N_{\ell+1}} \\ \\ \neg((F_\ell(J^N_{|\ell})^N, J^N_{\ell+1}))_{\ell+1} & \text{otherwise.} \end{cases}$$

**Remarks**

1. Restricted monotonicity (RM) is a monotonicity condition with a restricted domain: it is required only when the single switch of judge $i$ from $\neg p_k$ to $p_k$ leaves his/her judgment consistent (as implied by $\tilde{J}^i \in \mathscr{J}$).
2. Independence of past deliberations (IPD) requires that the aggregated decision on issue $I_{k+1}$ depend only on the profile of judgments on this issue and on previous decisions on the issues $(I_1, \ldots, I_k)$ (but not on the profiles of judgments that led to those decisions).
3. Local neutrality (LN) requires neutrality between $p_{k+1}$ and $\neg p_{k+1}$ only when there is unanimity of judgment on previous issues and when both $p_{k+1}$ and $\neg p_{k+1}$ are consistent with previous decisions. In addition, it imposes the anonymous (arbitrary) tie-breaking rule in favor of $p_k$.

In preparation for our main characterization theorem we first characterize the aggregation procedure for the case of a single issue ($k = 1$) by modifying May's (1952) axiomatization of the majority rule. While May's model allows for the neutrality between two alternatives, in our model, the choice is between a proposition and its negation that must be single-valued, and no neutrality is possible (in May's notation the values of the decision function are in $\{-1, 1\}$ rather than $\{-1, 0, 1\}$.)

We consider the case of $N = \{1, \ldots, n\}$ players and two alternatives, $p$ and $\neg p$. Each player chooses one alternative. *Majority voting with anonymous tie-breaking* (MVAT) is defined as follows:

- If $n$ is odd then the majority alternative is selected by the group.
- If $n = 2k$ and exactly $k$ members choose $p$, then $p$ is chosen; otherwise, the majority alternative is chosen.

Denote $d(i) = 1$ if $i$ chooses $p$, and $d(i) = -1$ if $i$ chooses $\neg p$. Let $d = (d(1),...,d(n))$. A *voting rule* (VR) is a function $f : \{1,-1\}^N \to \{1,-1\}$. Obviously, MVAT can be written as a voting rule. It satisfies the following axioms.

(AN) *Anonymity*. $f(d(1),...,d(n)) = f(d(t(1)),...,d(t(n)))$ for all permutations $t$ of $N$.

(M) *Monotonicity*. $[d(i) = d^*(i)\ \forall i \neq j,$ and $d(j) > d^*(j)] \models f(d) \geq f(d^*)$.

(LN) *Limited neutrality*. i) If $n$ is odd then $f(-d) = -f(d)$ for all $d$. ii) If $n$ is even, $n = 2k$, and $|\{i : d(i) = 1\}| = k$, then $f(d) = f(-d) = 1$; otherwise, $f(-d) = -f(d)$.

**Theorem 2** *There is a unique VR f that satisfies (AN), (M), and (LN) and it is MVAT.*

*Proof* This is actually a slight modification of May's characterization but it can be directly proved as follows. Call a coalition of players "winning" if when all its members vote 1 then society's vote is also 1. By anonymity and monotonicity the simple game of MVAT is $(n,k)$ where $k$ is in $\{0,...,n\}$, that is, a game in which a coalition is winning if and only if it has at least $k$ members. Limited neutrality now yields the final characterization. ∎

We are now ready to state our characterization theorem for SAP.

**Theorem 3** *There is one and only one aggregation function F satisfying the axioms (AN), (RA), (RM), (IPD), and (LN). It is the sequential aggregation procedure (SAP).*

*Proof* SAP satisfies all five axioms. Let $F$ be a judgment aggregation function satisfying the axioms.

- Since $F$ satisfies the *restricted agenda* property (RA), $F$ is sequential and we have to show that for each issue $I_k$ (formally by induction of $k$) $F$ coincides with SAP.
- For $k = 1$, axioms (AN), (RM), and (LN) lead, by Theorem 2, to majority voting with an anonymous tie-breaking rule (MVAT) in favor of $p_1$.
- Assume that $F$ coincides with SAP for an agenda of up to $k$ issues and let us prove it for the $k+1$-th issue. Given a profile $J^N$ with $k+1$ issues:
  - If $F_k(J^N) \models p_{k+1}$ or $F_k(J^N) \models \neg p_{k+1}$, then by consistency $F_{k+1}(J^N) = p_{k+1}$ or $F_{k+1}(J^N) = \neg p_{k+1}$ respectively and hence $F$ coincides with SAP on the $k+1$-th issue.
  - Otherwise both $(F_k(J^N), p_{k+1})$ and $(F_k(J^N), \neg p_{k+1})$ are consistent.
    By the (IPD) axiom, $F(J^N) = F_{k+1}((F_k(J^N))^N, J^N_{k+1})$ and again (as for $k = 1$), by (AN), (RM), and (LN), this implies that the $k+1$-th issue is decided by MVAT, as in SAP, completing the proof. ∎

*Remark 7* Note that when we applied (in Theorem 3) the MVAT à la May, we had full domain, both of $J^N_1$ for the first step $k = 1$ and of $J^N_{k+1}$ in the induction.

**Independence of the axioms**

For each of the five axioms we show an aggregation function not satisfying that axiom but satisfying all four other axioms.

(AN) Dictatorship satisfies all axioms except (AN).

(RAP) Let $\sigma^*$ be the permutation of the issues $\{I_1,\ldots,I_K\}$ given by $\sigma^*(I_k) = I_{K-k+1}$, for $k = 1,\ldots,K$. Let $F$ be SAP and consider the following aggregation function $F^*$ defined by

$$F^*(J^N) = F(\sigma^*(J^N))$$

where $\sigma^*(J^N)$ is obtained from the profile $J^N$ by reordering the issues according to the permutation $\sigma^*$. The function $F^*$ satisfies (AN), (LN), (RM), and (IPD) since SAP, $F$, satisfy these axioms. However, $F^*$ does not satisfy (RAP) as can be seen in the following Doctrinal Paradox:

Considering the three issues $\{(p,\neg p),(q,\neg q),(g,\neg g)\}$ with $g \Leftrightarrow p \wedge q$ and the judgment profile $J^N$ of three judges given by

Issues

| | $p$ | $\neg p$ | $q$ | $\neg q$ | $g$ | $\neg g$ |
|---|---|---|---|---|---|---|
| Judge 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| Judge 2 | 1 | 0 | 0 | 1 | 0 | 1 |
| Judge 3 | 0 | 1 | 1 | 0 | 0 | 1 |

Then, $F^*(J^N) = (p,\neg q,\neg g)$ but $F^*(J^N$ restricted to $\{(p,\neg p),(q,\neg q)\}) = (p,q)$.

(LN) Let $F$ be our SAP and let $\tilde{F}$ be the same procedure except that for a free issue $(p_k,\neg p_k)$,

$$\tilde{F}_k(J^N) = \begin{cases} \neg p_k & \text{if } |\{i|J_k^i = \neg p_k\}| > \frac{2}{3}n \\ p_k & \text{otherwise.} \end{cases}$$

This $\tilde{F}$ satisfies all axioms except (LN).

(RM) Let $F$ be our SAP and let $\hat{F}$ be the same procedure except that for a free issue $(p_k,\neg p_k)$,

$$\hat{F}_k(J^N) = \begin{cases} p_k & \text{if } |\{i|J_k^i = \neg p_k\}| \text{ is even} \\ \neg p_k & \text{otherwise.} \end{cases}$$

This $\hat{F}$ satisfies all axioms except (RM).

(IPD) Consider $F^*$, which is the same as SAP except that the anonymous tie-breaking rule in a free item $(p_k,\neg p_k)$ for $k > 1$ is determined by the first issue profile $J_1^N$ in the following way:

if $n = 2m$ and $|\{i|J_k^i = p_k\}| = m$, then

- $F_k^*(J^N) = p_k$ If $|\{i|J_1^i = F_1^*(J^N)\}| > \frac{2}{3}n$.
- $F_k^*(J^N) = \neg p_k$ if $|\{i|J_1^i = F_1^*(J^N)\}| \leq \frac{2}{3}n$.

This $F^*$ satisfies all axioms except (IPD).

## Appendix

Proof of *Claim 1* (page 14):

The issue $I_{10}$ is not relevant to the issue $I_{44}$, that is, $I_{10} \notin R(I_{44})$.

*Proof* The proof is by straightforward verification noticing that $\neg q_{10} = q_1^c$, $\neg q_{01} = q_1'^c$, $\neg q_{kk} = q_k^c \cup q_k'^c$, and using the entailments established in Example 8.

- $q_{10} \wedge q_{01} = q_1 \cup q_1' \not\models I_{44}$.
- $q_{10} \wedge \neg q_{01} = \neg q_1' \not\models I_{44}$.
- $\neg q_{10} \wedge q_{01} = \neg q_1 \not\models I_{44}$.
- $\neg q_{10} \wedge \neg q_{01} = \emptyset \not\models I_{44}$.

- $q_{10} \wedge q_{22} = q_2' \not\models I_{44}$.
- $q_{10} \wedge \neg q_{22} = q_1 \cup \neg q_2' \not\models I_{44}$.
- $\neg q_{10} \wedge q_{22} = q_2 \not\models I_{44}$.
- $\neg q_{10} \wedge \neg q_{22} = \{f, g, h, m\} \not\models I_{44}$.

- $q_{10} \wedge q_{33} = q_1 \cup q_3' \not\models I_{44}$.
- $q_{10} \wedge \neg q_{33} = \neg q_3' \not\models I_{44}$.
- $\neg q_{10} \wedge q_{33} = \{c, f, g\} \not\models I_{44}$.
- $\neg q_{10} \wedge \neg q_{33} = \neg q_3 \not\models I_{44}$.

We proceed to check the implications of the triples of issues involving $I_{10}$.

- Propositions from $I_{10}, I_{01}, I_{22}$.
  - $q_{10} \wedge q_{01} \wedge q_{22} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge \neg q_{22} = q_1 \cup q_1' \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge q_{22} = q_2' \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge \neg q_{22} = \{f', g', h', m'\} \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge q_{22} = q_2 \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge \neg q_{22} = \{f, g, h, m\} \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} \wedge q_{22} = \emptyset \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} \wedge \neg q_{22} = \emptyset \not\models I_{44}$.
- Propositions from $I_{10}, I_{01}, I_{33}$.
  - $q_{10} \wedge q_{01} \wedge q_{33} = q_1 \cup q_1' \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge \neg q_{33} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge q_{33} = \{c', f', g'\} \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge \neg q_{33} = \neg q_3'^c \not\models I_{44}$.

  - $\neg q_{10} \wedge q_{01} \wedge q_{33} = \{c, f, g\} \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge \neg q_{33} = \neg q_3^c \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} \wedge q_{33} = \emptyset \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} \wedge \neg q_{33} = \emptyset \not\models I_{44}$.
- Propositions from $I_{10}, I_{22}, I_{33}$.
  - $q_{10} \wedge q_{22} \wedge q_{33} = \{c'\} \models I_{44}$,

    but this does not imply the relevance of $q_{10}$ to $q_{44}$ since $q_{22} \wedge q_{33} = \{c'\} \models I_{44}$.
  - $q_{10} \wedge q_{22} \wedge \neg q_{33} = \{d', e'\} \models I_{44}$,

    but this does not imply the relevance of $q_{10}$ to $q_{44}$ since $q_{22} \wedge \neg q_{33} = \{d', e'\} \models I_{44}$.
  - $q_{10} \wedge \neg q_{22} \wedge q_{33} = \{a, b, a', b', f', g'\} \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{22} \wedge \neg q_{33} = \{h', m'\} \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{22} \wedge q_{33} = \{c\} \models I_{44}$,

    but this does not imply the relevance of $q_{10}$ to $q_{44}$ since $q_{22} \wedge q_{33} = \{c\} \models I_{44}$.
  - $\neg q_{10} \wedge q_{22} \wedge \neg q_{33} = \{d, e\} \models I_{44}$,

    but this does not imply the relevance of $q_{10}$ to $q_{44}$ since $q_{22} \wedge \neg q_{33} = \{d, e\} \models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{22} \wedge q_{33} = \{a, b, f, g\} \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{22} \wedge \neg q_{33} = \{h, m\} \not\models I_{44}$.

Finally, we check the implications of the quadruples of issues involving $I_{10}$.

  - $q_{10} \wedge q_{01} \wedge q_{22} \wedge q_{33} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge q_{22} \wedge \neg q_{33} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge \neg q_{22} \wedge q_{33} = \{a, b, a', b'\} \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge \neg q_{22} \wedge \neg q_{33} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge q_{22} \wedge q_{33} = \{c'\} \models I_{44}$,

    but this does not imply the relevance of $q_{10}$ to $q_{44}$ since $\neg q_{01} \wedge q_{22} \wedge q_{33} = \{c'\} \models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge q_{22} \wedge \neg q_{33} = \{d', e'\} \models I_{44}$,

    but this does not imply the relevance of $q_{10}$ to $q_{44}$ since $\neg q_{01} \wedge q_{22} \wedge \neg q_{33} = \{d', e'\} \models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge \neg q_{22} \wedge q_{33} = \{f', g'\} \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge \neg q_{22} \wedge \neg q_{33} = \{h', m'\} \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge q_{22} \wedge q_{33} = \{c\} \models I_{44}$,

    but this does not imply the relevance of $q_{10}$ to $q_{44}$ since $q_{01} \wedge q_{22} \wedge q_{33} = \{c\} \models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge q_{22} \wedge \neg q_{33} = \{d, e\} \models I_{44}$,

    but this does not imply the relevance of $q_{10}$ to $q_{44}$ since $q_{01} \wedge q_{22} \wedge \neg q_{33} = \{d, e\} \models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge \neg q_{22} \wedge q_{33} = \{f, g\} \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge \neg q_{22} \wedge \neg q_{33} = \{h, m\} \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} = \emptyset$, eliminating the remaining four cases $\neg q_{10} \wedge \neg q_{01} \wedge \{\pm q_{22}\} \wedge \{\pm q_{33}\}$. ∎

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