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THE HEBREW UNIVERSITY OF JERUSALEM

UNIQUENESS OF OPTIMAL STRATEGIES IN CAPTAIN LOTTO GAMES

By

NADAV AMIR

Discussion Paper # 687 (June 2015)

מרכז פדרמן לחקר הרציונליות

THE FEDERMANN CENTER FOR THE STUDY OF RATIONALITY

Feldman Building, Edmond J. Safra Campus,
Jerusalem 91904, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

Uniqueness of Optimal Strategies in Captain Lotto Games *

Nadav Amir †

Abstract

We consider the class of two-person zero-sum allocation games known as Captain Lotto games (Hart 2014). These are Colonel Blotto type games in which the players have capacity constraints. We consider the game with non-strict constraints, and with strict constraints. We show in most cases that when optimal strategies exist, they are necessarily unique. When they don't exist, we characterize the pointwise limit of the cumulative distribution functions of ϵ-optimal strategies.

1 Introduction

A continuous *Captain Lotto* game is a two-person zero-sum game. Each player is given a nonnegative real number, say, a for player A and b for player B, together with "caps" c_A and c_B. Player A chooses (a distribution of) a nonnegative random variable X with values bounded from above by c_A, i.e., $0 \leq X \leq c_A$, and expectation $E(X) = a$, and player B chooses (a distribution of) a nonnegative random variable Y with values bounded from above by c_B, i.e., $0 \leq Y \leq c_B$, and expectation $E(Y) = b$. The payoff function (from B to A) is

$$H(X,Y) = P(X > Y) - P(X < Y) \tag{1.1}$$

with X and Y independent. Thus c_A and c_B serve as upper bounds – "caps" – on the two players. We assume that $c_A \geq a$ and $c_B \geq b$ (otherwise the set of strategies would be empty). We denote this game by $\Lambda_{c_A,c_B}(a,b)$. When there are no caps, i.e., $c_A = c_B = \infty$, this game is known as the *continuous General Lotto* game. The name Captain Lotto is due to the players' upper bounds, called *caps.* The case $a = b$ of the General Lotto game is solved by Bell and Cover (1980, Sect. 2); see also Myerson (1993) and Lizzeri (1999). The solution of the nonsymmetric case, $a > b$, is given by Sahuguet and Persico (2006); see also Hart (2008, Appendix). Sahuguet and Persico use this game to model an electoral competition between political parties. In their model voters perceive the parties as differing in valence (which is interpreted as competence in administrative

*This research has received funding from the European Research Council under the European Community's Seventh Framework Program(FP7/2007-2013) / ERC grant agreement n [249159]. The author would like to express his deep gratitude to his advisor, Sergiu Hart. The author would also like to thank Nir Gadish and the anonymous referees for useful comments and discussions.

†N. Amir, Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel E-mail: nadav.amir1@mail.huji.ac.il

tasks, ability to make efficient government decisions, etc.). The parties choose how to allocate electoral promises among voters. The party with the greater valence is believed to be able to generate more resources, and therefore is able to make electoral promises under less stringent budget constraints.

These games have a predecessor, a discrete game called the *Colonel Blotto* game. In this game two players simultaneously allocate battalions (say, A for the first player and B for the second player, where A and B are positive integers) across K battlefields. In a given battlefield, the player with the greater amount of battalions wins the battlefield and earns $1/K$. The payoff is then the sum of the payoffs across all battlefields. The Colonel Blotto game originated with Borel (1921), where the game is solved for the case of $A = B = 1$ and $K = 3$. Many papers have dealt with this problem. For a recent survey of the literature see Roberson (2006).

We may view the pure strategies of our players (K-partitions of the amount of battalions that each player has) as distributions of integer-valued random variables with expectations $A/K = a$ and $B/K = b$, respectively. When adding the assumption that the K battlefields are indistinguishable, since they are chosen with equal probability, the payoff function becomes exactly (1.1). Then, by removing the requirements that the strategies be derived from probability distributions on K-partitions, and that they be integer-valued, thereby allowing any nonnegative random variables with the given expectations, we obtain the continuous General Lotto game. For more details see Hart (2008).

Hart (2014) discusses the value, optimal strategies and ϵ-optimal strategies of the continuous Captain Lotto game. The purpose of the present paper is to study the uniqueness of these strategies. In the continuous General Lotto game, i.e., when there are no caps, the optimal strategies are indeed unique; see Sahuguet and Persico (2006) and Hart (2008). Thus it is natural to ask whether uniqueness holds when there are finite caps. We show here that when the players' caps are equal, the optimal strategies of the two players are unique. When the caps are different, in some cases the optimal strategies for both players remain unique, but in other cases an optimal strategy is not necessarily guaranteed for one of the players. The analysis then focuses on ϵ-optimal strategies, and we characterize their limit[1] when $\epsilon \to 0$. In Section 2 we present known facts and prove some useful lemmas. In Section 3 we treat the case of equal caps, i.e., $c_A = c_B$. In Section 4 we treat the case of unequal caps, and in Section 5 we present additional results for the case of unequal caps with strict constraints.

2 Preliminaries

In this section we rule out some trivial cases and prove useful lemmas.

Consider $\Lambda_{c_A,c_B}(a,b)$. The $b = 0$ case is trivial since player B has only one possible strategy[2], namely, 1_0. As long as $a > 0 = b$ the value of the game is 1 and player A's set of optimal strategies is $\{X \mid 0 < X \leq c_A\}$. The same arguments hold for the $b > a = 0$ case. The game is even simpler in the $a = b = 0$ case: the value is 0 and both players can play only 1_0. Thus we will assume throughout this paper that $\min\{a,b\} > 0$.

[1]In terms of the pointwise limit of their cumulative distribution function (CDF).

[2]1_d denotes the distribution that puts mass 1 on the point d.

Every nonnegative random variable X satisfies

$$E(X) = \int_0^\infty P(X \geq x)dx = \int_0^\infty P(X > x)dx \tag{2.1}$$

(see, e.g., Billingsley, 1986 (21.9)). Since $H(X, Y) = P(X > Y) - P(Y > X)$, we easily obtain

$$1 - 2P(Y \geq X) \leq H(X, Y) \leq 2P(X \geq Y) - 1. \tag{2.2}$$

lemma 1. *For every nonnegative random variable $X \geq 0$ and every*[3] $t > 0$,

1. $$H(U_{[0,2t]}, X) \geq 1 - \frac{E(X)}{t}.$$

2. $X \leq 2t$ *if and only if*

$$H(U_{[0,2t]}, X) = 1 - \frac{E(X)}{t}.$$

remark 1. *Part (1) of this lemma is proved in Hart (2008) (see (10)–(11)). Since the proof is short and we use this lemma repeatedly, we include its proof here.*

Proof.

$$P(X \geq U_{[0,2t]}) = \frac{1}{2t}\int_0^{2t} P(X \geq x)dx \leq \frac{1}{2t}\int_0^\infty P(X \geq x)dx = \frac{E(X)}{2t}$$

$$H(U_{[0,2t]}, X) \geq 1 - 2P(X \geq U_{[0,2t]}) \geq 1 - 2\frac{E(X)}{2t} = 1 - \frac{E(X)}{t},$$

which proves the first part of our lemma. For the second part we first assume that $X \leq 2t$. Thus,

$$\begin{aligned} H(U_{[0,2t]}, X) &= P(U_{[0,2t]} > X) - P(U_{[0,2t]} < X) \\ &= 1 - P(X \geq U_{[0,2t]}) - P(X > U_{[0,2t]}) \\ &= 1 - \frac{1}{2t}\int_0^{2t} P(X \geq x) + P(X > x)dx \\ &= 1 - \frac{1}{2t}\int_0^\infty P(X \geq x) + P(X > x)dx \\ &= 1 - \frac{1}{2t}2E(X) = 1 - \frac{E(X)}{t}. \end{aligned}$$

For the other direction, assume $H(U_{[0,2t]}, X) = 1 - E(X)/t$. Using this assumption and (2.1) gives us

$$1 - \frac{E(X)}{t} = H(U_{[0,2t]}, X) = 1 - \frac{1}{2t}\int_0^{2t} P(X \geq x) + P(X > x)dx$$

[3] Throughout this paper we identify a random variable with its distribution. We denote the uniform distribution on the interval $[c, d]$ by $U_{[c,d]}$.

$$\geq 1 - \frac{1}{2t}\int_0^{2t} 2P(X \geq x)dx$$
$$= 1 - \frac{1}{t}\int_0^{2t} P(X \geq x)dx$$
$$\geq 1 - \frac{1}{t}\int_0^{\infty} P(X \geq x)dx = 1 - \frac{E(X)}{t}.$$

This implies that

$$\int_{2t}^{\infty} P(X \geq x)dx = 0;$$

thus $P(X \geq x) = 0$ for almost every $x \in [2t, \infty)$. However, the left-hand side (LHS) is a nonincreasing function and so we obtain that $P(X > x) = 0$ for every $x \in [2t, \infty)$, i.e., $X \leq 2t$. □

When both players have the same cap, i.e., $c_A = c_B = c$, and, without loss of generality we assume $a \geq b$, the value of the $\Lambda_{c,c}(a,b)$ game is $1 - b/a$. This is part of Theorem 2 below, which is proved in Hart (2014). Using this yields the following:

lemma 2. *Let $0 < a \leq c$. X is an optimal strategy of player A in $\Lambda_{c,c}(a,b)$ for some $0 < b \leq a$ if and only if X is an optimal strategy of player A in $\Lambda_{c,c}(a,t)$ for all $t \in [b,a]$.*

Proof. Assume X is an optimal strategy of player A in $\Lambda_{c,c}(a,b)$ for some $0 < b \leq a$. Let $t \in [b,a]$ and let Y be some strategy of player B in $\Lambda_{c,c}(a,t)$. Define $Y^b = (1 - b/t)1_0 + (b/t)Y$ with $E(Y^b) = b$. Since X is optimal we obtain

$$1 - \frac{b}{a} \leq H(X, Y^b) = \left(1 - \frac{b}{t}\right) H(X, 1_0) + \frac{b}{t} H(X,Y) \leq 1 - \frac{b}{t} + \frac{b}{t} H(X,Y)$$

$$1 - \frac{t}{a} = \frac{t}{b}\left(\frac{b}{t} - \frac{b}{a}\right) \leq H(X,Y).$$

Thus X is optimal in $\Lambda_{c,c}(a,t)$.

The other direction is trivial. □

remark 2. *Note that in Lemma 2 c may be greater than $2a$ (and even infinite).*

corollary 1. *If X is optimal in $\Lambda_{c,c}(a,b)$ for $a \geq b$ then X is optimal in $\Lambda_{c,c}(a,a)$.*

lemma 3. *Let X be an optimal strategy of player A in $\Lambda_{c,c}(a,b)$, and let*[4] *$Y = \alpha 1_{t_1} + (1-\alpha)1_{t_2}$ be some strategy of player B with $0 \leq t_1 < b < t_2 \leq c$ and $\alpha = (t_2 - b)/(t_2 - t_1)$. Then*

$$\frac{H(X, 1_{t_1}) - v}{t_1 - b} \leq \frac{H(X, 1_{t_2}) - v}{t_2 - b}$$

where $v = val\Lambda_{c,c}(a,b)$.

[4]The sum of random variables refer to mixture distributions.

Proof. X is optimal; thus

$$v \leq H(X,Y) = \frac{t_2 - b}{t_2 - t_1} H(X, 1_{t_1}) + \frac{b - t_1}{t_2 - t_1} H(X, 1_{t_2}).$$

Multiplying by $(t_2 - t_1)/[(t_2 - b)(b - t_1)]$ yields

$$\frac{t_2 - t_1}{(t_2 - b)(b - t_1)} v \leq \frac{H(X, 1_{t_1})}{b - t_1} + \frac{H(X, 1_{t_2})}{t_2 - b}$$

and so

$$\frac{H(X, 1_{t_1}) - v}{t_1 - b} \leq \frac{H(X, 1_{t_2}) - v}{t_2 - b}.$$

□

Finally, we cite a version of Helly's selection theorem, and prove a useful outcome.

theorem 1 (**Helly's selection theorem**). *For every sequence F_n of distribution functions there exists a subsequence F_{n_k} and a nondecreasing, right-continuous function F such that $\lim_{k \to \infty} F_{n_k}(x) = F(x)$ at continuity points x of F.*

(see, e.g., Billingsley, 1986 (Theorem 25.9)). Note that since F is nondecreasing, the set of points where F is not continuous is at most countable. An application of the diagonal method gives a subsequence of F_n that converges for every[5] $t \in \mathbb{R}$. We also note that it is sufficient to assume that F_n is a sequence of nondecreasing functions, that are uniformly bounded, i.e., there exists some $0 < M \in \mathbb{R}$ with $|F_n(x)| < M$ for every $x \in U$ and every $n \in \mathbb{N}$.

lemma 4. *Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative random variables, and assume $\lim_{n \to \infty} E(X_n) = 0$. Then for every[6] $t > 0$, $\lim_{n \to \infty} F_{X_n}(t) = 1$.*

Proof. Let $t_0 > 0$. We wish to show that

$$\lim_{n \to \infty} F_{X_n}(t_0) = 1.$$

It is sufficient to show that every converging subsequence of $F_{X_n}(t_0)$ convergence to 1. For convenience assume $F_{X_n}(t_0)$ converges, and let $l \in \mathbb{R}$ be its limit.

Observe that for every $n \in \mathbb{N}$ $0 \leq F_{X_n}(t) \leq 1$ for every t, and that $F_{X_n}(t)$ is a nondecreasing function. By Helly's selection theorem (Theorem 1), there exists a subsequence $F_{X_{n_k}}(t)$ of $F_{X_n}(t)$, and a function $F(t)$ with

$$\lim_{k \to \infty} F_{X_{n_k}}(t) = F(t)$$

for every $t \in \mathbb{R}$.

Using (2.1) and the dominated convergence theorem gives us,

$$\int_0^\infty 1 - F(t) dt = \int_0^\infty \lim_{k \to \infty} \left(1 - F_{X_{n_k}}(t)\right) dt = \lim_{k \to \infty} \int_0^\infty \left(1 - F_{X_{n_k}}(t)\right) dt$$

[5] $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ denotes the set of natural numbers.

[6] We denote the cumulative distribution function (CDF) of a random variable X by $F_X(t)$.

$$= \lim_{k\to\infty} \int_0^\infty P(X_{n_k} > t)dt = \lim_{k\to\infty} E(X_{n_k}) = 0.$$

Thus, $1 - F(t) = 0$, i.e., $F(t) = 1$, for almost every $t \geq 0$. However, as a pointwise limit of nondecreasing functions $F(t)$ is a nondecreasing function, and so $F(t) = 1$ for every $t > 0$.

Finally,

$$l = \lim_{n\to\infty} F_{X_n}(t_0) = \lim_{k\to\infty} F_{X_{n_k}}(t_0) = F(t_0) = 1.$$

□

3 Equal Caps

In this section we will consider the continuous Captain Lotto game $\Lambda_{c_A,c_B}(a,b)$ when both players have the same cap, i.e., $c_A = c_B = c$. We first recall the result of Hart (2014) on the optimal strategies (Theorem 2) and then prove their uniqueness (Theorem 3).

theorem 2 (Hart 2014, Theorem 2). *Let $c \geq a, b > 0$. Then*

$$val\Lambda_{c,c}(a,b) = val\Lambda(a,b) = \frac{a-b}{\max\{a,b\}}.$$

Without loss of generality let $a \geq b$; optimal strategies are

$$X^\star = \begin{cases} U_{[0,2a]} & c > 2a \\ \frac{c-a}{a} U_{[0,2(c-a)]} + \frac{2a-c}{a} 1_c & c \leq 2a \end{cases}$$

for player A, and

$$Y^* = \begin{cases} \left(1 - \frac{b}{a}\right) 1_0 + \frac{b}{a} U_{[0,2a]} & c > 2a \\ \left(1 - \frac{b}{a}\right) 1_0 + \frac{b}{a} \left(\frac{c-a}{a} U_{[0,2(c-a)]} + \frac{2a-c}{a} 1_c\right) & c \leq 2a \end{cases}$$

for player B.

remark 3.
1. *When $c \geq 2a$ player A's optimal strategy is $U_{[0,2a]}$. When $c < 2a$ this strategy is no longer feasible since X^* must satisfy $X^* \leq c < 2a$. As a result, instead of playing $U_{[0,2a]}$, player A replaces the uniform distribution on the interval $[2(c-a), 2a]$, which is a symmetric interval around c, with an atom at c (of the same mass).*

2. *Player B, the weak player, imitates whatever player A does "as much as he can", namely, with probability b/a, and gives up, i.e., chooses 0, with the remaining probability $1 - b/a$. Player A plays the same strategy regardless of player B's expectation b, as long as $a \geq b$. When $b > a$ the players switch roles: player B becomes the strong player and player A, who is the weak player this time, imitates player B.*

theorem 3. *For both players, the optimal strategy of Theorem 2 is the unique optimal strategy.*

Proof. When $c > 2a$ the uniqueness proof is identical to the one in the continuous General Lotto game $\Lambda(a, b)$; see Hart (2008, Appendix).

Assume now that $c \leq 2a$; thus

$$X^* = \frac{c-a}{a} U_{[0,2(c-a)]} + \frac{2a-c}{a} 1_c$$

$$Y^* = \left(1 - \frac{b}{a}\right) 1_0 + \frac{b}{a} \left(\frac{c-a}{a} U_{[0,2(c-a)]} + \frac{2a-c}{a} 1_c\right).$$

First, notice that the $c = a$ case is trivial. Player A has only one feasible strategy, $X^* = 1_a$. By playing anything less than c player B loses. Thus his goal is to play "as much c" as possibly. Therefore, his unique optimal strategy is $Y^* = (1 - b/a)1_0 + (b/a)1_a$. These are the exact expressions we were hoping for. The $c = b$ case is even simpler since $c = b \Longrightarrow c = b \leq a \leq c \Longrightarrow a = b = c$, causing each player's set of strategies to be exactly $\{1_c\}$, and the value of the game to be $0 = 1 - b/a$. Henceforth we may assume $c > a$. Our goal is to show that the above optimal strategies are unique. We divide the proof into two subsections, one for each of the two players, and each subsection is split into two cases: $c = 2a$ and $c < 2a$.

3.1 Uniqueness of the Optimal Strategy of Player A

3.1.1 The $c = 2a$ Case

Assume $c = 2a$, and so $X^* = U_{[0,2a]}$. Let X^0 be an optimal strategy of player A in $\Lambda_{c,c}(a, b)$. Let $Y = \alpha 1_{t_1} + (1 - \alpha)1_{t_2}$, where $\alpha = (t_2 - a)/(t_2 - t_1)$ and $0 \leq t_1 < a < t_2 \leq 2a = c$. It is immediate that $E(Y) = a$; thus Y is allowed as a strategy of player B in $\Lambda_{c,c}(a, a)$. From Lemma 2 we conclude that $H(X^0, Y) \geq 0$ (since the lemma guarantees that X^0 is an optimal strategy of player A in the symmetric game $\Lambda_{c,c}(a, a)$).

Denote $h(t) = H(X^0, 1_t)$. Using Lemma 3, together with the fact that $b = a$, yields

$$\frac{h(t_2)}{a - t_2} \leq \frac{h(t_1)}{a - t_1}.$$

There exists some $\lambda \in \mathbb{R}$ satisfying

$$\frac{h(t_2)}{a - t_2} \leq \lambda \leq \frac{h(t_1)}{a - t_1}$$

for every $0 \leq t_1 < a < t_2 \leq 2a$. Thus $\lambda(a - t) \leq h(t)$ for every $t \in [0, 2a]$, excepting, perhaps, $t = a$. However, the inequality is also true for $t = a$ since 1_a is a feasible strategy for player B in $\Lambda_{c,c}(a, a)$.

Notice that

$$h(t) = H(X^0, 1_t) = P(X^0 > t) - P(X^0 < t) = P(X^0 > t) + P(X^0 \geq t) - 1;$$

thus $h(t)$ is a nonincreasing function. By integration and the use of (2.1) we obtain

$$\int_0^{2a} h(t)dt = \int_0^{2a} [P(X^0 > t) + P(X^0 \geq t) - 1]dt = E(X^0) + E(X^0) - 2a = 0.$$

However, we also have $\int_0^{2a} \lambda(a-t)dt = 0$; thus

$$\int_0^{2a} h(t)dt = \int_0^{2a} \lambda(a-t)dt$$

and we conclude that $h(t) = \lambda(a-t)$ almost everywhere. Hence $\lambda \geq 0$ (since $h(t)$ is nonincreasing). Since both functions are nonincreasing, $\lambda(a-t) \leq h(t)$ for every $t \in [0, 2a]$, and the LHS is continuous, we obtain $h(t) = \lambda(a-t)$ everywhere, excepting, perhaps, $t = 0$.

claim 1. *X^0 has no atom at $t = 2a$.*

Proof. Denote $p = P(X^0 = 2a)$. We have $h(2a) = P(X^0 > 2a) + P(X^0 \geq 2a) - 1 = p - 1$ and $h(2a^-) = 2p - 1$. Since $h(t) = \lambda(a-t)$ for every $t \in (0, 2a]$, $h(t)$ is left continuous in $t = 2a$, and thus $h(2a) = h(2a^-)$, which yields $p = 0$. □

Using this new information will enable us to find λ.

$$\lambda(-a) = h(2a) = P(X^0 > 2a) + P(X^0 \geq 2a) - 1 = -1 \implies \lambda = \frac{1}{a}$$

and thus

$$\forall t \in (0, 2a] \; h(t) = 1 - \frac{t}{a}.$$

Notice that for $t = 0$ we have $1 = \lambda a \leq h(0) \leq 1$, and so $h(t) = 1 - t/a$ everywhere. Then,

$$2P(X^0 > t) - 1 \leq h(t) = P(X^0 > t) + P(X^0 \geq t) - 1 \leq 2P(X^0 \geq t) - 1$$

$$2P(X^0 > t) \leq h(t) + 1 \leq 2P(X^0 \geq t)$$

$$P(X^0 > t) \leq 1 - \frac{t}{2a} \leq P(X^0 \geq t);$$

thus $X^0 = U_{[0,2a]}$ and X^* is unique.

3.1.2 The $c < 2a$ Case

Assume $c < 2a$. Let X^0 be an optimal strategy of player A in $\Lambda_{c,c}(a, b)$. According to Lemma 2, X_0 is also optimal in $\Lambda_{c,c}(a, a)$. We express $X^0 = \alpha Z + (1-\alpha)1_c$, where $Z < c$, $E(Z) = z$, and $\alpha = (c-a)/(c-z)$, and for every $Y \in [0, c]$ with $E(Y) = a$, we express $Y = \beta W + (1-\beta)1_c$ with $W < c$, $E(W) = w$ and $\beta = (c-a)/(c-w)$. Conversely, for every $W < c$ with $E(W) = w \leq a$, there is β such that $Y = \beta W + (1-\beta)1_c$ satisfies $E(Y) = a$ and $Y \in [0, c]$. Since X^0 is optimal, we have

$$0 \leq H(X^0, Y) \implies 0 \leq \alpha\beta H(Z, W) + \alpha(1-\beta)(-1) + (1-\alpha)\beta$$

$$\frac{1}{\beta} - \frac{1}{\alpha} = \frac{\alpha - \beta}{\alpha\beta} \leq H(Z, W).$$

Notice that since we have already assumed $c > a$, we can indeed divide the equation by $\alpha, \beta > 0$. Substituting these into the above inequality with their explicit form yields

$$\frac{z-w}{c-a} = \frac{c-w}{c-a} - \frac{c-z}{c-a} = \frac{1}{\beta} - \frac{1}{\alpha} \leq H(Z, W) \tag{3.1}$$

By combining (3.1) above and the fact that $H(Z,W)$ is always less than or equal to 1 (see (1.1)), we conclude that $z - w \leq c - a$. The $W = 1_0$ case yields $z \leq c - a$.

We claim that $z = c - a$. Assume $z < c - a$. Let $W = U_{[0,2w]}$, where $z < w < c - a$ (W is allowed because $2w < 2(c-a) \leq c$). Notice that according to the $c > 2a$ case of Theorem 2, W is optimal in $\Lambda_{c,c}(z,w)$; thus

$$H(Z,W) \leq \frac{z-w}{w}. \tag{3.2}$$

By (3.1) and (3.2) we obtain $w \geq c - a$, a contradiction. So $z = c - a$ and $\alpha = (c-a)/a$.

Summary: $z = c - a$ and for every $W < c$ with $E(W) = c - a$ we have $0 \leq H(Z,W)$ (see (3.1)).

We wish to show that Z is optimal in $\Lambda_{c,c}(c-a, c-a)$. Let $\tilde{Y} = \gamma W + (1-\gamma)1_c$, where $W < c$, $E(W) = w$, $E(\tilde{Y}) = c - a$, and $\gamma = a/(c-w)$.

$$H(Z,\tilde{Y}) = \frac{a}{c-w} H(Z,W) + \left(1 - \frac{a}{c-w}\right) H(Z,1_c).$$

Using (3.1) (we can do so since $w \leq c - a < a$ and $W < c$), $z = c - a$ and $Z < c$ yields

$$H(Z,\tilde{Y}) \geq \frac{a}{c-w}\frac{c-a-w}{c-a} + \left(1 - \frac{a}{c-w}\right)(-1) = \left(\frac{a}{c-a} - 1\right)\frac{c-a-w}{c-w}.$$

By $2a > c$ and $c - a \geq w$ we obtain $H(Z,\tilde{Y}) \geq 0 = \text{val}\Lambda_{c,c}(c-a, c-a)$. This means that Z is optimal in the symmetric game $\Lambda_{c,c}(c-a,c-a)$; thus $Z = U_{[0,2(c-a)]}$ (since $2(c-a) < c$, we are in the $c > 2a$ case of Theorem 2, and as we saw in this theorem, in such a case the optimal strategy is unique). We conclude that $X^0 = X^*$. Thus

$$X^* = \frac{c-a}{a} U_{[0,2(c-a)]} + \frac{2a-c}{a} 1_c$$

is a unique optimal strategy of player A in $\Lambda_{c,c}(a,b)$.

3.2 Uniqueness of the Optimal Strategy of Player B

3.2.1 The $c = 2a$ Case

We turn now to player B and first assume $c = 2a$. Let Y^0 be an optimal strategy for player B in $\Lambda_{c,c}(a,b)$. Let $X = \alpha 1_{t_1} + (1-\alpha)1_{t_2}$ be a strategy of player A, where $0 \leq t_1 < a < t_2 \leq c$. Taking expectation arguments gives us $\alpha = (t_2 - a)/(t_2 - t_1)$.

Denote $g(t) = H(1_t, Y^0)$. Using Lemma 3 (we switch between players A and B), and $\text{val}\Lambda_{c,c}(a,b) = 1 - b/a$, gives us

$$\frac{\left(1 - \frac{b}{a}\right) - g(t_2)}{a - t_2} \leq \frac{\left(1 - \frac{b}{a}\right) - g(t_1)}{a - t_1}.$$

There exists some $\lambda \in \mathbb{R}$ with

$$\left(1 - \frac{b}{a}\right)\frac{1}{a-t_2} - \frac{g(t_2)}{a-t_2} \leq \lambda \leq \left(1 - \frac{b}{a}\right)\frac{1}{a-t_1} - \frac{g(t_1)}{a-t_1}$$

for every $t_1 \in [0,a)$ and $t_2 \in (a,c]$. For every $t \neq a$ we obtain that $g(t) \leq \lambda(t-a)+1-b/a$. However, since 1_a is a feasible strategy for player A we also obtain this inequality for $t=a$; thus

$$g(t) \leq \lambda(t-a)+1-\frac{b}{a} \ \forall t \in [0,2a]. \tag{3.3}$$

By

$$-g(t) = H(Y^0, 1_t) = P(Y^0 > t) + P(Y^0 \geq t) - 1 \tag{3.4}$$

we conclude that $g(t)$ is a weakly increasing. Moreover, by integration and the use of (2.1) we obtain

$$\textstyle\int_0^{2a} -g(t)dt = 2b-2a \quad \Rightarrow \quad \int_0^{2a} g(t)dt = 2(a-b)$$

$$\int_0^{2a} \left(\lambda(t-a)+1-\frac{b}{a}\right) dt = \left(1-\frac{b}{a}\right) 2a = 2(a-b).$$

Recalling (3.3) yields $g(t) = \lambda(t-a)+1-b/a$ almost everywhere. Hence $\lambda \geq 0$ (since $g(t)$ is a weakly increasing function). Due to the fact that $g(t)$ and $\lambda(t-a)+1-b/a$ are both weakly increasing, the latter is continuous, and $g(t) \leq \lambda(t-a)+1-b/a$ for every t, the equality holds for every t, excepting, perhaps, $t=0$.

claim 2. *Y^0 has no atom at $c=2a$.*

Proof. Denote $q = P(Y^0 = c)$. Using (3.4):

$$g(2a^-) = 1-2q$$
$$g(2a) = 1-q,$$

and by continuity $q=0$. ($g(t)$ is left continuous in $2a$ since $g(t) = \lambda(t-a)+1-b/a$ for all $t \in (0,2a]$.) □

Then,

$$1 = g(2a) = \lambda a + 1 - \frac{b}{a}$$

implies $\lambda = b/a^2$; hence

$$g(t) = \frac{b}{a^2}(t-a)+1-\frac{b}{a} = 1-\frac{b}{a}-\frac{b}{a}\left(1-\frac{t}{a}\right).$$

Using (3.4) immediately yields $P(Y^0 > t) \leq (1-g(t))/2 \leq P(Y^0 \geq t)$, i.e.,

$$P(Y^0 > t) \leq \frac{b}{a}\left(1-\frac{t}{2a}\right) \leq P(Y^0 \geq t).$$

Thus

$$Y^0 = \left(1-\frac{b}{a}\right)1_0 + \frac{b}{a}U_{[0,2a]} = Y^*,$$

and so the uniqueness of the optimal strategy of player B in the $c=2a$ case is proved.

3.2.2 The $c < 2a$ Case

Assume now that $c < 2a$. Let Y^0 be an optimal strategy of player B. We will show that $Y^0 = Y^*$. We may express $Y^0 = \beta Z + (1-\beta)1_c$, where $0 \le Z < c$, $E(Z) = z$, and $\beta = (c-b)/(c-z)$ (since $E(Y^0) = b$ we have $z \le b$). If $\beta = 0$ then $b = c = a$, and we have already discussed this situation at the beginning of our proof. Thus we may assume $\beta > 0$. Since both X^* and Y^0 are optimal, using the first part of Lemma 1 gives us

$$1 - \frac{b}{a} = H(X^*, Y^0)$$

$$= \frac{c-a}{a}\beta H(U_{[0,2(c-a)]}, Z) + \frac{c-a}{a}(1-\beta)(-1) + \left(1 - \frac{c-a}{a}\right)\beta$$

$$\ge \frac{c-a}{a}\beta\left(1 - \frac{z}{c-a}\right) + \beta - \frac{c-a}{a} = 1 - \frac{b}{a},$$

which implies that $H(U_{[0,2(c-a)]}, Z) = 1 - z/(c-a)$. By using the second part of Lemma 1, we conclude that $Z \le 2(c-a)$.

This guarantees $z \le 2(c-a)$. Assume $z > c-a$, and define $X^Z = \alpha Z + (1-\alpha)1_c$, where $\alpha = (c-a)/(c-z)$, a strategy of player A. By letting player A play X^Z against Y^0 we obtain

$$H(X^Z, Y^0) = \alpha\beta H(Z, Z) + \alpha(1-\beta)(-1) + (1-\alpha)\beta$$

$$= \beta - \alpha = \frac{c-b}{c-z} - \frac{c-a}{c-z} = \frac{a-b}{c-z} > \frac{a-b}{a},$$

which contradicts Y^0's optimality. Thus $z \le c-a$.

We wish to show that Z is optimal for player B in $\Lambda_{2(c-a),2(c-a)}(c-a, z)$ (remember that $Z \le 2(c-a)$ and $z \le c-a$). Let W be a strategy of player A in $\Lambda_{2(c-a),2(c-a)}(c-a, z)$. We define a strategy of player A in the game $\Lambda_{c,c}(a, b)$ using W: $X^W = \alpha W + (1-\alpha)1_c$, where $\alpha = (c-a)/a$ (thus $E(X^W) = a$ since $E(W) = c-a$). Due to Y^0's optimality, we get

$$H(X^W, Y^0) = \alpha\beta H(W, Z) + \beta - \alpha \le 1 - \frac{b}{a}$$

$$H(W, Z) \le \frac{1}{\alpha\beta}\left(1 - \frac{b}{a}\right) + \frac{1}{\beta} - \frac{1}{\alpha}$$

$$= \frac{a}{c-a}\frac{c-z}{c-b}\frac{a-b}{a} + \frac{c-z}{c-b} - \frac{a}{c-a}$$

$$= 1 - \frac{z}{c-a} = \text{val}\Lambda_{2(c-a),2(c-a)}(c-a, z).$$

Thus, Z is an optimal strategy for player B in $\Lambda_{2(c-a),2(c-a)}(c-a, z)$. We already know that for every $z \le c-a$ we have a unique optimal strategy for player B in $\Lambda_{2(c-a),2(c-a)}(c-a, z)$:

$$Z = \left(1 - \frac{z}{c-a}\right)1_0 + \frac{z}{c-a}U_{[0,2(c-a)]}$$

(see the $c = 2a$ case of player B in this theorem).

Define $\tilde{Y} : [0, \min\{b, c-a\}] \to \{Y \mid Y \text{ is a strategy of player B in } \Lambda_{c,c}(a,b)\}$, such that:

$$\tilde{Y}(z) = \frac{c-b}{c-z}\left(\left(1 - \frac{z}{c-a}\right)1_0 + \frac{z}{c-a}U_{[0,2(c-a)]}\right) + \frac{b-z}{c-z}1_c. \tag{3.5}$$

$\tilde{Y}(z)$ is well defined since $z \leq c-a$ and $z \leq b$. Notice that $Y^0 = \tilde{Y}(z)$ for some $z \in [0, min\{b, c-a\}]$, and that $Y^* = \tilde{Y}(z_0)$ for

$$z_0 = \frac{\frac{b}{a}\frac{c-a}{a}(c-a)}{1 - \frac{b}{a} + \frac{b}{a}\frac{c-a}{a}}. \tag{3.6}$$

Thus it remains to show that Y^0 is given by substituting (3.6) into (3.5).

We denote $\gamma = 1 - z/(c-a)$. Let $X = \alpha V + (1-\alpha)1_c$ be a strategy of player A, where $0 < V \leq 2(c-a)$, $E(V) = v$, and $\alpha = (c-a)/(c-v)$. Using the fact that $V > 0$ and the second part of Lemma 1:

$$H(X, Y^0) = \beta\gamma H(X, 1_0) + \beta(1-\gamma)H(X, U_{[0,2(c-a)]}) + (1-\beta)H(X, 1_c)$$

$$= \beta\gamma + \beta(1-\gamma)\left(\alpha H(V, U_{[0,2(c-a)]}) + (1-\alpha)H(1_c, U_{[0,2(c-a)]})\right) + (1-\beta)(-\alpha)$$

$$= \beta\gamma + \beta(1-\gamma)\alpha\left(\frac{v}{c-a} - 1\right) + \beta(1-\gamma)(1-\alpha) + (1-\beta)(-\alpha).$$

With some development of this equation we obtain

$$H(X, Y^0) = \frac{a-b}{c-v} + \beta\gamma\left(1 - \frac{a}{c-v}\right).$$

From Y^0's optimality we obtain

$$\frac{a-b}{c-v} + \beta\gamma\left(1 - \frac{a}{c-v}\right) \leq 1 - \frac{b}{a},$$

and so

$$\beta\gamma\left(1 - \frac{a}{c-v}\right) \leq \frac{a-b}{a}\left(1 - \frac{a}{c-v}\right). \tag{3.7}$$

As we know, $0 < V \leq 2(c-a)$, and $E(V) = v$. Thus we may choose $v < c-a$, which yields $1 - a/(c-v) > 0$, or $v > c-a$, which yields $1 - a/(c-v) < 0$. Applying these two possibilities to (3.7) yields $\beta\gamma = 1 - b/a$. By substituting $\gamma = 1 - z/(c-a)$, and $\beta = (c-b)/(c-z)$, we obtain

$$\frac{c-b}{c-z}\left(1 - \frac{z}{c-a}\right) = \frac{a-b}{a},$$

which implies

$$z = \frac{\frac{b}{a}\frac{c-a}{a}(c-a)}{1 - \frac{b}{a} + \frac{b}{a}\frac{c-a}{a}} = z_0.$$

Thus $Y^0 = \tilde{Y}(z_0) = Y^*$, and we are done. □

4 Unequal Caps

In this section we consider the game $\Lambda_{c_A,c_B}(a,b)$ in the case of unequal caps. Thus $c_A \neq c_B$ and, without loss of generality, $c_A > c_B$ (note that we no longer assume $a \geq b$).

Clearly, all the cases where $c_A > c_B \equiv c$ are equivalent, as any value above the upper bound c of player B has the same effect, and so player A would want to use the lowest possible such value. We denote this by c^+, which stands for an infinitesimal $c + \delta > c$, and consider the corresponding game $\Lambda_{c^+,c}(a,b)$. In addition, note that any strategy of player A can be expressed as $X = \alpha Z + (1-\alpha)W$, where $0 \leq Z \leq c$, $c < W$, $E(Z) = z$, $E(W) = w > c$, and $\alpha = (w-a)/(w-z)$. Since every strategy of player B realizes $Y \in [0,c]$, W's distributions has no effect whatsoever on the payoff function, and so we may assume that, without loss of generality, $W = 1_{c+\delta}$ with $\delta > 0$.

Let $\epsilon > 0$. We call a strategy X^* of player A ϵ-optimal if it satisfies

$$H(X^*, Y) \geq \text{val}\Lambda_{c^+,c}(a,b) - \epsilon$$

for every Y of player B. Similarly, we call a strategy Y^* of player B ϵ-optimal if it satisfies

$$H(X, Y^*) \leq \text{val}\Lambda_{c^+,c}(a,b) + \epsilon$$

for every X of player A.

We first recall the result of Hart (2014) on the value, optimal strategies and ϵ-optimal strategies of the $\Lambda_{c_A,c_B}(a,b)$ game.

theorem 4 (Hart 2014, Theorem 4). *Let $c_A > c_B = c > 0$, and let $0 \leq a \leq c_A$ and $0 \leq b \leq c$. Then*

$$val\Lambda_{c_A,c_B}(a,b) = \begin{cases} 0 & \text{if } a = b = 0 \\ \frac{a-b}{\max\{a,b\}} & \text{if } 0 < \max\{a,b\} \leq \frac{c}{2} \\ 1 - \frac{4b(c-a)}{c^2} & \text{if } b \leq \frac{c}{2} \text{ and } \frac{c}{2} \leq a \leq c \\ \frac{2a}{c} - 1 & \text{if } \frac{c}{2} \leq b \leq c \text{ and } a \leq c \\ 1 & \text{if } a \geq c. \end{cases}$$

Optimal strategies X^* and Y^* of player A and player B, respectively, are as follows.

(i) *When $a = b = 0$:*

$$X^* = Y^* = 1_0.$$

(ii) *When $b \leq a \leq c/2$ and $a > 0$:*

$$X^* = U_{[0,2a]},$$

$$Y^* = \left(1 - \frac{b}{a}\right) 1_0 + \frac{b}{a} U_{[0,2a]}.$$

(iii) *When $a \leq b \leq c/2$ and $b > 0$:*

$$X^* = \left(1 - \frac{a}{b}\right) 1_0 + \frac{a}{b} U_{[0,2b]},$$

$$Y^* = U_{[0,2b]}.$$

(iv) *When*[7] $b \le c/2 < a \le c:$

$$X^* = \frac{2c^+ - 2a}{2c^+ - c} U_{[0,c]} + \frac{2a - c}{2c^+ - c} 1_{c^+},$$

$$Y^* = \frac{c - 2b}{c} 1_0 + \frac{2b}{c} U_{[0,c]}.$$

(v) *When* $c/2 < b \le c$ *and* $a \le c:$

$$X^* = \frac{c^+ - a}{c^+} 1_0 + \frac{a}{c^+} 1_{c^+},$$

$$Y^* = \frac{c - b}{b} U_{[0,2c-2b]} + \frac{2b - c}{b} 1_c.$$

(vi) *When* $a > c:$ *Any feasible* X *with values* $> c$ *(for example,* $X^* = 1_a$*) is optimal, and any* Y *is optimal.*

remark 4. *We rule out some trivial cases:*

- *When* $a = b = 0:$ *The only possible strategy for both players is* 1_0.
- *When* $a > c:$ *In this case by playing* 1_a *player A guarantees* $H(1_a, Y) = 1$ *for every* Y *of player B. Thus* $val\Lambda_{c^+,c}(a, b) = 1$*, every strategy* X *of player A with* $X > c$ *is optimal, and every strategy of player B is optimal.*

For the rest of this section, we will assume $a \le c$**, and, as pointed out in Section 2,** $\min\{a, b\} > 0$.

In Section 4.1 we show that the optimal strategies of Theorem 4 in the case of $0 < \max\{a, b\} \le c/2$ are unique. In Sections 4.2 and 4.3, while for player B we still discuss whether his(her) optimal strategies are unique, for player A the discussion changes somewhat. We cannot talk about uniqueness of ϵ-optimal strategies. However, as we will see, given a sequence X_n, where X_n is an ϵ_n-optimal strategy, with $\lim_{n\to\infty} \epsilon_n = 0$, we characterize all of its CDFs' pointwise limits. Note that even as we look at $\Lambda_{c^+,c}(a, b)$, all the results of this section hold when $c_A = \infty$.

4.1 Cases *(ii)* and *(iii)*: $0 < \max\{a, b\} \le c/2$.

theorem 5. *The optimal strategies of Theorem 4 in cases (ii) and (iii) are the unique optimal strategies for both players.*

Proof. When $c > \max\{2a, 2b\}$, uniqueness follows from the same proof as in Hart (2008, Appendix). When $c = \max\{2a, 2b\}$, the uniqueness of Y^* derives from its uniqueness in $\Lambda_{c,c}(a, b)$ (see Theorem 3), and so all that remains to show is that X^* is unique as an optimal strategy of player A. We divide the proof into two subcases: $c = 2a > 2b$ and $c = 2b \ge 2a$.

First assume $c = 2a > 2b$.

Let X^0 be an optimal strategy of player A. We express $X^0 = \alpha Z + (1-\alpha)1_{c+\delta}$, where $0 \le Z \le c$ with $E(Z) = z$ and $\alpha = (c + \delta - a)/(c + \delta - z)$. We wish to show that $\alpha = 1$. Assume $\alpha < 1$, and notice that $\alpha < 1 \Leftrightarrow z < a$.

[7]Here and in the next case, X^* is an ϵ-optimal strategy, where c^+ stands for $c + \delta$ for a small enough $\delta > 0$.

If $z < b$, then $Y = U_{[0,2b]}$ is an optimal strategy of player B in $\Lambda_{c^+,c}(z,b)$ (this is the $c > \max\{2a, 2b\}$ case of this theorem). Thus,

$$1 - \frac{b}{a} \leq H(X^0, U_{[0,2b]}) = \alpha H(Z, U_{[0,2b]}) + 1 - \alpha,$$

which implies,

$$1 - \frac{b}{a}\frac{1}{\alpha} \leq H(Z, U_{[0,2b]}) \leq \frac{z}{b} - 1.$$

It is easy to verify that $\alpha \geq (c-a)/(c-z)$, and so

$$1 - \frac{b}{a}\frac{c-z}{c-a} \leq \frac{z}{b} - 1.$$

Rearranging yields,

$$\left(2 - \frac{z}{b}\right)\frac{c-a}{c-z} \leq \frac{b}{a}.$$

By multiplying by $ab(c-z)$, and substituting $c = 2a$, we obtain: $(2b-z)a^2 \leq b^2(2a-z)$. A closer look reveals that

$$(2b-z)a^2 \leq b^2(2a-z) = b^2(2a-2b) + b^2(2b-z)$$
$$(2b-z)(a^2-b^2) \leq 2b^2(a-b)$$
$$(2b-z)(a+b) \leq 2b^2$$
$$2ab \leq z(a+b).$$

Since $z < b < a$, we have $2ab \leq z(a+b) < 2ab$, a contradiction. Thus $z \geq b$.

Let $Y = \beta Z + (1-\beta)1_0$ with $\beta = b/z$ (we use the same Z from the strategy X^0). Y is allowed for player B since $0 \leq Z \leq c$ and $z \geq b$. By the optimality of X^0, we get

$$1 - \frac{b}{a} \leq H(X^0, Y) = \alpha(1-\beta)H(Z, 1_0) + (1-\alpha)H(1_{c+\delta}, Y)$$
$$\leq \alpha(1-\beta) + 1 - \alpha.$$

(The last inequality is obtained by using the fact that $H(*,*) \leq 1$ always). Rearranging yields,

$$1 - \frac{b}{a}\frac{1}{\alpha} \leq 1 - \beta,$$

and again using $\alpha \geq (c-a)/(c-z)$ and substituting $\beta = b/z$ yields,

$$1 - \frac{b}{a}\frac{c-z}{c-a} \leq 1 - \frac{b}{z}$$
$$(c-a)a \leq (c-z)z.$$

Keeping in mind that $a = c/2$, and that $c/2$ is the maximum of the function $(c-x)x$, we obtain that $z = c/2$. However, $z < a = c/2$, a contradiction. Thus $\alpha = 1$ and our optimal strategy X^0 satisfies $0 \leq X^0 \leq c$.

Since X^0 satisfies $0 \leq X^0 \leq c$, $E(X^0) = a$, and $H(X^0, Y) \geq 1 - b/a$ for all $Y \in [0,c]$ with $E(Y) = b$, X^0 is optimal in $\Lambda_{c,c}(a,b)$. Using Theorem 3 gives us $X^0 = X^*$(since $c = 2a > 2b$). This concludes the proof for the case of $c = 2a > 2b$.

Now, assume $c = 2b \geq 2a$. Let X^0 be an optimal strategy of player A, and again we express $X^0 = \alpha Z + (1-\alpha)1_{c+\delta}$, where $0 \leq Z \leq c$ with $E(Z) = z$ and $\alpha = (c+\delta-a)/(c+\delta-z)$. Again we wish to show that $\alpha = 1$, which is equivalent to $z = a$. $Y = U_{[0,c]}$ is an optimal strategy of player B in $\Lambda_{c^+,c}(a, c/2)$ for any $a \leq c/2$, in particular for our z. Thus,

$$\frac{2a}{c} - 1 \leq H(X^0, Y) \leq \alpha H(Z, Y) + 1 - \alpha \leq \alpha\left(\frac{2z}{c} - 1\right) + 1 - \alpha$$

which implies $z \geq a$ and so $z = a$. Thus, X^0 is optimal in $\Lambda_{c,c}(a, c/2)$, and from Theorem 3 we are done. □

4.2 Case *(iv)*: $0 < b \leq c/2 < a \leq c$.

In this case of Theorem 4 we first prove the uniqueness of player B's optimal strategy (Theorem 6). Then, turning to player A, we see that he(she) has no optimal strategies in this case (Theorem 7). However, given a sequence of ϵ_n-optimal strategies X_n with $\lim_{n\to\infty} \epsilon_n = 0$, we characterize the pointwise limits of its CDFs'. We divide the discussion for player A into three parts:

- $0 < b < c/2 < a < c$ (Theorem 8).
- $0 < b = c/2 < a < c$ (Theorem 9).
- $0 < b \leq c/2 < a = c$ (Theorem 10).

theorem 6. *When $0 < b \leq c/2 < a \leq c$, the optimal strategy of Theorem 4 is the unique optimal strategy for player B.*

Proof. Let Y^0 be an optimal strategy of player B in $\Lambda_{c^+,c}(a, b)$, where $0 < b \leq c/2 < a \leq c$. Let X be some strategy of player A in $\Lambda_{c^+,c}(c/2, b)$ (thus $E(X) = c/2$). We define $X_a = \alpha X + (1-\alpha)1_{c+\delta}$ with $\alpha = (c+\delta-a)/(c+\delta-c/2)$. Thus $E(X_a) = a$ and thus, by the optimality of Y^0, we obtain

$$\begin{aligned} 1 - \frac{4b(c-a)}{c^2} &\geq H(X_a, Y^0) = \alpha H(X, Y^0) + (1-\alpha)H(1_{c+\delta}, Y^0) \\ &= \frac{c+\delta-a}{c+\delta-\frac{c}{2}} H(X, Y^0) + 1 - \frac{c+\delta-a}{c+\delta-\frac{c}{2}}. \end{aligned}$$

The above is True for every $\delta > 0$, and so we have

$$1 - \frac{2b}{c}\frac{2(c-a)}{c} \geq \frac{2(c-a)}{c} H(X, Y^0) + 1 - \frac{2(c-a)}{c}.$$

Thus,

$$1 - \frac{2b}{c} \geq H(X, Y^0).$$

According to Theorems 4 and 5, since $b \leq c/2$, Y^0 is an optimal strategy of player B in $\Lambda_{c^+,c}(c/2, b)$. Thus $Y^0 = (2b/c)U_{[0,c]} + (1 - 2b/c)1_0 = Y^*$, and Y^* is unique. □

theorem 7. *When $0 < b \leq c/2 < a \leq c$, player A has no optimal strategies.*

Proof. Assume that player A has an optimal strategy denoted by X^0. Notice, $0 < b < a$ and $c/2 < a \leq c$. Thus,

$$1 - \frac{4b(c-a)}{c^2} > 1 - \frac{b}{a}.$$

If $X^0 \in [0, c]$, then X^0 is a feasible strategy for player A in $\Lambda_{c,c}(a, b)$, and also guarantees a value greater than val$\Lambda_{c,c}(a, b)$ - a contradiction. We can express $X^0 = \alpha Z + (1-\alpha)1_{c+\delta}$, where $Z \in [0, c]$, $E(Z) = z$, $\alpha < 1$ and $\delta > 0$. Since X^0 is optimal, for every Y of player B we have

$$1 - \frac{4b(c-a)}{c^2} \leq H(X^0, Y) = \alpha H(Z, Y) + 1 - \alpha$$

Substituting $\alpha = (c + \delta - a)/(c + \delta - z)$ and some rearranging yields,

$$1 - \frac{4b(c-a)}{c^2} \frac{c+\delta-z}{c+\delta-a} \leq H(Z, Y). \tag{4.1}$$

Finally, it is easy enough to verify that

$$\frac{c+\delta-z}{c+\delta-a} \leq \frac{c-z}{c-a}$$

and we obtain,

$$1 - \frac{4b(c-z)}{c^2} \leq H(Z, Y).$$

This inequality holds for every Y of player B, and in particular for player B's optimal strategy in $\Lambda_{c,c}(z, b)$. Thus,

$$1 - \frac{4b(c-z)}{c^2} \leq \frac{z-b}{\max\{z, b\}}. \tag{4.2}$$

If $z \geq b$, since $b > 0$, (4.2) is equivalent to

$$\left(\frac{c}{2}\right)^2 \leq z(c-z),$$

and so $z = c/2$. Thus Z is a strategy of player A in $\Lambda_{c,c}(c/2, b)$. Notice (4.1) is true for every Y of player B in $\Lambda_{c,c}(c/2, b)$, in particular for an optimal strategy. Thus,

$$1 - \frac{4b(c-a)}{c^2} \frac{c+\delta-c/2}{c+\delta-a} \leq 1 - \frac{2b}{c},$$

which is equivalent to $a \leq c/2$ - a contradiction.

Assume that $z < b$. Rearranging (4.2) yields

$$2bc(c-2b) \leq z(c+2b)(c-2b).$$

If $b < c/2$, we obtain $2bc \leq z(c+2b)$. However, $z(c+2b) < 2cz$, thus $2bc < 2cz$, and we get $b < z$ - a contradiction. We conclude that as long as $b < c/2$, player A has no optimal strategies.

When $z < b = c/2$, again we have equality in (4.2), and this time Z is an optimal strategy for player A in $\Lambda_{c,c}(z, c/2)$. Substituting $b = c/2$ and Y with an optimal strategy of player B in (4.1), gives us

$$1 - \frac{2(c-a)}{c}\frac{c+\delta-z}{c+\delta-a} \le 1 - \frac{2(c-z)}{c},$$

which is equivalent to $a \le z$. Thus $z < b = c/2 < a \le z$ - a contradiction. Thus, player A has no optimal strategies in $\Lambda_{c^+,c}(a,b)$ when $0 < b \le c/2 < a \le c$. $\square$

We turn now to the characterization of player A's ϵ-optimal strategies.

theorem 8. *Assume $0 < b < c/2 < a < c$, and let X_n be a sequence of ϵ_n-optimal strategies of player A with $\lim_{n\to\infty} \epsilon_n = 0$. Then*

$$\lim_{n\to\infty} F_{X_n}(t) = \begin{cases} 0 & t < 0 \\ \frac{2(c-a)}{c}\frac{t}{c} & 0 \le t \le c \\ 1 & c < t. \end{cases}$$

Proof. Let X_n be a sequence as described in the Theorem. For every $n \in \mathbb{N}$ we express $X_n = \alpha_n Z_n + (1-\alpha_n)1_{c+\delta_n}$, where $Z_n \in [0,c]$, $E(Z_n) = z_n$, $\delta_n > 0$ and $\alpha_n = (c+\delta_n-a)/(c+\delta_n-z_n)$. For every Y of player B we have,

$$1 - \frac{4b(c-a)}{c^2} - \epsilon_n \le H(X_n, Y) = \alpha_n H(Z_n, Y) + 1 - \alpha_n.$$

Rearranging this equation and using $\alpha_n \ge (c-a)/(c-z_n)$, yields

$$1 - \frac{4b(c-a)}{c^2}\frac{c-z_n}{c-a} - \frac{c-z_n}{c-a}\epsilon_n \le H(Z_n, Y),$$

and since $z_n \ge 0$ always, we obtain

$$1 - \frac{4b(c-z_n)}{c^2} - \frac{c}{c-a}\epsilon_n \le H(Z_n, Y) \tag{4.3}$$

for every $n \in \mathbb{N}$ and every Y of player B (when using (4.3), we will omit the constant that multiplies ϵ_n)[8].

It is easy enough to verify that

$$1 - \frac{4b(c-z)}{c^2} \ge \frac{z-b}{\max\{z,b\}}, \tag{4.4}$$

with strong inequality when $z < b$. Indeed, when $z \ge b$, since $b > 0$ (4.4) is equivalent to $(c/2)^2 \ge z(c-z)$, which is always true. When $z < b$, since $b < c/2$ (4.4) is equivalent to $z(2b+c) < 2bc$, which is also true.

By substituting Y with an optimal strategy of player B in $\Lambda_{c,c}(z_n, b)$ in (4.3) we get,

$$1 - \frac{4b(c-z_n)}{c^2} - \epsilon_n \le H(Z_n, Y) \le \frac{z_n - b}{\max\{z_n, b\}}.$$

Using (4.4) yields,

$$0 \le 1 - \frac{4b(c-z_n)}{c^2} - \frac{z_n - b}{\max\{z_n, b\}} \le \epsilon_n. \tag{4.5}$$

[8]Throughout we will not insist on writing constants that multiply our ϵ's, since these arguments vanish when we take the limit.

lemma 5.

$$\lim_{n\to\infty} z_n = \frac{c}{2}.$$

Proof. It is sufficient to show that every converging subsequence of z_n converges to $c/2$. Let z_{n_k} be a converging subsequence of z_n (there exists one since z_n is blocked). For convenience we look at z_n. Denote $z_0 = \lim_{n\to\infty} z_n$.

Taking the limit $n \to \infty$ in (4.5) yields equality in (4.4) for $z = z_0$, thus

$$1 - \frac{4b(c-z_0)}{c^2} = \frac{z_0 - b}{\max\{z_0, b\}}. \tag{4.6}$$

Since (4.4) has a strong inequality when $z_0 < b$, we conclude that $z_0 \geq b$. If $z_0 = b$, (4.6) yields $b = c/2$, a contradiction. Thus $z_0 > b$, and in this case (4.6) yields $z_0 = c/2$. □

lemma 6.

$$\lim_{n\to\infty} \delta_n = 0,$$

$$\lim_{n\to\infty} \alpha_n = \frac{2(c-a)}{c}.$$

Proof. We begin by showing that if X_n is ϵ_n-optimal in $\Lambda_{c^+,c}(a,b)$, then it is ϵ_n-optimal in $\Lambda_{c^+,c}(a,c/2)$. Let Y be a strategy of player B in $\Lambda_{c^+,c}(a,c/2)$. Define $Y_b = (1-2b/c)1_0 + (2b/c)Y$. For all $n \in \mathbb{N}$,

$$1 - \frac{4b(c-a)}{c^2} - \epsilon_n \leq H(X_n, Y_b) = \left(1 - \frac{2b}{c}\right) H(X_n, 1_0) + \frac{2b}{c} H(X_n, Y).$$

Keeping in mind that $H(*,*) \leq 1$ always, by rearranging the above we get

$$1 - \frac{2(c-a)}{c} - \frac{c}{2b}\epsilon_n \leq H(X_n, Y).$$

Thus, X_n is also ϵ_n-optimal in $\Lambda_{c^+,c}(a,c/2)$ (to be precise, it is $(c/2b)\epsilon_n$-optimal, but this will not matter). Let $Y = U_{[0,c]}$.

$$1 - \frac{2(c-a)}{c} - \epsilon_n \leq H(X_n, U_{[0,c]}) = \alpha_n\left(\frac{2z_n}{c} - 1\right) + 1 - \alpha_n$$

(The equality is given from Lemma 1 since $Z_n \in [0,c]$). Rearranging yields,

$$1 - \frac{2(c-a)}{c}\frac{1}{\alpha_n} - \frac{1}{\alpha_n}\epsilon_n \leq \frac{2z_n}{c} - 1.$$

Since $1/\alpha_n \leq (c-z_n)/(c-a)$ and $z_n \geq 0$, we have,

$$\frac{2z_n}{c} - 1 - \frac{c}{c-a}\epsilon_n \leq 1 - \frac{2(c-a)}{c}\frac{1}{\alpha_n} - \frac{c}{c-a}\epsilon_n \leq \frac{2z_n}{c} - 1.$$

Taking the limit $n \to \infty$ and remembering Lemma 5 yields,

$$\lim_{n\to\infty} \alpha_n = \frac{2(c-a)}{c};$$

thus, the expression for α_n and Lemma 5 also yield,

$$\lim_{n\to\infty} \delta_n = 0.$$

□

Define

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{2(c-a)}{c}\frac{t}{c} & 0 \le t \le c \\ 1 & c < t. \end{cases}$$

We wish to show that $\lim_{n\to\infty} F_{X_n}(t) = F(t)$ for every $t \in \mathbb{R}$. By Helly's selection theorem (Theorem 1), $F_{X_n}(t)$ has a pointwise converging subsequence $F_{X_{n_k}}(t)$.

lemma 7.

$$\lim_{k\to\infty} F_{X_{n_k}}(t) = F(t).$$

Proof.

- When $t < 0$,

$$F_{X_{n_k}}(t) = P(X_{n_k} \le t) = 0 \xrightarrow[k\to\infty]{} 0.$$

- When $c < t$,

$$F_{X_{n_k}}(t) = P(X_{n_k} \le t) = \alpha_{n_k} P(Z_{n_k} \le t) + (1-\alpha_{n_k})P(c+\delta_{n_k} \le t)$$
$$= \alpha_{n_k} + (1-\alpha_{n_k})P(c+\delta_{n_k} \le t) \xrightarrow[k\to\infty]{} 1.$$

  This is true because when $k \to \infty$, $\alpha_{n_k} \to 2(c-a)/c$ according to Lemma 6, and $P(c+\delta_{n_k} \le t) \to 1$ according to Lemma 4.

- When $0 \le t \le c$,

$$F_{X_{n_k}}(t) = P(X_{n_k} \le t) = \alpha_{n_k} P(Z_{n_k} \le t).$$

  Since $\alpha_{n_k} \to 2(c-a)/c$ when $k \to \infty$, it is sufficient to show that $\lim_{k\to\infty} P(Z_{n_k} \le t) = t/c$ for every $t \in [0,c]$. Notice that since $F_{X_{n_k}}(t)$ converges pointwise, and α_{n_k} converges, we know that $\lim_{k\to\infty} P(Z_{n_k} \le t)$ exists for every $t \in [0,c]$. Thus we can make do with finding a subsequence of $P(Z_{n_k} \le t)$ that converges to t/c for every $t \in [0,c]$.

  As we saw in Lemma 6, X_{n_k} is also ϵ_{n_k}-optimal in $\Lambda_{c^+,c}(a, c/2)$, and so for every $Y \in [0,c]$ with expectation $c/2$ we have,

$$1 - \frac{2(c-a)}{c} - \epsilon_{n_k} \le H(X_{n_k}, Y) = \alpha_{n_k} H(Z_{n_k}, Y) + 1 - \alpha_{n_k}.$$

  When remembering that $1/\alpha_{n_k} \le (c - z_{n_k})/(c-a)$ and $z_{n_k} \ge 0$, rearranging yields,

$$1 - \frac{2(c-z_{n_k})}{c} - \epsilon_{n_k} \le H(Z_{n_k}, Y).$$

  Let $Y = \beta 1_{t_1} + (1-\beta)1_{t_2}$, where $0 \le t_1 < c/2 < t_2 \le c$ and $\beta = (t_2 - c/2)/(t_2 - t_1)$. Denote $m_k = 1 - 2(c - z_{n_k})/c - \epsilon_{n_k}$, and $h_k(t) = H(Z_{n_k}, 1_t)$. Notice $\lim_{k\to\infty} m_k = 0$ since z_{n_k} converges to $c/2$. Furthermore,

$$h_k(t) = P(Z_{n_k} > t) + P(Z_{n_k} \ge t) - 1;$$

  thus $h_k(t)$ is a nonincreasing function, and $-1 \le h_k(t) \le 1$. Without loss of generality assume $h_k(t)$ converges pointwise (from Helly's Selection

Theorem we know that it has a pointwise converging subsequence, and as we mentioned before, we will make do with finding a subsequence of $P(Z_{n_k} \leq t)$ that converges pointwise to t/c).

Using the same ideas of Lemma 3, we conclude that there exists a real constant λ_k with

$$\frac{H(Z_{n_k}, 1_{t_1}) - m_k}{t_1 - \frac{c}{2}} \leq \lambda_k \leq \frac{H(Z_{n_k}, 1_{t_1}) - m_k}{t_2 - \frac{c}{2}}. \tag{4.7}$$

We obtain,

$$\lambda_k \left(t - \frac{c}{2} \right) + m_k \leq h_k(t)$$

for all $t \in [0, c]$ (for $t = c/2$ we get the above by letting player B play $Y = 1_{c/2}$).

Notice that (4.7) guarantees that the sequence λ_k is blocked, thus has a converging subsequence. For convenience we look at λ_k, and let λ be its limit. For every $k \in \mathbb{N}$ define

$$g_k(t) := \lambda_k \left(t - \frac{c}{2} \right) + m_k,$$

and for every $t \in [0, c]$,

$$G(t) := \lim_{k \to \infty} g_k(t) = \lambda \left(t - \frac{c}{2} \right).$$

Since $g_k(t) \leq h_k(t)$ for every $t \in [0, c]$, we also have $G(t) \leq \lim_{k \to \infty} h_k(t)$. By the dominated convergence theorem and (2.1),

$$0 = \int_0^c G(t)dt \leq \int_0^c \lim_{k \to \infty} h_k(t)dt = \lim_{k \to \infty} \int_0^c h_k(t)dt = \lim_{k \to \infty} (2z_{n_k} - c) = 0,$$

thus $G(t) = \lim_{k \to \infty} h_k(t)$ for almost every $t \in [0, c]$. Since $\lim_{k \to \infty} h_k(t)$ is a nonincreasing function (as a pointwise limit on nonincreasing functions), $G(t)$ is nonincreasing, and $\lambda \leq 0$.

Since $G(t) \leq \lim_{k \to \infty} h_k(t)$ for every $t \in [0, c]$, $G(t) = \lim_{k \to \infty} h_k(t)$ for almost every $t \in [0, c]$, both are nonincreasing, and $G(t)$ is continuous, we conclude that $G(t) = \lim_{k \to \infty} h_k(t)$ for every $0 < t \leq c$.

lemma 8.

$$\lambda = \frac{-2}{c}.$$

Proof. Denote $p_k = P(Z_{n_k} = c)$. By the definition of $h_k(t)$ we have that $h_k(c) = p_k - 1$. Since $h_k(t)$ converges for every $t \in [0, c]$, we conclude that $\lim_{k \to \infty} p_k$ exists and denote in by p. $\lim_{k \to \infty} h_k(t)$ is left continuous at c, and so

$$p - 1 = \lim_{k \to \infty} p_k - 1 = \lim_{k \to \infty} h_k(c) = \lim_{s \to 0} \lim_{k \to \infty} h_k(c - s) =$$

$$\lim_{k \to \infty} \lim_{s \to 0} h_k(c - s) = \lim_{k \to \infty} 2p_k - 1 = 2p - 1$$

(it is easy to verify that indeed the order of the limits can be changed). We obtain $p = 0$, and since $\lambda(c/2) = G(c) = \lim_{k \to \infty} h_k(c) = -1$, we get $\lambda = -1/(c/2)$. □

Since $G(t) = \lim_{k\to\infty} h_k(t)$ for every $0 < t \le c$, we obtain

$$G(t) = \lim_{k\to\infty} h_k(t) = 1 - \frac{t}{c/2}.$$

Notice,

$$2P(Z_{n_k} > t) - 1 \le h_k(t) \le 2P(Z_{n_k} \ge t) - 1.$$

Taking the limit $k \to \infty$ gives us $P(Z_{n_k} \le t) \to t/c$ for every $0 < t \le c$. For $t = 0$ we obtain $P(Z_{n_k} \le 0) \to 0$ out of monotonicity considerations. □

Now that we have proven Lemma 7, i.e., $\lim_{k\to\infty} F_{X_{n_k}}(t) = F(t) = t/c$ for every pointwise converging subsequence of $F_{X_n}(t)$, we would like to show that for every $t \in [0, c]$

$$\lim_{n\to\infty} F_{X_n}(t) = F(t) = \frac{t}{c}.$$

Let $t_0 \in \mathbb{R}$. The sequence $F_{X_n}(t_0)$ is blocked and thus has a converging subsequence $F_{X_{n_k}}(t_0)$. From Helly's Selection Theorem the sequence $F_{X_{n_k}}(t)$ has a pointwise converging subsequence $F_{X_{n_{k_l}}}(t)$. Lemma 7 guarantees that $F_{X_{n_{k_l}}}(t)$ converges to $F(t)$ for every t. Thus $F_{X_{n_k}}(t_0)$ converges to $F(t_0)$. This is true for every converging subsequence of $F_{X_n}(t_0)$, and so $\lim_{n\to\infty} F_{X_n}(t_0) = F(t_0)$. □

theorem 9. *Assume $0 < b = c/2 < a < c$, and let X_n be a sequence of ϵ_n-optimal strategies with of player A $\lim_{n\to\infty} \epsilon_n = 0$. Then for every pointwise converging subsequence $F_{X_{n_k}}$ of F_{X_n}, it holds that $\lim_{k\to\infty} F_{X_{n_k}}(t) = F_{z_0}(t)$, where $z_0 \in [0, c/2]$ and*[9]

$$F_{z_0}(t) = \begin{cases} 0 & t < 0 \\ \in \left[0, \frac{c-a}{c-z_0}\left(1 - \frac{2z_0}{c}\right)\right] & t = 0 \\ \frac{c-a}{c-z_0}\left(1 - \frac{2z_0}{c} + \frac{2z_0}{c}\frac{t}{c}\right) & 0 < t \le c \\ 1 & c < t. \end{cases}$$

Proof. Express $X_n = \alpha_n Z_n + (1-\alpha_n)1_{c+\delta_n}$, where $Z_n \in [0, c]$, $E(Z_n) = z_n$, $\delta_n > 0$ and $\alpha_n = (c + \delta_n - a)/(c + \delta_n - z_n)$. For every Y of player B we have,

$$1 - \frac{2(c-a)}{c} - \epsilon_n \le H(X_n, Y) = \alpha_n H(Z_n, Y) + 1 - \alpha_n.$$

Rearranging this equation and using $\alpha_n \ge (c-a)/(c-z_n)$ and $z_n \ge 0$, yields

$$H(Z_n, Y) \ge 1 - \frac{2(c-a)}{c}\frac{1}{\alpha_n} - \epsilon_n \ge 1 - \frac{2(c-z_n)}{c} - \epsilon_n. \tag{4.8}$$

(Notice we omit the constants that multiplies ϵ_n). Since $b = c/2$, we have

$$1 - \frac{2(c-z)}{c} \ge \frac{z - \frac{c}{2}}{\max\{z, \frac{c}{2}\}}. \tag{4.9}$$

By (4.8) and (4.9), we conclude that Z_n is an ϵ_n-optimal strategy in $\Lambda_{c,c}(z_n, c/2)$.

[9]We write "$\in [\alpha, \beta]$" to mean that the value of F_{z_0} there must belong to the interval $[\alpha, \beta]$; in Lemma 10 we will show that in fact every $\gamma \in [\alpha, \beta]$ and every $z_0 \in [0, c/2]$ are attained for some subsequence.

lemma 9.

$$0 \le \liminf_{n\to\infty} z_n \le \limsup_{n\to\infty} z_n \le \frac{c}{2}.$$

Proof. Let z_{n_k} be a subsequence of z_n with $\lim_{k\to\infty} z_{n_k} = z_0$. Since $z_n \ge 0$ always, we have $z_0 \ge 0$. It remains to show that $z_0 \le c/2$.

Assume the opposite, i.e., $z_0 > c/2$. There exists some large enough K with $z_{n_k} > c/2$ for every $k > K$. Since Z_{n_k} is an ϵ_{n_k}-optimal strategy in $\Lambda_{c,c}(z_{n_k}, c/2)$, by letting player B play optimally we obtain,

$$1 - \frac{2(c - z_{n_k})}{c} - \epsilon_{n_k} \le H(Z_{n_k}, Y) \le 1 - \frac{c}{2z_{n_k}}.$$

Thus,

$$0 \le \frac{c/2}{z_{n_k}} - \frac{2(c - z_{n_k})}{c} \le \epsilon_{n_k} \xrightarrow[k\to\infty]{} 0,$$

a contradiction. Thus $z_0 \le c/2$. □

For convenience assume $F_{X_n}(t)$ converges pointwise to $F(t)$. Let z_{n_k} be a converging subsequence of z_n, and let $z_0 \in [0, c/2]$ be its limit. Observe that $F_{X_{n_k}}(t)$ also converges pointwise to $F(t)$. We wish to show that $F(t) = F_{z_0}(t)$, i.e.,

$$F(t) = \begin{cases} 0 & t < 0 \\ \in \left[0, \frac{c-a}{c-z_0}\left(1 - \frac{2z_0}{c}\right)\right] & t = 0 \\ \frac{c-a}{c-z_0}\left(1 - \frac{2z_0}{c} + \frac{2z_0}{c}\frac{t}{c}\right) & 0 < t \le c \\ 1 & c < t. \end{cases}$$

Using (4.8) and letting B play optimally yields,

$$1 - \frac{2(c - z_{n_k})}{c} - \epsilon_{n_k} \le 1 - \frac{2(c-a)}{c}\frac{1}{\alpha_{n_k}} - \epsilon_{n_k} \le \frac{z_{n_k} - \frac{c}{2}}{\max\{z_{n_k}, \frac{c}{2}\}}.$$

When taking $k \to \infty$, we have $z_{n_k} \to z_0$ and $\epsilon_{n_k} \to 0$, thus

$$\alpha_{n_k} \longrightarrow \frac{c-a}{c-z_0},$$

$$\delta_{n_k} \longrightarrow 0.$$

Now,

- When $t < 0$,

$$F_{X_{n_k}}(t) = P(X_{n_k} \le t) = 0 \xrightarrow[k\to\infty]{} 0.$$

- When $c < t$,

$$\begin{aligned} F_{X_{n_k}}(t) = P(X_{n_k} \le t) &= \alpha_{n_k} P(Z_{n_k} \le t) + (1 - \alpha_{n_k}) P(c + \delta_{n_k} \le t) \\ &= \alpha_{n_k} + (1 - \alpha_{n_k}) P(c + \delta_{n_k} \le t) \xrightarrow[k\to\infty]{} 1. \end{aligned}$$

  The latter is true since α_{n_k} converges, and $P(c + \delta_{n_k} \le t)$ converges to 1 (Lemma 4).

- When $0 < t \le c$, since $F_{X_{n_k}}(t) = \alpha_{n_k} F_{Z_{n_k}}(t)$, and $\alpha_{n_k} \to (c-a)/(c-z_0)$, it is sufficient to show that

$$F_{Z_{n_k}}(t) \xrightarrow[k\to\infty]{} 1 - \frac{2z_0}{c} + \frac{2z_0}{c}\frac{t}{c}.$$

All we need to do is notice that Z_{n_k} is ϵ_{n_k}-optimal for player A in $\Lambda_{c,c}(z_{n_k}, c/2)$, and repeat the same methods of Lemma 7.

- When $t = 0$, out of monotonicity considerations we understand that

$$\lim_{k\to\infty} F_{X_{n_k}}(0^-) \le \lim_{k\to\infty} F_{X_{n_k}}(0) \le \lim_{k\to\infty} F_{X_{n_k}}(0^+),$$

thus

$$0 \le \lim_{k\to\infty} F_{X_{n_k}}(0) \le \frac{c-a}{c-z_0}\left(1 - \frac{2z_0}{c}\right).$$

□

lemma 10. *For every $z_0 \in [0, c/2]$ and every*

$$y_0 \in \left[0, \frac{c-a}{c-z_0}\left(1 - \frac{2z_0}{c}\right)\right],$$

there exists an ϵ-optimal strategy X^* with $\lim_{\epsilon\to 0} F_{X^*}(0) = y_0$.

Proof. Let $z_0 \in [0, c/2]$ and let

$$y_0 \in \left[0, \frac{c-a}{c-z_0}\left(1 - \frac{2z_0}{c}\right)\right].$$

There exists $0 \le \beta \le 1$ with $y_0 = \beta 0 + (1-\beta)(c-a)(c-z_0)(1-2z_0/c)$.

When $z_0 > 0$, define

$$X_1 = \frac{c+\delta-a}{c+\delta-z_0}\left(\left(1 - \frac{2z_0}{c}\right)1_0 + \frac{2z_0}{c}U_{[0,c]}\right) + \frac{a-z_0}{c+\delta-z_0}1_{c+\delta},$$

$$X_2 = \frac{c+\delta-a}{c+\delta-z_0}\left(\frac{c-2z_0}{c-2\delta}1_\delta + \frac{2z_0-2\delta}{c-2\delta}U_{[0,c]}\right) + \frac{a-z_0}{c+\delta-z_0}1_{c+\delta}.$$

We claim that these are ϵ-optimal strategies for player A when δ is small enough.

For X_1, take

$$0 < \delta < \epsilon\frac{c}{2}\frac{c-z_0}{a-z_0}.$$

Then for every Y of player B we have,

$$H(X_1, Y) = \frac{c+\delta-a}{c+\delta-z_0}H(\left(1 - \frac{2z_0}{c}\right)1_0 + \frac{2z_0}{c}U_{[0,c]}, Y) + \frac{a-z_0}{c+\delta-z_0}H(1_{c+\delta}, Y)$$

$$\ge \frac{c+\delta-a}{c+\delta-z_0}\left(1 - 2\frac{c-z_0}{c}\right) + \frac{a-z_0}{c+\delta-z_0} \ge 1 - 2\frac{c-a}{c} - \epsilon.$$

(The first inequality holds since $(1-2z_0/c)1_0 + (2z_0/c)U_{[0,c]}$ is optimal for player A in $\Lambda_{c,c}(z_0, c/2)$ when $z_0 \le c/2$, and the second inequality holds due to the choice of δ).

Showing that X_2 is an ϵ-optimal strategy for player A is not much more complex. It relies on the fact that $(c-2z_0)/(c-2\delta)1_\delta + (2z_0-2\delta)/(c-2\delta)U_{[0,c]}$ is an ϵ-optimal strategy for player A in $\Lambda_{c,c}(z_0, c/2)$ when $z_0 \le c/2$. (Though our choice of δ will be smaller).

Now,

$$\lim_{\epsilon\to 0} F_{X_1}(0) = \frac{c-a}{c-z_0}\left(1-\frac{2z_0}{c}\right),$$

$$\lim_{\epsilon\to 0} F_{X_2}(0) = 0.$$

Define $X^* = \beta X_2 + (1-\beta)X_1$. It holds that X^* is an ϵ-optimal strategy of player A as a convex combination of ϵ-optimal strategies of player A, and that $\lim_{\epsilon\to 0} F_{X^*}(0) = y_0$.

For the case of $z_0 = 0$ we consider the ϵ-optimal strategies

$$X_1 = \left(1-\frac{a}{c+\delta}\right)1_0 + \frac{a}{c+\delta}1_{c+\delta},$$

and

$$X_2 = \left(1-\frac{a-\delta}{c}\right)1_\delta + \frac{a-\delta}{c}1_{c+\delta}.$$

□

theorem 10. *Assume $0 < b \le c/2 < a = c$, and let X_n be a sequence of ϵ_n-optimal strategies of player A with $\lim_{n\to\infty}\epsilon_n = 0$. Then*

$$\lim_{n\to\infty} F_{X_n}(t) = \begin{cases} 0 & t \le c \\ 1 & c < t. \end{cases}$$

Proof. Express $X_n = \alpha_n Z_n + (1-\alpha_n)1_{c+\delta_n}$, where $Z_n \in [0,c]$, $E(Z_n) = z_n$, $\delta_n > 0$ and $\alpha_n = \delta_n/(c+\delta_n-z_n)$. For every Y of player B we have,

$$1-\epsilon_n \le H(X_n, Y).$$

In particular, for $Y = (1-b/c)1_0 + (b/c)1_c$ we have,

$$1-\epsilon_n \le \left(1-\frac{b}{c}\right)H(X_n, 1_0) + \frac{b}{c}H(X_n, 1_c).$$

$H(*,*) \le 1$, thus rearranging and omitting constants that multiply ϵ_n yield

$$1-\epsilon_n \le H(X_n, 1_c).$$

Now,

$$1-\epsilon_n \le H(X_n, 1_c) = P(X_n > c) - P(X_n < c),$$

and so,

$$-\epsilon_n \le \alpha_n - \epsilon_n = 1 - P(X_n > c) - \epsilon_n \le -P(X_n < c) \le 0.$$

When $n\to\infty$, we have $\epsilon_n \to 0$, thus

$$\lim_{n\to\infty} \alpha_n = 0,$$

and so,

$$\lim_{n\to\infty} \delta_n = 0.$$

Let

$$F(t) = \begin{cases} 0 & t \le c \\ 1 & c < t. \end{cases}$$

We now show that $\lim_{n\to\infty} F_{X_n}(t) = F(t)$ for every t.

- When $t < 0$,

$$F_{X_n}(t) = P(X_n \le t) = 0 \xrightarrow[n\to\infty]{} 0.$$

- When $0 \le t \le c$,

$$0 \le F_{X_n}(t) = P(X_n \le t) = \alpha_n P(Z_n \le t) \le \alpha_n \xrightarrow[n\to\infty]{} 0.$$

- When $c < t$,

$$F_{X_n}(t) = P(X_n \le t) = \alpha_n + (1-\alpha_n)P(c+\delta_n \le t) \xrightarrow[n\to\infty]{} 1.$$

  (This is an outcome of $\delta_n \to 0$ when $n \to \infty$, and Lemma 4).

□

4.3 Case *(v)*: $c/2 < b \le c$ and $0 < a \le c$.

We begin this case of Theorem 4 by characterizing some of player B's optimal strategies (Theorem 11 and Lemma 11).

theorem 11. *In the $\Lambda_{c^+,c}(a,b)$ game, where $c/2 < b$ and $0 < a \le c$, Player B's set of optimal strategies contains*[10] *$conv\{\Omega\}$, where*

$$\Omega = \left\{ Y^w = \beta W + (1-\beta)1_c \,\middle|\, W = \frac{c-w}{w}U_{[0,2(c-w)]} + \frac{2w-c}{w}1_c,\ w \in [\frac{c}{2}, b],\ \beta = \frac{c-b}{c-w} \right\}.$$

Proof. Recall from Theorem 4 that the value of the game is $\text{val}\Lambda_{c^+,c}(a,b) = 2a/c - 1$, and that $Y^b = (c-b)/bU_{[0,2(c-b)]} + (2b-c)/b1_c$ is an optimal strategy of player B.

Let $Y^w \in \Omega$ with $w \in [c/2, b]$. If $w > c/2$, then, as for Y^b, $W = (c-w)/wU_{[0,2(c-w)]} + (2w-c)/c1_c$ is an optimal strategy of player B in $\Lambda_{c^+,c}(a,w)$, and thus $H(X,W) \le 2a/c - 1$. If $w = c/2$, then $W = U_{[0,c]}$ and $H(X,W) \le 2a/c - 1$, by Lemma 1. Thus $H(X,W) \le 2a/c - 1$ for every strategy X of player A, for all $w \in [\frac{c}{2}, b]$. Furthermore, notice that 1_c is an optimal strategy of player B in $\Lambda_{c^+,c}(a,c)$, and so again we obtain $H(X, 1_c) \le 2a/c - 1$ for every strategy X of player A.

Next, let X be some strategy of player A,

$$H(X,Y^w) = \beta H(X,W) + (1-\beta)H(X,1_c)$$
$$\le \beta\left(\frac{2a}{c} - 1\right) + (1-\beta)\left(\frac{2a}{c} - 1\right)$$

[10] $conv\{\Omega\}$ is the convex hull of the set Ω.

$$\leq \frac{2a}{c} - 1;$$

thus every $Y^w \in \Omega$ is an optimal strategy of player B, and so every strategy in $conv\{\Omega\}$. □

Unfortunately, in this last case we were unable to completely characterize player B's set of optimal strategies. However, the next lemma provides us with further knowledge of the optimal strategies of player B. It shows that an optimal strategy of player B is uniform in the sense that it is independent of player A's given expectation.

lemma 11. *If Y^0 is an optimal strategy of player B in $\Lambda_{c^+,c}(a,b)$, where $0 < a \leq c$ and $\frac{c}{2} < b$, then it is also optimal for player B in $\Lambda_{c^+,c}(t,b)$ for all $t \in [0,c]$.*

Proof. Let X be some strategy of player A in $\Lambda_{c^+,c}(t,b)$.

When $a \leq t$ we define the strategy $X^a = (1-\alpha)1_0 + \alpha X$ with $\alpha = a/t$.

$$1 - 2\frac{c-a}{c} \geq H(X^a, Y^0) = (1-\alpha)H(1_0, Y^0) + \alpha H(X, Y^0)$$

$$\geq (1-\alpha)(-1) + \alpha H(X, Y^0),$$

and so,

$$H(X, Y^0) \leq \frac{1}{\alpha}\left(\frac{2a}{c} - \alpha\right) = \frac{2t}{c} - 1 = 1 - 2\frac{c-t}{c}.$$

Hence Y^0 is an optimal strategy of player B in $\Lambda_{c^+,c}(t,b)$.

When $t \leq a$ we define $X^a = \alpha X + (1-\alpha)1_{c+\delta}$ with $\alpha = (c+\delta-a)/(c+\delta-t)$, and $\delta > 0$.

$$1 - 2\frac{c-a}{c} \geq H(X^a, Y^0) = \frac{c+\delta-a}{c+\delta-t}H(X, Y^0) + \left(1 - \frac{c+\delta-a}{c+\delta-t}\right)H(1_{c+\delta}, Y^0).$$

This is true for every $\delta > 0$, thus

$$1 - 2\frac{c-a}{c} \geq \frac{c-a}{c-t}H(X, Y^0) + 1 - \frac{c-a}{c-t},$$

and so,

$$1 - 2\frac{c-t}{c} \geq H(X, Y^0).$$

Thus, again, Y^0 is an optimal strategy of player B in $\Lambda_{c^+,c}(t,b)$. □

We now turn to player A. We first show that player A has no optimal strategies in this case (Theorem 12). However, given a sequence of ϵ_n-optimal strategies X_n with $\lim_{n\to\infty} \epsilon_n = 0$, we characterize the pointwise limits of its CDFs'. We divide the discussion for player A into two parts:

- $0 < a < c$ and $c/2 < b \leq c$ (Theorem 13).
- $0 < a = c$ and $c/2 < b \leq c$ (Theorem 14).

theorem 12. *When $0 < a \leq c$ and $c/2 < b \leq c$, player A has no optimal strategies.*

Proof. Assume the opposite, i.e., that player A has an optimal strategy X^*. When $a > 0$ and $b > c/2$, it is easy to verify that

$$\frac{a-b}{\max\{a,b\}} < 1 - \frac{2(c-a)}{c}. \tag{4.10}$$

If $X^* \in [0,c]$, then according to (4.10) in $\Lambda_{c,c}(a,b)$ it guarantees a greater value than the value of the game - a contradiction. Thus we can express $X^* = \alpha Z + (1-\alpha)1_{c+\delta}$, where $Z \in [0,c]$, $E(Z) = z$, $\alpha = (c+\delta-a)/(c+\delta-z) < 1$ and $\delta > 0$. By arguments that we have seen before we can obtain,

$$1 - \frac{2(c-z)}{c} \leq 1 - \frac{2(c-a)}{c}\frac{c+\delta-z}{c+\delta-a} \leq H(Z,Y) \tag{4.11}$$

for every Y of player B, and in particular an optimal strategy of player B in $\Lambda_{c,c}(z,b)$. Thus,

$$1 - \frac{2(c-z)}{c} \leq \frac{z-b}{\max\{z,b\}},$$

which is a contradiction to (4.10), unless $z = 0$.

Assume $z = 0$. Then $Z = 1_0$. Using a strategy of player B Y with $H(1_0, Y) = -1$ in (4.11) yields

$$1 - \frac{2(c-a)}{c}\frac{c+\delta}{c+\delta-a} \leq -1,$$

which is equivalent to $a \leq 0$ - a contradiction. Thus player A has no optimal strategies. ☐

theorem 13. *Assume $0 < a < c$ and $c/2 < b \leq c$, and let X_n be a sequence of ϵ_n-optimal strategies of player A with $\lim_{n\to\infty} \epsilon_n = 0$. Then*[11]

$$\lim_{n\to\infty} F_{X_n}(t) = \begin{cases} 0 & t < 0 \\ \in \left[0, 1-\frac{a}{c}\right] & t = 0 \\ 1 - \frac{a}{c} & 0 < t \leq c \\ 1 & c < t. \end{cases}$$

Proof. Express $X_n = \alpha_n Z_n + (1-\alpha_n)1_{c+\delta_n}$, where $Z_n \in [0,c]$, $E(Z_n) = z_n$, $\delta_n > 0$ and $\alpha_n = (c+\delta_n-a)/(c+\delta_n-z_n)$. For every Y of player B we have,

$$1 - \frac{2(c-a)}{c} - \epsilon_n \leq H(X_n, Y).$$

By rearranging and using an optimal strategy of player B and other repeated arguments we obtain,

$$1 - \frac{2(c-z_n)}{c} - \epsilon_n \leq 1 - \frac{2(c-a)}{c}\frac{1}{\alpha_n} - \epsilon_n \leq \frac{z_n - b}{\max\{z_0, b\}}. \tag{4.12}$$

Let z_{n_k} be a converging subsequence of z_n, and let z_0 be its limit. By (4.12) we have,

$$1 - \frac{2(c-z_0)}{c} \leq \frac{z_0 - b}{\max\{z_0, b\}},$$

[11] As in Lemma 10, we will see in Remark 5 that every $y \in [0, 1-a/c]$ is attained for some subsequence.

which is possible only when $z_0 = 0$ (since $b > c/2$). Since z_{n_k} is an arbitrary converging subsequence of z_n, we conclude that z_n converges to 0.

Now, taking $n \to \infty$ in (4.12) yields

$$\alpha_n \xrightarrow[n\to\infty]{} 1 - \frac{a}{c},$$

and so,

$$\delta_n \xrightarrow[n\to\infty]{} 0.$$

Let

$$F(t) = \begin{cases} 0 & t < 0 \\ \in \left[0, 1 - \frac{a}{c}\right] & t = 0 \\ 1 - \frac{a}{c} & 0 < t \le c \\ 1 & c < t. \end{cases}$$

We claim that $\lim_{n\to\infty} F_{X_n}(t) = F(t)$ for every t.

- When $t < 0$,

$$F_{X_n}(t) = P(X_n \le t) = 0 \xrightarrow[n\to\infty]{} 0.$$

- When $0 < t \le c$,

$$F_{X_n}(t) = P(X_n \le t) = \alpha_n P(Z_n \le t)$$

  Lemma 4 gives us that $P(Z_n \le t)$ converges to 1, and since $\lim_{n\to\infty} \alpha_n = 1 - a/c$, we get,

$$F_{X_n}(t) \xrightarrow[n\to\infty]{} 1 - \frac{a}{c}.$$

- When $c < t$,

$$F_{X_n}(t) = P(X_n \le t) = \alpha_n + (1 - \alpha_n)P(c + \delta_n \le t) \xrightarrow[n\to\infty]{} 1.$$

  This is true thanks to $\lim_{n\to\infty} \alpha_n = 1 - a/c$, $\lim_{n\to\infty} \delta_n = 0$, and Lemma 4.

- When $t = 0$, monotonicity arguments yield

$$0 \le \lim_{n\to\infty} F_{X_n}(0) \le 1 - \frac{a}{c}.$$

□

remark 5. *As in Lemma 10, for every $y_0 \in [0, 1 - a/c]$ we can find an ϵ-optimal strategy of player A, X^*, with $\lim_{\epsilon\to 0} F_{X^*}(0) = y_0$. We do so by taking X^* to be a convex combination of two ϵ-optimal strategies, X_1 and X_2, with $\lim_{\epsilon\to 0} F_{X_1}(0) = 1 - a/c$ and $\lim_{\epsilon\to 0} F_{X_2}(0) = 0$. The above X_1 and X_2 can be*

$$X_1 = \left(1 - \frac{a}{c+\delta}\right) 1_0 + \frac{a}{c+\delta} 1_{c+\delta},$$

$$X_2 = \left(1 - \frac{a-\delta}{c}\right) 1_\delta + \frac{a-\delta}{c} 1_{c+\delta}.$$

theorem 14. *Assume $0 < a = c$ and $c/2 < b \leq c$, and let X_n be a sequence of ϵ_n-optimal strategies of player A with $\lim_{n\to\infty} \epsilon_n = 0$. Then*

$$\lim_{n\to\infty} F_{X_n}(t) = \begin{cases} 0 & t \leq c \\ 1 & c < t. \end{cases}$$

The proof of this theorem is identical to the proof of Theorem 10.

remark 6. *We would like to summarize this section with a notation. When player A does not have any optimal strategy, it holds that every ϵ-optimal strategy X of player A satisfies $P(X > c) > 0$. In such cases, we can formally write the limit X^* of ϵ_n-optimal strategies as follows.*

1. *When $0 < b < c/2 < a < c$,*

$$X^* = \frac{2c^+ - 2a}{2c^+ - c} U_{[0,c]} + \frac{2a - c}{2c^+ - c} 1_{c^+}.$$

2. *When $0 < b = c/2 < a < c$, X^* can be any convex combination of X_1 and X_2, where*

$$X_1 = \frac{c-a}{c-z_0}\left[\left(1 - \frac{2z_0}{c}\right) 1_0 + \frac{2z_0}{c} U_{[0,c]}\right] + \frac{a - z_0}{c - z_0} 1_{c^+},$$

$$X_2 = \frac{c-a}{c-z_0}\left[\left(1 - \frac{z_0 - 0^+}{\frac{c}{2} - 0^+}\right) 1_{0^+} + \frac{z_0 - 0^+}{\frac{c}{2} - 0^+} U_{[0,c]}\right] + \frac{a - z_0}{c - z_0} 1_{c^+},$$

and $0 \leq z_0 \leq c/2$.

3. *When $0 < a < c$ and $c/2 < b \leq c$, X^* can be any convex combination of X_1 and X_2, where*

$$X_1 = \left(1 - \frac{a}{c^+}\right) 1_0 + \frac{a}{c^+} 1_{c^+},$$

$$X_2 = \left(1 - \frac{a - 0^+}{c^+ - 0^+}\right) 1_{0^+} + \frac{a - 0^+}{c^+ - 0^+} 1_{c^+}.$$

4. *When $0 < b \leq c$ and $a = c$,*

$$X^* = \frac{0^+}{c^+ - z} Z + \frac{c - z}{c^+ - z} 1_{c^+}$$

for any $Z \leq c$ with $E(Z) = z$.

When replacing c^+ with $c + \delta$ and 0^+ with δ, all of the above strategies become ϵ-optimal (for a small enough δ)), and we obtain that indeed every limit we suggested in the theorems of this section is attained.

5 Unequal Caps: Additional Results

In this section we consider the case of strict constraints $\Lambda_{c_A^-,c_B^-}(a,b)$; i.e., a strategy X of player A must realize $X < c_A$ and a strategy Y of player B must realize $Y < c_B$. The case of $c_A = c_B = c$ is interesting since it includes cases in which the $\Lambda_{c^-,c^-}(a,b)$ game has no value (see Hart 2014, Proposition 3). Here we assume $c_A > c_B = c$. Since playing anything above $c_B = c$ has the same effect, the greatest value that player A uses is c. Thus, we may denote this game by $\Lambda_{c,c^-}(a,b)$, and indeed all the results of this section hold when $c_A = \infty$.

We first recall the result of Hart (2014).

theorem 15 (Hart 2014, Theorem 5). *Let $c_A \geq c_B = c > 0$, and let $0 \leq a \leq c_A$ and $0 \leq b < c$. Then*

$$val\Lambda_{c_A,c_B^-}(a,b) = val\Lambda_{c,c^-}(a,b) = val\Lambda_{c^+,c}(a,b),$$

which is given in Theorem 4 (Hart 2014, Theorem 4). Moreover, the optimal strategies of $\Lambda_{c_A,c_B}(a,b)$ given there yield optimal strategies in $\Lambda_{c_A,c_B^-}(a,b)$ with the following modifications:

1. *In all cases but the last we require that $a < c$, and, in case (vi): when $a \geq c$ any feasible X with values $\geq c$ is optimal, and any Y is optimal.*

2. *In cases (iv) and (v): the atom of X^* at c^+ is moved to c, and the atom of Y^* at c is moved to*[12] *c^-.*

In this section as before we assume $\min\{a,b\} > 0$, and we also add the assumption that $\max\{a,b\} < c$ (otherwise the game is trivial). We will present our results, however the proofs of all following theorems are somewhat similar to the proofs we have seen so far, or use the same methods, and thus we only give them in Appendix 5.

When $c \geq \max\{2a, 2b\}$, we divide our results into three:

1. $c > \max\{2a, 2b\}$ - Theorem 16.

2. $c/2 = a > b$ - Theorem 17.

3. $c/2 = b \geq a$ - Theorem 18.

Both players have a unique optimal strategy in the $\Lambda_{c^+,c}(a,b)$ game. When considering strict constraints the optimal strategies of this case, given by Theorem 15, remain unique as long as $b < c/2$ (Theorems 16, and 17). However, when $c/2 = b \geq a$, while player B's optimal strategy remains unique, player A no longer has a unique optimal strategy, but a complete set. We fully characterize it in Theorem 18.

theorem 16. *In the $\Lambda_{c,c^-}(a,b)$, where $c > \max\{2a, 2b\}$, unique optimal strategies are:*

$$X^* = \begin{cases} U_{[0,2a]} & a \geq b \\ (1 - \frac{a}{b})1_0 + \frac{a}{b}U_{[0,2b]} & b > a \end{cases}$$

[12] In case (v) Y^* is an ϵ-optimal strategy, where c^- stands for $c - \delta$ for a small enough $\delta > 0$.

for player A, and

$$Y^* = \begin{cases} (1-\frac{b}{a})1_0 + \frac{b}{a}U_{[0,2a]} & a \geq b \\ U_{[0,2b]} & b > a \end{cases}$$

for player B.

theorem 17. *In the $\Lambda_{c,c^-}(a,b)$ game, where $b < a = c/2$, unique optimal strategies are:*

$$X^* = U_{[0,2a]}$$

for player A, and

$$Y^* = (1-\frac{b}{a})1_0 + \frac{b}{a}U_{[0,2a]}$$

for player B.

theorem 18. *In the $\Lambda_{c,c^-}(a,b)$ game, where $a \leq b = c/2$, player B has a unique optimal strategy*

$$Y^* = U_{[0,c]},$$

and player A's set of all optimal strategies consists of

$$M = \left\{ X(z) = \frac{c-a}{c-z}\left[\left(1-\frac{2z}{c}\right)1_0 + \frac{2z}{c}U_{[0,c]}\right] + \frac{a-z}{c-z}1_c | 0 \leq z \leq a \right\}.$$

The remaining case $c < \max\{2a, 2b\}$, we divide into three parts:

1. $b = c/2 < a$ - Theorem 19.
2. $b < c/2 < a$ - Theorem 20.
3. $c/2 < b$ and $a < c$ - Theorem 21.

Note that if we use the notation given in Remark 6, then we can use the same logic of Theorem 15, and claim that optimal strategies for player A are obtained by moving atoms at c^+ to c, and 0^+ to 0; while ϵ-optimal strategies for player B are obtained by moving atoms at c to c^-. If player B's optimal strategy in $\Lambda_{c^+,c}(a,b)$ does not involve an atom at c, then this strategy is optimal for player B in $\Lambda_{c,c^-}(a,b)$. All of this is given with greater detail in the following Theorems.

theorem 19. *In the $\Lambda_{c,c^-}(a,b)$ game, where $b = c/2 < a < c$, player B has a unique optimal strategy*

$$Y^* = U_{[0,c]},$$

and player A's set of all optimal strategies consists of

$$M = \left\{ X(z) = \frac{c-a}{c-z}\left[\left(1-\frac{2z}{c}\right)1_0 + \frac{2z}{c}U_{[0,c]}\right] + \frac{a-z}{c-z}1_c | 0 \leq z \leq \frac{c}{2} \right\}.$$

theorem 20. *In the $\Lambda_{c,c^-}(a,b)$ game, where $b < c/2 < a$, unique optimal strategies are*

$$Y^* = \frac{2b}{c}U_{[0,c]} + (1-\frac{2b}{c})1_0$$

for player B, and

$$X^* = \frac{2a-c}{c}1_c + \frac{2c-2a}{c}U_{[0,c]}$$

for player A.

theorem 21. *In the $\Lambda_{c,c^-}(a,b)$ game, where $c/2 < b < c$ and $a < c$, player A has a unique optimal strategy*

$$X^* = \left(1 - \frac{a}{c}\right) 1_0 + \frac{a}{c} 1_c.$$

and player B's set of all ϵ-optimal strategies contains conv$\{\Omega\}$, where

$$\Omega = \left\{ Y(w) = \beta W + (1-\beta) 1_{c-\delta} | W = \frac{c-\delta-w}{w-\delta} U_{[0,2(c-w)]} + \frac{2w-c}{w-\delta} 1_{c-\delta},\ w \in [\frac{c}{2}, b],\ \beta = \frac{c-\delta-b}{c-\delta-w} \right\}.$$

Unfortunately, in this last theorem, as in the corresponding case of $\Lambda_{c^+,c}(a,b)$, we were unable to fully characterize the limit points of player B's ϵ-optimal strategies.

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A Appendix

In this appendix we prove Theorems 16, 17, 18, 19, 20, and 21. We begin with a lemma.

lemma 12. *If X is an optimal strategy for player A in $\Lambda_{c,c^-}(a,b)$ with $P(X = c) = 0$, and*

$$val\Lambda_{c,c}(a,b) \leq val\Lambda_{c,c^-}(a,b),$$

then X is an optimal strategy for player A in $\Lambda_{c,c}(a,b)$.

Proof. Let X^0 be an optimal strategy of player A in $\Lambda_{c,c^-}(a,b)$ with $P(X^0 = c) = 0$, and let Y be some strategy of player B in $\Lambda_{c,c}(a,b)$. Express $Y = \gamma W + (1-\gamma)1_c$, with $E(W) = w$ and $\gamma = (c-b)/(c-w)$.

If $\gamma = 1$, then optimality in $\Lambda_{c,c^-}(a,b)$ gives us,

$$\text{val}\Lambda_{c,c}(a,b) \leq \text{val}\Lambda_{c,c^-}(a,b) \leq H(X^0, Y).$$

Assume $\gamma < 1$. For every $\delta > 0$ small enough, define $Y_\delta = \gamma_\delta W + (1-\gamma_\delta)1_{c-\delta}$ with $\gamma_\delta = (c-b-\delta)/(c-w-\delta)$. By X^0's optimality,

$$\text{val}\Lambda_{c,c^-}(a,b) \leq H(X^0, Y_\delta) = \gamma_\delta H(X^0, W) + (1-\gamma_\delta)H(X^0, 1_{c-\delta})$$

$$\leq \gamma_\delta H(X^0, W) + (1-\gamma_\delta)[2P(X^0 \geq c-\delta) - 1].$$

Taking the $\delta \to 0^+$ limit, and remembering that $P(X^0 \geq t)$ is left continuous yields,

$$\text{val}\Lambda_{c,c^-}(a,b) \leq \gamma H(X^0, W) + (1-\gamma)[2P(X^0 \geq c) - 1] = H(X^0, Y)$$

(The last equality is given by Y's definition and by $P(X^0 = c) = 0$). Since

$$\text{val}\Lambda_{c,c}(a,b) \leq \text{val}\Lambda_{c,c^-}(a,b),$$

X^0 is an optimal strategy of player A in $\Lambda_{c,c}(a,b)$. □

A.1 Proof of Theorem 16

For both players, optimality follows from Theorem 15, and uniqueness follows from the same proof as in Hart (2008, Appendix).

A.2 Proof of Theorem 17

For both players optimality follows from Theorem 15. Y^*'s uniqueness derives from its uniqueness in $\Lambda_{2a,2a}(a,b)$. We are left with X^*'s uniqueness.

Let X^0 be an optimal strategy of player A. We express $X^0 = \alpha Z + (1-\alpha)1_c$, where $0 \leq Z < c$ with $E(Z) = z$, and $\alpha = (c-a)/(c-z)$. We wish to show that $\alpha = 1$. Assume $\alpha < 1$, and notice $\alpha < 1 \Leftrightarrow z < a$.

If $z < b$, then by Theorem 16 $Y = U_{[0,2b]}$ is an optimal strategy of player B in $\Lambda_{c,c^-}(z,b)$. Thus,

$$1 - \frac{b}{a} \leq H(X^0, U_{[0,2b]}) = 1 - \alpha + \alpha H(Z, U_{[0,2b]}) \leq 1 - \alpha + \alpha(\frac{z}{b} - 1)$$

$$(2 - \frac{z}{b})\alpha \leq \frac{b}{a}.$$

By substituting $\alpha = (c-a)/(c-z)$, $c = 2a$, and multiplying by $ab(c-z)$ we obtain: $(2b-z)a^2 \leq b^2(2a-z)$.

$$(2b-z)a^2 \leq b^2(2a-z) = b^2(2a-2b) + b^2(2b-z)$$

$$(2b-z)(a^2-b^2) \leq 2b^2(a-b)$$

$$(2b-z)(a+b) \leq 2b^2$$

$$(2b - z)a + 2b^2 - zb \leq 2b^2$$
$$2ab \leq z(a + b).$$

Since $z < b < a$, we have $2ab \leq z(a + b) < 2ab$ - a contradiction. We conclude that $z \geq b$.

Let $Y = \beta Z + (1 - \beta)1_0$ with $\beta = b/z$ (we use the same Z as in player A's strategy X^0). Y is indeed allowed for player B since $0 \leq Z < c$ and $z \geq b$. Now,

$$1 - \frac{b}{a} \leq H(X^0, Y) = \alpha(1 - \beta)H(Z, 1_0) + 1 - \alpha \leq \alpha(1 - \beta) + 1 - \alpha.$$

Rearranging and substituting $\alpha = (c - a)/(c - z)$, $\beta = b/z$ yields,

$$\frac{c - a}{c - z}\frac{b}{z} = \alpha\beta \leq \frac{b}{a},$$

and so,

$$(\frac{c}{2})^2 = (c - a)a \leq (c - z)z.$$

However, $z < a = \frac{c}{2}$, which leads to a contradiction. Thus, $\alpha = 1$ and X^0 satisfies $0 \leq X^0 < c$, i.e., $P(X^0 \geq c) = 0$.

Notice that

$$\text{val}\Lambda_{c,c^-}(a, b) = \text{val}\Lambda_{c,c}(a, b) = 1 - \frac{b}{a};$$

thus we can obtain by Lemma 12 that X^0 is optimal for player A in $\Lambda_{c,c}(a, b)$. Since X^* is the unique optimal strategy for player A in $\Lambda_{c,c}(a, b)$ (Theorem 3), we have $X^0 = X^*$.

A.3 Proof of Theorem 18

Denote $X(a) = X^* = (1 - 2a/c)1_0 + (2a/c)U_{[0,c]} \in M$. By Theorem 15 X^* and Y^* are optimal for players A and B, respectively. As is Theorem 17, Y^*'s uniqueness derives from its uniqueness in $\Lambda_{c,c}(a, c/2)$, and so we turn to prove that M is the set of all optimal strategies of player A.

Let $X(z) \in M$, and let Y be some strategy of player B.

$$H(X(z), Y) = \frac{c - a}{c - z}H((1 - \frac{2z}{c})1_0 + \frac{2z}{c}U_{[0,c]}, Y) + \frac{a - z}{c - z}H(1_c, Y). \quad \text{(A.1)}$$

Since $Y < c$ we have $H(1_c, Y) = 1$. Moreover, the strategy $(1 - 2z/c)1_0 + (2z/c)U_{[0,c]}$ is optimal for player A in $\Lambda_{c,c^-}(z, c/2)$, where $z \leq c/2$. Applying these to (A.1) yields,

$$H(X(z), Y) \geq \frac{c - a}{c - z}(\frac{2z}{c} - 1) + \frac{a - z}{c - z} = \frac{2a}{c} - 1.$$

Thus, every strategy $X(z) \in M$ is optimal for player A.

We now wish to show that every optimal strategy of player A equals $X(z) \in M$ for some $0 \leq z \leq a$. Note that for $a \leq b = c/2$,

$$\text{val}\Lambda_{c,c^-}(a, \frac{c}{2}) = \text{val}\Lambda_{c,c}(a, \frac{c}{2}) = \frac{2a}{c} - 1.$$

Let X^0 be some optimal strategy of player A. If $P(X^0 = c) = 0$, by Lemma 12 and Theorem 3 we obtain, $X^0 = (1 - 2a/c)1_0 + (2a/c)U_{[0,c]} = X(a) \in M$. If $P(X^0 = c) > 0$, express $X^0 = \alpha Z + (1-\alpha)1_c$ with $Z < c$, $E(Z) = z$ and $\alpha = (c-a)/(c-z)$. Since $P(X^0 = c) > 0$, we have $z < a$. Let Y be some strategy of player B.

$$1 - 2\frac{c-a}{c} \leq H(X^0, Y) = \alpha H(Z, Y) + 1 - \alpha.$$

Rearranging and substituting $\alpha = (c-a)/(c-z)$ yields,

$$1 - 2\frac{c-z}{c} \leq H(Z, Y);$$

thus Z is an optimal strategy of player A in $\Lambda_{c,c^-}(z, c/2)$ with $z < a \leq c/2$, and $P(Z = c) = 0$. By Lemma 12 and Theorem 3 we obtain, $Z = (1 - 2z/c)1_0 + (2z/c)U_{[o,c]}$, and so

$$X^0 = \frac{c-a}{c-z}\left[\left(1 - \frac{2z}{c}\right)1_0 + \frac{2z}{c}U_{[0,c]}\right] + \frac{a-z}{c-z}1_c = X(z) \in M,$$

and we are done.

A.4 Proof of Theorem 19

Denote $X^* = X(c/2) = (2c-2a)/cU_{[0,c]} + (2a-c)/c1_c$. The optimality of X^* and $Y^* = U_{[0,c]}$ for players A and B, respectively, derives from Theorem 15.

We turn to prove that Y^* is unique, and begin with a lemma.

lemma 13. *Let Y^0 be an optimal strategy of player B in $\Lambda_{c,c^-}(a, b)$, where $b \leq c/2 < a$. Then for every X of player A in $\Lambda_{c,c^-}(c/2, b)$ we have,*

$$H(X, Y^0) \leq 1 - \frac{2b}{c}.$$

Proof. Let Y^0 be an optimal strategy of player B in $\Lambda_{c,c^-}(a, b)$, where $b \leq c/2 < a$, and let X be some strategy of player A in $\Lambda_{c,c^-}(c/2, b)$. Define $X^a = \alpha X + (1-\alpha)1_c$ with $\alpha = 2(c-a)/c$, and so $E(X^a) = a$. Since Y^0 is optimal we have,

$$1 - 2\frac{c-a}{c}\frac{2b}{c} \geq H(X^a, Y^0) = \alpha H(X, Y^0) + 1 - \alpha.$$

Rearranging and substituting $\alpha = 2(c-a)/c$ yields,

$$H(X, Y^0) \leq 1 - \frac{2b}{c}.$$

□

Let Y^0 be an optimal strategy of player B. Using the above lemma gives us that Y^0 is an optimal strategy of player B in $\Lambda_{c,c^-}(c/2, c/2)$. According to Theorem 18, $Y^0 = U_{[0,c]} = Y^*$, and so Y^* is unique.

Turning to player A, let $X(z) \in M$. As we saw for $z = c/2$, $X(c/2) = X^*$ is an optimal strategy of player A. Assume $z < c/2$. According to Theorem 18 the strategy $(1-2z/c)1_0 + (2z/c)U_{[0,c]}$ is optimal for player A in $\Lambda_{c,c^-}(z, c/2)$; thus,

$$H(X(z), Y) = \frac{c-a}{c-z} H(\left(1 - \frac{2z}{c}\right) 1_0 + \frac{2z}{c} U_{[0,c]}, Y) + \frac{a-z}{c-z} H(1_c, Y)$$

$$\geq \frac{c-a}{c-z}(\frac{2z}{c} - 1) + \frac{a-z}{c-z} = \frac{2a}{c} - 1.$$

And so, every strategy $X(z) \in M$ is optimal.

Now, let X^0 be some optimal strategy of player A. We wish to show that $X^0 = X(z)$ for some $0 \leq z \leq c/2$.

lemma 14. *An optimal strategy of player A in $\Lambda_{c,c^-}(a, c/2)$, where $c/2 < a < c$, must have an atom at c, i.e., $P(X^0 = c) > 0$.*

Proof. Assume $P(X^0 = c) = 0$. Note that

$$\text{val}\Lambda_{c,c}(a, \frac{c}{2}) = \text{val}\Lambda_{c,c^-}(a, \frac{c}{2}) = 1 - 2\frac{c-a}{c}.$$

By Lemma 12 we obtain that X^0 is also optimal for player A in $\Lambda_{c,c}(a, c/2)$, where $c/2 < a < c$. By Theorem 3, $X^0 = (c-a)/aU_{[0,2(c-a)]} + (2a-c)/a1_c$, which means that $P(X^0 = c) > 0$, a contradiction. □

Express $X^0 = \alpha Z + (1-\alpha)1_c$ with $Z < c$, $E(Z) = z$ and $\alpha = (c-a)/(c-z)$. From the above lemma $\alpha < 1$, which is equivalent to $z < a$.

Let Y be some strategy of player B. Substituting $\alpha = (c-a)/(c-z)$ in

$$1 - 2\frac{c-a}{a} \leq H(X^0, Y) = \alpha H(Z, Y) + 1 - \alpha,$$

and rearranging yields,

$$1 - 2\frac{c-z}{c} \leq H(Z, Y);$$

thus Z is optimal for player A in $\Lambda_{c,c^-}(z, c/2)$.

Note that $Z < c$. If $c/2 < z$, Lemma 14 guarantees that $P(Z = c) > 0$ - a contradiction. Thus $z \leq c/2$. According to Theorem 18, since Z is optimal for player A, it holds that

$$Z = \frac{c-z}{c-t}\left[\left(1 - \frac{2t}{c}\right) 1_0 + \frac{2t}{c} U_{[0,c]}\right] + \frac{z-t}{c-t} 1_c,$$

for some $t \in [0, z]$. However, Z has no atom at c, and so $t = z$, and

$$Z = \left(1 - \frac{2z}{c}\right) 1_0 + \frac{2z}{c} U_{[0,c]};$$

thus

$$X^0 = \frac{c-a}{c-z}\left[\left(1 - \frac{2z}{c}\right) 1_0 + \frac{2z}{c} U_{[0,c]}\right] + \frac{a-z}{c-z} 1_c = X(z)$$

with $0 \leq z \leq c/2$. $X^0 = X(z) \in M$ and we are done.

A.5 Proof of Theorem 20

Optimality of X^* and Y^* for players A and B, respectively, derives from Theorem 15.

We turn to prove that Y^* is unique. Let Y^0 be an optimal strategy of player B in $\Lambda_{c,c^-}(a,b)$, where $b < c/2 < a$. By Lemma 13, Y^0 is an optimal strategy of player B in $\Lambda_{c,c^-}(c/2,b)$. Since $b < c/2$, we can use Theorem 17; thus $Y^0 = (1-2b/c)1_0 + (2b/c)U_{[0,c]} = Y^*$, and Y^* is unique.

Turning to X^*'s uniqueness, let X^0 be an optimal strategy of player A in $\Lambda_{c,c^-}(a,b)$, where $b < c/2 < a < c$, and let Y be some strategy of player B in $\Lambda_{c,c^-}(a,c/2)$. Define $Y^b = (1-2b/c)1_0 + (2b/c)Y$.

$$1 - 2\frac{c-a}{c}\frac{2b}{c} \le H(X^0, Y^b) = (1 - \frac{2b}{c})H(X^0, 1_0) + \frac{2b}{c}H(X^0, Y)$$

$$\le 1 - \frac{2b}{c} + \frac{2b}{c}H(X^0, Y),$$

and so,

$$1 - 2\frac{c-a}{c} \le H(X^0, Y);$$

thus X^0 is optimal for player A in $\Lambda_{c,c^-}(a,c/2)$, and according to Theorem 19:

$$X^0 = X(z) = \frac{c-a}{c-z}\left[\left(1 - \frac{2z}{c}\right)1_0 + \frac{2z}{c}U_{[0,c]}\right] + \frac{a-z}{c-z}1_c \qquad \text{(A.2)}$$

for some $z \in [0, \frac{c}{2}]$.

claim 3. $P(X^0 = 0) = 0$.

Proof. Express $X^0 = (1-\alpha)1_0 + \alpha W$, with $W > 0$, $E(W) = w$ and $\alpha = a/w$. We wish to show that $\alpha = 1$. Assume the opposite, i.e., $\alpha < 1$.

$$1 - 2\frac{c-a}{c}\frac{2b}{c} \le H(X^0, Y^*) = (1-\alpha)\frac{2b}{c}(-1) + (1 - \frac{2b}{c})\alpha + \alpha\frac{2b}{c}H(W, U_{[0,c]}).$$

Rearranging yields,

$$\frac{c}{2b}(1-\alpha) + (\frac{2a}{c} - 1) \le \alpha H(W, U_{[0,c]}).$$

However, $H(W, U_{[0,c]}) \le 2w/c - 1$ (Lemma 1); thus,

$$\frac{c}{2b}(1-\alpha) + (\frac{2a}{c} - 1) \le \alpha(\frac{2w}{c} - 1) = \frac{2a}{c} - \alpha,$$

which yields,

$$\frac{c}{2b}(1-\alpha) \le 1 - \alpha.$$

Since $\alpha < 1$, we obtain $c/2 \le b$ - a contradiction. □

Now, using the fact that $P(X^0 = 0) = 0$ together with (A.2) gives us $z = c/2$. Thus $X^0 = X(c/2) = X^*$, and we are done.

A.6 Proof of Theorem 21

The optimality of X^* and ϵ-optimality of $Y^* = Y(b)$ derives from Theorem 15.

We begin with X^*'s uniqueness. Let X^0 be some optimal strategy of player A, and express $X^0 = \alpha Z + (1-\alpha)1_c$, with $Z < c$, $E(Z) = z$ and $\alpha = (c-a)/(c-z)$. It is sufficient to show that $z = 0$. Assume the opposite, i.e., $z > 0$.

Let Y be some strategy of player B. Then,

$$1 - 2\frac{c-a}{c} \leq H(X^0, Y) = \alpha H(Z, Y) + 1 - \alpha.$$

Rearranging and substituting $\alpha = (c-a)/(c-z)$ yields,

$$1 - 2\frac{c-z}{c} \leq H(Z, Y);$$

thus Z is an optimal strategy of player A in $\Lambda_{c,c^-}(z,b)$, where $c/2 < b$ and $z > 0$.

Note that in this case we have the following **strong inequality**,

$$\text{val}\Lambda_{c,c}(z,b) = \frac{z-b}{\max\{z,b\}} < 1 - 2\frac{c-z}{c} = \text{val}\Lambda_{c,c^-}(z,b).$$

Since $Z < c$, repeating the steps of Lemma 12 gives us that for every strategy Y of player B in $\Lambda_{c,c}(z,b)$,

$$\text{val}\Lambda_{c,c}(z,b) = \frac{z-b}{\max\{z,b\}} < H(Z, Y).$$

Meaning Z, a feasible strategy for player A in $\Lambda_{c,c}(z,b)$, guarantees a greater value than the value of the game - a contradiction. We conclude that $z = 0$, and $X^0 = X^*$.

We turn to player B. Since a convex combination of ϵ-optimal strategies is ϵ-optimal, it is sufficient to show that every strategy in Ω is ϵ-optimal.

We know that for $c/2 < b$, the strategy

$$Y^* = Y(b) = \frac{c-\delta-b}{b-\delta} U_{[0,2(c-b)]} + \frac{2b-c}{b-\delta} 1_{c-\delta}$$

is ϵ-optimal for player B in $\Lambda_{c,c^-}(a,b)$, and $val\Lambda_{c,c^-}(a,b) = 2a/c - 1$. We conclude that for every $w \in (c/2, b]$, the strategy

$$W = \frac{c-\delta-w}{w-\delta} U_{[0,2(c-w)]} + \frac{2w-c}{w-\delta} 1_{c-\delta}$$

is ϵ-optimal for player B in $\Lambda_{c,c^-}(a,w)$, and $val\Lambda_{c,c^-}(a,w) = \frac{2a}{c} - 1$. Using Theorems 18 and 19, we may add $w = c/2$ to this conclusion (as a matter of fact when $w = c/2$, W is optimal). Moreover, for a small enough $\delta > 0$, we can use Markov's inequality and obtain,

$$P(X \geq c-\delta) \leq \frac{a}{c-\delta} \leq \frac{a}{c} + \frac{\epsilon}{2}.$$

Let $Y(w) \in \Omega$. Then,

$$Y(w) = \beta W + (1-\beta)1_{c-\delta},$$

with $w \in [c/2, b]$ and $\beta = (c - \delta - b)/(c - \delta - w)$. Let X be some strategy of player A in $\Lambda_{c,c^-}(a, b)$. Then,

$$H(X, Y(w)) = \beta H(X, W) + (1 - \beta)H(X, 1_{c-\delta})$$

$$\leq \beta(\frac{2a}{c} - 1 + \epsilon) + (1 - \beta)\left[2P(X \geq c - \delta) - 1\right]$$

$$\leq \beta(\frac{2a}{c} - 1 + \epsilon) + (1 - \beta)\left[2\left(\frac{a}{c} + \frac{\epsilon}{2}\right) - 1\right]$$

$$= \frac{2a}{c} - 1 + \epsilon;$$

thus $Y(w)$ is ϵ-optimal for player B and we are done.