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Sequential aggregation judgments: Logical derivation of relevance relation

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Abstract Following Dietrich (2014) we consider using choice by plurality voting (CPV) as a judgment aggregation correspondence. We notice that a result of Roberts (1991) implies that CPV is axiomatically characterized by anonymity, neutrality, unanimity, and (Young's) reinforcement. Following List (2004) and Dietrich (2015) we construct a sequential voting procedure of judgement aggregation which satisfies rationality, anonymity, unanimity, and independence of irrelevant propositions (with respect to a relevance correspondence that does not satisfy transitivity). We offer a tentative characterization for this aggregation procedure.

Introduction

We have two goals in this paper. The first is to argue that, practically, rules for judgment aggregation are sequential. The second is to describe and axiomatize a special sequential judgment aggregation rule. Starting with the first goal, let us consider the Doctrinal Paradox (see Example 1): The three judges must first decide whether proposition p (= the contract is valid) is true. (They might as well ask first whether proposition q (=the contract has been violated) is true which will only lead to a permutation of their two decisions.) Finally, they apply the law $p \wedge q$ if and only if g (=the defendant is guilty) to decide whether or not g is true. We argue that, in general, a group of people cannot decide simultaneously on two (non trivial) binary choices by majority rule. This is because majority decision takes time for communication, discussion, and persuasion. Furthermore, we insist throughout the entire paper that binary choices will be resolved by majority rule if necessary (and not, for example, by forming subcommittees). Moreover, in doing so, we have just followed the tradition in the judgment aggregation literature (see, for example, Dietrich (2014)).

We are, of course, not the first to consider sequential aggregation. The first to do this, as far as we know, is List (2004). Conceptually, we use the same ideas: The first proposition is determined by the majority rule, and we proceed by induction: If propositions $p_1, \ldots, p_k$ were chosen for $k \geq 1$, then we check whether $p_1 \wedge, \ldots, \wedge p_k \Rightarrow q$ for some q in the issue I_{k+1}. If the answer is positive, then we choose q. Otherwise decide the $(k+1)$-th proposition by a simple majority rule. We immediately obtain anonymity , rationality. and unanimity. The main differences between our approach and List's are the following. (1) Except towards the end of his paper, List incorporates in his algorithm the assumption of proposition-wise independence (PI) (also called *independence of irrelevant alternatives*). As a result, his conclusions are mainly negative. We use the weaker assumption of *independence of irrelevant propositions* (IIP) due to Dietrich (2015), and thus we are able to obtain positive results. (2) List is also interested in path independence of his algorithm, that is independence of the collective judgment of the ordering of the issues (which may be arbitrary to some extent). We have in mind a parliament or a cabinet (or, more generally, a committee) that have to resolve a stream of issues which arrive sequentially one after the other. Thus, the issues in our model are conceived to be temporally ordered. This seems to us to be a useful model though, perhaps, not the most general one.

We now turn to briefly describe the contents of our paper. We start with the basic definitions that are relevant to the standard model of judgment aggregation. In Section 2 we adapt to the standard model a result of Roberts (1991) which yields an axiomatization of *choice by plurality voting* (CPV). His work relies on earlier works of Young (1975) and Richelson (1978). The axioms for CPV are anonymity, neutrality, unanimity, and reinforcement. Section 3 presents our rule as described above. The relationship to the doctrinal paradox is illustrated. We proceed with a modification of Dietrich's concept of relevance relation. In Dietrich (2015)

PI is weakened to independence of irrelevant propositions (IIP) which is derived from an arbitrary relevance relation $R(P)$ on the agenda. To eliminate arbitrariness we insist that the relevance relation is derived from the agenda. First we restrict ourselves only to implications; however this works only for two issues. Then we devise a (quite sophisticated) intuitive relevance relation such that our sequential aggregation procedure (SAP) satisfies IIP with respect to it. We conclude in Section 5 with a characterization of SAP. The important properties of SAP are: Rationality, anonymity, unanimity, reinforcement, IIP, and restricted agenda property (to be defined later).

1 The model

Let $N = \{1,2,\ldots,n\},\ n \geq 2,$ be a set of players and let $\mathscr{L}$ be a propositional language on a given (countable) set of atoms, endowed with the following functions: For $p \in \mathscr{L}$, $\neg p$ (not p) (with $\neg p \neq p$ and $\neg\neg p = p$), $p_1 \wedge p_2$ (both p_1 and p_2 are true), $p_1 \vee p_2$ (p_1 or p_2 is true) and $p_1 \Rightarrow p_2$ (p_1 implies p_2).

Definition 1

- An *issue* is a pair of propositions $I = (p, \neg p)$.
- An *agenda* is a finite set of propositions partitioned into issues:

$$A = \{I_1, I_2, \ldots\} = \{(p_1, \neg p_1), (p_2, \neg p_2), \ldots\}$$

Definition 2

- A *judgment* is s set $J \subset A$ such that $p \in J \Rightarrow \neg p \notin J$.
- A judgment J is *complete* if $I = \{p, \neg p\} \subset A$ implies that either $p \in J$ or $\neg p \in J$.
- A judgment J is *consistent* if it has no (logical) contradictions that is, in the semantic model for propositional logic (Dietrich 2014), $\bigwedge J = \bigcap_{p \in J} p \neq \emptyset$. (or more generally if $J \in \mathscr{C}$ where $\mathscr{C} = \mathscr{C}_A$ is a specified collection of consistent subsets of A).
- A judgment J is *rational* if it is complete and consistent.
- Denote by $\mathscr{J}$ the set of all complete and consistent (rational) judgments.

Definition 3 A *judgement aggregation problem* (JAP) is $(N, A, \neg, \mathscr{J})$ where N is the set of players, $A \subseteq \mathscr{L}$ is the agenda and $\mathscr{J}$ is the set of all consistent and complete subsets of A .

Definition 4 Let $(N, A, \neg, \mathscr{J})$ be a JAP. An *aggregation function* (AF) is a function $F : \mathscr{J}^N \to \mathscr{J}$.

Special case: *Preference aggregation*.

Given a set $S = \{a, b, \ldots\}$ of social alternatives, the propositions are of the form $a \succ b$ (or $a \succeq b$), A judgment of a player is his (complete or incomplete, weak or strict) preference order on the set of social alternatives, and consistency is imposed by the acyclicity of the (strict) preferences.

Example 1 The Doctrinal Paradox: The the situation described in the Doctrinal Paradox our AF provides a complete and consistent aggregation and the *'paradox'* is just a manifestation of the fact that the resulting aggregated judgement depends on the order in which the issues are decided.
Consider three judges deliberating on the following issues:

- p – The contract is legally valid.
- q – The defendant has broken the contract.
- g – The defendant is liable.
- By law, $g \Leftrightarrow p \wedge q$.

Assume that the judgements of the three judges are those given in the following table (where 1 indicates that the proposition is true and 0 indicates that it is false):

Issues

	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1

Aggregation of propositions by simple majority voting yields:

Issues

	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1
	1	**0**	**1**	**0**	**0**	**1**

This aggregated judgment is **inconsistent** as p and q are accepted but yet $\neg g$ is also accepted. In other word the "Paradox" is that:

- By the *Premise-base-rule*:
  p and q are accepted and hence the verdict is g – *guilty*, while
- By the *Conclusion-base-rule*:
  $\neg g$ is accepted by majority and the verdict *NOT guilty*.

2 Choice by plurality voting (CPV)

Definition 5 Let $(N, A, \neg, \mathscr{J})$ be a JAP (judgment aggregation problem). A *judgment aggregation correspondence* (JAC) is a function $F : \mathscr{J}^N \to 2^{\mathscr{J}}$, assigning to each judgment profile a set of judgments.

Definition 6 *Choice by plurality voting (CPV)* is the aggregation correspondence F defined by:

$$F(J^N) = \{J^i, i \in N : J^i \in J^N \text{ and } \mid j' : J^{j'} = J^j \mid \leq \mid j' : J^{j'} = J^i \mid, \forall j \in N\}$$

In words: Given a judgment profile, the AC chooses those judgments in the profile that are shared by the largest number of judges. This aggreation choice shares the following properties:

- *Anonymity*: For all profiles $J^N \in \mathscr{J}^N$ and for all permutations π of $N = \{1, 2, \ldots, n\}$,
  $$F(J^{\pi(1)}, \ldots, J^{\pi(n)}) = F(J^1, \ldots, J^n)$$
- *Neutrality*: For all permutations σ of $\mathscr{J}$ and for all profiles $J^N \in \mathscr{J}^N$,
  $$F(\sigma(J^1), \ldots, \sigma(J^n)) = \sigma(F(J^1, \ldots, J^n))$$
- *Unanimity*: For all judgments $J \in \mathscr{J}$,
  $$F(J, \ldots, J) = \{J\}$$
- *Reinforcement*: Let $(N, A, \neg, \mathscr{J})$ and $(M, A, \neg, \mathscr{J})$ two judgment aggregation problems with the same judgment set and disjoint sets of judges, N and M; $N \cap M = \emptyset$.
  If $F(J^N) \cap F(J^M) \neq \emptyset$, then (in the JAP $(N \cup M, A, \neg, \mathscr{J})$),
  $$F(J^N, J^M) = F(J^N) \cap F(J^M)$$

Theorem 1 *The choice by plurality voting is the only judgment aggregation correspondence that satisfies anonymity, neutrality, unanimity and reinforcement.*

Proof This follows readily from Robert's work (Roberts (1991)) who, following Young (1975 and Richelson (1978) considered a choice function (or correspondence) from an abstract set X of alternatives and any number of voters: $f : \bigcup_{n=1}^{\infty} X^n \to P_0(X)$, where $P_0(X)$ is the set of nonempty subsets of X. Roberts provided several sets of axioms characterizing the CPV correspondence in his abstract aggregated choice model. Our characterization theorem is a special case of Roberts' results for $X = \mathscr{J}$ stating that our stated properties *anonymity, neutrality, unanimity* and *reinforcement*, characterize the CPV correspondence, (Theorem 3 (case 4) in Roberts (1991)). ∎

Example 2 (The Doctrinal Paradox revisited) For the classical example of the Doctrinal Paradox,

Issues

	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1

We have $F(pqg, p\neg q\neg g, \neg pq\neg g) = \{pqg, p\neg q\neg g, \neg pq\neg g\}$

In other words, the judgment of each of the judges can be chosen.

Consider now the following variant of the situation with five judges:

Issues

	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	1	0	1	0
Judge 3	1	0	0	1	0	1
Judge 4	0	1	1	0	0	1
Judge 5	0	1	0	1	0	1

We note that the same "paradox" persists but now, $F(J^N) = \{pqg\}$. In particular, the verdict is *Guilty*!

Consider now the following variant of the situation with five judges:

Issues

	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	1	0	0	1	0	1
Judge 4	0	1	1	0	0	1
Judge 5	0	1	1	0	0	1

Again, the same "paradox" persists but now, $F(J^N) = \{p\neg q\neg g, \neg pq\neg g\}$. In particular, the verdict is *Not Guilty*!

3 Sequential Aggregation Procedure (*SAP*)

Consider the following aggregation function, which we call a Sequential Aggregation Procedure (*SAP*):
Let $k = \#A/2$, and denote by A_k an agenda of size $2k$ (i.e. k propositions/formulas with their negations, also called *issues*). We define F by induction on k. Let $J^N \in \mathscr{J}^N$. For $k = 1$, i.e. $A_1 = \{p, \neg p\}$, choose a formula by majority rule (with anonymous tie-breaking). Assume now that F has been defined for $k \geq 1$ and consider an agenda with $k+1$ pairs: $A_{k+1} = \{(p_1, \neg p_1), \ldots, (p_k, \neg p_k), (p_{k+1}, \neg p_{k+1})\}$. Consider the truncated profile J_k^N, the restriction of J^N to $A_k = \{(p_1, \neg p_1), \ldots, (p_k, \neg p_k)\}$. Let $F(J_k^N) = \{q_1, ..., q_k\}$. We assume that $F(J_k^N)$ is consistent, and we distinguish the following possibilities.

1. $q_1 \wedge q_2 \wedge, \ldots, \wedge q_k \Rightarrow p_{k+1}$, then we choose p_{k+1} at the $(k+1)$-th stage.
2. $q_1 \wedge q_2 \wedge, \ldots, \wedge q_k \Rightarrow \neg p_{k+1}$, then we choose $\neg p_{k+1}$ at the $(k+1)$-th stage.
3. Otherwise we choose from $\{p_{k+1}, \neg p_{k+1}\}$ by majority rule with anonymous tie-breaking.

Remark 1 We emphasize that the foregoing *SAP* depends on the order of introducing the members of A that we have chosen. Different orderings yield different aggregators. as is the case in the well-known Doctrinal Paradox.

Example 3 (The Doctrinal Paradox revisited) For the classical example of the Doctrinal Paradox,

Issues

	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1

If we apply our *SAP* for the order of issues (p,q,r) we obtain:

Issues

	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1
SAP(J)	**1**	**0**	**1**	**0**	**1**	**0**

That is, the aggregate judgement is *pqg* (in particular, the defendant is liable).

If the order of issues is (p,g,q) we obtain:

Issues

	p	$\neg p$	g	$\neg g$	q	$\neg q$
Judge 1	1	0	0	0	1	0
Judge 2	1	0	1	0	0	1
Judge 3	0	1	0	1	1	0
SAP(J)	**1**	**0**	**0**	**1**	**0**	**1**

That is, the aggregate judgement is $p\neg g\neg q$ (in particular, the defendant is *not* liable). The same aggregated judgement is obtained for the order (g,p,q), while the orders (q,g,p) and (g,q,p) yield $q\neg g\neg p$.

We shall argue that in each aggregation problem there is a natural order in which the issues are deliberated. In this example p and then q seem to be the natural temporal order. However even when the order is given, the aggregation procedure is vulnerable to manipulation. For example, in the above described situation judge 3, who thinks that the contract is invalid ($\neg p$) and therefore thinks that the defendant is not liable, may dishonestly vote for $\neg q$ in order to reach the verdict "not liable" ($\neg g$).

The following is readily verified:

Proposition 1 *The SAP shares the following properties:*

1. *Rationality: The aggregated $F(J^N)$ is consistent and complete.*
2. *Anonymity.*
3. *Unanimity.*
4. *Restricted Agenda property: If $k' \leq k$ and $J_{k'}$ denotes the restriction of the judgment J (of k issues) to the first k' issues then:*

$$F(J^N_{k'}) = (F(J^N))_{k'}, \ \ \forall J^N \in \mathscr{J}^N.$$

4 *Relevance Relation: from IIA to IIP*

The most crucial axiom in Arrow's impossibility theorem is the IIA – *independence of Irrelevant Alternatives*. The analogue axiom for judgment aggregation would be PI *proposition-wise independence*. It is easily observed that this axiom is too strong, and it readily yield an impossibility tesult. Any attempt to obtain positive results must go through weakening this axiom. Such a weakening was suggested by Dietrich (2015) who replaced PI by IIP – *Independence of irrelevant propositions*, with respect to an abstract given relevance relation. We adopt this idea but attempt to derive the *relevance relation* from the agenda: We will derive a "natural" *relevance relation* between propositions in the agenda and show that our proposed aggregation function satisfies the IIP (*Independence of Irrelevant Propositions*).

Definition 7 A *relevance relation* R is a reflexive and acyclic binary relations on the propositions of the agenda A. "q is relevant to p" is denoted by qRp and for $p \in A$, the set $R(p) = \{q \in A | qRp\}$ is the set of propositions relevant to p.

Definition 8 (Independence of Irrelevant Propositions (IIP))
Given a (JAP) $(N, A_k, \neg, \mathscr{J})$, a judgment aggregation function $F : \mathscr{J}^N \to \mathscr{J}$ satisfies *independence of irrelevant propositions* (IIP) w.r.t. the relevance relation R, if for all $J_1^N, J_2^N \in \mathscr{J}^N$ and for all $p \in A$,
$\left[J_1^i \cap R(p) = J_2^i \cap R(p),\ \forall i\ \in N\right] \Longrightarrow \left[p \in F(J_1^N) \Leftrightarrow p \in F(J_2^N)\right]$

Example: If $R(p) = \{p\}$ for all $p \in A$, then the IIP is the proposition-wise independence PI.
The first natural attempt to derive a relevance relation from the agenda is:

Definition 9 (Relevance by direct implication)
Given an agenda A of k issues and a fixed order $A_k = \{(p_1, \neg p_1), \ldots, (p_k, \neg p_k)\}$, the relevance relation IM (*Implication*) is a correspondence $IM : A \to 2^A$ defined by,

$$q_h \in IM(p_j) \;\text{ if }\; h \le j \;\text{ and }\; [q_h \Rightarrow p_j \text{ or } q_h \Rightarrow \neg p_j]$$

Remark 2 We note that:

1. This relevance relation is *reflexive* $(p \in IM(p))$, *negation invariant* $(IM(p) = IM(\neg p))$ but it is *not transitive*.
2. This relevance relation is *not symmetric* that is, qRp does not imply pRq. Furthermore, for $q \neq p$, if qRp then pRq cannot hold even if $p \Rightarrow q$ since $h \le j$ excludes $j \le h$ for $j \neq h$. In other words, the issue $(p, \neg p)$ is irrelevant to the issue $(q, \neg q)$ even if $p \Rightarrow q$ sine it is decided *after*$(q, \neg q)$.

Nevertheless, for the case of two issues we have:

Proposition 2 *For $k = 1, 2$, the aggregation function F defined above satisfies independence of irrelevant propositions (IIP) w.r.t. the relevance relation $IM(\cdot)$*

Proof We have to prove that for each $j \le k,\ p \in \{p_j, \neg p_j\}$, and all $J_1^N, J_2^N \in \mathscr{J}^N$,

$$J_1^i \cap IM(p) = J_2^i \cap IM(p),\ \forall i\ \in N \Longrightarrow \left[p \in F(J_1^N) \Leftrightarrow p \in F(J_2^N)\right].$$

1. For $k = 1,\ A_1 = \{p, \neg p\}$ and $IM(p) = \{p\}$. By our assumption $p \in J_1^i$ if and only if $p \in J_2^i$ for all $i \in N$. As p is admitted to the collective choice set by majority rule, $p \in J_1$ if and only if $p \in J_2$ where $J_t = F(J_t^N),\ t = 1, 2$.
2. For $k = 2,\ A_2 = \{p_1, \neg p_1; p_2, \neg p_2\}$. By 1. we have only to consider the second issue. So let $p \in \{p_2, \neg p_2\}$. We distinguish the following cases.
   2.1. $IM(p) = \{p\}$ (and thus $IM(\neg p) = \{\neg p\}$). Then $F(J_1^N)$ and $F(J_2^N)$ are determined by majority rule. As $IM(p) = \{p\}$ and $p \in J_1^i$ if and only if $p \in J_2^i$ for all $i \in N$, it follows that $p \in J_1$ if and only if $p \in J_2$.
   2.2. There is $q \in \{p_1, \neg p_1\}$ such that $q \Rightarrow p$. By our assumptions $q \in J_1^i$ if and only if $q \in J_2^i$. Thus, the first element in our choice (i.e. the first issue) is determined uniquely (by 1.). Hence the second element is also determined uniquely (by our assumptions, as it is implied by the first).

2.3. There is $q \in \{p_1, \neg p_1\}$ such that $q \Rightarrow \neg p$, then $q \in IM(p)$. By our assumptions we have the same choice for both profiles in the first issue and therefore the same selection for the second issue in both profiles . This completes the proof. ∎

Unfortunately, Proposition 2 cannot be extended for $k > 2$. Furthermore, the following example shows that for $k > 2$, our aggregation function *SAP* cannot satisfy the IIP w.r.t. any relevance relation between two propositions based only on binary implications between the propositions or their negations.

Example 4 Let $N = \{1,2,3\}$ be the set of judges. It will be convenient to adopt a semantic model (see e.g. F. Dietrich (2014) section 2) in which the propositions are subsets of a finite set $\Omega = \{a_1, a_2, \ldots, a_8\}$ and the negation of a proposition $P \subset \Omega$ is its complement w.r.t. Ω; $\neg P = \Omega \setminus P$. The *implication* " $\Rightarrow$" is represented by set *inclusion* " $\subset$", the *conjunction* "$\wedge$" is represented by *intersection* "$\cap$" and the *disjunction* "$\vee$" is represented by set *union* "$\cup$".
Consider now the following three propositions and their negations:

$$\begin{aligned} P_1 &= \{a_1, a_2, a_5, a_6\} & \neg P_1 &= \{a_3, a_4, a_7, a_8\} \\ P_2 &= \{a_1, a_3, a_7, a_8\} & \neg P_2 &= \{a_2, a_4, a_5, a_6\} \\ P_3 &= \{a_1, a_4, a_7, a_8\} & \neg P_3 &= \{a_2, a_3, a_5, a_6\} \end{aligned}$$

First, observe that there is no implication relation between any distinct two of the propositions and their negations that is, for any $p \in \{P_1, P_2, P_3, \neg P_1, \neg P_2, \neg P_3\}$, $IM(P) = \{P\}$.
Next we see that $P_1 \wedge P_2 \Rightarrow P_3$, $\neg P_1 \wedge \neg P_2 \Rightarrow P_3$, and $P_1 \wedge \neg P_2 \Rightarrow \neg P_3$.

For the order of propositions (P_1, P_2, P_3) our aggregation function yields:

$$F\left((P_1, P_2, P_3), (P_1, \neg P_2, \neg P_3), (\neg P_1, P_2, P_3)\right) = (P_1, P_2, P_3),$$

as P_1 and P_2 are decided by majority rule and P_3 is determined by $P_1 \wedge P_2 \Rightarrow P_3$.

Changing P_2 in the judgment of the third voter to $\neg P_2$ yields:

$$F\left((P_1, P_2, P_3), (P_1, \neg P_2, \neg P_3), (\neg P_1, \neg P_2, P_3)\right) = (P_1, \neg P_2, \neg P_3),$$

since P_1 and $\neg P_2$ are decided by majority rule and then $\neg P_3$ is determined since $P_1 \wedge \neg P_2 \Rightarrow \neg P_3$. This contradicts the IIP since P_2 is irrelevant to P_3.

In view of our last example, if our objective is to have our aggregation function F satisfy IIP, we must introduce a relevance relation of a wider range than that of simple implication.

Definition 10 Let $p \in I_h$, $h > 1$ and $q \in I_j$, $j < h$. The proposition q is relevant to the proposition p (notation qR^*p) if there exist a set of issues $(I_\ell)_{\ell \in L}$, where $L \subset \{1, \ldots, h-1\}$ may be empty (in which case, by convention $\wedge_{\ell \in \emptyset} q_\ell = \mathscr{L}$), and $q_\ell \in I_\ell$, $\ell \in L$ such that one of the following holds:

$$\begin{aligned} q \wedge_{\ell \in L} q_\ell &\Rightarrow p \\ \wedge_{\ell \in L} q_\ell &\not\Rightarrow p \end{aligned}$$

or

$$\begin{aligned} q \wedge_{\ell \in L} q_\ell &\Rightarrow \neg p \\ \wedge_{\ell \in L} q_\ell &\not\Rightarrow \neg p \end{aligned}$$

As us usual we denote the relevance correspondence by: $R^*(p) = \{q : qR^*p\}$. By definition, R^* is negation invariant: $R^*(p) = R^*(\neg p)$. However, R^* may not be transitive. As the property of transitivity is, formally and substantially, needed for our analysis, we take the transitive closure of R^*.

Definition 11 The relevance relation R is the the transitive closure of the relevance relation R^* in Definition 10 extended by the convention $p \in R(p)$ for all $p \in \mathscr{L}$.

The following corollary will be used in our proofs in the sequel.

Corollary 1 *For any $p \in I_h$ and any restricted consistent judgement J_{h-1} the following holds:*

$$\wedge J_{h-1} \Rightarrow p(or\ \neg p) \ \ \text{if and only if} \ \ \wedge_{q \in J_{h-1} \cap R(p)} q \Rightarrow p(or\ \neg p)$$

Proof The 'if' part follows since $\wedge J_{h-1} \subset \wedge_{q \in J_{h-1} \cap R(p)} q$.

For the 'only if' part assume that $\wedge J_{h-1} \Rightarrow p(\text{or } \neg p)$ and $\wedge_{q \in J_{h-1} \cap R(p)} q \not\Rightarrow p(\text{or } \neg p)$. By removing one by one the propositions in $J_{h-1} \setminus R(p)$ from the conjunction $\wedge J_{h-1}$ there must be a first stage in which when removing a $\tilde{q}$, the implication $\Rightarrow p$ (or $\Rightarrow \neg p$) no longer holds implying that $\tilde{q} \in R(p)$ in contradiction to $\tilde{q} \in J_{h-1} \setminus R(p)$. ∎

Proposition 3 *Our aggregation function F satisfies IIP w.r.t. the above defined relevance relation R.*

Proof Let $J_1^N, J_2^N \in \mathscr{J}^N$ and let $p \in I_h$. We have to prove that if $J_1^i \cap R(p) = J_2^i \cap R(p)$ for all $i \in N$, then $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$. Actually we will prove a stronger result namely: Under the same conditions $F(J_1^N) \cap R(p) = F(J_2^N) \cap R(p)$ that is, not only that $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$ but also $q \in F(J_1^N)$ if and only if $q \in F(J_2^N)$ for all $q \in R(p)$. In other words, if $J_1^i \cap R(p) = J_2^i \cap R(p)$ for all $i \in N$, then not only the appearance of p is the same in both $F(J_1^N)$ and $F(J_2^N)$ but this is true for all propositions relevant to p.

The proof is by induction on h. The case $h = 1$ follows from our assumptions, the reflexivity of $R(\cdot)$, and the definition of our F. Let $h > 1$ and assume by induction that the claim is true for $j = 1, \ldots, h-1$.

Note first that from the transitivity of R we have $q \in R(p) \Rightarrow R(q) \subset R(p))$ and therefore from

$$J_1^i \cap R(p) = J_2^i \cap R(p), \ \forall i \in N$$

we also have (by intersecting both sides with $R(q)$),

$$J_1^i \cap R(q) = J_2^i \cap R(q), \ \forall i \in N, \ \forall q \in R(p)$$

and therefore by the induction hypothesis,

$$F(J_1^N) \cap R(q) = F(J_2^N) \cap R(q), \ \ \forall q \in I_j, \ j < h, \ q \in R(p).$$

and hence,

$$F(J_1^N)_{h-1} \cap R(p) = F(J_2^N)_{h-1} \cap R(p) \tag{1}$$

We now distinguish two cases:

1. If $\wedge F(J_1^N)_{h-1} \Rightarrow p$. In this case, it must also be that $\wedge F(J_2^N)_{h-1} \Rightarrow p$.
   Indeed by Corollary 1 we have $\wedge_{q \in F(J_1^N)_{h-1} \cap R(p)} q \Rightarrow p$ and equation (1) $\wedge_{q \in F(J_2^N)_{h-1} \cap R(p)} q \Rightarrow p$, applying Corollary 1 again we have $\wedge F(J_2^N)_{h-1} \Rightarrow p$.
   Similarly if $\wedge F(J_1^N)_{h-1} \Rightarrow \neg p$ then also $\wedge F(J_2^N)_{h-1} \Rightarrow p$.
   It follows that in this case the *SAP* chooses p (or $\neg p$)) in both J_1^N and J_2^N. combined with equation (1) we get $F(J_1^N) \cap R(p) = F(J_2^N) \cap R(p)$.
2. If $\wedge F(J_1^N)_{h-1} \not\Rightarrow p$ and $\wedge F(J_1^N)_{h-1} \not\Rightarrow \neg p$, then again by Corollary 1 and equation (1) (by the same argument as in part 1.) we also have $\wedge F(J_2^N)_{h-1} \not\Rightarrow p$ and $\wedge F(J_2^N)_{h-1} \not\Rightarrow \neg p$. Hence the issue $(p, \neg p)$ is decided by simple majority voting in both profiles. Since for all $i \in N$, $p \in J_1^i$ if and only if $p \in J_2^i$, we get $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$. Combined with equation (1) we get $F(J_1^N) \cap R(p) = F(J_2^N) \cap R(p)$. completing the proof. ∎

5 Characterization of the SAP

The properties od SAP established so far are:

- SAP is an aggregation function $F : \mathscr{J}^N \to \mathscr{J}$ which implies:
  - Full domain.
  - Rationality (SAP is *complete* and *consistent*).
- Anonymity.
- Unanimity.
- Reinforcement.
- Restricted agenda property.
- IIP with respect to the above defined relevance relation (Definition 11).

For the next property we define the notion of the *marginal* of a JAF. Let $(N,A,\neg,\mathscr{J})$ be a JAP with an ordered agenda $A = \{I_1,\ldots,I_K\}$. Let $J = (x_1,\ldots,x_K) \in \mathscr{J}$. For $1 \le k \le K$ we denote $J(k) = (x_1,\ldots,x_k)$. For a profile of individual judgments J^N and $1 \le k \le K$ let $J^N(k) = (J^1(k),\ldots,J^n(k))$. Given $J^N \in \mathscr{J}^N$, let $k < K$, a sequential judgment aggregation function, F, consider the social judgment $F((J^1(k),y^1),\ldots,(J^n(k),y^n)) = J(k+1) = (J(k),y)$ where $y^i \in I_{k+1}$ for all $i \in N$. This defines a social choice function $E : I_{k+1}^N \to I_{k+1}$ given by $E(y^1,...,y^n;J^N,k,F) = y$, which is called the *marginal* of F at (J^N,k).

Definition 12 $E(\cdot,J^N,k,F)$ is of *maximal range* if there is no rational sequential JAF F^* such that

1. F^* coincides with F on $\{I_1,\ldots,I_k\}$, and
2. The range of $E(\cdot,J^N,k,F^*)$ strictly contains the range of $E(\cdot,J^N,k,F)$.

A sequential JAF F is *maximal* if $E(\cdot,J^N,k,F)$ is of maximal range for all J^N and $1 \le k \le K$.

We readily have:

- The SAP is maximal.

Definition 13 A sequential JAF F is *majoritarian* if for every profile of individual judgments J^N and every $k < K$, if the range of $E(\cdot,J^N,k,F)$ is $\{p_{k+1},\neg p_{k+1}\}$ then $E(\cdot,J^N,k,F)$ coincides with the majority rule.

We have:

- The SAP is majoritarian.

Clearly, we can replace majoritarianism by the three axioms which determine the majority rule. We are now able to state.

Theorem 2 *There is a unique sequential JAF that satisfies rationality, anonymity, unanimity, reinforcement, independence of irrelevant propositions (with respect to the relevance relation in Definition 11) and restricted agenda property that is maximal and majoritarian, and it is the sequential aggregation procedure (SAP).*

Proof As these properties follow readily from the definition of the SAP, we have only to prove uniqueness. We prove it by induction on k, the number of issues. Let F be a JAF that satisfies the properties of Theorem 2.

1. For $k = 1$, by unanimity, F has maximal range. As F is majoritarian, it coincides with the majority rule.
2. Assume that the theorem is true for k issues, $k \ge 1$, and let us prove it for $k+1$.
   We first observe that if F is a JAF for the (ordered) issues $(I_1,\ldots,I_K)$ satisfying the properties of Theorem 2, then, for any $k \le K$, the the restriction of F to $(I_1,\ldots,I_k)$ also satisfies the properties of Theorem 2 (and thus we can use the theorem in the induction hypothesis). Indeed anonymity, unanimity, reinforcement and the restricted agenda property, are carried over by the restricted agenda property. The restriction is majoritarian by definition and it is maximal since (as follows from the definition), if $E(\cdot,J^N,k,F)$ is maximal then so is

$E(\cdot, J^N, k', F)$ for any $k' \leq k$. Finally the restriction satisfies IIP since this property is stated 'for all $p \in I_k$ for all $1 \leq k \leq K$'.
Now let J^N be a profile for $k+1$ issues. Denote by F^* our SAP. By restricted agenda property $F(J^N_k) = (F(J^N))_k$. Also, by the induction hypothesis $F(J^N_k) = F^*(J^N_k)$. Let $F(J^N_k) = \{q_1, \ldots, q_k\}$. We distinguish the following cases:

(2.1) $q_1 \wedge q_2 \wedge, \ldots, \wedge q_k \Rightarrow p$ for some p in I_{k+1}. Then p is chosen by rationality by F, and thus F coincides with F^* in the $(k+1)$-th step.

(2.2) It is not true that $q_1 \wedge q_2 \wedge, \ldots, \wedge q_k \Rightarrow p$ for some p in I_{k+1}. As F is maximal and majoritarian, $E(\cdot, J^N, k, F)$ coincides with the majority rule. ∎

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