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A GENERALIZED SECRETARY PROBLEM

By

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A GENERALIZED SECRETARY PROBLEM

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Abstract

A new Secretary Problem is considered, where for fixed k and m one wins if at some time $i = m(j-1)+1$ up to jm one selects one of the j best items among the first jm items, $j = 1, \dots, k$. Selection is based on relative ranks only. Interest lies in small k values, such as $k = 2$ or 3. This is compared with the classical rule, where one wins if one of the k best among the $n = km$ items is chosen. We prove that the win probability in the new formulation is always larger than in the classical one. We also show, for $k = 2$ and 3 that one stops sooner in the new formulation. Numerical comparisons are included.

Keywords: Secretary Problem; Optimal Stopping Rule; Time Dependent Win Probability; Relative Rank

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1. Introduction

In the classical secretary problem there are n candidates for a job, and the goal is to maximize the probability of picking the best one, based only on sequentially observing the relative ranks of the candidates. It is assumed that the candidates are exchangeable and rankable without ties, and that a candidate that is passed by is no longer available.

As observed by [3] and [2], this goal is rather ambitious. Under the same assumptions they suggest, for a fixed $k \geq 1$, that the goal be generalized to that of maximizing the probability of choosing one of the k best. They discuss in detail the case of $k = 2$. We

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hereafter refer to this case as the classical problem. The classical problem is considered in detail in [1].

Now consider the following question: There are altogether 200 candidates. You are given the choice between playing the game in the classical problem for $k = 2$ or playing the following game, henceforth referred to as the new problem: You win if you either stop within the first 100 candidates and you picked the best from among the first 100, or you choose not to stop until after observing the 100 candidates and then you win if you pick one of the two best from among all 200 candidates. Which of the two games would you rather play, i.e., which has a higher probability of winning?

The answer is not obvious. It can be argued easily that picking one of the two overall best must have a higher probability. Note that there are situations where you win in the classical problem and not in the new one, and vice versa. For example, you might have stopped within the first 100 observations with relative rank of 1, which turned out to be the second best in the first 100 and in all 200. Stopping at this observation is a win in the classical formulation, but not in the new formulation. On the other hand, you might have stopped with an observation that is best among the first 100, but is not one of the two best among the 200. This outcome provides a win in the new setting, but not in the classical setting.

Clearly, 100 and 200 can be replaced by m and $2m$, respectively, $m \geq 2$. Being the best among m should be approximately equivalent to being one of the two best among $2m$, but in the optimal solution for the new problem, one might be motivated to stop sooner. Like in the classical formulation, $k = 2$ can be replaced by general $k > 1$.

When comparing the classical and new formulations we consider that the number of candidates is $n = km$. The new "Generalized Secretary Problem" can thus be described as follows: For a given $m \geq 2$ and $k \geq 1$, find the optimal rule and payoff (probability of winning) in a situation where if you stop at a time between $(j-1)m+1$ and jm and the item you select is among the j best in the first jm items, $j \leq k$, then that constitutes a win.

The method of finding the optimal rules is, as always, by backward induction (i.e., dynamic programming). We denote the respective probability of winning for the classical and new formulations by $W(k,n)$ and $W^*(k,m)$. The aim of the paper is to make statements about the probability of winning and characteristics of the optimal rules in these two formulations.

The paper is organized as follows. Some notation and basic results appear in Section 2. Section 3 is devoted to the case where k is any fixed number. The main result is $W(k,n) < W^*(k,m)$, where $n = mk$. An interesting conjecture which appears to be true by computation is that one stops no later in the new formulation than in the classical one for all k. Although we have not shown this in general, we are able to prove this in Section 4 and Section 5 when $k = 2$ and $k = 3$ respectively. These sections also include results about the rules and the probability of winning when $m \to \infty$. Section 6 provides some computational results and insights into the behavior of the rules and probability of winning for the two formulations.

2. Preliminaries

In this section, we develop the two formulations that we consider throughout the paper, providing the notation and some key relations that are useful in the ensuing sections. We consider first the classical formulation. The assumptions are:

1. There is a known horizon of n items.

2. A win occurs if one of the top k items out of all n items is chosen.

3. When item i is observed, its relative rank among the first i items is given.

4. All $n!$ permutations of the ranks of the n items are equally likely.

5. Once an item is passed it is no longer available.

We let $P(r,i,k,n)$ denote the probability that the item with relative rank r at observation (time) i is among the best k at the end of the horizon n. Consider the random variable, X_i, which is the number of items of the top k among the n items that are in the first i items. It follows that an item with relative rank $r \le k$ at position i is a winner if $X_i \ge r$. Since X_i has a hypergeometric distribution this yields

$$P(r,i,k,n) = \frac{\sum_{j=r}^{k} \binom{i}{j}\binom{n-i}{k-j}}{\binom{n}{k}}. \tag{1}$$

This implies the obvious fact that $P(r,i,k,n)$ decreases in r.

Since $E(X_i) = ik/n$ and $E(X_i) = \sum_{j=1}^{k} P(X_i \geq j)$, it follows that

$$\sum_{r=1}^{k} P(r,i,k,n) = \frac{ik}{n}. \tag{2}$$

It is straightforward to obtain the solution to the problem by dynamic programming. We begin with $P(r,n,k,n) = 1$ if $r \leq k$ and 0 otherwise. Note that it is only necessary to keep track of these values for $r \leq k$. Since the item with rank r at i is either rank r or $r+1$ at $i+1$ it follows that

$$P(r,i,k,n) = \frac{r}{i+1} P(r+1,i+1,k,n) + \frac{i+1-r}{i+1} P(r,i+1,k,n). \tag{3}$$

We let the probability of winning given that we did not stop before item i be denoted by $W(i,k,n)$. The ultimate probability of winning is $W(k,n) \equiv W(1,k,n)$. The values of W are also available by backward induction. First note that $W(n,k,n) = \frac{k}{n}$. The optimal solution is obtained by considering when relative rank r satisfies $P(r,i,k,n) > W(i+1,k,n)$. Let

$$j(i) = \arg\max_{r} \{P(r,i,k,n) > W(i+1,k,n)\}.$$

If $P(r,i,k,n) < W(i+1,k,n)$ for all r then $j(i) = 0$. The $j(i)$ provide the rule that we stop at time i only when the relative rank does not exceed $j(i)$. Since the $j(i)$ are non-decreasing because $P(r,i,k,n)$ increases in i and $W(i,k,n)$ is non-increasing in i, the rule can alternatively be described by thresholds $r(j,k,n)$; $1 \leq j \leq k$, where we stop with relative rank j at time i if and only if $i \geq r(j,k,n)$.

Since we only stop at item i if its relative rank r does not exceed $j(i)$ and all relative ranks at i are equally likely, this yields

$$W(i,k,n) = \frac{1}{i}\left[\sum_{r=1}^{j(i)} P(r,i,k,n) + (i-j(i))W(i+1,k,n)\right]. \tag{4}$$

We now turn to the new version of the problem. The formulation is the same as in the classical one, except that the horizon of interest depends on when one stops. To this end, assume that $n = mk$ and divide the items in blocks:

$$S_j = \{i \mid (j-1)m + 1 \leq i \leq jm\}, \quad j = 1,\ldots,k.$$

If one stops in block j, that is a time in S_j, then one wins if the item is among the best j in the more limited horizon of jm.

We have the analogous definitions to (1) and (2) of

$$P^*(r,i,k,m) = \frac{\sum_{s=r}^{j} \binom{i}{s}\binom{mj-i}{j-s}}{\binom{mj}{j}} \quad \text{if} \quad i \in S_j. \tag{5}$$

Note that i determines S_j and hence the horizon of jm, therefore, $P^*(r,i,k,m) = P(r,i,j,jm)$ and so it follows that

$$\sum_{r=1}^{j} P^*(r,i,k,m) = \frac{i}{m}. \tag{6}$$

Similarly,

$$P^*(r,i,k,m) = \frac{r}{i+1}P^*(r+1,i+1,k,m) + \frac{i+1-r}{i+1}P^*(r,i+1,k,m) \tag{7}$$

if $(j-1)m+1 \leq i < jm$ as in (3), and $P^*(r,jm,k,m) = 1$ if $r \leq j$ and 0 otherwise for $j = 1,\ldots,k$. We also have a similar recursion for the probability of winning. If we let the probability of winning if we have not stopped before i be denoted by $W^*(i,k,m)$ then defining

$$j^*(i) = \arg\max_r \,\{P^*(r,i,k,m) > W^*(i+1,k,m)\},$$

we obtain

$$W^*(i,k,m) = \frac{1}{i}\left[\sum_{r=1}^{j^*(i)} P^*(r,i,k,m) + (i - j^*(i))W^*(i+1,k,m)\right]. \tag{8}$$

Note that if $P^*(r,i,k,n) < W^*(i+1,k,m)$ for all r then $j^*(i) = 0$. The stopping rule can be described alternatively with thresholds within each block similar to the discussion above.

We study the behavior of W and W^* as well as $j(i)$ and $j^*(i)$ in the results that follow. We first consider results for general k in the next section. We then focus on the two most practical cases of $k = 2$ and $k = 3$ in the sections that follow. There are two primary results that we consider among others. We first show that the probability of winning in the new version exceeds the probability of winning in the classical version, that is, $W^*(k,m) > W(k,n)$ where $n = mk$. We also conjecture that $j(i) \leq j^*(i)$. This is tantamount to saying that in the optimal solution one always stops at least as early in the new formulation as in the classical formulation. Although the second conjecture appears to be true in general from the many examples we computed, the proof for the general case is elusive. We prove this result for the cases of most interest where $k = 2$ and $k = 3$.

3. General Results

In this section we first turn to the way in which the probability of winning changes with problem size. If we fix k and let the number of observations grow it is intuitive that

Lemma 1. $W(k,n) < W(k,n-1)$.

Proof. Consider the n-observation problem where the optimal choice is already made. We now describe a rule for the $n-1$ problem that is suboptimal. Delete one observation at random to create an $n-1$ problem. With probability $(n-1)/n$ the item eliminated was not the item chosen by the n-rule. In this case the item chosen has a probability that exceeds $W(k,n)$ to be among the k best among the remaining $n-1$ items. The strict inequality follows since the item chosen by the n-rule might not have been one of the k best among the n , but may have been one of the best k among the $n-1$ items, after we eliminated one item. If the item deleted at random was the item chosen by the n-rule (and this has probability $1/n$) use the optimal rule to solve the $n-1$ problem. We thus get

$$W(k,n-1) > \frac{n-1}{n}W(k,n) + \frac{1}{n}W(k,n-1)$$

which implies that $W(k,n-1) > W(k,n)$.

A similar argument implies that $W^*(k,m) < W^*(k,m-1)$.

It is immediate that $W(k,n)$ increases in k for fixed n. It might seem obvious that the same is true for the new formulation, that is, $W^*(k,m)$ increases in k , but it is not as immediate, as n also grows with k for fixed m. But a suboptimal rule for the $k+1$ problem is to proceed according to rule for the k problem for the first k blocks and if we did not stop we have the added opportunity of stopping in the last (i.e., $k+1$) block hence the result follows. We now turn to showing the main theorem in this section, namely

Theorem 1. *$W^*(k,m) > W(k,n)$ where $n = mk$.*

Proof. We will show by backward induction that conditions 1. and 2. below hold:

1. $W(i,k,n) \leq W^*(i,k,m)$. We shall show that there is equality for $i \in S_k$ and otherwise the inequality is strict.

2. There exists an $r^*(i)$ such that $P(r,i,k,n) \leq P^*(r,i,k,m)$ for $1 \leq r \leq r^*(i)$ and $P(r,i,k,n) \geq P^*(r,i,k,m)$ for $r^*(i) < r \leq k$.

All we need to show is that $W(k,n) \equiv W(1,k,n) < W^*(k,m) \equiv W^*(1,k,m)$ to prove the Theorem, but by showing condition 1. we are showing more.

We consider the optimal decision rule for the solution to the classical problem which stops at time i if the relative rank at i does not exceed $j(i)$.

The probability of winning at time point i in the classical solution is given recursively in (3) where $j(i)$ denotes the optimum relative rank at which one should stop in this formulation. But for the new formulation,

$$W^*(i,k,m) \geq \frac{1}{i}\left[\sum_{r=1}^{j(i)} P^*(r,i,k,m) + (i - j(i))W^*(i+1,k,m)\right]. \qquad (9)$$

as $j(i)$ might not be the optimal relative rank at which to stop at time i in the new formulation.

We now proceed to prove conditions 1. and 2. by backward induction. The proof relies on the observation that relative rank r at $i-1$ becomes either relative rank r or $r+1$ at time i with respective probabilities $\frac{i-r}{i}$ and $\frac{r}{i}$. It also relies on the observation that if condition 1. holds at time point i and condition 2. holds at time point $i-1$, then condition 1. holds at time point $i-1$ from $W(i-1,k,n)$ in (4) and $W^*(i-1,k,m)$ in (9). This follows from $\sum_{r=1}^{j_{i-1}} P^*(r,i-1,k,m) \geq \sum_{r=1}^{j_{i-1}} P(r,i-1,k,n)$ because condition 2. holds at $i-1$ and the fact that the sums of these terms in the two formulations over all values from 1 to k are equal.

First consider $i \in S_k$. Conditions 1. and 2. hold because the two formulations are identical problems for all time points in S_k. Now consider, $i = (k-1)m$. Condition 2.

holds because $P(r,(k-1)m,k,m)=1$ if $r \le k-1$ and otherwise it is zero. Condition 1. follows from condition 2. as described above.

We now assume that conditions 1. and 2. hold for $i \in S_{k-1}$ with $i > (k-2)m+1$. We now consider time point $i-1$. Consider proving condition 2. for $i-1$ (from which condition 1. follows). For any $r \le r^*(i)-1$, since $P(r,i,k,n) \le P^*(r,i,k,m)$ and similarly for $r+1$ then $P(r,i-1,k,n) \le P^*(r,i-1,k,m)$. Similarly for any $r \ge r^*(i)+1$, it follows that $P(r,i,k,n) \ge P^*(r,i,k,m)$ and similarly for $r+1$. The only ambiguous case as to which is larger is at $r = r^*(i)$. But if the probability is larger in the new formulation then $r^*(i-1) = r^*(i)$ and condition 2 holds and if it is larger in the classical formulation then $r^*(i-1) = r^*(i)-1$ and condition 2 holds. Since at $i=(k-1)m$, $P(r,i,k,n) < P^*(r,i,k,m) = 1$, for $r \le k-1$ this implies that $W(i,k,n) < W^*(i,k,m)$, and so, for $i \in S_{k-1}$, the inequalities are strict. Proceed through the remaining blocks of time points S_{k-2} and so on in a like fashion.

We showed that the probability of winning is greater in the new formulation than in the classical formulation for any k and m. There is a second conjecture that relates these two formulations:

Conjecture $j^*(i) \ge j(i)$ for all i for any k and m

This implies that we stop with at least as high a relative rank in the new method as we do in the classical method. This conjecture is non-intuitive. The largest relative rank for which we stop at item i is the largest relative rank for which the probability that this item is a winner exceeds the probability of winning if we follow the optimal strategy from item $i+1$ and thereafter. But the probability of winning with low relative rank at i is higher for the new method than the classical, but the probability of winning if one continues is also higher.

Since obviously one does not stop in the new method at $i \in S_j$, if the relative rank exceeds j, a necessary condition is to show that

Lemma 2. *If $i \in S_j$, then $j(i) \le j$.*

Proof. Since $j(i)$ is non-decreasing in i it is sufficient to show that

$$P(j,(j-1)m,k,n) < W((j-1)m+1,k,n). \tag{10}$$

The proof relies on considering the random variable R_j which is the relative rank at

jm of an item that has relative rank j at $(j-1)m$.

$$P(R_j = s) = \frac{\binom{s-1}{j-1}\binom{jm-s}{(j-1)m-j}}{\binom{jm}{(j-1)m}} = \frac{\binom{s-1}{s-j}\binom{jm-s}{m+j-s}}{\binom{jm}{m}}.$$

It is easy to show that $P(R_j = j) \leq [(m-1)/m]^m$ which increases from $\frac{1}{4}$ to e^{-1} as m goes from 2 to ∞. It is also easy to show that $P(R_j = j+1) > P(R_j = j)$.

We will prove the lemma by induction beginning with $j = k$ and going backwards to $j = 2$. If $j = k$, it has already been shown elsewhere that $W((k-1)m+1, k, n) > \frac{1}{2}$ when $k = 2$ and hence for all k as $W((k-1)m+1, k, n)$ is clearly increasing in k. Furthermore, $P(k, (k-1)m, k, n) = P(R_k = k) < e^{-1} < \frac{1}{2}$.

Assume (10) holds for j. Consider $j-1$. The result is trivially true if $P(j, jm, k, n) \leq W(jm+1, k, n)$ because an item with relative rank of j at $(j-1)m$ must have relative rank of at least j at jm. So assume that $P(j, jm, k, n) > W(jm+1, k, n)$. But

$$\begin{aligned} P(j, (j-1)m, k, n) &= \sum_{s=j}^{min(j+m,k)} P(R_j = s)P(s, jm, k, n) \\ &= P(R_j = j)P(j, jm, k, n) + P(R_j = j+1)P(j+1, jm, k, n) \\ &+ \sum_{s=j+2}^{min(j+m,k)} P(R_j = s)P(s, jm, k, n). \end{aligned} \tag{11}$$

To obtain a lower bound for $W((j-1)m+1, k, n)$ consider the suboptimal rule where we stop at the first $i \in [(j-1)m+1, jm]$ at which the observation is superior to the item with relative rank j at $i = (j-1)m$. Hence if we stop and $R_j = s$, since the relative rank of the item at i is s , the relative rank of the observation we stopped at must be $s-1$ or smaller. If on the other hand , $R_j = j$ so we do not stop, we continue with the optimal rule to obtain a probability of winning of $W(jm+1, k, n)$. If $R_j = j+1$ then we do stop and the probability of winning exceeds $P(j, jm, k, n)$. Therefore,

$$\begin{aligned} W((j-1)m+1, k, n) &\geq P(R_j = j)W(jm+1, k, n) + P(R_j = j+1)P(j, jm, k, n) \\ &+ \sum_{s=j+2}^{(min(j+m,k)} P(R_j = s)P(s-1, jm, k, n). \end{aligned} \tag{12}$$

Since $P(s-1, jm, k, n) \geq P(s, jm, k, n)$ we only need to consider the first two terms in (11) and (12) to show that $P(j, (j-1)m, k, n) < W((j-1)m+1, k, m)$.

The first two terms in (11) can be written as

$$P(R_j = j)A + P(R_j = j+1)B$$

where $A = P(j, jm, k, n)$ and $B = P(j+1, jm, k, n)$.

The first two terms in (12) can be written as

$$P(R_j = j)C + P(R_j = j+1)A$$

where $C = W(jm+1, k, n)$. But $P(R_j = j+1) > P(R_j = j)$, $C > B$ by induction and $A > C$ by assumption. Hence the induction step follows.

4. k=2

In this section, we consider the case where an item must ultimately be one of the two best for a win to occur in the classical formulation. Hence in the new formulation, if we stop at observation m or before the item we choose must be the best among the first m and if we stop after the m^{th} item, it must be one of the two best among all of the $n = 2m$ items. Since $k = 2$ throughout this section, we denote the ultimate probability of winning in the new and classical settings as $W^*(m)$ and $W(m)$ respectively. There is a literature on this problem in the classical setting. Most notably, in [2] it is shown that $lim_{m\to\infty} W(m) = .5736$. It is also shown in that paper that as m goes to infinity the optimal rule is to stop with relative rank of 1 when $i/2m$ reaches $\approx .347$ and to stop when the relative rank is 1 or 2 when $i/2m$ reaches $2/3$.

We provide two results in this section. The first result finds the optimal cutoffs and value of $\lim_{m\to\infty} W^*(m)$ for the new formulation. Of course, the cutoff at which we stop with relative rank of 2 (and hence 1 as well) is the same as in the classical formulation. We show that the optimal rule is to stop with relative rank of 1 when $i/2m$ reaches $\approx .3149$ as $m \to \infty$. This supports the assertion that one stops no later in the new formulation than in the classical formulation. The second result is to show that $j^*(i) \le j(i)$ for all i and m. In the process of proving the first result we show that $\lim_{m\to\infty} W^*(m) = 0.6298 > \lim_{m\to\infty} W(m) = 0.5736$. Of course, we showed this inequality holds in general (in terms of k and m) in Section 3.

The aim is to find the cutoffs when m is large. To this end, we begin with some preliminaries. Let $Q(j, i, t, n)$ be the probability that relative rank j at i has rank t at n.

It follows from Section 2 that $Q(1,i,1,n) = \frac{i}{n} \to y$ as $n \to \infty$; $Q(1,i,2,n) = \frac{i}{n}\frac{n-i}{n-1} \to y(1-y)$; $Q(2,i,2,n) = \frac{i}{n}\frac{i-1}{n-1} \to y^2$; $Q(1,i,1,n) + Q(1,i,2,n) + Q(2,i,2,n) \to 2y$.

The dynamic programming approach yields $j(i) = 2$, that is stop with relative rank 1 or 2, as long as $y > x$, where x solves

$$\int_{y=x}^{1} 2y\frac{x^2}{y^3}dy = x^2, \tag{13}$$

the solution of which is $x = 2/3$. Hence we stop with relative rank 1 or 2 in both formulations if $i = \frac{4}{3}m$ when m is large.

To see that (13) holds note that x^2 is the approximate probability of winning at $i = \lfloor xn \rfloor$ if the relative rank is 2. The probability that the item with relative rank of 1 or 2 occurs first at item $j > i$ is

$$\frac{2}{j}\prod_{s=i}^{j-1}\frac{s-2}{s} = \frac{2(i-2)(i-1)}{j(j-2)(j-1)} \approx \frac{2x^2}{ny^3}.$$

Since we changed variable of integration to go from x to 1, the $\frac{1}{n}$ is absorbed. Further, we must divide by 2 as we have equal probability that we stop with relative ranks of 1 and 2. Thus x is the "break even value" for accepting or rejecting relative rank 2, where $i = 2mx$.

To check whether we should accept relative rank of 1 all the way down to $i = m+1$ we must check the cumulative probability if we do, with the win probability if observation $m+1$ has relative rank of 1, which is $\frac{3}{4}$. As we saw, the win probability for the part where we pick an item if its relative rank is 2 is x^2, with $x = \frac{2}{3}$, i.e. $\frac{4}{9}$. This is provided that we did not stop earlier. Thus we must see whether

$$\int_{y=1/2}^{2/3}(2y-y^2)\frac{1/2}{y^2}dy + \left(1 - \frac{1}{2}\int_{1/2}^{2/3}\frac{1}{y^2}dy\right)\frac{4}{9} < \frac{3}{4}. \tag{14}$$

The derivation of the first term in (14) is similar to that on the l.h.s. of (13). The numerical value in the bracket is the probability of not stopping before $(4/3)m$, in which case the win probability is $(2/3)^2$. So the value of the l.h.s. of (14) is $\frac{1}{4} + \log(4/3)$ which is smaller than $3/4$. Thus one should stop with relative rank 1 in either formulation for $m < i \le 2m$, and also for relative rank of 2 for $i > (4/3)m$. It is easily seen from [2] that the probability of winning if one has not stopped at m and we stop for the first time with relative rank of 2 at T_2 is

$$W(m+1,2m) = \sum_{i=m+1}^{T_2} \frac{1}{i-1} - \frac{T_2-(m+1)}{4m-2} + (T_2-2)\left[\frac{1}{T_2-1} - \frac{1}{2m-1}\right]$$

and asymptotically, as $m \to \infty$ (with $T_2 = 4m/3$)

$$W(m+1,2m) = \log(4/3) - \frac{1}{12} + 1 - \frac{2}{3} \approx .5377.$$

Now, to find the smallest i for which one should stop with relative rank of 1 in the new formulation to be denoted by i^*, the probability we do not stop with relative rank of 1 is the probability that the best of the first m is before i^*. Hence the probability we do stop is $1 - \frac{i^*-1}{m}$. The win probability if we stop with relative rank 1 for the first time at or after i^* but no later than time m is denoted by

$$V(i^*) = (i^*-1)\sum_{i=i^*}^{m} \frac{1}{i(i-1)}\frac{i}{m} = \frac{i^*-1}{m}\sum_{i=i^*}^{m}\frac{1}{i-1}.$$

If we let $\frac{i^*}{m} \to \gamma$ as $m \to \infty$, the above yields that the probability of stopping is $1-\gamma$ and the win probability is $-\gamma\log(\gamma)$. To find the optimal γ one must solve for γ that maximizes

$$f(\gamma) = -\gamma\log\gamma + \gamma[\log(4/3) + 1/4].$$

Taking derivatives and solving for γ yields

$$\gamma^* = e^{-(1+[\log(4/3)+1/4])} = \frac{4}{3}e^{-3/4} = 0.6298.$$

The total win probability also turns out to be $\gamma^* = 0.6298$.

It is evident by comparing the results we obtained for the cutoffs when m goes to infinity for the new formulation with the corresponding results for the classical formulation, that we stop with relative rank 1 sooner in the new formulation. In the lemma below we show that this result is true in general. Specifically,

Lemma 3. *$j^*(i) \geq j(i)$ for all i and m when $k = 2$.*

Proof. It is sufficient to show that the smallest i for which we stop with relative rank 1 is no larger in the new formulation than in the classical formulation. Assume the lemma is not true. Then there exists an i, $1 < i < m$, for which it is best to stop with relative rank 1 in the classical formulation but not in the new formulation. Let j

be the largest such i for which it is true. This implies that we stop with relative rank of 1 in both formulations at $j+1$. Furthermore, note that j is strictly less than m as it always the case that we stop with relative rank of 1 at m in the new formulation. The proof relies on certain observations:

1. If we do not stop before j, the probability we stop at t where $j \leq t \leq m$ is $j/[t(t-1)]$. This applies to both formulations as we are waiting for relative rank of 1.

2. If we do not stop at j, the probability we do not stop at m or before is j/m.

3. If we stop at j, the probability of winning is $P^*(1,j,1,m) = j/m$ for the new formulation and $P(1,j,2,2m) = \frac{j}{2m} + \frac{j}{2m}\frac{2m-j}{2m-1} = \frac{j}{m} - \frac{j(j-1)}{2m(2m-1)}$ for the classical formulation.

4. We have shown that the probability of winning if we have not stopped at m is $W(m+1,2m) > .5$

Since it is assumed that it is better to stop with relative rank of 1 in the classical formulation,

$$\frac{j}{m} - \frac{j(j-1)}{2m(2m-1)} \geq \sum_{t=j+1}^{m} \frac{j}{t(t-1))}\left(\frac{t}{m} - \frac{t(t-1)}{2m(2m-1)}\right) + \frac{j}{m}W(m+1,2m). \quad (15)$$

Since it is better not to stop at j with relative rank of 1 in the new formulation.

$$\frac{j}{m} \leq \sum_{t=j+1}^{m} \frac{j}{t(t-1)}\frac{t}{m} + \frac{j}{m}W(m+1,2m). \quad (16)$$

After trivial algebra and realizing that the r.h.s. of (15) includes the same terms as the r.h.s. of (16), these two equations imply that $m-j \geq j-1$, that is $j \leq \frac{m+1}{2}$. Since the probability of winning if we do not stop before m , $W(m+1,2m) = .5333$, as $m \to \infty$, and $W(m+1,2m)$ decreases in m if

$$\frac{j}{m} - \frac{j(j-1)}{2m(2m-1)} < .5333, \quad \text{for} \quad j \leq \frac{m+1}{2}, \quad (17)$$

we have a contradiction. We assumed that it is best to stop with relative rank 1 at j.

The left-hand side of (17) increases in j so we need to consider $j =< \frac{m+1}{2} >$. If m is even, $j = \frac{m}{2}$ and (17) is immediate. If m is odd, $j = \frac{m+1}{2}$ and therefore the left-hand side of (17) is

$$g(m) = \frac{m+1}{2} - \frac{(m+1)(m-1)}{8m(2m-1)} = \frac{1}{2} + \frac{8m - m^2 - 3}{8m(2m-1)}.$$

It follows that $g(m) < \frac{1}{2}$ when $m \geq 9$. Therefore, we only need to consider $m = 3, 5, 7$. But $g(3) = .6$ and $W(4,6) = \frac{19}{30} > .6$; $g(5) = \frac{8}{15}$ and $W(6,10) = \frac{53}{90} > \frac{8}{15}$; $g(7) = \frac{46}{91} <$ .5333.

5. k=3

In this section, we prove the results that parallel the discussion in the previous section where $k = 2$. We first find the solutions when $m \to \infty$. The development of these results is similar to those of the previous section. It relies on the preliminary remarks from the previous section and the following additional observations:

1. $Q(2,i,3,n) = \frac{2i(i-1)(n-i)}{n(n-1)(n-2)} \to 2y^2(1-y)$ where $\lim_{n\to\infty} \dfrac{i}{n} = y$.
2. $Q(3,i,3,n) = \frac{i(i-1)(i-2)}{n(n-1)(n-2)} \to y^3$.
3. $Q(1,i,1,n)+Q(1,i,2,n)+Q(1,i,3,n)+Q(2,i,2,n)+Q(2,i,3,n)+Q(3,i,3,n) \to 3y$.

We find the item above which one stops with relative rank of 3. The probability that we first stop at $j > i$ is

$$\frac{3}{j}\prod_{s=i}^{j-1}\frac{s-3}{s} \approx \frac{3}{n}\frac{x^3}{y^4}$$

where $x = \frac{i}{n}$ and $y = \frac{j}{n}$. The win probability if we stop with relative rank of 3 at $i = xn$ is x^3. Thus one must solve

$$x^3 \int_{y=x}^{1} \frac{3y}{y^4} dy = x^3$$

which yields, $\frac{3}{2}(x^{-2} - 1) = 1$. Hence $x = \sqrt{3/5} = .7746$. So the cutoff point where we stop with relative rank 2 is $3m\sqrt{3/5}$.

We first show that we stop with relative rank of 2 for all $i > 2m$. Since $Q(2,i,2,n) + Q(2,i,3,n) \to (\frac{2}{3})^2 + 2(\frac{2}{3})^2(1-\frac{2}{3}) = \frac{20}{27}$, the probability of winning if we stop with relative rank of 2 at $i = 2m$ is approximately $\frac{20}{27}$. If we do not stop with relative rank of 2 at $i = 2m$, the win probability, V, is composed of two terms: probability of winning before time $3m\sqrt{3/5}$ with relative rank of 1 or 2 and probability of not stopping by time $3m\sqrt{3/5}$ and winning with relative rank of 3 or less. The first probability is

$$\int_{2/3}^{\sqrt{3/5}} \frac{3y-y^3}{2}\frac{(2/3)^2 2}{y^3}dy = .230705.$$

Since the probability of not stopping is $\frac{(2/3)^2}{\sqrt{3/5}^2} = \frac{20}{27}$, the second term is $\frac{20}{27}(\frac{3}{5})^{3/2} =$ .344265. But $V = .230705 + .344265 = .574970 < 20/27$ which implies that we stop with relative rank of 2 or less for all $i > 2m$. To see how far down we need to go and still stop with relative rank of 2 we must solve

$$\int_{y=x}^{1} 2y\frac{x^2}{y^3}dy + \left(1 - \int_{y=x}^{1} \frac{2x^2}{y^3}\right) V = x^2.$$

This yields $2x - 2x^2 + x^2V = x^2$ or $x = \frac{2}{3-V} = C = .824732$. Thus one should stop with relative rank of 2 from $1.64946m$ to $2m$. To see how far down we should pick relative rank of 1 we must solve for x and see if $x \geq .5$ in the equation

$$\frac{x}{C}C^2 + x\int_{y=x}^{C} (2y-y^2)\frac{1}{y^2}dy = 2x - x^2.$$

The above equation yields

$$C + \int_{y=x}^{C} \left(\frac{2}{y} - 1\right) dy = 2 - x.$$

In solving for x one obtains $x = .50045$. Thus one should pick a relative rank of 1 not all the way down to m but rather beginning slightly above at $i = 1.0009m$. The expected value when stopping is $2x - x^2 = .75045$. Thus from 1 to m one should stop (asymptotically) with relative rank of 1 when $i \geq e^{.75045-1}m = .7792m$ with win probability of the same amount of .7792.

We consider the probability of winning if we do not stop before S_k, in the limit, that is $\lim_{m\to\infty} W((k-1)m+1, km)$. We observed earlier that for fixed k, $W((k-1)m+1, km)$ decreases in m. To this end, one stops with relative rank k when

$$x^k \int_{y=x}^{1} \frac{ky}{y^{k+1}}dy = x^k.$$

The solution is $x = B = \left(\frac{k}{2k-1}\right)^{1/(k-1)}$. Hence one stops with relative rank of k at all $i > kmB$ and the win probability if one does not stop earlier is $B^k = \left(\frac{k}{2k-1}\right)^{k/(k-1)} \to$ $1/2$ as $k \to \infty$.

We already showed that we do not stop with relative rank k at $(k-1)m$. Hence to find $\lim_{m\to\infty} W((k-1)m+1, km)$, let $A = \frac{k-1}{k}$.

So the probability we win because we stop with relative rank $k-1$ between A and B and win is

$$A^{k-1}\int_{y=A}^{B} \frac{ky-y^k}{y^k}dy = \frac{k}{k-2}\Big[A - A^{k-1}B^{-(k-2)}\Big] - A^{k-1}B + A^k.$$

The probability we win because we do not stop at B is $\left(\frac{A}{B}\right)^{k-1}B^k = A^{k-1}B$.

Thus the total probability of winning becomes

$$\frac{k-1}{k-2} - \frac{k-1}{k-2}\Big(\frac{k-1}{k}\Big)^{k-1}\Big(\frac{2k-1}{k}\Big)^{\frac{k-2}{k-1}} + \Big(\frac{k-1}{k}\Big)^{k} \stackrel{k\to\infty}{\longrightarrow} 1 - e^{-1}.$$

Note that the left-hand side of the above expression when $k = 3$ is $V = .5749$.

Assertion: For $k = 3$ one stops at least as early in the new formulation as compared to the classical formulation.

Proof. We want to show that one stops earlier with relative rank of 2 in the new formulation than in the classical formulation. We consider the difference in the probability of winning in these two formulations if we stop with relative rank of 2, which we denote by $DS(i, m)$ and the difference in probability of winning if we do not stop at i which is denoted by $DC(i, m)$. It is sufficient to show that $DS(i, m) \geq DC(i, m)$. The difference of interest, which depends only on i and m is then,

$$DS(i,m) = \frac{\binom{i}{2}}{\binom{2m}{2}} - \frac{\binom{i}{2}(3m-i)+\binom{i}{3}}{\binom{3m}{3}} = \frac{\binom{i}{2}}{3m}\left\{\frac{8mi-9m^2-m-4i+2}{(2m-1)(3m-1)(3m-2)}\right\}.$$

.

We now consider the difference if we continue, $DC(i, m)$. Since we are going to take the difference between the new and classical formulation we can ignore the amount that is obtained if we do not stop by $i = 2m$. Since we are considering the effect from $i+1$ to $2m$, the rule to stop in both cases is whether the relative rank is 1 or 2. Since the probability of stopping at $j > i$ can be found to be $\frac{2i(i-1)}{j(j-1)(j-2)}$, that implies that

$$DC(i,m) = \sum_{j=i+1}^{2m}\frac{2i(i-1)}{j(j-1)(j-2)}\frac{j}{2m} - \sum_{j=i+1}^{2m}\frac{2i(i-1)}{j(j-1)(j-2)}\frac{\frac{j}{m} - \frac{\binom{j}{3}}{\binom{3m}{3}}}{2}.$$

This makes use of the fact that the sum of the probabilities that relative rank 1 or 2 in the new setting produces a win is j/m and the sum of the probabilities that relative ranks 1, 2 and 3 in the classical setting produces a win is j/m. The above expression reduces to

$$DC(i,m) = \sum_{j=i+1}^{2m} \frac{2i(i-1)}{j(j-1)(j-2)} \frac{\binom{j}{3}}{2\binom{3m}{3}} = 2\frac{\binom{i}{2}(2m-i)}{3m(3m-1)(3m-2)}.$$

The values of i for which $DS(i,m) > DC(i,m)$ are necessarily values of i for which we stop no later for the new formulation. If we ignore like terms in $DS(i,m)$ and $DC(i,m)$ this is equiavent to considering when

$$\frac{8mi - 9m^2 - m - 4i + 2}{2m-1} > 2(2m-i).$$

The above inequality reduces to the values of i such that

$$i > \frac{17m^2 - 3m - 2}{12m - 6} = \gamma(n) = \frac{17n^2 - 9n - 18}{36n - 54}.$$

What remains to show is that if i does not exceed the right hand side then we do not stop with relative rank of 2 in the classical formulation. But, from [4] we only stop at time i with relative rank of 2 in the classical formulation, if

$$c(i) = \frac{6(i-1)(n-i) + 5(i-1)(i_3-3)}{(n-1)(n-2)} + \frac{3(i-1)}{i_3-2} \geq 6$$

where i_3 is the smallest integer for which the optimal rule in the classical problem is to stop with relative rank of 3 or less. What we need to show is that if $i \leq \gamma(n)$, then $c(i) < 6$.

The first term in $c(i)$, $\frac{6(i-1)(n-i)}{(n-1)(n-2)} < 2$ when $n \geq 6$ as the numerator is maximized when $i = (n+1)/2$. Hence we only need to show that $(i-1)\left\{\frac{5(x-1)}{(n-1)(n-2)} + \frac{3}{x}\right\} < 4$ when $i \leq \gamma(n)$ where $x \equiv i_3 - 2$. Let the term in brackets be denoted by $\phi(x)$. It is straightforward to verify that $\phi''(x)$ is positive and that $\phi(.6n-1.2) = \phi(n-1) = \frac{8}{n-1}$. But it was shown by [4] that $.77n \leq i_3 \leq n$ so x must be in the range of $.6n - 1.2$ to $n-1$.

It suffices to show that $(\gamma(n)-1)\frac{8}{n-1} < 4$. The above inequality holds for $n \geq 4$.

We now turn to showing that the first time we stop with relative rank of 1 in the new formulation occurs no later than the first time we stop with relative rank of 1 in the classical formulation. Let $i = \lfloor \gamma m \rfloor$. We only need to consider $i \in S_2$. It is clear that $P(1, i, 3, 3m) - W(i+1, 3, 3m)$ increases in i and similarly for $P^*(1, i, 3, m) - W^*(i+1, 3, m)$ as the probability of winning with relative rank 1 increases in i and the probability of winning if we continue past i decreases in i.

We showed that there is a γ_1 ($\gamma_1 \approx 1.01$) such that $\lim_{m\to\infty} P(1, \lfloor \gamma_1 m \rfloor, 3, 3m) - W(\lfloor \gamma_1 m \rfloor + 1, 3, 3m) = 0$. We also showed that there is a $\gamma_1^* < \gamma_1$ ($\gamma_1^* \approx 1.0009$) such that $\lim_{m\to\infty} P^*(1, \lfloor \gamma_1^* m \rfloor, 3, m) - W^*(\lfloor \gamma_1^* m \rfloor + 1, 3, m) = 0$.

Consider γ_0 such that $\gamma_1^* < \gamma_0 < \gamma_1$ (e.g., $\gamma_0 = 1.005m$). There exists an m_0 such that for all $m > m_0$

$$P(1, \lfloor \gamma_0 m \rfloor, 3, 3m) < W(\lfloor \gamma_0 m \rfloor + 1, 3, 3m)$$

and

$$P^*(1, \lfloor \gamma_0 m \rfloor, 3, m) > W^*(\lfloor \gamma_0 m \rfloor + 1, 3, m),$$

as $i = \lfloor \gamma_0 m \rfloor > \lfloor \gamma_1^* m \rfloor$ and $i = \lfloor \gamma_0 m \rfloor < \lfloor \gamma_1 m \rfloor$. This implies that for all $m > m_0$, the optimal rule in the new formulation is to stop with relative rank of 1 as early as $i = \gamma_0 m$ while in the classical formulation the optimal rule is to stop with relative rank of 1 for the first time at an $i > \gamma_0 m$.

If $m = 200$, $1.005m = 200$. But $P^*(1, 201, 3, 200) = .7531 > W^*(202, 3, 200) = .7518$ and $P(1, 201, 3, 600) = .7066 < W(202, 3, 600) = .7095$. Hence the optimal rule is to stop at time 201 in the new formulation, but not in the classical formulation.

It is easy to check that for $m < 200$ one stops with relative rank 1 at $m + 1$ in the new formulation and we already showed that one does not stop with relative rank 1 at m in the classical version. In fact the smallest m for which we do not stop with relative rank 1 at $m + 1$ in the new formulation is when $m = 788$. Hence, $m = 788$ is the first smallest m for which there is a "hole" in the new formulation in that one stops with relative rank 1 at m and then at an $m + a$, $a \geq 2$, but not at $m + 1$.

Remark We already showed in the classical formulation that the optimal rule at $i = jm$ is to stop with relative rank that is j or less. Consider $k = 3$ at $i = m$. Could it be that we stop with relative rank of 1 at time m in the classical formulation? The answer is no. We showed that $W(3m)$ decreases as a function of m to .708. But $P(1, m, 3, 3m)$, the probability that rank 1 at m wins, is $1 - \binom{2m}{3} / \binom{3m}{3}$ which decreases to $19/27 = .7037$. It is easy to verify that for all $m \geq 40$, $P(1, m, 3, 3m) < .708$ which implies one does not stop with relative rank of 1 at time m in the classical formulation for all $m \geq 40$. But one can verify by computer that the optimal solution in the classical problem is to not stop with relative rank 1 at time m for all $m < 40$.

6. Computer Results

In this section, we illustrate the results from previous sections. In Table 1, we provide the optimal rule and win probabilities for both formulations for $k = 2$ for selected values of m.

	Classical Formulation			New Formulation		
m	$r(1)$	$r(2)$	W	$r^*(1)$	$r^*(2)$	W^*
5	4	8	.6367	4	8	.7033
10	8	14	.6042	7	14	.6656
25	18	34	.5858	17	34	.6436
50	35	67	.5796	32	67	.6367
100	70	134	.5766	64	134	.6333
1000	695	1334	.5739	631	1334	.6302
Limit	$.6940m$	$4m/3$	.5736	$.6298m$	$4m/3$	.6298

TABLE 1: Optimal rules and win probabilities for $k = 2$

The table indicates that the optimal rule and win probability for finite m which was obtained by computation, converges rapidly to the asymptotically optimal rule found in the paper. The two main results: 1. That the win probability is greater in the new formulation and 2. that one stops at least as early in the optimal rule in the new versus classical formulation is evident from the table.

We now consider $k = 3$ in Table 2. If m is not that large, $m < 788$, we always stop with relative rank 1 in block 2 and hence there is no cutoff, that is $r_2^*(1)$. But if $m = 1000$ in the new formulation the optimal rule in block 2 is to stop with relative rank 1 at $i = 1002$. In block 1 we also stop with relative rank 1 from $i = 780$ to 1000. Hence there is a hole; we stop with relative rank 1 at $i = 1000$ and at $i = 1002$ but not at $i = 1001$.

	Classical Formulation				New Formulation				
m	$r(1)$	$r(2)$	$r(3)$	W	$r_1^*(1)$	$r_2^*(1)$	$r^*(2)$	$r^*(3)$	W^*
5	6	10	12	.7620	5		9	12	.8456
10	11	18	24	.7349	9		17	24	.8133
25	26	45	49	.7188	20		42	59	.7921
50	51	89	118	.7134	40		83	118	.7856
100	102	177	233	.7108	79		166	233	.7824
1000	1011	1761	2325	.7085	780	1002*	1650	2325	.7795
Limit	$1.01m$	$1.76m$	$2.32m$	.708	$.779m$	$1.001m$	$1.65m$	$2.32m$	.779

TABLE 2: Optimal rules and win probabilities for $k = 3$

The conclusions are otherwise the same in $k = 3$ as described for $k = 2$. We see that $W^* - W$ tends to be bigger when $k = 3$ as compared to $k = 2$ even if the sample sizes are calibrated to be the same.

To see what happens for larger k we consider $k = 5$ and $m = 100$ in Table 3. We computed the optimal rules and winning probabilities for other k and m and found them to be substantively the same.

The rule for the classical problem is to stop with relative rank j for all times $i \geq r(j)$ where: $r(1) = 163$, $r(2) = 256$, $r(3) = 325$, $r(4) = 381$ and $r(5) = 433$ with win probability $W = .8621$.

The rule for the new problem is more complicated as cutoffs need to be specified within each of the five blocks (See Table 3).

Block	$r^*(1)$	$r^*(2)$	$r^*(3)$	$r^*(4)$	$r^*(5)$	W^*
5				401	433	
4		301	315	367		
3	201	235	284			
2	143	191				
1	93					
						.9220

TABLE 3: Optimal rule and win probability for the new formulation for $k = 3$ and $m = 100$

As is apparent from Table 3, the rule for the new formulation is more complicated as each block of 100 observations needs to be considered separately. The rule in block 5, (i.e., time points 401 to 500) is to stop with relative ranks 1 to 4 from time 401 to 432 and stop with relative ranks of 1 to 5 thereafter. The result in block 4 says that we stop with relative rank of 1 or 2 throughout the time period 301 to 400. in addition, we stop with relative rank 3 from time 315, and relative rank 4 from time 367 until 400. If we put the results of block 4 and block 5 together it implies that we stop with relative rank 4 from time 367 until the end of the time horizon. If we consider block 3 we see that there is a hole for relative rank 3. We stop with relative rank 3 for the time period from 284 to 300 from block 3 and then again from 315 to the end of the time horizon from blocks 4 and 5. We do not stop with relative rank 3, however, from time 301 to 314.

The main result that the win probability is always greater (by about .05 in this example) in the new formulation holds in this case. The conjecture that we stop no later in the new formulation than in the classical formulation, which we proved for the cases $k = 2$ and $k = 3$, also holds in this case with $k = 5$. We stop with relative rank 5 at the same time point in both formulations. We stop with relative rank 4 at time 381 in the classical formulation and time 367 in the new formulation. We stop with relative rank 3 from time 325 in the classical formulation and from 315 in the new formulation (as well as for earlier time periods). We stop with relative rank 2 from time 256 in the classical formulation and from time 235 in the new formulation (as well as for earlier time periods). Finally, we stop with relative rank 1 from time 163 in the classical formulation and from time 143 in the new formulation (as well as the earlier time points 93 to 100).

Finally, we showed in the course of proving Theorem 1 (see equation (9)) that if the optimal rule for the classical formulation is used in both the classical formulation and the new formulation, the probability of winning is higher in the new formulation. This is illustrated in Table 4. It is also apparent from the cases we ran that using the optimal rule from the classical problem on the new problem produces win probabilities that are closer to the optimal rule for the new problem than the optimal rule on the classical problem.

k	m	pwin opt new	pwin subopt new	pwin opt classical
2	10	0.6656	0.6599	0.6046
2	25	0.6436	0.6413	0.5858
2	50	0.6367	0.6345	0.5796
2	100	0.6333	0.6303	0.5766
2	1000	0.6302	0.6270	0.5739
3	10	0.8113	0.7996	0.7349
3	25	0.7921	0.7685	0.7188
3	50	0.7856	0.7585	0.7134
3	100	0.7824	0.7534	0.7108
3	1000	0.7795	0.7488	0.7085
5	10	0.9401	0.9344	0.8781
5	25	0.9279	0.9176	0.8674
5	50	0.9240	0.9113	0.8639
5	100	0.9220	0.9085	0.8621
5	1000	0.9202	0.9054	0.8605

TABLE 4: Win probabilities for the optimal rules for the new and classical formulation

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