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**ANALYZING GAMES WITH AMBIGUOUS
PLAYER TYPES USING THE
MINTHENMAX DECISION MODEL**

By

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Analyzing Games with Ambiguous Player Types Using the MINTHENMAX Decision Model

(work in progress)

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Abstract

In many common interactive scenarios, participants lack information about other participants, and specifically about the preferences of other participants. In this work, we model an extreme case of incomplete information, which we term *games with type ambiguity*, where a participant lacks even information enabling him to form a belief on the preferences of others. Under type ambiguity, one cannot analyze the scenario using the commonly used Bayesian framework, and therefore one needs to model the participants using a different decision model.

To this end, we present the MINTHENMAX decision model under ambiguity. This model is a refinement of Wald's MiniMax principle, which we show to be too coarse for games with type ambiguity. We characterize MINTHENMAX as the finest refinement of the MiniMax principle that satisfies three properties we claim are necessary for games with type ambiguity. This prior-less approach we present here also follows the common practice in computer science of worst-case analysis.

Finally, we define and analyze the corresponding equilibrium concept, when all players follow MINTHENMAX. We demonstrate this equilibrium by applying it to two common economic scenarios: coordination games and bilateral trade. We show that in both scenarios, an equilibrium in pure strategies always exists, and we analyze the equilibria.

Keywords: Decision making under ambiguity, Games with ambiguity, Wald's MiniMax principle, MINthenMAX decision model, MINthenMAX equilibrium

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Contents

1	Introduction	1
2	Model	7
3	Axiomatization of MINthenMAX	11
4	Bilateral Trade	15
5	Coordination Games	18
6	Summary and Future Directions	23
	References	25
A	Proof of Lemma 3 (Best response is well defined)	28
B	Proof of Theorem 4 (MIN-NE existence)	28
C	Proof of Lemma 11 (Axiomatization)	29
D	Proof of Proposition 12 (Axiomatization of MIN)	31
E	Proof of Lemma 13 (Bilateral trade games)	32
F	Proof of Lemma 16 (Coordination games)	36

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1 Introduction

In many common interactive scenarios participants lack information about other participants, and specifically about the preferences of other participants. The extreme case of such partial information scenario is termed *ambiguity*,¹ and in our case ambiguity about the preferences of other participants. In these scenarios, not only does a participant not know the preferences of other participants, but he cannot even form a belief on them (that is, he lacks the knowledge to form a probability distribution over preferences). Hence, one cannot analyze the scenario using the Bayesian framework, which is the common practice for analyzing partial-information scenarios, and new tools are needed.² Similarly, in the computer science literature, algorithms, agents, and mechanisms are often analyzed without assuming a distribution on the input space or on the environment. In this work, we define and analyze equilibria under ambiguity about the the game. In particular, we concentrate on equilibria of *games with type ambiguity*, i.e., games with ambiguity about the other players’ preferences, namely, their *type*. Our equilibrium definition is based on a refinement of Wald’s MiniMax principle, which corresponds to the common practice in computer science of worst-case analysis.

In Section 2, we define a general model of games with ambiguity, similar to Harsanyi’s model of games with incomplete information [17],³ and derive from it the special case of *games with type ambiguity*. In this model, the knowledge of player i on player j is represented by a set of types \mathcal{T} . Player i knows that the type of player j belongs to \mathcal{T} , but has no prior distribution on this set, and no information that can be used to construct one. Our model also enables us to apply the extensive literature on knowledge, knowledge operators, and knowledge hierarchy to ambiguity scenarios.

Next, we present a novel model for decision making under ambiguity: MINthenMAX preferences. We characterize MINthenMAX in the general framework of decision making under partial information, and show MINthenMAX is the unique finest preference that satisfies a few natural properties. Specifically, we claim that these properties are satisfied by rational players in games with type ambiguity, and hence that MINthenMAX is the right tool of analysis.

Finally, we derive the respective equilibrium concept, dubbed MINthenMAX-NE, and present some of its properties, both in the context of the general model of games with type ambiguity and in two common economic scenarios.

¹In decision theory literature, the terms “ambiguity,” “pure ambiguity,” “complete ignorance,” “uncertainty” (as opposed to “risk”), and “Knightian uncertainty” are used interchangeably to describe this case of unknown probabilities.

²Clearly, if a player has information that he can use to construct a belief about the others, we expect the player to use it. In this work, we study the extreme case in which one has no reason to assume the players hold such a belief.

³As described, for example, in [23, Def. 10.37 p. 407].

Wald’s MiniMax principle

A common model for decision making under ambiguity is the MiniMax principle presented by Wald [30], which we refer to as MIN preferences (as distinct from the MINthenMAX preferences that we present later).⁴ In the MIN model, similarly to worst-case analysis in computer science, the preference of the decision maker over actions is based solely on the set of possible outcomes. E.g., in games with type ambiguity, the possible outcomes are the consequence of playing the game with the possible types of the other players. An action a is preferred to another action b if the worst possible outcome (for the decision maker) of taking action a is better than the worst outcome of taking action b . This generalizes the classic preference maximization model: if there is no ambiguity, there is a unique outcome for each action, and the MIN decision model coincides with preference maximization. The MIN model has been used for analyzing expected behavior in scenarios of decision making under ambiguity. For example, ambiguity about parameters of the environment, such as the distribution of prizes (the multi-prior model) [16] and ambiguity of the decision maker about his own utility [9]. The MIN model was also applied to define higher-order goals for a DM, like regret minimization [27, Ch. 9] which is applying MIN when the utility of the DM is the regret comparing to other possible actions. In addition, MIN has been used for analyzing interactive scenarios with ambiguity, e.g., first-price auctions under ambiguity both of the bidders and of the seller about the ex-ante distribution of bidders’ values [8], and for designing mechanisms assuming ambiguity of the players about the ex-ante distribution of the other players’ private information [31], or assuming they decide according the regret minimization model [18].

As we show shortly, the MIN decision model is too coarse and offers too little predictive value in some scenarios involving ambiguity about the other players’ types. We show a natural scenario (a small perturbation of the Battle of the Sexes game [21, Ch. 5 Sec. 3]) in which almost all action profiles are Nash equilibria according to MIN. Hence, we are looking for a refinement of the MIN model that breaks indifference in some reasonable way in cases in which two actions result in equivalent worst outcomes. In Section 3, we show that naïvely breaking indifference by applying MIN recursively⁵ does not suit scenarios with ambiguity about the other players’ types either. We present two game scenarios, and we claim that they are equivalent in a very strong sense: a player cannot distinguish between these two scenarios, even if he has enough information to know the outcomes of all of his actions. Hence, we claim that a rational player should act the same way in these two scenarios. Yet, we show that if a player follows the recursive MIN decision model he plays differently in the two scenarios. In general, we claim that a decision model for scenarios with type ambiguity should not be susceptible to this problem, i.e., it should instruct the player to act the same way in scenarios if the player cannot distinguish between them. Otherwise, when a decision

⁴Wald’s principle measures actions by their losses rather than by gains like we do here. Hence, Wald dubbed this principle, which aims to maximize the worst (minimal) gain, the MiniMax principle while we dub it MIN.

⁵I.e., when the decision maker faces two actions that have equivalent worst outcomes, he decides according to the second-worst outcome.

rule depends on information which is not visible to the DM, we find it to be ill-defined.⁶

The MINthenMAX decision model

In this work we suggest a refinement of Wald’s MiniMax principle that is not susceptible to the above-mentioned problems, and which we term MINthenMAX. According to MINthenMAX the decision maker (DM) picks an action having an optimal worst outcome (just like under MIN), and breaks indifference according to the best outcome. We characterize MINthenMAX as the unique finest refinement of MIN that satisfies three desired properties (Section 3): monotonicity in the outcomes, state symmetry, and independence of irrelevant information. *Monotonicity in the outcomes* is a natural rationality assumption stating that the DM (weakly) prefers an action a to an action b if in every state of the world (in our framework, a state is a vector of types of the other players), action a results in an outcome that is at least as good as the outcome of action b in this state. *State symmetry* asserts that the decision should not depend on the names of the states and should not change if the names are permuted. *Independence of irrelevant information* asserts that the DM should not be susceptible to the irrelevant information bias, describes above. That is, the DM’s decision should depend only on state information that is relevant to his utility. Specifically, it requires that if two states of the world have the same outcomes for each of the actions, the distinction between the two should be irrelevant for the DM, and his preference over actions should not change in case he considers these two states as a single state. We show that these properties characterize the family of preferences that are determined by only the worst and the best outcomes of the actions. Moreover, we show that MINthenMAX is the finest refinement of MIN in this family: for any preference P that satisfies the three properties, if P is a refinement of MIN,⁷ then MINthenMAX is a refinement of P .

Equilibrium under MINthenMAX preferences

In Section 2, we define MIN-NE to be the Nash equilibria under MIN preferences, that is, the set of action profiles in which each player best-responds to the actions of other players, and similarly we define MINthenMAX-NE to be the Nash equilibria under MINthenMAX preferences.

We show that for every game with ambiguity, a MIN-NE in mixed strategies always exists (Thm. 4). On the other hand, we show that there are generic games with ambiguity in which the set of MIN-NE is unrealistic and too large to be useful. This holds even for cases in which the ambiguity is symmetric (all players have the same partial knowledge) and is only about the other players’ preferences. Here once again is our motivation for studying the equilibria under MINthenMAX. On the other hand, we present a simple generic two-player game with type ambiguity for which no MINthenMAX-NE exists. We note that since

⁶This assumption is with accordance to our assumption of ambiguity which in particular assumes there is no information on the world except the information on outcomes.

⁷That is, for any two actions a and b , if a is strongly preferred to b according to MIN, then a is strongly preferred to b according to P too.

MINthenMAX is the unique finest refinement of MIN (which satisfies some properties), the equilibria of a game with ambiguity under any other refinement of MIN is a super-set of the set of MINthenMAX-NE. Hence, one can think of MINthenMAX-NE as the set of equilibria that do not depend on assumptions on the tie-breaking rule over MIN applied by the players.

We show that the problem of finding a MIN-NE is a *PPAD-complete* problem [25, 26], just as finding a Nash equilibrium when there is no ambiguity.⁸

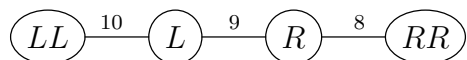
Applications of MINthenMAX-NE to economic scenarios

To understand the benefits of analysis using the MINthenMAX model, we apply the equilibrium concept MINthenMAX-NE to two well studied economic scenarios – coordination games and bilateral trade games – while introducing ambiguity. We show that in both scenarios, a MINthenMAX-NE in pure strategies always exists, and we analyze these equilibria.

Coordination games

In coordination games, the players simultaneously choose a location for a common meeting. All players prefer to choose a location that maximizes the number of players they meet, but they differ in their tie-breaking rule, i.e., their preference over the possible locations. This is a generalization of the game battle of the sexes [21, Ch. 5, Sec. 3], and it models economic scenarios where the players need to coordinate a common action, like agreeing on a meeting place, choosing a technology (e.g., cellular company), and locating a public good when the cost is shared, as well as Schelling’s focal point experiments [28, pp. 54–57]: the parachuters’ problem and the meeting in NYC problem.

Take for example, a linear street with four possible meeting locations: LL , L , R , and RR (see figure for the distances between the locations), and two players who choose locations simultaneously in an attempt to meet each other.



Both players prefer the meeting to take place, but have different preferences over the meeting places (and both are indifferent about their action if the meeting does not take place). We also assume that the preference of each player is determined by the distance between the meeting place and his initial position.⁹

Next, consider a scenario in which each player might be positioned in any one of the four locations, but he does not know the position of the other player; i.e., both have ambiguity about the type of the other player (and hence about his preference).

⁸When there is no ambiguity, a Nash equilibrium is also a MIN-NE of the game and vice-versa. Hence, finding a MIN-NE is a *PPAD-hard* problem. We show it belongs to *PPAD* and so adding ambiguity does not make the problem harder.

⁹For example, the preference of a player whose position is L is $L \succ R \succ LL \succ RR$.

For clarity, in this example we use *position* for the initial position of a player determining his preference, and *location* for the actions and outcomes.

First, we notice that profiles in which both players choose, regardless of their types, the same location are MINthenMAX-NE of the game. From the perspective of Player 2, if all the types of Player 1 choose the same location, then choosing this location and meeting Player 1 for sure is strictly preferred (regardless of the position of Player 2) to any other choice, which would surely result in no meeting. The analysis for Player 1 is identical, hence these profiles are MINthenMAX-NE.

Next, consider a case in which Player 1 goes either to LL or to RR (i.e., the locations chosen by the types of Player 1 are these two locations). From Player 2's perspective, all of his actions are equivalent in terms of their worst outcome, as there is always a possibility of not meeting Player 1 (when facing a type of Player 1 who chose a different location). Thus, Player 2 chooses according to the best outcome for him. Only the actions LL and RR result in a possibility to meet Player 1, i.e., to meet one of the types of Player 1. Hence Player 2 strictly prefers LL and RR to L and R , and so he will choose between LL and RR according to his preference over them. Following this reasoning, we show that the profile in which each type of each of the players goes to the location closest to him out of LL and RR is MINthenMAX-NE, and that any other profile in which one of the players plays LL and RR (i.e., the locations played by his types are these two locations) is not MINthenMAX-NE.

This simple scenario also demonstrates the drawback of using the MIN model to analyze games with type ambiguity. When Player 1 goes to either LL or RR , Player 2 is indifferent between the worst outcomes of all of his actions, and so, according to the MIN model, Player 2 is indifferent between all of his actions. Particularly, Player 2 of type LL is indifferent between playing LL (which is his own position), playing RR (the location farthest from him), and playing L (in which he is certain not to meet Player 1). This seems highly unrealistic: we would expect a rational Player 2 of type LL to prefer playing LL to RR or L . When using the MIN model to analyze equilibria we get that almost all profiles are MIN-NE¹⁰ including, for instance, the profile in which each type goes to the location **farthest** from him among LL and RR .

We show that in general, for every coordination game with type ambiguity, the pure Nash equilibria of the no-ambiguity case in which all players choose the same unique location are also MINthenMAX-NE.¹¹ Note that the set of equilibria when there is no ambiguity does not depend on the players' types (which are only a tie-breaking rule between two locations having the same number of other players). Yet, we get that ambiguity about the types gives rise to new equilibria, which we characterize in Section 5. We show that an equilibrium is uniquely defined by a set of meeting locations (LL and RR in the example above) to be the action profile in which each type of each player chooses his optimal location in the set. Finally, we also characterize the equilibria for several cases in which we assume a natural homogeneity constraint on the type sets. The constraint we choose, taking the type sets to be single-peaked consistent w.r.t. a line, restricts the ambiguity about the other players'

¹⁰In general, for a coordination game over m locations and n players s.t. each of them has at least t types, more than $(1 - 1/m^{t-1})^n$ -fraction of the pure action profiles are MIN-NE of the game.

¹¹In general, if an action profile a is a Nash equilibrium for all the possible combinations of types, then a is also a MINthenMAX-NE, since we expect the players to follow the action profile when the information about the types is irrelevant to their decision.

preferences in a natural way, and hence its impact on the set of equilibria is informative for the study of equilibria under type ambiguity.

Bilateral trade

The second scenario we analyze is bilateral trade. These are two-player games between a seller owning an item and a buyer who would like to purchase the item. Both players are characterized by the value they attribute to the item (their respective willingness to accept and willingness to pay). In the mechanism that we analyze, both players simultaneously announce a price and if the price announced by the buyer is higher than the one announced by the seller, then a transaction takes place and the price is the average of the two.¹² For simplicity, we assume that a player has the option not to participate in the trade.¹³

When there is no ambiguity, an equilibrium that includes a transaction consists of a single price, which is announced by both players. When there is ambiguity about the values, we show that, in addition to the single-price equilibrium, a new kind of equilibrium emerges. For instance, consider the case in which the value of the buyer can be any value between 20 and 40 and the value of the seller can be any value between 10 and 30. First, we notice that there are equilibria that are based on one price as above,¹⁴ but in any such equilibrium there will be types (either of the buyer or of the seller) that will prefer not to participate. If, for example, the price is 25 or higher, then there are types of the buyer that value the item at less than this price and will prefer not to participate; similarly, for prices below 25 there are possible sellers who value the item at more than 25 and hence will prefer not to participate. We can show further a MINthenMAX-NE with two prices, 15 and 35, in which both players participate regardless of their type: the seller announces 35 if his value is higher than 15 and 15 otherwise, and the buyer announces 35 if his value is higher than 35 and 15 otherwise.¹⁵ In this profile, a buyer who values the item at more than 35 prefers buying the item at either price to not buying it, and hence he best-responds by announcing 35 and buying the item for sure. A buyer who values the item at less than 35 prefers buying the item at 15 to not buying it, and not buying the item to buying it at 35. The best worst-case outcome he can guarantee is not buying the item (e.g., by announcing any value between 15 and his value). Based on the worst outcome the buyer is indifferent between these announcements. Hence, in choosing between these announcements according to the best-outcome (i.e., meeting a seller who announces 15), he best-responds by announcing 15. A similar analysis shows that also the seller best-responds in this profile.

We characterize the set of MINthenMAX-NE for bilateral trade games, and in particular we show that for every bilateral trade game, an equilibrium consists of at most two

¹²Our result also holds for a more general case than setting the price to be the average.

¹³This option is equivalent to the option of the seller declaring an extremely high price that will not be matched (and similarly for the buyer).

¹⁴An equilibrium in which both players choose (as a function of their value) either to announce a price common to both or not to participate.

¹⁵I.e., the seller announces the lower of the two if both are acceptable to him, and the higher otherwise; and the buyer announces the higher of the two if both are acceptable to him, and the lower otherwise.

prices. As a corollary we characterize the cases for which there exists a full-participation MINthenMAX-NE, i.e., equilibria in which both players choose to participate regardless of their value (but their bid in these equilibria might depend on the value).

2 Model

We derive our model for games with type ambiguity as a special case of a more general model of games with ambiguity. A *game with ambiguity*¹⁶ is a vector

$$\langle \mathcal{N}, (\mathcal{A}^i)_{i \in \mathcal{N}}, \Omega, (u^i)_{i \in \mathcal{N}}, (\mathcal{T}^i)_{i \in \mathcal{N}} \rangle$$

where:

- $\langle \mathcal{N}, (\mathcal{A}^i)_{i \in \mathcal{N}} \rangle$ is an n -player game form. That is, \mathcal{N} is a finite set of *players* $N = \{1, \dots, n\}$; \mathcal{A}^i is a finite set of *actions* of player i , and we denote by \mathcal{A} the set of *action profiles* $\times_{i \in \mathcal{N}} \mathcal{A}^i$.
- Ω is a finite set of *states of the world*.
- $u^i: \Omega \times \mathcal{A} \rightarrow \mathfrak{R}$ is a *utility function* for player i that specifies his utility from every state of the world and profile of actions. We identify u^i with its linear extension to mixed actions, $u^i: \Omega \times \Delta(\mathcal{A}^i) \rightarrow \mathfrak{R}$, where $\Delta(\mathcal{A}^i)$ is the set of mixed actions over \mathcal{A}^i .¹⁷
- $\langle \mathcal{N}, \Omega, (\mathcal{T}^i)_{i \in \mathcal{N}} \rangle$ is an Aumann model of incomplete information. That is, \mathcal{T}^i is a partition of Ω to a finite number of partition elements ($\Omega = \dot{\cup}_{t^i \in \mathcal{T}^i} t^i$). We refer to $t^i \in \mathcal{T}^i$ as a *type* of player i .¹⁸

The above is commonly known by the players. A game proceeds as follows.

- Nature chooses (arbitrarily) a state of the world $\omega \in \Omega$.
- Each player is informed (only) about his own partition element $t^i \in \mathcal{T}^i$ satisfying $\omega \in t^i$.
- The players play their actions simultaneously: Player i , knowing his type t^i , selects a (mixed) action $a^i \in \Delta(\mathcal{A}^i)$.
- Every player gets a payoff according to u : Player i gets $u^i(t, a)$, where $a = (a^1, a^2, \dots, a^2)$ is the action profile and $t = (t^1, t^2, \dots, t^2)$ is the type profile.

Notice that the difference between this model and the standard model of games with incomplete information [17] (e.g., as described in [23, Def. 10.37 p. 407]) is that in the latter it is assumed that the players also have posterior distributions on t^i (or equivalently, they have subjective prior distributions on Ω).

¹⁶For simplicity we define a finite game but the definitions extend to infinite cases as well. Our definitions of preferences and results also extend under minor technical assumptions.

¹⁷An implicit assumption here is that the players hold vNM preferences, that is, they evaluate a mixed action profile by its expectation. This does not restrict the modeling of preferences under ambiguity. Using the terminology of Anscombe and Aumann [2], we distinguish between roulettes and horse races.

¹⁸For a full definition of Aumann's model and its descriptive power, see, e.g., [4, 5] and [23, Def. 9.4 p. 323]. As described in [5] this model is equivalent to defining \mathcal{T}^i using signal functions and to defining them using knowledge operators (i.e., the systematic approach).

In this work we are interested in *games with type ambiguity*. In these games the states of the world are types vectors $\Omega \subseteq \times_{i=1}^n \mathcal{T}^i$, i.e., the unknown information can be represented as information on the types, and in particular any two states of the world are distinguishable by at least one player.¹⁹ For this restricted model, we justify our choice of MINthenMAX preferences. Note that we prove the existence of a mixed MIN-NE (Thm. 4) for every game with ambiguity.

A strategy of a player states his action for each of his types $\sigma^i: \mathcal{T}^i \rightarrow \Delta(\mathcal{A}^i)$. Given a type profile $t = (t^1, \dots, t^n)$ and a strategy profile $\sigma = (\sigma^1, \dots, \sigma^n)$, we denote by t^{-i} the types of the players besides player i and by $\sigma^{-i}(t^{-i})$ their actions under t and σ . I.e., $\sigma^{-i}(t^{-i}) = (\sigma^1(t^1), \dots, \sigma^{i-1}(t^{i-1}), \sigma^{i+1}(t^{i+1}), \dots, \sigma^n(t^n))$. We note that the utility of a given type of player i is only affected by that actions taken by other players and not by the actions of the other types player i . Hence, we assume a player chooses his action after knowing his type and not ex-ante beforehand, and model player i 's choice of action (best-responding to the others) as a series of independent problems, one for each of his types, of choosing an action. We refer to these problems as the decision process carried out by a type.

2.1 Preferences under ambiguity

Decision theory ([21, Ch. 13], [15]) deals with scenarios in which a single decision maker (DM) needs to choose an action from a given set \mathcal{A} when his utility from an action $a \in \mathcal{A}$ depends also on an unknown state of the world $\omega \in \Omega$, and so his preference is represented by a utility function $u: \mathcal{A} \times \Omega \rightarrow \mathfrak{R}$. Player i (of type t^i) looks for a response (an action) to a profile σ^{-i} . This response problem is of the same format as the DM problem: he needs to choose an action while not knowing the state of the world ω (the types of his opponents t^{-i} and their actions $\sigma^{-i}(t^{-i})$ are derived from ω).

We define the two preference orders over actions, MIN and MINthenMAX, in the framework of Decision Theory. We define them by defining the pair-wise comparison relation, and it is easy to see that this relation is indeed an order. The first preference we define corresponds to Wald's MiniMax decision rule [30].

Definition 1 (MIN preference).

A DM strongly prefers an action a to an action a' according to MIN, if the worst outcome when playing a is preferred to the worst outcome of playing a' :²⁰

$$\min_{\omega \in \Omega} u(a, \omega) > \min_{\omega \in \Omega} u(a', \omega).$$

These preferences follow the same motivation as worst-case analysis of computer science (where a designer needs to choose an algorithm or a system to use and the expected en-

¹⁹For Bayesian settings, this assumption is without loss of generality, because we can unify two indistinguishable states and replace them by the respective lottery, without changing the preferences. Since here we assume no posterior distribution, this assumption is indeed constraining.

²⁰The MIN preference is representable by a utility function $U(a) = \min_{\omega \in \Omega} u(a, \omega)$.

vironment in unknown in advance²¹). MIN preference can also be justified as an extreme ambiguity aversion; judging an action by the worst possible outcome ignoring the probability of this outcome. In Section 3 (Prop. 12) we show an axiomatization of MIN preference as the unique preference in the ambiguity aversion maxmin model presented by Gilboa and Schmeidler [16] (i.e., that satisfies the axioms they present), which satisfies two additional axioms we present here.

The second preference we introduce is a refinement of the MIN preference, as it breaks ties in cases where MIN states indifference between actions.

Definition 2 (MINthenMAX preference).

A DM strongly prefers an action a to an action a' according to MINthenMAX, if either $\min_{\omega \in \Omega} u(a, \omega) > \min_{\omega \in \Omega} u(a', \omega)$ or he is indifferent between the two respective worst outcomes and he prefers the best outcome of playing a to the best outcome of playing a' :²²

$$\begin{cases} \min_{\omega \in \Omega} u(a, \omega) = \min_{\omega \in \Omega} u(a', \omega) \\ \max_{\omega \in \Omega} u(a, \omega) > \max_{\omega \in \Omega} u(a', \omega). \end{cases}$$

Returning to our framework of games with ambiguity, we define the corresponding *best response* (BR) correspondences: MIN-BR and MINthenMAX-BR. The best response of a (type of a) player is a function that maps any action profile of the other players to the actions that are optimal according to the preference. It is easy to see that a best response according to MINthenMAX is also a best response according to MIN; that is, MINthenMAX-BR is a refinement of MIN-BR. We show that the two best response notions are well defined and exist for any (finite) game.²³

Lemma 3.²⁴

The following best response correspondences are non-empty: pure MIN-BR, mixed MIN-BR, pure MINthenMAX-BR, and mixed MINthenMAX-BR.

2.2 Equilibria under ambiguity

Next we define the corresponding (interim) Nash equilibrium (NE) concepts as the profiles of strategies in which each type best-responds to the strategies of the other players. From the definition of MINthenMAX it is clear that any equilibrium according to MINthenMAX is also an equilibrium according to MIN. Hence we regard MINthenMAX-NE as an equilibrium-selection notion or a refinement of MIN-NE, in cases in which we find MIN-NE to be unrea-

²¹Note that also the commonly used competitive-ratio is actually applying the MIN preference when we define $u(a, \omega)$ not to be the performance of an algorithm a on an input ω , but the ratio between this performance and the performance of an optimal all-knowing algorithm.

²²The MINthenMAX preference is not representable by a utility function, for the same reason that the lexicographic preference over \mathbb{R}^2 is not representable by a utility function [22, Ch. 3.C, p. 46].

²³This lemma can be easily extended to the case of an infinite number of states by assuming some structure on the action set and the utility function.

²⁴Proof can be found in Appendix A.

sonable. Our main theorem for this section is showing that any game with ambiguity has an equilibrium according to MIN (MIN-NE) in mixed strategies.²⁵

Theorem 4.

Every game with ambiguity has a MIN-NE in mixed actions.

Proof sketch. (The full details of the proof can be found in Appendix B)

We take \mathbb{S} to be the set of all profiles of mixed strategies of the types and define the following set-valued function $F : \mathbb{S} \rightarrow \mathbb{S}$. Given a strategy profile s , $F(s)$ is the product of the best responses to s (according to MIN) of the different types. We prove the existence of a mixed MIN-NE by applying Kakutani’s fixed point theorem [20] to F . A fixed point of F is a profile s satisfying $s \in F(s)$; i.e., each type best-responds to the others in the profile s , and hence s is a MIN-NE. \square

Since the existence of MIN-NE is the result of applying Kakutani’s fixed point theorem to the best response function,²⁶ we get as a corollary the complexity of the problem of finding MIN-NE.

Corollary 5.

The problem of finding a MIN-NE is in PPAD [25, 26]. Moreover, it is a PPAD-complete problem since a special case of it, namely, finding a Nash equilibrium, is a PPAD-hard problem [11].

Next we show that there are games that with no equilibrium according to MINthenMAX. We show that this is true even for a simple generic game: a two-player game with *type ambiguity on one side only*.

Lemma 6.

There are games for which there is no MINthenMAX-NE.

Proof.

Let G be the following two-player game with two actions for each of the players. The row

player’s utility is

	L	R
T	0	0
B	-1	1

. The column player is one of two types: either having utility

²⁵Throughout this paper, unless stated otherwise, when we refer to MIN-NE and MINthenMAX-NE we mean equilibria in pure actions.

²⁶The best response correspondence can be computed in polynomial time.

$$\begin{aligned} \text{BR}(s) &= \text{argmax}_{\sigma^i} \min_{\omega \in \Omega} u(\omega, \sigma^i, s^{-i}(t^{-i}(\omega))) \\ &= \text{argmax}_{\sigma^i} \min_{\omega \in \Omega} \mathbb{E}_{a \sim \sigma^i} C_{a,\omega} \quad \text{for } C_{a,\omega} = u(\omega, a, s^{-i}(t^{-i}(\omega))). \end{aligned}$$

The maximal value a player can guarantee himself, $v^* = \max_{\sigma^i} \min_{\omega \in \Omega} \mathbb{E}_{a \sim \sigma^i} C_{a,\omega}$, is the solution to the following program which is linear in v and σ^i

$$\max v \text{ s.t. } \forall \omega \quad \mathbb{E}_{a \sim \sigma^i} C_{a,\omega} \geq v,$$

that can be solved in polynomial time. Given v^* , $\text{BR}(s)$ is the intersection of $|\Omega|$ hyperplanes of the form $\mathbb{E}_{a \sim \sigma^i} C_{a,\omega} \geq v^*$.

	<i>L</i>	<i>R</i>
<i>T</i>	0	1
<i>B</i>	0	-2

or

	<i>L</i>	<i>R</i>
<i>T</i>	0	2
<i>B</i>	0	-1

(and the row player does not know which).

Then, in the unique MIN-NE the first type of the column player mixes $\frac{1}{2}L + \frac{1}{2}R$, the second type of the column player plays *R*, and the row player mixes $\frac{2}{3}T + \frac{1}{3}B$ (all his mixed actions give him a worst-case payoff of 0). But this is not a MINthenMAX-NE since the row player prefers to deviate to playing *B* for the possibility of getting 1, and hence the game does not have MINthenMAX-NE in mixed strategies.²⁷ \square

3 Axiomatization of MINthenMAX

In this section we justify using equilibria under MINTHENMAX preferences for the analysis of games with type ambiguity. To do so, we present three properties for decision making under ambiguity and characterize MINTHENMAX as the finest refinement of MIN that satisfies them (Thm. 10). We claim that these properties are necessary for modeling decision making under ambiguity about the other players' types. In doing so, we justify our application of MINTHENMAX-NE.

3.1 The decision-theoretic framework

Let Ω be a finite set of states of the world. We characterize a preference, i.e., a total order, of a decision maker (DM) over the action set \mathcal{A} where an action is a function $a: \Omega \rightarrow \mathfrak{R}$ that yields a utility for each state of the world.²⁸

Our first two properties are natural and we claim that any reasonable preference under ambiguity should satisfy them. The first property we present is a basic rationality assumption: monotonicity. It requires that if an action a results in a higher or equal utility than an action b in all states of the world, then the DM should weakly prefer a to b .

²⁷Technical comment: The reason Kakutani's theorem cannot be applied here (besides its result being wrong) is twofold:

- The best-response set is not convex:

Consider a player who has two possible pure actions, *T* and *B*, and his utility (as a function of the

action of the opponent) is

	<i>L</i>	<i>M</i>	<i>R</i>
<i>T</i>	0	1	2
<i>B</i>	0	2	1

. Next, consider he faces one of three types of his opponent

who play the three actions, respectively. He is indifferent between his two actions (both give him 0 in the worst case and 2 in the best case), but strictly prefers the two pure actions to any mixture of the two (giving him less than 2 in the best case).

- The best-response function is not upper semi-continuous:

In the example in the lemma, when the row player faces one type that plays the pure strategy *R* and another type that mixes $(\frac{1}{2} + \epsilon)L + (\frac{1}{2} - \epsilon)R$, his unique best response is to play *T* for any $\epsilon > 0$, but to play *B* for $\epsilon = 0$.

²⁸Replacing \mathfrak{R} with any other ordered set would not change our results.

Axiom 7 (Monotonicity).

For any two actions a and b , if $a(\omega) \geq b(\omega)$ for all $\omega \in \Omega$, then either the DM is indifferent between the two or he prefers a to b .

The second property, state symmetry, states that the DM should choose between actions based on properties of the actions and not of the states. I.e., if we permute the states' names, his preference should not change. Since we can assume that the states themselves have no intrinsic utility beyond the definition of the actions, this property formalizes the property that the DM, due to the ambiguity about the state, should satisfy the *Principle of Insufficient Reason* and treat the states symmetrically.²⁹

Axiom 8 (State symmetry).

For any two actions a and b and a bijection $\psi: \Omega \rightarrow \Omega$, if a is preferred to b , then $a \circ \psi$ is preferred to $b \circ \psi$ ($a \circ \psi(\omega)$ is defined to be $a(\psi(\omega))$), i.e., the outcome of the action a in the state $\psi(\omega)$).

The last property we present is independence of irrelevant information. This property requires that if the DM considers one of the states of the world as being two states, by way of considering some new parameter, his preference should not change. We illustrate the desirability of this property for games with type ambiguity using the following example. Consider the following variant of the Battle of the Sexes game between Alice and Bob, who need to decide on a joint activity: either a Bach concert (B) or a Stravinsky concert (S). Taking the perspective of Alice, assume that she faces one of two types of Bob: Bob^B whom she expects to choose B , or Bob^S whom she expects to choose S . Assume that Alice prefers

B , and so her valuation of actions is

	Bob^B	Bob^S
$B :$	2	0
$S :$	0	1

(0 if they do not meet and 2 or 1

if they jointly go to a concert). But there might be other information Alice does not know about Bob. For example, it might be that in case Bob prefers (and chooses) S , Alice also does not know his favorite soccer team.³⁰ So she might actually conceive the situation as

	Bob^B	$Bob^{S,*}$	$Bob^{S,\dagger}$
$B :$	2	0	0
$S :$	0	1	1

. Since this new soccer information is irrelevant to the game,

it should not change the action of a rational player. Notice that if Alice chooses according to the recursive MIN rule we described in the introduction, she will choose according to the second-worst outcome and hence choose B in the first scenario and S in the second scenario. We find a decision model of a rational player which is susceptible to this problem to be an ill-defined model.

²⁹Note that this property rules out any subjective expectation maximization preference, except for expectation under the uniform distribution.

³⁰Of course, his favorite soccer team is clear in case he prefers Bach.

Axiom 9 (Independence of irrelevant information).

Let a and b be two actions on Ω s.t. $a \preceq b$, and let $\hat{\omega} \in \Omega$ be a state of the world. Define a new state space $\Omega' = \Omega \dot{\cup} \{\hat{\omega}\}$ and let a' and b' be two actions on Ω' satisfying $a'(\omega) = a(\omega)$ and $b'(\omega) = b(\omega)$ for all states $\omega \in \Omega \setminus \{\hat{\omega}\}$, $a'(\hat{\omega}) = a'(\hat{\omega}) = a(\hat{\omega})$, and $b'(\hat{\omega}) = b'(\hat{\omega}) = b(\hat{\omega})$. Then $a' \preceq b'$.

We show that $\text{MIN}_{\text{THENMAX}}$ is the finest refinement of MIN that satisfies the above three axioms.³¹

Theorem 10.

$\text{MIN}_{\text{THENMAX}}$ is the unique preference that satisfies

- Monotonicity.
- State symmetry.
- Independence of irrelevant information.
- It is a refinement of MIN .³²
- It is the finest preference that satisfies the above three properties. That is, it is a refinement of any preference that satisfies the above properties.

We claim that the three axioms are necessary for modeling a rational decision making under type ambiguity: Monotonicity is a basic rationality axiom, and the other two capture that the DM does not have any additional information distinguishing between the states of the world besides the outcomes of his actions. Following Wald, one could define the family of all refinements of MIN that satisfy the axioms, and analyze the equilibria when all players follow models from this family. Showing that $\text{MIN}_{\text{THENMAX}}$ is a refinement of any of the preferences in this family, says that the set of equilibria when all players follow models from this family, must include all equilibria for the case when the players follow the $\text{MIN}_{\text{THENMAX}}$ model (i.e., all $\text{MIN}_{\text{THENMAX-NE}}$). Moreover, $\text{MIN}_{\text{THENMAX-NE}}$ are the only profiles which are equilibria whenever all players follow models from this family. We interpret this result as robustness of the $\text{MIN}_{\text{THENMAX-NE}}$ notion: these are the equilibria an outside party can expect (e.g., the self-enforcing contracts he can offer to the players), while not knowing the exact preferences of the players.

Proof of Theorem 10.

In order to prove this theorem we first prove that any preference that satisfies the first three properties can be defined using the worst (minimal) and best (maximal) outcomes of the actions.³³

³¹Notice that the MIN preference satisfies these properties.

³²That is, for any two actions a and b , if a is strongly preferred to b by a DM holding a MIN preference, then a is strongly preferred to b by a DM holding a $\text{MIN}_{\text{THENMAX}}$ preference.

³³Arrow and Hurwicz [3] showed a similar result for decision rules. They defined four properties (A–D) and showed that under these properties the decision rule can be defined using the worst and best outcomes only. Their properties are of a similar nature to the properties we present: Property A in [3] derives that the decision rule is derived by a preference ([22, Prop. 1.D.2, p. 13]), and Properties B, C, and D are of a similar flavor to the properties of state symmetry, independence of irrelevant information, and monotonicity in the outcomes, respectively. In order to avoid defining the framework of decision rules, and because our proof is simple and different from theirs, we prove Lemma 11 directly and do not rely on their result.

Lemma 11.³⁴

Any preference that satisfies monotonicity, state symmetry, and independence of irrelevant information can be defined as a function of the worst and the best outcomes of the actions.

Using this lemma, we turn to prove the characterization of MINTHENMAX.

It is easy to verify that MINTHENMAX indeed satisfies the first four properties. We note that uniqueness is an immediate consequence of uniqueness of a finest refinement and prove that indeed MINTHENMAX is a finest refinement. Let P be an arbitrary refinement of MIN that satisfies Monotonicity, state symmetry, and Independence of Irrelevant Information. Let a and b be two actions s.t. a DM holding a P preferences strongly prefers a to b , and we'll show that a is preferred to b also by a DM holding a MINTHENMAX preference. Applying Lemma 11, we get that P can be defined using the minimal and maximal outcomes. We denote the respective minimal and maximal outcomes of a by m_a and M_a , and of b by m_b and M_b .

Assume for contradiction that a DM holding a MINTHENMAX preference weakly prefers b to a . Since P is a refinement of MIN, it cannot be that $m_b > m_a$, and hence we get that $m_b = m_a$ and $M_b \geq M_a$. Both P and MINTHENMAX can be defined as a function of the minimal and maximal outcome, so with no loss of generality we can assume that a and b result in the same minimal outcome m_a in some state ω_m , result in their respective maximal outcomes in the same state ω_M , and result in the same intermediate outcomes in all other states. Since P is monotone, we get that a DM holding a P preference weakly prefers b to a and by that get a contradiction. \square

As a corollary of Lemma 11, we get in addition an axiomatization of Wald's MIN rule as the unique rule that satisfies both the above three properties and the natural axioms of Gilboa and Schmeidler [16]: certainty independence, continuity, monotonicity, and uncertainty aversion.

Proposition 12.³⁵

When there are at least three states of the world ($|\Omega| \geq 3$), MIN is the unique preference over \mathfrak{R}^Ω (i.e., actions that return cardinal outcomes) that satisfies

- *Let a and b be two actions, $c \in \mathfrak{R}$, and $\alpha \in (0, 1)$. Then for the two actions a' and b' defined by $a'(\omega) = \alpha \cdot a(\omega) + (1 - \alpha) \cdot c$ and $b'(\omega) = \alpha \cdot b(\omega) + (1 - \alpha) \cdot c$ for all $\omega \in \Omega$:*

$$a \succ b \iff a' \succ b'.$$

- *Let a , b , and c be three actions s.t. $a \succ b \succ c$. Then, there exists a scalar $\alpha \in (0, 1)$ and an action f_α defined by $f_\alpha(\omega) = \alpha \cdot a(\omega) + (1 - \alpha) \cdot c(\omega)$ for all $\omega \in \Omega$ s.t. $f_\alpha \succ b$, and there exists a scalar $\beta \in (0, 1)$ and an action f_β defined by $f_\beta(\omega) = \beta \cdot a(\omega) + (1 - \beta) \cdot c(\omega)$ for all $\omega \in \Omega$ s.t. $f_\beta \prec b$.*

³⁴This lemma is proved in Appendix C

³⁵This lemma is proved in Appendix D

- For any two actions a and b s.t. $a \sim b$ (i.e., $a \succcurlyeq b$ and $b \succcurlyeq a$), it holds that $c_\alpha \succcurlyeq a$ for any action c_α defined by $c_\alpha(\omega) = \alpha \cdot a(\omega) + (1 - \alpha) \cdot b(\omega)$ for all $\omega \in \Omega$, for some $\alpha \in [0, 1]$.
- *Monotonicity.*
- *State symmetry.*
- *Independence of irrelevant information.*

4 Bilateral Trade

In order to demonstrate this new notion of equilibrium, MINTHENMAX-NE, we apply it to two economic scenarios that have type ambiguity. For both of them we show a MINTHENMAX-NE in pure strategies always exists and analyze these equilibria. The first scenario we analyze is bilateral trade games. Bilateral trade is one of the most basic economic models, which captures many common scenarios. It describes an interaction between two players, a *seller* and a *buyer*. The seller has in his possession a single indivisible item that he values at v_s (e.g., the cost of producing the item), and the buyer values the item at v_b . We assume that both values are private information, i.e., each player knows only his own value, and we would like to study the cases in which the item changes hands in return for money, i.e., a transaction occurs.³⁶ Chatterjee and Samuelson [10] presented bilateral trade as a model for negotiations between two strategic agents, such as settlement of a claim out of court, union-management negotiations, and of course a model for negotiation on transaction between two individuals and a model for trade in financial products. The important feature the authors note is that an agent, while certain of the potential value he places on a transaction, has only partial information concerning its value for the other player.³⁷ Bilateral trade is also of a theoretical importance, and moreover a multi-player generalization of it, *double auction*,³⁸ These models have been used as a tool to get insights into how to organize trade between buyers and sellers, as well as to study how prices in markets are determined.

In this section we assume that there is ambiguity about the players' values (their types), and we study trading mechanisms, i.e., procedures for deciding whether the item changes hands, and how much the buyer pays for it. We assume that the players are strategic, and hence a mechanism should be analyzed according to its expected outcomes in equilibrium.

We concentrate on a family of simple mechanisms (a generalization of the bargaining rules of Chatterjee and Samuelson [10]): the seller and the buyer post simultaneously their respective bids, a_s and a_b , and if $a_s \leq a_b$ the item is sold for $x(a_s, a_b)$, for x being a known monotone function satisfying $x(a_s, a_b) \in [a_s, a_b]$. For ease of presentation, we add to the action sets of both players a “no participation” action \perp , which models the option of a

³⁶Another branch of the literature on bilateral trade studies the process of bargaining (getting to a successful transaction). Since we would like to study the impact of ambiguity, we restrict our attention to the outcome.

³⁷For instance, in haggling over the price of a used car, neither buyer nor seller knows the other's walk-away price.

³⁸In a double auction [14], there are several sellers and buyers, and we study mechanisms and interactions matching them to trading pairs.

player not to participate in the mechanism; i.e., there is no transaction whenever one of the players plays \perp . This simplifies the presentation by grouping together profiles in which a player chooses extreme bids that would not be matched by the other player. Hence, the utilities of a seller of type v_s and a buyer of type v_b from an action profile (a_s, a_b) are (w.l.o.g., we normalize the utilities of both players to zero in the case where there is no transaction):

$$u_s(v_s; a_s, a_b) = \begin{cases} a_s \leq a_b & x(a_s, a_b) - v_s \\ a_s > a_b & 0 \\ a_s = \perp \vee a_t = \perp & 0 \end{cases}$$

$$u_b(v_b; a_s, a_b) = \begin{cases} a_s \leq a_b & v_b - x(a_s, a_b) \\ a_s > a_b & 0 \\ a_s = \perp \vee a_t = \perp & 0. \end{cases}$$

Under full information (i.e., the values v_s and v_b are commonly known), there is essentially only one kind of equilibrium: the *one-price equilibrium*. If $v_s \leq v_b$, the equilibria in which there is a transaction are all the profiles (a_s, a_b) s.t. $a_s = a_b \in [v_s, v_b]$ (i.e., the players agree on a price), and the equilibria in which there is no transaction are all profiles in which both players choose not to participate, regardless of their type.

Introducing type ambiguity, we define the seller type set V_s and the buyer type set V_b , where each set holds the possible valuations of the player for the item. We show that under type ambiguity, there are at most three kinds of equilibria, and we fully characterize the equilibria set. We show that in addition to the above no-transaction equilibria and one-price equilibria, we get a new kind of equilibrium: the *two-price equilibrium*. In such an equilibrium, both the seller and the buyer participate regardless of their valuations, and bid one of two possible prices: p_L and p_H . For some type sets, namely V_s and V_b , these two prices are the only full-participation equilibria, i.e., equilibria in which both players choose to announce a price and participate, regardless of their value.

Lemma 13.³⁹

Let G be a bilateral trade game defined by a price function $x(a_s, a_b)$ and two type sets V_s and V_t , both having a minimum and maximum.⁴⁰ Then all the MINTHENMAX-NE of G are of one of the following classes:

1. **No-transaction equilibria** (These equilibria exist for any two sets V_s and V_b)
In these equilibria, neither the buyer nor the seller participates (i.e., they play \perp , or bid a too extreme bid for all types of the other player), regardless of their valuations.
2. **One-price equilibria** (These equilibria are defined only when $\min V_s \leq \max V_b$, i.e., when an ex-post transaction is possible.)

³⁹This result is also valid, and even more natural, for infinite type sets.

⁴⁰We state the result here for the case where both sets have a minimal valuation and have a maximal valuation. Dropping this assumption does not change the result in any essential way: some of the inequalities are changed to strict inequalities.

In a one-price equilibrium, both the seller and the buyer choose to participate for some of their types. It is defined by a price $p \in [\min V_s, \max V_b]$ s.t. the equilibrium strategies are:

The seller bids p for valuations $v_s \leq p$, and \perp otherwise (the second clause might be vacuously true).

The buyer bids p for valuations $v_b \geq p$, and \perp otherwise (the second clause might be vacuously true).

Hence, the outcome is

	Seller	
Buyer	Low: $v_s \leq p$	High: $v_s > p$
Low: $v_b < p$	no transaction	no transaction
High: $v_b \geq p$	p	no transaction

3. **Two-price equilibria** (These equilibria are defined only when $\min V_s \leq \min V_b$ and $\max V_s \leq \max V_b$, i.e., when there is a value for the seller s.t. an ex-post transaction is possible for any value of the buyer, and vice versa.)

In a two-price equilibrium, all types of both the seller and the buyer choose to participate, and their bids depend on their valuations. It is defined by two prices $p_L < p_H$ s.t.

$$\begin{cases} \min V_s \leq p_L < \max V_s \leq p_H \\ p_L \leq \min V_b < p_H \leq \max V_b \end{cases} \text{ and the equilibrium strategies are:}$$

The seller bids p_L for valuations $v_s \leq p_L$, and p_H otherwise.

The buyer bids p_H for valuations $v_b \geq p_H$, and p_L otherwise.

Hence, the outcome is

	Seller	
Buyer	Low: $v_s \leq p_L$	High: $v_s > p_L$
Low: $v_b < p_H$	p_L	no transaction
High: $v_b \geq p_H$	$x(p_L, p_H) \in (p_L, p_H)$	p_H

Proof sketch. (The full details of the proof can be found in Appendix E)

It is easy to verify that these profiles are indeed equilibria. We will prove that they are the only equilibria.

Assume for contradiction there is another MINTHENMAX-NE profile of actions (i.e., bids or \perp), namely, (a_s, a_b) . We define P_s to be the set of bids that are bid by the seller, i.e., $P_s = \{a_s(v_s) \mid a_s(v_s) \neq \perp\}$, and similarly we define $P_b = \{a_b(v_b) \mid a_b(v_b) \neq \perp\}$. Both these sets are not empty since (a_s, a_b) is not a no-transaction equilibrium. First, we notice that if P_s is of size one, i.e., whenever the seller participates he announces p ; then, if the buyer chooses to participate (based on his valuation), he chooses to match p in order to minimize the price (and vice versa). Analyzing the valuations for which they choose to participate proves that this profile is a one-price equilibrium. Now, assume that both these sets are of size at least two. If both players participate regardless of their value (never choose \perp), then the worst and best cases for a player are those of facing the highest and lowest bids of the other player. Hence his best response will be to match one of the two, and we get a two-price equilibrium. If the seller chooses whether to participate based on his value, i.e., there is a value v_s for which he chooses \perp , then the buyer cannot guarantee himself more than zero (for instance, if he meets v_s). Hence, he will choose one of the actions that guarantee him zero in the worst case (e.g., \perp), and choose among these actions according to their best

case (meeting the lowest-bidding type of the seller). Hence, given his value, the buyer either chooses \perp , or matches the lowest bidding type of the seller. This proves this equilibrium is either a no-transaction equilibrium or a one-price equilibrium. The case in which the buyer chooses whether to participate based on his value is symmetrical. \square

We find it interesting that the set of equilibria depends on the possible types of the players, and not on the price mechanism $x(a_s, a_b)$. In addition to the two classic equilibrium kinds, no-transaction equilibrium and one-price equilibrium, we get a new kind of equilibrium. We see that in this equilibrium each of the players announces one of two bids, which is tantamount to announcing whether his value is above some threshold or not. This decision captures the (non-probabilistic) trade-off a player is facing: whether to trade for sure, i.e., with all types of the other player, or to get a better price. For example., the buyer decides whether to bid the high price and buy the item for sure, taking the risk of paying more than his value; or whether to bid the low price and buy at a lower price, taking the risk of not buying at all. Since `MINTHENMAX` is a function of the worst-case and best-case outcomes only, it does not seem surprising that we get this dichotomous trade-off and at most two bids (messages) for each player in equilibrium.

This result might explain the emergence of market scenarios in which a participant needs to choose which one of two markets to attend, e.g., florists who choose whether to sell in a highly competitive auction or in an outside market, and he needs to choose between the two while not knowing the demand for that day. In a continuation work, we follow this story, and analyze double auctions with several buyers and sellers.

5 Coordination Games

In this section we study a second application of `MINTHENMAX-NE` to an economic scenario: analyzing coordination games with type ambiguity. These games model scenarios in which the participants prefer to coordinate their actions with others, e.g., due to positive externalities. Some examples are choosing a meeting place (a generalization the Battle of the Sexes game [21, Ch. 5, Sec. 3]), choosing a cellular company, and placing a public good or bad when the cost is shared. We analyze coordination games in which all players prefer to maximize the number of other players they coordinate with (while being indifferent about their identity), but they might differ in their tie-breaking rule between two maximizing actions.

Definition 14 (Coordination games with type ambiguity).⁴¹

A (finite) coordination game of n players over m *locations* is a game in which all players have the same set of actions of size m (and we refer to the actions as *locations*), and the preference of each player (his type) over the action profiles is defined by a strict (ordinal) preference α over the locations in the following way: Player i , holding a preference α over

⁴¹Most of the current literature (e.g., [28, pp. 54–74], [21, pp. 90–91], [24, pp. 15–16]) deal with two-player coordination games (games in which the best response of a player is to copy the other player’s action). We use the same name here for multi-player (generalized) coordination games, which capture the same kind of scenarios.

the locations, strongly prefers an action profile $a = (a^1, \dots, a^n) \in \mathcal{A}$ to an action profile $b = (b^1, \dots, b^n) \in \mathcal{A}$ if either he meets more players under a than under b ($|\{j \neq i \mid a^j = a^i\}| > |\{j \neq i \mid b^j = b^i\}|$) or if he meets the same (non-zero) number of players under both profiles and he prefers the meeting location in a to the one in b (a^i is preferred to b^i according to α). That is, the set of types of player i , \mathcal{T}^i , is a set of strict preferences over the m locations. In particular, a player is indifferent between the outcomes in which he does not meet any of the other players.

First, we note that the concept MIN-NE is a too coarse for analyzing coordination games; almost all pure action profiles of a (large enough) coordination game are MIN-NE.

Lemma 15.

Let G be a coordination game over m locations with n players, each of them having at least t types. Then more than $(1 - 1/m^{t-1})^n$ -fraction of the pure action profiles are MIN-NE of the game.

Specifically, a profile is a MIN-NE of G iff either there exists a location l s.t. all types of every player choose l in the profile, or there does not exist a location l and a player i s.t. all the types of player i choose l in the profile.

We analyze the refinement of MIN-NE, pure MINTHENMAX-NE,⁴² which we show always exists. We show that every MINTHENMAX-NE profile a is uniquely defined by the set of locations chosen in a ,

$$L(a) = \{l \mid \exists i \in \mathcal{N}, t^i \in \mathcal{T}^i \text{ s.t. player } i \text{ of type } t^i \text{ plays } l \text{ in the profile } a\}.$$

Lemma 16.⁴³

Let G be a coordination game and let L be a non-empty set of locations. There exists a MINTHENMAX-NE profile a s.t. $L(a) = L$ if and only if for all i the mapping $f^i : \mathcal{T}^i \rightarrow L$ that maps a type to his best location in L is onto.

Moreover, the action profile a in which every type of every player chooses his best location in L is the unique pure MINTHENMAX-NE that satisfies $L(a) = L$.

Abusing notation, we say that a location set L is a MINTHENMAX-NE (and shortly $L \in \text{MINTHENMAX-NE}(G)$) if there exists a MINTHENMAX-NE profile a s.t. $L(a) = L$. Two immediate corollary from the lemma are that there is always a pure MINTHENMAX equilibrium and that any location set $L \in \text{MINTHENMAX-NE}$ satisfies $|L| \leq \min_i |\mathcal{T}^i|$, and, in particular, if there is no ambiguity about the type of at least one player ($\exists i \text{ s.t. } |\mathcal{T}^i| = 1$), then the only MINTHENMAX-NE are those in which all types of all players choose the same location. Notice that for any vector of type sets $\{\mathcal{T}^i\}$ an action profile in which all types of all players choose the same location, which does not depend of the players' preferences, is always a MINTHENMAX-NE. In these profiles, any deviation of a (type of a) player results

⁴²Since any MINTHENMAX-NE is also a MIN-NE this can be interpreted as an equilibrium selection process.

⁴³This lemma is proved in Appendix F

in the deviator not meeting any of the other players and hence it is not beneficial for the deviator.

From now on we assume that $|\mathcal{T}^i| > 1$ for all i (there is a real ambiguity about the preference of each of the players), and study the equilibria (location sets) according to MINTHENMAX. We analyze the non-trivial equilibria that emerge due to type ambiguity. We call a profile p a *non-trivial profile* if $|L(p)| > 1$,⁴⁴ and call a location set a *non-trivial location set* if it includes at least two locations.

From the characterization above, we prove several easy properties of MINTHENMAX-NE.

Corollary 17.

Let G be a coordination game over m locations and n players with type sets \mathcal{T}^i .

- Let a be an equilibrium profile; then the set

$$\{l \mid \exists t^i \in \mathcal{T}^i \text{ s.t. player } i \text{ of type } t^i \text{ plays } l \text{ in the profile } a\}$$

is independent of i (i.e., in equilibrium all players choose the same set of actions).

- (Increasing ambiguity) Let G' be a coordination game over m locations and n players with type sets $\widehat{\mathcal{T}}^i$ satisfying $\mathcal{T}^i \subseteq \widehat{\mathcal{T}}^i$ for all i . Then, $\text{MINTHENMAX-NE}(G) \subseteq \text{MINTHENMAX-NE}(G')$.⁴⁵
- (MINTHENMAX-NE is downward closed) If $L \in \text{MINTHENMAX-NE}(G)$, then any $L' \subseteq L$ is also an equilibrium of G .
- (Irrelevant information)
 - Let L be an equilibrium of G and let $\widehat{\mathcal{T}}^i$ be the result of changing the preferences of the types of player i , while keeping the preferences over the locations in L . Then L is also an equilibrium of the coordination game over m locations and n players with type sets $\langle \mathcal{T}^{-i}, \widehat{\mathcal{T}}^i \rangle$.⁴⁶
 - Let L be an equilibrium of G and let G' be an extension of G to $m + 1$ locations s.t. for any player and any of his types the preference over the first m locations is the same as in G . Then L is also an equilibrium of G' .

5.1 Coordination games with single-peaked consistent preferences

Next, we study the equilibria of coordination games under type ambiguity for several special cases in which the type sets satisfy some natural constraints. In this work, we present cases in which for every player the set of preferences \mathcal{T}^i is single-peaked consistent with regard to a line.

Definition 18 (single-peaked consistent preferences with regard to a line [7]).

A preference α over a set of locations $S \subseteq \mathfrak{R}$ is said to be *single-peaked with regard to \mathfrak{R}* , if there exists a utility function $f: \mathfrak{R} \rightarrow \mathfrak{R}$ s.t. for any two locations y and z , y is preferred to z if $f(y) > f(z)$ (i.e., f represents α); the top-ranked location x^* in α is the unique

⁴⁴I.e., there exist at least two different locations, each of which is chosen by some type of some player.

⁴⁵In particular, the equilibria of the case with no ambiguity are also equilibria of any coordination game.

⁴⁶That is, the same type sets for all players except player i , and the perturbed type set for player i .

maximizer of f ; and for any two locations y and z , if $x^* < y < z$ or $x^* > y > z$, then y is preferred to z .

A set of preferences over a set of locations S is *single-peaked consistent w.r.t. a line*, if there exists an embedding function $e: S \rightarrow \mathfrak{R}$ (which we refer to as the *order* of the locations) s.t. for any preference α in the set, $e(\alpha)$ is single-peaked with regard to \mathfrak{R} .

For ease of presentation, we state the results for the case $S \subseteq \mathfrak{R}$ (and so the embedding is the identity function). For example, consider the scenario of deciding on locating a common good on a linear street. It is known that each player holds an ideal location (his location), and that his preference is monotone in the path between this ideal location and the common good. Yet, there might be ambiguity about the ideal location, or about the preference between locations that are not on the same side of the ideal location (e.g, the preference might be a function of properties of the path).

We interpret the single-peakedness assumption as constraining the type ambiguity in a natural way: Player i knows the order over the locations common to all types of player j , and hence he has some information he can use to anticipate the actions of other players. Note that a set of single-peaked consistent preferences might be single-peaked with regard to more than one order of the locations (i.e., an embedding).⁴⁷ In cases where \mathcal{T}^i is single-peaked consistent w.r.t. several orders of the locations (which can be thought of as a stronger constraint), our results hold w.r.t. any of the orders, giving rise to stricter characterizations. In the two examples to follow, we limit the ambiguity further (i.e., adding more information on players) in two ways.

The first case we analyze is when assuming no ambiguity about the players' ideal locations. That is, for every player there exists a location x^* (his ideal location) that is shared by all his types in \mathcal{T}^i . We show that in this case there can be at most two locations in the equilibrium set, one situated to the right of all ideal locations and one situated to the left of all of them ("right" and "left" w.r.t. the order of each player).

Lemma 20.

Let G be a coordination game with single-peaked consistent type sets \mathcal{T}^i s.t. there is no ambiguity about the players' ideal locations, and let x^i be player i 's ideal location. Then any (non-trivial) MINTHENMAX-NE L satisfies that there exist two different locations α and β s.t. $L = \{\alpha, \beta\}$ and for every player i and order $<^i$ s.t. \mathcal{T}^i is single-peaked consistent w.r.t. $<^i$ it holds that $\alpha <^i x^i <^i \beta$.

*This condition is tight. Any set L satisfying the above condition is a MINTHENMAX-NE of the game in case the type sets \mathcal{T}^i are rich enough.*⁴⁸

⁴⁷ **Example 19.** The preferences $1 \succ 2 \succ 3 \succ 4 \succ 5$, $2 \succ 1 \succ 3 \succ 4 \succ 5$, $2 \succ 3 \succ 1 \succ 4 \succ 5$, and $3 \succ 2 \succ 1 \succ 4 \succ 5$ are single-peaked w.r.t. the following four orders (and their inverse): $1-2-3-4-5$, $5-1-2-3-4$, $5-1-2-3-4$, and $5-4-1-2-3$.

Escoffier et al. [13] proved that for any number of locations n and $r \leq 2^{n-1}$, there exist $\frac{1}{r}2^{n-1}$ different preferences that are single-peaked consistent w.r.t. r different orders (for the tight bounds see [13]).

⁴⁸In particular, if for all players the type set \mathcal{T}^i contains all single-peaked preferences with ideal location x^i w.r.t. to some order (and only them), then the above characterizes all (non-trivial) MINTHENMAX-NE.

In particular, in case $\cup_i \mathcal{T}^i$ is single-peaked consistent w.r.t. an order $<$, any (non-trivial) MINTHENMAX-NE L satisfies that there exist two different locations α and β s.t. $L = \{\alpha, \beta\}$ and $\alpha < \min_i x^i \leq \max_i x^i < \beta$.

Proof.

Notice that the result is equivalent to requiring that for every player i there be at most one location α in L s.t. $\alpha <^i x^i$ and at most one location β in L s.t. $x^i <^i \beta$. Assume for contradiction that either there are two locations in L to the left of x^i or two locations to the right of x^i . Then one of them must be on the path between x^i and the other location, and hence preferred to the other location regardless of the type of player i , in contradiction to L being an equilibrium.

In order to prove tightness, given a set $L = \{\alpha, \beta\}$ as above, for any player i there are single-peaked preferences with x^i at the top in which α is preferred to β , and there are single-peaked preferences with x^i at the top in which β is preferred to α . If the set \mathcal{T}^i includes both a type that prefers α to β and a type that prefers β to α , then the mapping $f^i : \mathcal{T}^i \rightarrow L$ of a type to his preferred location in L is onto. Hence, if this condition is satisfied by all type sets, then L is a MINTHENMAX-NE. \square

We see that under this homogeneity constraint, the (non-trivial) equilibria are constrained to be two extreme locations. This result can also be interpreted as the power of extreme players in scenarios where there is a common agreement on the order. Note that when the types are single-peaked w.r.t. several orders, the equilibria should satisfy the above characterization w.r.t. all of them. For instance, consider the scenario of two players and type sets $\mathcal{T}^1 = \mathcal{T}^2 = \{2 \succ 1 \succ 3 \succ 4 \succ 5, 2 \succ 3 \succ 1 \succ 4 \succ 5\}$ (a subset of the set of preferences in Example 19). These preferences are single-peaked w.r.t. the order 5-1-2-3-4, but $L = \{4, 5\}$ is not a MINTHENMAX-NE, albeit satisfying the property of the lemma because both types prefer 4 to 5 and will not choose 5, and indeed $\{4, 5\}$ does not satisfy the property w.r.t. the order 1-2-3-4-5.

The second case we analyze is having ambiguity about the ideal location only, while assuming a structure on the preferences (in a sense, this case is the complement of the first case). A simple example of such a structure is that of Euclidean preferences. A Euclidean preference is uniquely derived from its ideal location by ordering the locations according to their distance from the ideal location. Notice that this restriction too can be stated as a homogeneity constraint on the preferences given a common embedding: it can be stated as a common metric on the location set that is shared by all preferences.

Lemma 21.

Let G be a coordination game on the real line s.t. all players have Euclidean preferences. Then a (non-trivial) location set $L = \{l_1 < l_2 < \dots < l_k\}$ is a MINTHENMAX-NE if and only if for every player i there are types derived by possible ideal locations for him $x_1 < x_2 < \dots < x_k$ s.t. for $t = 1, 2, \dots, k - 1$

$$x_t < \frac{l_t + l_{t+1}}{2} < x_{t+1},$$

*i.e., the median between l_t and l_{t+1} is between x_t and x_{t+1} .*⁴⁹

A special case of the lemma is where there are exactly two types of each player that are derived from two possible ideal locations $x^i < y^i$. In this scenario we get that a (non-trivial) location set L is a MINTHENMAX-NE if and only if there exist two locations α and β s.t. $L = \{\alpha, \beta\}$ and $m = \frac{1}{2}(\alpha + \beta)$ is (strictly) between x^i and y^i for all players. Hence, a (non-trivial) MINTHENMAX-NE exists if and only if the intersection of the segments (x^i, y^i) is non-empty.

Proof.

\Rightarrow : Let $L = \{l_1 < l_2 < \dots < l_k\}$ be an equilibrium and let i be a player. Since L is an equilibrium, there are types of player i ,⁵⁰ $x_1 < x_2 < \dots < x_k$, s.t. player i chooses l_t when his type is x_t . In particular, for $t < k$ his preference between l_t and l_{t+1} when his type is x_t is different from his preference when his type is x_{t+1} . Hence, x_t and x_{t+1} lie on different sides of the median between l_t and l_{t+1} .

\Leftarrow : Following the same reasoning, it is easy to see that if L satisfies the property of the lemma, then player i of type x_t prefers l_t to any other location in L , and hence L is an equilibrium. \square

6 Summary and Future Directions

How people choose an action to take when possessing only partial information, and how they should choose their action, are basic questions in economics, and on them the definition of equilibrium is built (both as a prediction tool and as a self-enforcing contract). The main stream of game-theoretic literature assumes that economic agents are expectation maximizers (according to some objective or subjective prior) and, moreover, that there is some consistency between the players' priors (commonly, the common prior assumption).

In this work, we chose the inverse scenario and studied cases in which the players have no information on the state of the world. We defined a general framework of games with ambiguity following Harsanyi's model of games with incomplete information [17], and in particular, games with type ambiguity. We axiomatized a family of decision models under ambiguity that we claim a rational agent is expected to follow, and characterized the finest refinement of this family, MINTHENMAX. This family can be interpreted as all *rational* models of decision in cases of (extreme) ambiguity that follow Wald's MiniMax principle, defined by the different ways to decide in cases in which the MiniMax principle is mute. In many scenarios with type ambiguity Wald's MiniMax principle is too coarse, and so the corresponding equilibrium (MIN-NE) has almost no predictive power. The way to justify a selection process from these equilibria is to refine the players' preference, that is, assume they act according to a decision model in this family. We showed MINTHENMAX is the

⁴⁹In the proof we do not use the fact that all type sets \mathcal{T}^i are Euclidean w.r.t. the same embedding, and actually the condition in the lemma can be weakened to hold only with respect to each of the embeddings of \mathcal{T}^i .

⁵⁰We identify the types with their ideal locations.

unique model that follows Wald’s MiniMax principle and breaks all possible instances of indifference (without violating the rationality axioms we assumed). Finally, we studied the respective equilibrium notion, MINTHENMAX-NE , and applied it to two families of games: coordination games and bilateral trade games.

One might ask himself why to choose MINTHENMAX as the analysis tool, and not a different decision model in the family. First, we note that, just like the MIN model, the MINTHENMAX model has a simple and intelligible cognitive interpretation, and so it does not require a complex epistemological assumption on the players, which other models might impose. In addition, we note that $\text{MINTHENMAX-NE}(G)$, i.e., the equilibria of G under the MINTHENMAX , are in fact also equilibria of G under any profile of rational refinements of Wald’s MiniMax principle. Moreover, $\text{MINTHENMAX-NE}(G)$ can be equivalently defined as the set of all such robust equilibria of G . For instance, these are the profiles an outside actor can suggest as a self-enforcing contract, even if he does not know the exact preferences of the players.

This scenario of extreme ambiguity might seem unrealistic. Yet, we claim this model approximates many partial-information real-life scenarios better than the subjective expectation maximization model. Clearly, if players have information which they can use to construct a belief about the other players, and we expect they will use it, then the expectation maximization decision model is a better analysis tool. In intermediate scenarios, when players have some information but it is unreasonable to expect them to form a distribution over the world, it is reasonable to model the players as following one of the intermediate models for decision making under ambiguity,⁵¹ e.g., the multi-prior model (for an overview of such models, the interested reader is referred to [15]). It remains an open question what are the common features of equilibria under MINTHENMAX and equilibria under other decision models under ambiguity. By characterizing these common features, we would like to analyze the sensitivity of the equilibria we’ve found in this work to the specific analysis tool. One could also justify the model we presented, via justifying the axioms, for other scenarios. E.g., for scenarios in which the players might have some information on the preferences of others, but due to extreme risk aversion or bounded rationality constraints, they follow Wald’s MiniMax model. Moreover, in cases in which one justified the axioms we presented (and mostly the invariance to irrelevant information axiom), e.g., on cognitive grounds, we get that the MINTHENMAX-NE is the right analysis tool for the same reasons we presented above.

In order to study further the notion of equilibrium under MINTHENMAX-NE , we hope to analyze other families of games, e.g., finer cases of coordination games when adding homogeneity constraints that are either knowledge on other players, or intra-player agreement. In addition, we see few further directions of research.

Variance of information between types: In the examples we analyzed the information of a player did not depend on his type (in other words, a player cannot deduce from his type any information on the feasible types of others). The model we presented in Sec. 2 includes

⁵¹We refer to them as intermediate since they do not satisfy the principle of indifference, and hence they differentiate between the states of the world.

more general scenarios. We have preliminary results, which we omitted here, for Schelling's Homeowner-Burglar game [28, p. 207]. For this game, we got similar predictions to the prediction of Schelling, replicating the power of partial-knowledge of high degrees.

Mechanism design: In a continuation work, we extend the result regarding bilateral trade, to characterization of incentive compatible, individually rational, deterministic mechanisms for bilateral trade. We show that (essentially) the implementable allocation rules, are those that are implemented using the price announcement mechanisms we analyzed. Also here, implementability under MINTHENMAX-NE can be interpreted as robust implementability under ambiguity, i.e., implementability without assuming a specific decision model, but analyzing the profiles that are equilibria for any profile of decision models of the players from the family we characterized. Another direction of research is analyzing what a designer can gain by adding ambiguity to a mechanism, and specifically whether it is possible to increase the participation of the players, similarly to the full-participation result we proved.

Information update: The main drawback of modeling the knowledge of a player using a set of types for other players (or, in the general case, of states of the world) instead of a richer structure, is that there is no reasonable way to define information update in this model. This prevents us to extend this work to two natural directions: analysis of extensive form games (e.g., when the players play in turns, learning the type of each other as the evolve), and value of information (the question of how much a player should invest in order to decrease his ambiguity). In the decision theory literature, there are several non-Bayesian information update rules, e.g., Dempster-Shafer [12, 29] and Jeffrey update rule [19]. These rules usually assume a finer representation of knowledge than the representation we had in this work, but we think that after basing a rational decision model in our simplified knowledge representation, it should not be hard to extend the decision model to these finer knowledge models.

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A Proof of Lemma 3 (Best response is well defined)

This lemma can be easily extended to infinite number of states case by assuming some structure on the actions set and the utility function.

Lemma.

The following best response correspondences are non-empty: pure MIN-BR, mixed MIN-BR, pure MINTHENMAX-BR, and mixed MINTHENMAX-BR.

Proof.

We prove the lemma in the more general decision theory framework. We show that if there are finitely many states of the world and finitely many pure actions, there is always at least one optimal mixed action. Since the number of players and the number of types of each player are finite, a player faces one of a finite number of profiles (states of the world) and we get the desired result.

The existence of an optimal pure action is trivial since there are finitely many pure actions. Next, we prove the existence of optimal mixed actions. The set of optimal actions according to MIN $\text{OptAct}_{\text{MIN}}$ is

$$\text{OptAct}_{\text{MIN}} = \underset{\sigma \in \Delta(\mathcal{A})}{\text{argmax}} \min_{\omega \in \Omega} u(\sigma, \omega).$$

For every state of the world $\omega \in \Omega$, $u(\sigma, \omega)$ is a continuous function in σ (it is a linear transformation). $\min_{\omega \in \Omega} u(\sigma, \omega)$ is also continuous in σ as the minimum for finitely many continuous functions. Hence, $\underset{\sigma \in \Delta(\mathcal{A})}{\text{argmax}} \min_{\omega \in \Omega} u(\sigma, \omega)$ is non-empty as the maxima of a continuous function over a simplex. Moreover, $\text{OptAct}_{\text{MIN}}$ is a compact set.

Similarly, the set of optimal actions according to MINTHENMAX

$$\text{OptAct}_{\text{MINTHENMAX}} = \underset{a \in \text{OptAct}_{\text{MIN}}}{\text{argmax}} \max_{\omega \in \Omega} u(a, \omega)$$

is non-empty since $\text{OptAct}_{\text{MIN}}$ is compact and $\max_{\omega \in \Omega} u(a, \omega)$ is continuous in a . \square

B Proof of Theorem 4 (MIN-NE existence)

Theorem.

Every game with ambiguity has a MIN-NE in mixed actions.

Proof.

We prove the existence of a mixed MIN-NE by applying Kakutani's fixed point theorem [20]⁵² to the following set \mathbb{S} and set-valued function F . We set $\mathbb{S} \subseteq \mathfrak{R}^K$ to be the set of all profiles of mixed strategies of the types,⁵³ i.e., a cross product of the corresponding

⁵²Let $\mathbb{S} \subseteq \mathfrak{R}^n$ be a non-empty compact convex set. Let $F : \mathbb{S} \rightarrow \mathbb{S}$ be a set-valued upper semi-continuous function on \mathbb{S} such that $F(s)$ is non-empty and convex for all $s \in \mathbb{S}$. Then F has a fixed point, i.e., a point $s \in \mathbb{S}$ such that $s \in F(s)$.

⁵³for K being the sum of the number of types over the players.

simplexes. Given a strategy profile s , we define $F(s)$ to be the product of the best response correspondences of the different types to the profile s .

Clearly, \mathbb{S} is a convex compact non-empty set. We proved (Lemma 3) that the best response always exists and hence $F(s)$ is non-empty for all $s \in \mathbb{S}$. We show that the best response correspondence of a type is an upper semi-continuous function (and hence also F) by applying Berge's Maximum theorem [6, Thm. 2, p. 116]⁵⁴ to (where $t^{-i}(\omega)$ are the types player i is facing according to ω , and $s^{-i}(t^{-i}(\omega))$ are their actions according to s)

$$f(\sigma^i, s) = \min_{\omega \in \Omega} u(\omega, \sigma^i, s^{-i}(t^{-i}(\omega))).$$

We notice that the best response of a type of player i to a strategy profile $s \in \mathbb{S}$ is the maxima over his mixed strategies σ^i of $f(\sigma^i, s)$; that u is a linear function in σ^i and in s ; and that $f(\sigma^i, s)$ is a continuous function in s and σ^i . Hence, we get that the best response is an upper semi-continuous correspondence.

Next, we show that for any profile s and a type of player i , the best response set is convex, and hence also $F(s)$ is a convex set. In order to prove the convexity of the best response set, let σ and τ be two best responses and let

$$m = \min_{\omega \in \Omega} u(\omega, \sigma^i, s^{-i}(t^{-i}(\omega)))$$

be their worst-case value. For any convex combination ζ of σ and τ , on one hand we get that

$$\min_{\omega \in \Omega} u(\omega, \zeta^i, s^{-i}(t^{-i}(\omega))) \leq m$$

from the optimality of σ , and on the other hand

$$u(\omega, \zeta^i, s^{-i}(t^{-i}(\omega))) \geq m \text{ for all } \omega \in \Omega.$$

since we assumed this lower-bound both on σ and on τ . Hence,

$$\min_{\omega \in \Omega} u(\omega, \zeta^i, s^{-i}(t^{-i}(\omega))) = m$$

and ζ is a best response.

By applying Kakutani's fixed point theorem we get that there is a profile p s.t. $p \in F(p)$. That is, in the profile p every type best-responds to the others, so p is a MIN-NE. \square

C Proof of Lemma 11 (Axiomatization)

Lemma.

Any preference that satisfies monotonicity, state symmetry, and independence of irrelevant information can be defined as a function of the worst and the best outcomes of the actions.

⁵⁴If $f: X \times Y \rightarrow \mathfrak{R}$ is a continuous function, then the mapping $\mu: X \rightarrow Y$ defined by $\mu(x) = \operatorname{argmax}_{y \in Y} f(y)$ is an upper semi-continuous mapping. (The statement of the theorem is taken from [1, Lemma 17.31, p. 570])

Proof.

For this proof it is easier to use the following property that is equivalent (when Ω is finite) to state symmetry.

Axiom.

For any action a and bijection $\psi: \Omega \rightarrow \Omega$, the DM is indifferent between a and $a \circ \psi$.

Let a and a' be two actions over the states set Ω s.t. $\min_{\omega \in \Omega} a(\omega) = \min_{\omega \in \Omega} b(\omega)$ and $\max_{\omega \in \Omega} a(\omega) = \max_{\omega \in \Omega} b(\omega)$. We show that the DM is indifferent between the two, and notice that this will prove the lemma. Assuming this claim, given any two pairs of actions: a and a' that have the same minimal and maximal outcome, and b and b' that have the same minimal and maximal outcome, a is preferred to b if and only if a' is preferred to b' , because the DM is indifferent between a and a' , and between b and b' .

Assume for contradiction that (w.l.o.g) the DM strictly prefers a' to a . Since the preference satisfies state symmetry, we can assume that a and a' are co-monotone, that is, for any two states ω and ω' , the actions satisfy that $(a(\omega) - a(\omega'))(a'(\omega) - a'(\omega')) \geq 0$. We denote by m and M the minimal and maximal outcomes of a and by ω_m and ω_M the two respective states of world.

	ω_m	other states	ω_M
a	m	$\in [m, M]$	M
a'	m	$\in [m, M]$	M

We define two new actions over Ω . An action b that results in m in all states besides ω_M and M otherwise, and an action b' that results in M in all states besides ω_m and m otherwise.

	ω_m	other states	ω_M
a	m	$\in [m, M]$	M
b	m	m	M
a'	m	$\in [m, M]$	M
b'	m	M	M

Due to monotonicity the DM (weakly) prefers a to b , and b' to a' , and hence the DM strictly prefers b' to b . Next, we define an auxiliary space $\hat{\Omega} = \{\omega_m, \omega_o, \omega_M\}$ and two actions on it c and c' by unifying the middle states into one as follows:

	ω_m	ω_o	ω_M
c	m	m	M
c'	m	M	M

Since the preference satisfies Independence of Irrelevant Information we get that the DM strictly prefers c' to c . Now we define a new action c'' (see below), and by the state symmetry property we get that also c'' is strictly preferred to c .

	ω_m	ω_o	ω_M
c	m	m	M
c'	m	M	M
c''	M	M	m

Using a collapsing argument similar to the one above, we get that for the following collapsed space and two actions

	ω_m	ω_M
d	m	M
d''	M	m

d'' is strictly preferred to d , but this contradicts the state symmetry property. \square

D Proof of Proposition 12 (Axiomatization of MIN)

Proposition.

Let X be a set of outcomes and let Y be the set of distributions over X with finite supports, and we identify X with the Dirac distributions $\{y \in Y \mid y(x) = 1 \text{ for some } x \in X\}$.

Let Ω be a finite set of states of the world s.t. $|\Omega| \geq 3$ and let $L = Y^\Omega$ (i.e., actions which in each state of the world return a lottery over X). For $f, g \in L$ and $\alpha \in [0, 1]$, we define $\alpha f + (1 - \alpha)g$ as the state-wise compound lottery, that is, the action that returns for each state $\omega \in \Omega$ the lottery $\alpha f(\omega) + (1 - \alpha)g(\omega) \in Y$.

We will denote by $L_c \subset L$ the constant functions in L , and by $L_o = X^\Omega \subset L$ the functions that return pure outcomes.

Let \succsim be a preference order over L that satisfies Gilboa-Schmeidler Axioms A.2-A.5 [16]

- A.2: Certainty independence: For all f, g in L and h in L_c and for all $\alpha \in (0, 1)$:

$$f \succ g \iff \alpha g + (1 - \alpha)h \succ \alpha f + (1 - \alpha)h.$$

- A.3: Continuity: For all f, g and h in L : if $f \succ g$ and $g \succ h$ then there are α and β in $(0, 1)$ such that $\alpha f + (1 - \alpha)h \succ g$ and $g \succ \beta f + (1 - \beta)h$.
- A.4: Monotonicity: For all f and g in L : if $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$ then $f \succsim g$.
- A.5: Uncertainty aversion: For all $f, g \in L$ and $\alpha \in (0, 1)$: $f \succsim g$ and $g \succsim f$ implies $\alpha f + (1 - \alpha)g \succsim f$.

and satisfies state symmetry and independence of irrelevant information on L_o .

Then \succsim is represented by a function $u: X \rightarrow \mathfrak{R}$ such that for any $f, g \in Y$

$$f \succsim g \iff \min_{\omega \in \Omega} \mathbb{E}_{x \sim f(\omega)} [u(x)] \geq \min_{\omega \in \Omega} \mathbb{E}_{x \sim g(\omega)} [u(x)].$$

Proof.

Let \succsim be a preference order which satisfies the above. Gilboa and Schmeidler [16] proved that \succsim is represented by a function $u: X \rightarrow \mathfrak{R}$ and a non-empty, closed and convex set C of probability measures on Ω such that for any $f, g \in L$

$$f \succsim g \iff \min_{P \in C} \mathbb{E}_{\omega \sim P} \left[\mathbb{E}_{x \sim f(\omega)} [u(x)] \right] \geq \min_{P \in C} \mathbb{E}_{\omega \sim P} \left[\mathbb{E}_{x \sim g(\omega)} [u(x)] \right]$$

and in particular for any $a, b \in L_o$

$$a \succsim b \iff \min_{P \in C} \mathbb{E}_{\omega \sim P} [u(a(\omega))] \geq \min_{P \in C} \mathbb{E}_{\omega \sim P} [u(b(\omega))].$$

Let l and h be two outcomes s.t. the DM strictly prefers the action that gives him h in all states of the world over the action that gives him l in all states of the world, and so $u(h) > u(l)$. We denote $u(h)$ and $u(l)$ by u_h and u_l , respectively.

For every $\omega \in \Omega$, we define $\epsilon_\omega = \min_{P \in C} P(\omega)$, and we define the actions l_ω and h_ω by $l_\omega(\omega') = \begin{cases} l & \omega = \omega' \\ h & \text{otherwise} \end{cases}$ and $h_\omega(\omega') = \begin{cases} h & \omega = \omega' \\ l & \text{otherwise} \end{cases}$. From Lemma 11 we get that the preference on L_0 can be defined as a function of the worst and the best outcomes of the actions, and so the DM is indifferent between l_{ω^\star} and h_{ω^\dagger} for every $\omega^\star, \omega^\dagger \in \Omega$. Hence,

$$u_l + (h_h - u_l) \sum_{\omega \neq \omega^\star} \epsilon_\omega \leq \min_{P \in C} \mathbb{E}_{\omega \sim P} [u(l_{\omega^\star}(\omega))] = \min_{P \in C} \mathbb{E}_{\omega \sim P} [u(h_{\omega^\dagger}(\omega))] = u_l + (u_h - u_l) \epsilon_{\omega^\dagger},$$

and $\sum_{\omega \neq \omega^\star} \epsilon_\omega \leq \epsilon_{\omega^\dagger}$. Hence, since $|\Omega| > 2$ we get that $\epsilon_\omega = 0$ for all $\omega \in \Omega$.

Next we prove that for any action $a \in L_0$ it holds that

$$\min_{P \in C} \mathbb{E}_{\omega \sim P} [u(a(\omega))] = \min_{\omega \in \Omega} u(a(\omega)).$$

The DM is indifferent between a and the action a' that gives him $\operatorname{argmax}_{\omega \in \Omega} u(a(\omega))$ on one state ω^\star and $\operatorname{argmin}_{\omega \in \Omega} u(a(\omega))$ on all other states. Hence,

$$\begin{aligned} \min_{P \in C} \mathbb{E}_{\omega \sim P} [u(a(\omega))] &= \min_{P \in C} \mathbb{E}_{\omega \sim P} [u(a'(\omega))] \\ &= \min_{\omega \in \Omega} u(a(\omega)) + (\max_{\omega \in \Omega} u(a(\omega)) - \min_{\omega \in \Omega} u(a(\omega))) \cdot \min_{P \in C} P(\omega^\star) \\ &= \min_{\omega \in \Omega} u(a(\omega)). \end{aligned}$$

We get that for any action $a \in L_0$ it holds that $\min_{P \in C} \mathbb{E}_{\omega \sim P} [u(a(\omega))] = \min_{\omega \in \Omega} [u(a(\omega))]$, and so the Dirac distribution concentrated on ω belongs to C for every state of the world $\omega \in \Omega$. Hence for any $f, g \in Y$:

$$\begin{aligned} f \succcurlyeq g &\iff \min_{P \in C} \mathbb{E}_{\omega \sim P} [\mathbb{E}_{x \sim f(\omega)} [u(x)]] \geq \min_{P \in C} \mathbb{E}_{\omega \sim P} [\mathbb{E}_{x \sim g(\omega)} [u(x)]] \\ &\iff \min_{\omega \in \Omega} \mathbb{E}_{x \sim f(\omega)} [u(x)] \geq \min_{\omega \in \Omega} \mathbb{E}_{x \sim g(\omega)} [u(x)]. \quad \square \end{aligned}$$

E Proof of Lemma 13 (Bilateral trade games)

Lemma.

Let G be a bilateral trade game defined by a price function $x(a_s, a_b)$ and two type sets V_s and V_t , both having a minimum and maximum.⁵⁵ Then all the MINTHENMAX-NE of G are of one of the following classes:

1. **No-transaction equilibria** (These equilibria exist for any two sets V_s and V_b)

In these equilibria, neither the buyer nor the seller participates (i.e., they play \perp , or bid a too extreme bid for all types of the other player), regardless of their valuations.

⁵⁵We state the result here for the case where both sets have a minimal valuation and have a maximal valuation. Dropping this assumption does not change the result in any essential way: some of the inequalities are changed to strict inequalities.

2. **One-price equilibria** (These equilibria are defined only when $\min V_s \leq \max V_b$, i.e., when an ex-post transaction is possible.)

In a one-price equilibrium, both the seller and the buyer choose to participate for some of their types. It is defined by a price $p \in [\min V_s, \max V_b]$ s.t. the equilibrium strategies are:

The seller bids p for valuations $v_s \leq p$, and \perp otherwise (the second clause might be vacuously true).

The buyer bids p for valuations $v_b \geq p$, and \perp otherwise (the second clause might be vacuously true).

Hence, the outcome is

	Seller		
	Buyer	Low: $v_s \leq p$	High: $v_s > p$
Low: $v_b < p$		no transaction	no transaction
High: $v_b \geq p$		p	no transaction

3. **Two-price equilibria** (These equilibria are defined only when $\min V_s \leq \min V_b$ and $\max V_s \leq \max V_b$, i.e., when there is a value for the seller s.t. an ex-post transaction is possible for any value of the buyer, and vice versa.)

In a two-price equilibrium, all types of both the seller and the buyer choose to participate, and their bids depend on their valuations. It is defined by two prices $p_L < p_H$ s.t.

$\begin{cases} \min V_s \leq p_L < \max V_s \leq p_H \\ p_L \leq \min V_b < p_H \leq \max V_b \end{cases}$ and the equilibrium strategies are:

The seller bids p_L for valuations $v_s \leq p_L$, and p_H otherwise.

The buyer bids p_H for valuations $v_b \geq p_H$, and p_L otherwise.

Hence, the outcome is

	Seller		
	Buyer	Low: $v_s \leq p_L$	High: $v_s > p_L$
Low: $v_b < p_H$		p_L	no transaction
High: $v_b \geq p_H$		$x(p_L, p_H) \in (p_L, p_H)$	p_H

Proof.

We will prove that these profiles are equilibria (MINTHENMAX-NE) and that they are the only equilibria.

- No-transaction profiles

In these profiles for at least one of the players does not participate (\perp , or bids an extreme bid) regardless of his type.

A player, facing a profile in which all the types of the other player do not participate, is indifferent between all his actions (all of them result in no-transaction for sure), and hence he best-responds. Facing any other profile, at least for some of his types (e.g., the maximal value for the buyer, and minimal value of the seller), choose to participate. Hence, the only no-transaction profile that is an equilibrium, is when both players do not participate.

- One-price profiles

In these profiles, both the seller and the buyer choose to participate for some of their types, and when they choose to participate, both bid the same price p .

First, we prove that given a one-price profile as described in the lemma, it is indeed

an equilibrium.

Consider a seller of type v_s facing a profile in which there are types of the buyer that choose p and (maybe) others that choose not to participate.

- If $v_s \leq p$: He prefers bidding p to any bid $a_s < p$, because both bids will result in a transaction with the same types of the buyer, but a_s will result in a lower price (Monotonicity of MINTHENMAX).

He prefers bidding p to any bid $a_s > p$, because a_s will result in no-transaction with all types of the buyer, while p results in a profitable transaction with some of them (Monotonicity of MINTHENMAX).

We notice that since $p \geq \min V_s$ there are types of the seller that bid p .

- If $v_s > p$: Any bid that results in a transaction with some types of the buyer, will be at a price lower than v_s , and hence the transaction will be a losing one. Hence, he prefers not to participate at all (Monotonicity of MINTHENMAX).

Similarly, consider a buyer of type v_b facing a profile in which there are types of the seller that choose p and (maybe) others that choose not to participate.

- If $v_b < p$: Any bid that results in a transaction with some types of the seller, will be at a price higher than v_b , and hence the transaction will be a losing one. Hence, he prefers not to participate at all (Monotonicity of MINTHENMAX).

- If $v_b \geq p$: He prefers bidding p to any bid $a_b > p$, because both bids will result in a transaction with the same types of the seller, but a_s will result in a higher price (Monotonicity of MINTHENMAX).

He prefers bidding p to any bid $a_s < p$, because a_s will result in no-transaction with all types of the seller, while p results in a profitable transaction with some of them (Monotonicity of MINTHENMAX).

We notice that since $p \leq \max V_b$ there are types of the buyer that bid p .

Next, we prove that any single-price profile other than those described in the lemma, cannot be an equilibrium.

First we claim, that in equilibrium it cannot be that the prices of participating types of one of the sides vary, while all types of the other player bid the same. If there is a unique price p bid by participating types of the seller, based on the analysis above, all participating types of the buyer bid p . Similarly, if there is a unique price p bid by participating types of the buyer, all participating types of the seller bid p .

Hence, in a single-price profile, there exists a price p s.t. both some types of the seller and some types of the buyer choose p , and all other types (might be an empty set) choose not to participate.

If $p \in [\min V_s, \max V_b]$, by the analysis above, we see there is only one single-price equilibrium supporting p .

If $p \notin [\min V_s, \max V_b]$, by the analysis above, either all types of the buyer or all types of the seller choose not to participate. In either case this is a no-transaction profile.

- All-participating profiles

In these profiles, all types of both the seller and the buyer choose to participate (did not choose \perp).

First we prove that given an all-participating profile as described in the lemma as two-price equilibria, it is indeed an equilibrium.

Consider a seller of type v_s facing a profile in which all the types of the buyer participate, and let p_L be the infimum of the bids chosen by types of the buyer and p_H the supremum of the bids.

- If $v_s \leq p_L$: He prefers bidding p_L to any bid $a_s < p_L$, because both bids will result in a transaction with all types of the buyer, but a_s will result in a lower price with each of them (Monotonicity of MINTHENMAX).

He prefers bidding p_L to any bid $a_s > p_L$, because a_s will result in no-transaction with some types of the buyer, while p_L results in a non-losing transaction with all of them (MINTHENMAX is a refinement of MIN).

- If $v_s > p_L$ and $v_s \leq p_H$: He prefers bidding p_H to any bid $a_s < p_H$, because when facing the types of the buyer that bid $a_b \in [a_s, p_H]$, both bids will result in a transaction, but a_s will result in a lower price, and when facing other types of the buyer (that bid less than a_s), both result in no-transaction (Monotonicity of MINTHENMAX).

He prefers bidding p_H to any bid $a_s > p_H$, because a_s results in no-transaction with all types of the buyer, while p_H results in a non-losing transaction with some types of the buyer (Monotonicity of MINTHENMAX).

We notice that since in the profiles of the lemma satisfy $\min V_s \leq p_L < \max V_s \leq p_H$, all the types of the seller participate, there are types of the seller that bid p_L , and there are types that bid p_H .

Similarly, consider a buyer of type v_b facing a profile in which all the types of the seller participate, and let p_L be the infimum of the bids chosen by types of the seller and p_H the supremum of the bids.

- If $v_b < p_H$ and $v_b \geq p_L$: He prefers bidding p_L to any bid $a_b < p_L$, because a_b results in no-transaction with all types of the seller, while p_L results in a non-losing transaction with some types of the seller (Monotonicity of MINTHENMAX).

He prefers bidding p_L to any bid $a_b > p_L$, because when facing the types of the seller that bid $a_s \in [p_L, a_b]$, both bids will result in a transaction, but a_b will result in a higher price, and when facing other types of the seller (that bid more than a_b), both result in no-transaction (Monotonicity of MINTHENMAX).

- If $v_b \geq p_H$: He prefers bidding p_H to any bid $a_b < p_H$, because a_b will result in no-transaction with some types of the seller, while p_H results in a non-losing transaction with all of them (MINTHENMAX is a refinement of MIN).

He prefers bidding p_H to any bid $a_b > p_H$, because both bids will result in a transaction with all types of the seller, but a_b will result in a higher price with each of them (Monotonicity of MINTHENMAX).

We notice that since in the profiles of the lemma satisfy $p_L \leq \min V_b < p_H \leq \max V_b$, all the types of the buyer participate, there are types of the buyer that bid p_L , and there are types that bid p_H .

Next, we prove that any all-participating non single-price profile, other than the above,

cannot be an equilibrium.

In an all-participating non single-price profile, all the types of both the buyer and seller participate, and for both players, not all their types choose the same price.

If there is a type of the buyer whose value is lower than any of the prices bid by types of the seller, then this type prefers not to participate, so it cannot be an equilibrium. Any other type of the buyer bids either the supremum of the infimum of the prices bid by types of the seller. Given that types of the buyer bid one of two prices, if there is a type of the seller whose value is higher than both prices, then this type prefers not to participate, and the profile is not an equilibrium. Otherwise, the types of the seller also bid these two prices and we get the profile is of the type described in the lemma.

- Last, we show that all other profiles are not equilibria of the game.

Assume towards a contradiction that \bar{a} is an equilibrium profile of the game that is not one of the above profile types.

Clearly, for a match of any pair of types of the seller and the buyer, the utility of neither of them cannot be negative, because he can guarantee himself to get at least zero against all types of the other player by choosing \perp .

Assume there is a type v_s of the seller that chose \perp . Then for any participating type v_b of the buyer, v_b cannot guarantee himself to get a utility greater than zero (his utility from being matched with v_s) in the worst case. Hence, he breaks the tie between all the bids $a_b \leq v_b$, all giving him zero in the worst case, according to the best case which is being matched with the lowest bid of a participating type of the seller. Hence, we get a contradiction by showing the profile is a one-price profile.

Similarly, assume there is a type v_b of the buyer that chose \perp . Then for any participating type v_s of the seller, v_s cannot guarantee himself to get a utility greater than zero (his utility from being matched with v_b) in the worst case. Hence, he breaks the tie between all the bids $a_b \geq v_s$, all giving him zero in the worst case, according to the best case which is being matched with the highest bid of a participating type of the buyer. Hence, we get a contradiction by showing the profile is a one-price profile. \square

F Proof of Lemma 16 (Coordination games)

Lemma.

Let G be a coordination game and let L be a non-empty set of locations. There exists a MINTHENMAX-NE profile a s.t. $L(a) = L$ if and only if for all i the mapping $f^i : \mathcal{T}^i \rightarrow L$ that maps a type to his best location in L is onto.

Moreover, the action profile a in which every type of every player chooses his best location in L is the unique pure MINTHENMAX-NE that satisfies $L(a) = L$.

Proof.

The case $|L| = 1$ is trivial. Let l be the location in L . The only profile satisfying $L(a) = L$ is when all players choose l regardless of their type, and this is an equilibrium. The mappings f^i are onto, and hence the lemma is proved for this case.

From now on, we assume that $|L| \geq 2$. Given a profile a we define $L^i(a)$ to be the set of locations chosen in a by player i ,

$$L^i(a) = \{l \mid \exists t^i \in \mathcal{T}^i \text{ s.t. player } i \text{ of type } t^i \text{ plays } l \text{ in the profile } a\},$$

so $L(a) = \cup_{i \in \mathcal{N}} L^i(a)$.

\Rightarrow :

Assume there exists an equilibrium a s.t. $L(a) = L$.

First we claim that for all players, $|L^i(a)| > 1$. Assume towards a contradiction, there is a non-empty set of players S s.t. $i \in S \iff |L^i(a)| = 1$, and let p be a player in S and l the location chosen by p .

If $|S| = 1$, then for any of the other players the only action guaranteeing them to meet another player is choosing l , and hence all players choose l regardless of their type. I.e., $L(a) = \{l\}$ and we get a contradiction.

If $|S| > 1$, then any of the players can guarantee himself to meet another player on the worst case, and hence will choose a location in $\cup_{i \in S} L^i(a)$. Since all players aim to maximize the number of other players they meet on the worst case, it cannot be that two players in S choose differently (both maximize the number of other players from S that choose like them). Hence, $|\cup_{i \in S} L^i(a)| = 1$ and we get that $|L(a)| = 1$. Contradiction.

Hence, for all players, $|L^i(a)| > 1$. Player i of type $t^i \in \mathcal{T}^i$ cannot guarantee himself to meet other players, and hence he will choose the best location according to his type among the locations in $\cup_{j \neq i} L^j(a)$ that maximize the number of other players he might meet - $m(l) = \#\{j \neq i \mid l \in L^j(a)\}$.

Next, we prove that for all players $L^i(a) = L(a)$. Assume for contradiction there exists Player p s.t. $L^p(a) \subsetneq L(a)$. Let $l \in L^p(a)$ be a location chosen by p , and let l' be a location and p' a player s.t. $l' \in L^{p'}(a) \setminus L^p(a)$. Since Player p (of some type) chose l , we get that the number of players $j \neq p$ that chose l is at least as large as the number of players $j \neq p$ that chose the location l' . Taking the viewpoint of Player p' we get that the number of players $j \neq p'$ that chose l is strictly larger than the number of players $j \neq p'$ that chose the location l' . But this is a contradiction to $l' \in L^{p'}(a)$.

Hence, we get that in a , player i of type $t^i \in \mathcal{T}^i$ choose the best location according to his type among the locations in L (in all of them he meets no-one in the worst case and everyone else in the best case). That is, player i plays according to the function f^i , and $L = L^i = \text{Im}(f)$, so f is onto. Notice that we also get this profile is the unique equilibrium s.t. $L(a) = L$.

\Leftarrow :

Assume the functions f^i are onto. It is easy to verify that in the profile a in which player i plays according to f^i , player i best-responds (for all i). None of his actions guarantees him to meet some other player in the worst case; if he plays an action $a \notin L$, he does not meet another player even in the best case; if he plays an action $a \in L$, he meets all other players in the best case; hence, he best-responds playing according to f^i . \square