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THE DISTRIBUTION OF THE COMBINED LENGTH OF SPANNED CYCLES IN A RANDOM PERMUTATION

By

YANNAI A. GONCZAROWSKI

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מרכז לחקר הרציונליות

CENTER FOR THE STUDY OF RATIONALITY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

The Distribution of the Combined Length of Spanned Cycles in a Random Permutation

Yannai A. Gonczarowski*

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Abstract

For a random permutation π on $\{1, 2, \ldots, n\}$ for fixed n, and for $M \subseteq \{1, 2, \ldots, n\}$, we analyse the distribution of the combined length $L = L(\pi, M)$ of all cycles of π that contain at least one element of M. We give a simple, explicit formula for the probability of every possible value for L (backed by three proofs of distinct flavours), as well as closed-form formulae for its expectation and variance, showing that less than $\frac{1}{|M|+1}$ of the elements $1, \ldots, n$ are expected to be contained in cycles of π that are disjoint from M, with low probability for a large deviation from this fraction. We furthermore give a simple explicit formula for all rising-factorial moments of L. These results are applicable to the study of manipulation in matching markets.

Given a random permutation on a fixed finite set of objects, we are interested in the combined length of all cycles of the permutation that intersect a given subset of these objects (or alternatively, of all cycles of the permutation that are disjoint from this given subset); when the subset consists of a single point, this quantity is simply the well-studied length of the cycle that contains that point (for an analysis of this special case, see e.g. [1, p. 24]). The question of the distribution of this quantity arises during analysis of the limits of manipulation in matching markets; for more information, the interested reader is referred to [2]. We commence by precisely defining the problem at hand.

Definition 1. Throughout this paper, we use the following standard notation.

- $\mathbb{P} \triangleq \{1, 2, 3, \ldots\}$ [5]; throughout this paper, $k, \ell, \tilde{\ell}, m, \tilde{m}, n, \tilde{n}$ denote elements of $\mathbb{P}$.
- $[n] \triangleq \{1, 2, \ldots, n\}$ [5].
- $[m, n] \triangleq \{m, m+1, \ldots, n\}$ [5] . ($[m, n] = \emptyset$ if $m > n$.)
- $S_N \triangleq \{\pi : N \mapsto N \mid \pi \text{ is a bijection}\}$ — the set of permutations of a set N.
- $S_n \triangleq S_{[n]}$.
- Furthermore, we denote $n^{(k)} \triangleq n \cdot (n+1) \cdot \ldots \cdot (n+k-1) = \frac{(n+k-1)!}{(n-1)!}$.

*Einstein Institute of Mathematics and Center for the Study of Rationality, Hebrew University of Jerusalem, Israel. *Email*: yannai@gonch.name.

Definition 2 (Spanned Cycles)**.** Let $n \in \mathbb{P}$ and $\pi \in S_n$. For every $M \subseteq [n]$, we define

$$C_M^n(\pi) \triangleq \bigcup_{m \in M} \left\{ \pi^\ell(m) \mid \ell \in \mathbb{P} \right\} \supseteq M,$$

the set of all elements of all cycles of π that contain at least one element of M.

Given n and M, we study the distribution of $\left|C_M^n(\pi)\right|$, i.e. the combined length of all cycles of π that intersect M, for a random permutation π that is uniformly distributed in S_n. More formally, in the probability space $\left(S_n, 2^{S_n}, U(S_n)\right)$, consisting of S_n as sample space and with the uniform measure over possible outcomes, we study the distribution of the random variable $\left|C_M^n\right|$; we henceforth work in this space, and denote the outcome of the experiment underlying it by $\pi \in S_n$. We note that since $\pi \sim U(S_n)$, the distribution of $\left|C_M^n\right|$ is the same for sets $M \subseteq [n]$ of equal size, i.e. this distribution depends on M only through $|M|$; we thus consider, for ease of presentation, only subsets $M \subseteq [n]$ of the form $M = [m]$ for some $m \leq n$, and for the sake of succinctness define:

Definition 3. $L_m^n(\pi) \triangleq \left|C_{[m]}^n(\pi)\right| \in [m, n]$.

We now turn to state the main result of this paper.

Lemma 4 (Distribution of L_m^n)**.** *Let $m \leq n$.*

i. $\mathbf{Pr}\left[L_m^n = \ell\right] = \frac{\binom{\ell-1}{m-1}}{\binom{n}{m}}$, *for all* $\ell \in [m, n]$.

ii. $\mathbf{E}\left[L_m^n\right] = \frac{m \cdot (n+1)}{m+1}$.

iii. $\mathbf{E}\left[{L_m^n}^{(k)}\right] = \frac{m \cdot \left[(n+1)^{(k)}\right]}{m+k}$, *for all* $k \in \mathbb{P}$.

iv. $\mathbf{Var}\left[L_m^n\right] = \frac{m \cdot (n+1) \cdot (n-m)}{(m+1)^2 \cdot (m+2)}$.

Remark 5 (Equivalent formulations of Lemma 4(i))**.**

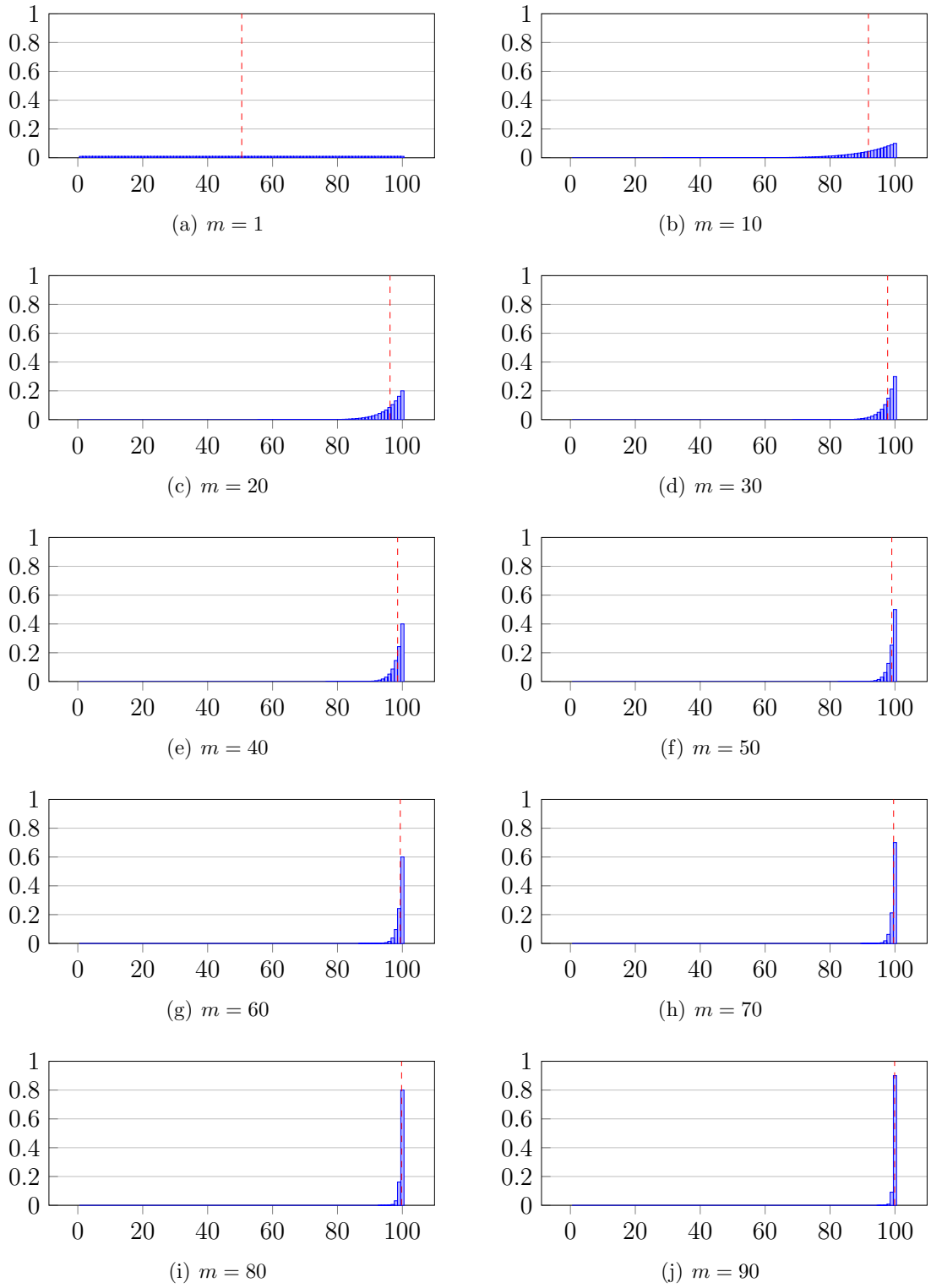
- $\mathbf{Pr}\left[L_m^n = \ell\right] = \frac{m}{n} \cdot \prod_{j=1}^{m-1} \frac{\ell - j}{n - j}$.
- $\left(\mathbf{Pr}\left[L_m^n = \ell\right]\right)_{\ell=m}^{n}$ *is the prefix of length* $n-m+1$ *of the* m*'th diagonal*[1] *of Pascal's triangle [3], normalized to sum-up to 1.*

Corollary 6. *The expected fraction of the elements of* $[n]$ *that are contained in cycles of* π *that are disjoint from* $[m]$ *is less than* $\frac{1}{m+1}$, regardless of the value of n. *Furthermore, the standard deviation of this fraction is less than* $\frac{1}{m+1}$ *as well.*

Corollary 7. $\mathbf{Pr}\left[C_{[m]}^n = [n]\right] = \frac{m}{n}$.

Corollary 6 shows that as m grows, $C_{[m]}^n$ quickly grows, *regardless of* n, to cover almost all of $[n]$, and its size L_m^n concentrates on large values (see also Fig. 1); nonetheless, Corollary 7 shows that the probability for $C_{[m]}^n$ to cover all of $[n]$ grows considerably slower in a sense, esp. for large n. This is demonstrated by the following example.

[1] The sequences known today as diagonals of Pascal's triangle are depicted as rows in Pascal's treatise.



(a) $m = 1$ (b) $m = 10$

(c) $m = 20$ (d) $m = 30$

(e) $m = 40$ (f) $m = 50$

(g) $m = 60$ (h) $m = 70$

(i) $m = 80$ (j) $m = 90$

Figure 1: Distribution of L_m^n for $n = 100$ and varying values of m. The (red) dashed vertical line marks the expectation.

Example 8. For $n = 1000$, $C^{1000}_{[100]}$ is expected to cover more than 99% of all elements (with std. dev. $\sigma < 1\%$), while the probability for it to cover all 1000 elements is $\frac{1}{10}$.

We present three proofs for Lemma 4(i). The first proof is a recursive one, calculating the distribution for m given the distribution for $m-1$. The second proof is an enumerative one, directly and succinctly proving the special case in which $\ell = n$ (i.e. Corollary 7), and proving the general case by reduction to this special case. The third proof, also enumerative, provides an interpretation of the nominator and the denominator of the r.h.s. of the equality in Lemma 4(i).

Probabilistic proof of Lemma 4(i). For all $m \leq \ell \leq n$, we define

$$p^n_m(\ell) \triangleq \mathbf{Pr}\big[L^n_m = \ell\big].$$

Throughout this proof, we make extensive use of the following well-known [4, p. 7] identity:

$$\forall m, n \in \mathbb{P} : \binom{n}{m} = \sum_{j=m-1}^{n-1} \binom{j}{m-1}, \tag{1}$$

obtained either inductively as in [4], or by conditioning upon the maximum element in the chosen set of m-out-of-n elements.

We prove, by induction on $\tilde{m}$, that $p^{\tilde{n}}_{\tilde{m}}(\tilde{\ell}) = \frac{\binom{\tilde{\ell}-1}{\tilde{m}-1}}{\binom{\tilde{n}}{\tilde{m}}}$ for all $\tilde{m} \leq \tilde{\ell} \leq \tilde{n}$.

Base: Let $\ell \leq n$. We show that the claim holds for $\tilde{m} = 1, \tilde{\ell} = \ell$, and $\tilde{n} = n$. We observe that $p^n_1(\ell)$ is simply the probability for the cycle of $\pi \sim U(S_n)$ that contains the element 1 to have length ℓ. It is well-established [1, p. 24] that the length of this cycle is uniformly distributed in $[n]$, yielding $p^n_1(\ell) = \frac{1}{n} = \frac{\binom{\ell-1}{0}}{\binom{n}{1}}$, as required.

Step: Let $1 < m \leq \ell \leq n$, and assume that the claim holds for $\tilde{n} = n$, $\tilde{m} = m-1$, and all $\tilde{\ell} \in [m-1, n]$; furthermore, assume that the base case holds whenever $\tilde{\ell} \leq \tilde{n} < n$. We claim that the following recurrence relation holds:

$$p^n_m(\ell) = p^n_{m-1}(\ell) \cdot \tfrac{\ell-m+1}{n-m+1} + \sum_{j=m-1}^{\ell-1} p^n_{m-1}(j) \cdot \tfrac{n-j}{n-m+1} \cdot p^{n-j}_1(\ell - j). \tag{2}$$

We justify Eq. (2) using the law of total probability, by conditioning upon the value of $j \triangleq L^n_{m-1} \in [m-1, n]$: If $j > \ell$, then obviously $L^n_m \geq j > \ell$ with probability 1. If $j = \ell$, then $L^n_m = \ell$ iff $m \in C^n_{[m-1]}$, which holds with probability $\frac{\left|C^n_{[m-1]} \setminus [m-1]\right|}{|[m,n]|} = \frac{\ell-m+1}{n-m+1}$. Otherwise, i.e. if $m-1 \leq j < \ell$, then $L^n_m = \ell$ iff both $m \notin C^n_{[m-1]}$ and $\left|C^n_{\{m\}}\right| = \ell - j$; the first condition holds with probability $\frac{\left|[n] \setminus C^n_{[m-1]}\right|}{|[m,n]|} = \frac{n-j}{n-m+1}$, and the second (conditioned upon the first) — with probability $p^{n-j}_1(\ell - j)$, since $\pi|_{[n]\setminus C^n_{[m-1]}}$, given $C^n_{[m-1]}$, is uniformly distributed in $S_{[n]\setminus C^n_{[m-1]}} \cong S_{n-j}$.

Plugging the induction hypotheses for $\tilde{m} = m-1$, and the base case for $\tilde{n} = n-j$, into Eq. (2), we obtain:

$$p^n_m(\ell) = \frac{\binom{\ell-1}{m-2}}{\binom{n}{m-1}} \cdot \frac{\ell-m+1}{n-m+1} + \sum_{j=m-1}^{\ell-1} \frac{\binom{j-1}{m-2}}{\binom{n}{m-1}} \cdot \frac{n-j}{n-m+1} \cdot \frac{1}{n-j} =$$

$$= \frac{1}{\binom{n}{m-1}\cdot(n-m+1)}\cdot\left(\binom{\ell-1}{m-2}\cdot(\ell-m+1)+\sum_{j=m-1}^{\ell-1}\binom{j-1}{m-2}\right)=$$
$$= \frac{1}{\binom{n}{m}\cdot m}\cdot\left(\binom{\ell-1}{m-2}\cdot(\ell-m+1)+\sum_{j=m-1}^{\ell-1}\binom{j-1}{m-2}\right)=$$
$$= \frac{1}{\binom{n}{m}\cdot m}\cdot\left((m-1)\cdot\binom{\ell-1}{m-1}+\sum_{j=m-1}^{\ell-1}\binom{j-1}{m-2}\right)= \qquad \text{By Eq. (1)}$$
$$= \frac{1}{\binom{n}{m}\cdot m}\cdot\left((m-1)\cdot\binom{\ell-1}{m-1}+\binom{\ell-1}{m-1}\right)=$$
$$= \frac{1}{\binom{n}{m}\cdot m}\cdot m\cdot\binom{\ell-1}{m-1}=\frac{\binom{\ell-1}{m-1}}{\binom{n}{m}},$$

and the proof by induction is complete. We note that by Eq. (1), we immediately verify that indeed $\sum_{\ell=m}^{n} p_m^n(\ell)=1$ for all $m\le n$. □

As mentioned above, before proceeding to prove the remaining parts of Lemma 4, we first present two additional, significantly different, proofs for Lemma 4(i). Both of these proofs, while of distinct flavours, make use of the following definition.

Definition 9. Let $n\in\mathbb{P}$. For every $\pi\in S_n$ and $k\le n$, by slight abuse of notation we denote by $\pi|_k\in S_k$ the permutation obtained by inspecting the cycle-structure representation of π and removing all elements of $[k+1,n]$ from it. More formally, for every $j\in[k]$, we define $\pi|_k(j)\triangleq\pi^{\ell_j^{\pi,k}}(j)$, where $\ell_j^{\pi,k}$ is the smallest positive integer s.t. $\pi^{\ell_j^{\pi,k}}(j)\in[k]$.

Example 10. If $\pi|_6=(365)(24)(1)$ (in cycle-structure representation), then the cycle-structure representation of π is of the form

$$\cdots(3\ldots6\ldots5\ldots)(2\ldots4\ldots)(1\ldots),$$

where the first ellipsis stands for zero or more cycles disjoint from the set [6], and each following ellipsis stands for zero or more consecutive elements within a cycle. (E.g. $\pi\triangleq(8)(3\ 10\ 11\ 6\ 5\ 7)(2\ 4)(1\ 9)\in S_{11}$ is of this form.) In fact, for every $j\in[6]$, the ellipsis immediately following j stands for precisely $\ell_j^{\pi,6}-1$ (as defined in Definition 9) elements, while the first ellipsis stands for a product of cycles of combined length $n-\sum_{j=1}^{6}\ell_j^{\pi,6}$.

Enumerative proof by reduction for Lemma 4(i). For every $m\le\ell\le n$, we define

$$\Pi_m^n(\ell)\triangleq\left\{\pi\in S_n\mid L_m^n(\pi)=\ell\right\}.$$

We show that $\left|\Pi_m^n(\ell)\right|=\binom{n-m}{\ell-m}\cdot m\cdot(\ell-1)!\cdot(n-\ell)!=n!\cdot\frac{\binom{\ell-1}{m-1}}{\binom{n}{m}}$. We first show this for the special case of $\ell=n$, i.e. we show that for all $m\le n$, the set $\Pi_m^n(n)$, of permutations on $[n]$ with all cycles intersecting $[m]$, is of size $m\cdot(n-1)!$.

Consider the following argument for the equality $|S_n|=n!$, tracing the construction of a permutation $\pi\in S_n$ by iteratively constructing $\pi|_1$, then $\pi|_2$, and so fourth until $\pi|_n=\pi$. Obviously, $\pi|_1=(1)$. To obtain $\pi|_2$ from $\pi|_1$, a 2-way choice is made: the

element 2 may be placed either (immediately) after 1 in its cycle, or in a new (singleton) cycle. To obtain $\pi|_3$, a 3-way choice is made: the element 3 may now be placed either after 1 in its cycle, after 2 in its cycle, or in a new cycle. More generally, to obtain $\pi|_k$ from $\pi|_{k-1}$, for $k \in [2,n]$, a k-way choice is made: the element k may be placed either after some element $j \in [k-1]$ in its cycle (more formally, setting $\pi|_k{}^{-1}(k) = j$ and $\pi|_k(k) = \pi|_{k-1}(j)$), or in a new cycle (i.e. having k a fixed point of $\pi|_k$). Thus, we obtain that there are $n!$ ways to construct a permutation $\pi \in S_n$, each resulting in a distinct outcome (as π uniquely determines $\pi|_k$ for all $k \leq n$), as required. We now note that construction of a permutation $\pi \in \Pi_m^n(n)$ may be undertaken in a very similar manner, with the only difference being that the elements of $[m+1,n]$ may not be placed in new cycles, thus reducing the choice for each $k \in [m+1,n]$ from a k-way one to a $(k-1)$-way one. By similar reasoning, we therefore obtain $\left|\Pi_m^n(n)\right| = m! \cdot m^{(n-m)} = m \cdot (n-1)!$.

We now move on to show the general case. A permutation $\pi \in \Pi_m^n(\ell)$ may be constructed as follows: First, choose a subset $I \subseteq [m+1,n]$ of size $\ell - m$ as the additional elements, in addition to $[m]$, of $C_{[m]}^n(\pi)$. ($\binom{n-m}{\ell-m}$ options.) Next, choose any permutation on $[m] \cup I$ in which all cycles intersect $[m]$ — this permutation constitutes the product of the cycles of π that intersect $[m]$. ($m \cdot (\ell-1)!$ options, by the above special case.) Finally, choose any permutation on $[n] \setminus ([m] \cup I)$ as the product of the remaining cycles of π, i.e. those that do not intersect $[m]$. ($(n-\ell)!$ options.) We thus obtain $\left|\Pi_m^n(\ell)\right| = \binom{n-m}{\ell-m} \cdot m \cdot (\ell-1)! \cdot (n-\ell)!$, as required. □

Direct enumerative proof for Lemma 4(i). Henceforth, we use the following convention when representing the cycle structure of any permutation: we write each cycle with its smallest element first, and write cycles in decreasing order of their first (i.e. smallest) element. E.g. the reader may verify that all cycle-structure representations in Example 10, and notably that of the general form (i.e. with ellipses) of π in that example, follow this convention. It is straightforward to check (see e.g. [5, Section 1.3], where a similar convention is used) that this representation is both unique, and unambiguous even if the parentheses are dropped. (Indeed, uniqueness implies unambiguity, since the number of ways to order $[n]$ in a row equals the number of permutations on $[n]$.)

Let $m \leq n$. For every $\pi \in S_n$, we denote by $\pi_{>m}$ the sequence consisting of the elements of $[m+1,n]$, ordered as in the cycle-structure representation (according to the above convention) of π. We claim that the mapping $\pi \mapsto \left(\pi|_m, (\ell_j^{\pi,m})_{j=1}^m, \pi_{>m}\right)$ is a bijection between S_n and $S_m \times \left\{(\ell_j)_{j=1}^m \in \mathbb{P}^m \mid \sum_{j=1}^m \ell_j \leq n\right\} \times S_{[m+1,n]}$, where by very slight abuse of notation we think of a permutation $\tau \in S_{[m+1,n]}$ as the sequence $\big(\tau(m+1), \ldots, \tau(n)\big)$. Under the notation of Example 10, $\pi|_m$ determines the general form of π w.r.t $[m]$, while $(\ell_j^{\pi,m})_{j=1}^m$ determine the number of elements each ellipsis stands for, and $\pi_{>m}$, given all of these, determines the exact content of each ellipsis (the unambiguity of the cycle-structure representation in the face of dropping parentheses is used when populating the first ellipsis). The reader who is not yet convinced of the validity of this bijection claim in general, may verify that this mapping is onto, and that the sizes of the domain and of the image (see the last equality of Eq. (3) below for the size of the second multiplicand) match.

Let $\ell \in [m+1,n]$. We observe that for every $\pi \in S_n$, by definition $L_m^n(\pi) = \sum_{j=1}^m \ell_j^{\pi,m}$

(see e.g. the suffix of Example 10). Thus, we have that for every $\sigma \in S_m$ and $\tau \in S_{[m+1,n]}$,

$$\left|\left\{\pi \in S_n \;\middle|\; \pi|_m = \sigma \;\&\; \pi_{>m} = \tau \;\&\; L_m^n(\pi) = \ell\right\}\right| = \left|\left\{(\ell_j)_{j=1}^m \in \mathbb{P}^m \;\middle|\; \textstyle\sum_{j=1}^m \ell_j = \ell\right\}\right| = \binom{\ell-1}{m-1}.$$

(For the calculation of the number of m-compositions of ℓ see, e.g. [5, Section 1.2].) For comparison, dropping the conditioning on $L_m^n(\pi)$ we have

$$\left|\left\{\pi \in S_n \;\middle|\; \pi|_m = \sigma \;\&\; \pi_{>m} = \tau\right\}\right| = \left|\left\{(\ell_j)_{j=1}^m \in \mathbb{P}^m \;\middle|\; \textstyle\sum_{j=1}^m \ell_j \leq n\right\}\right| = \binom{n}{m}, \qquad (3)$$

since such $(\ell_j)_{j=1}^m$ are in one-to-one correspondence with $(m+1)$-compositions of $n+1$, where the $(m+1)$'th element designates the successor of the remainder. Combining these, we obtain the slightly stronger result that

$$\mathbf{Pr}\left[L_m^n = \ell \;\middle|\; \pi|_m = \sigma \;\&\; \pi_{>m} = \tau\right] = \frac{\binom{\ell-1}{m-1}}{\binom{n}{m}},$$

for every choice of $\sigma \in S_m$ and $\tau \in S_{[m+1,n]}$. As the r.h.s. depends on neither σ nor τ, we have

$$\mathbf{Pr}\left[L_m^n = \ell\right] = \frac{\binom{\ell-1}{m-1}}{\binom{n}{m}},$$

as required. □

Finally, we prove the remaining parts of Lemma 4.

Proof of Lemma 4(ii to iv). We prove Part ii directly by definition of expectation:

$$\begin{aligned}
\mathbf{E}\left[L_m^n\right] &= \sum_{\ell=m}^{n} \mathbf{Pr}\left[L_m^n = \ell\right] \cdot \ell = \frac{1}{\binom{n}{m}} \cdot \left(\sum_{\ell=m}^{n} \binom{\ell-1}{m-1} \cdot \ell\right) = \\
&= \frac{1}{\binom{n}{m}} \cdot \left(m \cdot \sum_{\ell=m}^{n} \binom{\ell}{m}\right) = \qquad \text{By Eq. (1)} \\
&= \frac{1}{\binom{n}{m}} \cdot \left(m \cdot \binom{n+1}{m+1}\right) = \frac{m \cdot (n+1)}{m+1}.
\end{aligned}$$

More generally, all rising-factorial moments may be calculated in a similar manner:

$$\begin{aligned}
\mathbf{E}\left[L_m^{n\,(k)}\right] &= \sum_{\ell=m}^{n} \mathbf{Pr}\left[L_m^n = \ell\right] \cdot \ell^{(k)} = \frac{1}{\binom{n}{m}} \cdot \left(\sum_{\ell=m}^{n} \binom{\ell-1}{m-1} \cdot \ell^{(k)}\right) = \\
&= \frac{1}{\binom{n}{m}} \cdot \left(m^{(k)} \cdot \sum_{\ell=m}^{n} \binom{\ell+k-1}{m+k-1}\right) = \qquad \text{By Eq. (1)} \\
&= \frac{1}{\binom{n}{m}} \cdot \left(m^{(k)} \cdot \binom{n+k}{m+k}\right) = \frac{m \cdot \left[(n+1)^{(k)}\right]}{m+k}.
\end{aligned}$$

The rising-factorial moments give rise to calculation of the raw moments and of the central moments. The second raw moment, for instance, is given by

$$\mathbf{E}\left[L_m^{n\,2}\right] = \mathbf{E}\left[L_m^{n\,(2)}\right] - \mathbf{E}\left[L_m^n\right] =$$

$$
\begin{aligned}
&= \frac{m \cdot (n+1) \cdot (n+2)}{m+2} - \frac{m \cdot (n+1)}{m+1} = \\
&= m \cdot (n+1) \cdot \left(\frac{n+2}{m+2} - \frac{1}{m+1} \right) = \\
&= m \cdot (n+1) \cdot \frac{(m+1)(n+2) - (m+2)}{(m+2)(m+1)} = \\
&= \frac{m \cdot (n+1) \cdot (mn+n+m)}{(m+2)(m+1)},
\end{aligned}
$$

and thus the variance is given by

$$
\begin{aligned}
\mathbf{Var}\big[L_m^n\big] &= \mathbf{E}\big[{L_m^n}^2\big] - \mathbf{E}^2\big[L_m^n\big] = \\
&= \frac{m \cdot (n+1) \cdot (mn+n+m)}{(m+2)(m+1)} - \left(\frac{m \cdot (n+1)}{m+1} \right)^2 = \\
&= \frac{m \cdot (n+1)}{m+1} \cdot \left(\frac{mn+n+m}{m+2} - \frac{m \cdot (n+1)}{m+1} \right) = \\
&= \frac{m \cdot (n+1)}{m+1} \cdot \left(\frac{mn+n+m}{m+2} - \frac{mn+m}{m+1} \right) = \\
&= \frac{m \cdot (n+1)}{m+1} \cdot \frac{(m+1)(mn+n+m) - (m+2)(mn+m)}{(m+2)(m+1)} = \\
&= \frac{m \cdot (n+1)}{m+1} \cdot \frac{n-m}{(m+2)(m+1)} = \\
&= \frac{m \cdot (n+1) \cdot (n-m)}{(m+1)^2 \cdot (m+2)}.
\end{aligned}
$$

□

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