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CONSISTENT VOTING SYSTEMS REVISITED: COMPUTATION AND AXIOMATIC CHARACTERIZATION

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Consistent Voting Systems Revisited: Computation and Axiomatic Characterization

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Abstract

We add two results to the theory of consistent voting. Let *M* be the set of all survivors of some feasible elimination procedure. We prove that i) *M* can be computed in polynomial time for each profile of preferences and ii) *M* is characterized by anonymity, non-imposition, Maskin monotonicity, and additive blocking.

1. Introduction

The Gibbard-Satterthwaite theorem (Gibbard 1973 and Satterthwaite 1975) says that for every non-dictatorial social choice function whose range contains at least three alternatives, sincere voting is not a dominant strategy. As a corollary we obtain that for every non-dictatorial social choice function there exists a situation where either there exists no Nash equilibrium or the sincere outcome is not the unique Nash equilibrium outcome (see Theorem 1).Thus, one of the problems that a designer of voting schemes faces is strategic distortion of the outcome. This has led the author to introduce the class of exactly and strongly consistent social choice functions that resist distortion to a large extent (Peleg 1978). Indeed, for an exactly and strongly consistent social choice function the sincere outcome is always an outcome of a strong Nash equilibrium of the associated voting game. Of course, the sincere outcome may not be the unique outcome of a strong Nash equilibrium; however, it will always belong to the core of the relevant voting game. The foregoing paper has been followed by several investigations of the set of exactly and strongly consistent social choice functions: Dutta and Pattanaik (1978), Polishchuk (1978), Ishikawa and Nakamura (1980), Oren (1981), Kim and Roush (1981), Holzman (1986), and Peleg and Peters (2006). Also the books of Peleg (1984), Abdou and Keiding (1991), and Peleg and Peters (2010) have chapters devoted to consistent voting.

In this paper we address two problems that were left open by the foregoing literature: computability and axiomatization. In Section 3 we show that the values (i.e., output) of exactly and strongly consistent social choice functions may be computed in polynomial time. Exactly and strongly consistent social choice functions are constructed by means of feasible elimination procedures. Such procedures are determined by assigning blocking coefficients to the various alternatives. In Section 4 we consider the social choice correspondence that assigns to each profile of linear preferences the set of alternatives that survive some feasible elimination procedure (holding the blocking coefficients fixed). This correspondence is characterized in Theorem 5 by non-imposition, anonymity, Maskin monotonicity, and additive blocking.

2. Preliminaries

Let A be a set of m *alternatives*, $m \geq 2$, and let $N = \{1, \ldots, n\}$, $n \geq 2$, be a set of *voters*. Denote by L the set of all *linear orderings* of A. A *social choice function* (*SCF*) is a function $F: L^N \rightarrow A$. If $R^N \varepsilon L^N$ is a profile of preferences of the players and F is an *SCF*, then the pair (F, R^N) defines, in an obvious way, an ordinal n-person game in strategic form. $Q^N \varepsilon L^N$ is a *Nash equilibrium (NE)* of the game (F, R^N) if $F(Q^N) R^i F(P^i, Q^{N \setminus \{i\}})$ for all $i \varepsilon N$ and $P^i \varepsilon L$. An SCF F is *nonmanipulable* if for all $R^N \varepsilon L^N$, R^N itself is a Nash equilibrium of the game (F, R^N). An SCF F is *dictatorial* if there exists a player $d \varepsilon N$, a *dictator,* such that $F(R^N)R^d x$ for all $x \varepsilon A^*$ and all $R^N \varepsilon L^N$, where A^* is the range of F. The *Gibbard-Satterthwaite (G-S) theorem* tells us that if an SCF F is nonmanipulable and its range contains at least three alternatives, then F is dictatorial. Thus, if for example an SCF F is non-dictatorial and surjective and $m \geq 3$, then F is *manipulable*; that is, there exists a preference profile R^N that is not an NE of the game (F, R^N). If S is a set, then we denote by $P(S)$ the set of all subsets of S and by $P_0(S)$ the set of all nonempty subsets of S. A social choice *correspondence (SCC)* is a function $H: L^N \rightarrow P_0(A)$. We do not distinguish between the SCF F and the SCC H_F where $H_F(R^N) = \{F(R^N)\}$ for all $R^N \varepsilon L^N$. An SCC H is *Maskin monotonic* if it satisfies the following: let R^N and Q^N be in L^N and let $x \varepsilon H(Q^N)$. If for all $y \varepsilon A$ and $i \varepsilon N$ $x Q^i y$ implies $x R^i y$, then $x \varepsilon H(R^N)$. Let F be an SCF. For $R^N \varepsilon L^N$ denote

$$E(R^N) = \{Q^N : Q^N \text{ is an NE of } (F, R^N)\}.$$

We say that F is *distorted* if for some $R^N \varepsilon L^N$, $\{F(R^N)\}$ does not coincide with the set $F(E(R^N))$. We are now able to prove

Theorem 1. *If an SCF F is not distorted and its range contains at least three alternatives, then it is dictatorial.*

Proof. As F is not distorted it implements itself by Nash equilibria. Hence it is Maskin monotonic (see, e.g., Peleg 1984, Lemma 6.5.1). Thus, by Muller and Satterthwaite (1977) F is dictatorial. Q.E.D.

An SCC H is *Paretian* if for all x and y in A and $R^N \varepsilon L^N$, if $x \neq y$ and $y R^i x$ for all $i \varepsilon N$, then x is not in $H(R^N)$. A family of SCF's that are non-dictatorial and Paretian and not "easily" distorted was suggested in Peleg (1978). First we need a definition.

Definition 1. Let F be an SCF and let $R^N \varepsilon L^N$. A preference profile Q^N is a *strong* NE *(SNE)* of (F, R^N) if for every non-empty subset S of N and for every $P^S \varepsilon L^S$ there exists $i \varepsilon S$ such that $F(Q^N) R^i F(Q^{N \setminus S}, P^S)$.

Definition 2.A surjective SCF F is *exactly and strongly consistent (ESC)* if for every $R^N \varepsilon L^N$ there exists an SNE Q^N of (F, R^N) such that $F(Q^N) = F(R^N)$.

An SCC H is *anonymous* if for all $R^N \varepsilon L^N$ and for all permutations π of N, $H(R^1, \dots, R^n) = H(R^{\pi(1)}, \dots, R^{\pi(n)})$. We shall now describe a method for obtaining an important class of anonymous ESC SCF's.

Definition 3. Assume that $n + 1 \geq m$ and let $b{:}A \rightarrow N$ satisfy $\Sigma_{x \varepsilon A}\, b(x) = n + 1$. Let $R^N \varepsilon L^N$. A *feasible elimination procedure (f.e.p.)* is a sequence $(x_1, C_1; \dots ; x_{m-1}, C_{m-1}; x_m)$ such that

1) $C_{1,\dots,} C_{m-1}$ are pairwise disjoint and $\#C_j = b(x_j)$ for $j = 1, \dots, m - 1$.

2) $A = \{x_{1,\dots,} x_m\}$.

3) $x_k R^i x_j$ for $k = j + 1, \dots, m$, all $i \varepsilon C_j$, and $j = 1, \dots, m - 1$.

We remark that there is always at least one f.e.p. An alternative y is R^N- *maximal* if there exists an f.e.p. $(x_{1,} C_1; \dots ; x_{m-1,} C_{m-1}; y)$. We denote

$$M(R^N) = \{x \varepsilon A{:}\ x \text{ is } R^N - \text{maximal}\}.$$

M(.) is an anonymous and Paretian social choice correspondence. It may also be shown that it is Maskin monotonic (see, e.g., Peleg and Peters 2010, Theorem 9.3.6). Hence, it admits an anonymous, Paretian, and monotonic selection. (An SCF F is *monotonic* if it satisfies the following condition: if $R^N \varepsilon L^N$, $x = F(R^N)$, and Q^N is obtained from R^N by improving the position of x and leaving the relative positions of all other alternatives intact, then $x = F(Q^N)$.) This last fact is important in view of the following.

Theorem 2. *Every selection from M is ESC.*

(See, e.g., Peleg and Peters 2010, Theorem 9.2.6.) In order to formulate the converse to Theorem 2 we need the following definition.

Definition 4. A function $E{:}\ P(N) \rightarrow P(P_0(A))$ is an *effectivity function (EF)* if: *i)* $E(N) = P_0(A)$, *ii)* $A \varepsilon E(S)$ for every S in $P_0(N)$, and iii) $E(\emptyset) = \emptyset$.

With b we associate the EF E_b given by

$$B \varepsilon E_b(S) \text{ iff } \#S \geq \Sigma_{x \varepsilon A \setminus B}\, b(x) \text{ for all } B \varepsilon P_0(A) \text{ and } S \varepsilon P_0(N).$$

An EF is anonymous if $E(S)$ depends only on $\#S$ for all S. E_b is anonymous. An EF may be considered as the coalition function a la von Neumann and Morgenstern of some game form. In particular, if F is a surjective SCF then its EF E^F is defined by

$B \varepsilon E^F(S)$ *iff there exists* $R^S \varepsilon L^S$ *such that for all* $Q^{N \setminus S} \varepsilon L^{N \setminus S}$, $F(R^S, Q^{N \setminus S}) \varepsilon B$.

We are now able to formulate

Theorem 3. *An SCF F is a selection from M iff F is ESC and* $E^F = E_b$.

A player *i* is a *(weak) vetoer* with respect to an EF *E* if $E(\{i\}) \neq \{A\}$. Thus for a fixed *b(.)* we obtain by Theorem 3 all ESC SCF's with a fixed anonymous EF. It may be shown that by varying *b(.)* we obtain all ESC SCF's with anonymous EF and without vetoers (see Peleg and Peters 2010, Example 10.5.4).

3. A method for computing M(.)

We now prove that *M* may be computed in polynomial time.

Instance. A, N, $b: A \rightarrow N$ *with* $\Sigma_{y \varepsilon A} b(y) = n + 1$, $R^N \varepsilon L^N$, *and* $x \varepsilon A$.

Question. Is x in $M(R^N)$*?*

In order to describe our algorithm we need the following theorem.

Theorem 4. *x is in* $M(R^N)$ *iff there exist disjoint subsets of N* $S(y)$, $y \varepsilon A \setminus \{x\}$, *such that*

i)$x R^i y$ *for all* $i \varepsilon S(y)$, *and*

ii) $\# S(y) = b(y)$ *for all* $y \varepsilon A \setminus \{x\}$,

(see Peleg 1984, Lemma 5.3.5). Now for $y \varepsilon A \setminus \{x\}$ let $S(y) = \{i : x R^i y\}$. The computation of the sets *S(y)* is polynomial. We say that *y* "knows" *i* if $x R^i y$. . Thus, we have a bipartite graph with vertices in the union of $A \setminus \{x\}$ and *N*. By Theorem 4, *x* is in $M(R^N)$ iff the alternatives $y \varepsilon A \setminus \{x\}$ have disjoint harems of size *b(y)* for each *y*. This problem is well known to be equivalent to finding a maximum matching for the graph under consideration. The problem of finding a maximum matching is polynomial (see Hopcroft and Karp 1973). Repeating the foregoing procedure m times is still polynomial (in m and n). Sharp bounds are easily obtained.

Knowing *M(.)*, we may compute the following class of anonymous, Paretian, monotonic, and exactly and strongly consistent SCF's. Let $Q \varepsilon L$. Define an SCF *F* by

$$F(R^N) \text{ is the Q-maximum of } M(R^N) \text{ for every } R^N \varepsilon L^N.$$

Then *F* has all the foregoing properties and its values may be computed in polynomial time.

4. An axiomatic characterization of M(.)

We shall give in this section an axiomatic characterization of *M(.)*. First, we need some new concepts. Let $H: L^N \rightarrow P_0(A)$ be an SCC, let *S* be a nonempty subset of N, and let *B* be a nonempty proper subset of *A*. By $R^S(B)$ we shall denote any *S*-profile of preferences such that $x\, R^i\, y$ for all $x\ \varepsilon\ B$, $y\ \varepsilon\ A \setminus B$, and $i\ \varepsilon\ S$. The *blocking coefficient* of *B*, *b(B)*, is defined by

$$b(B) = \min \{\#S: H(R^S(A \setminus B), Q^{N \setminus S}) \text{ is contained in } A \setminus B \text{ for all } R^S(A \setminus B) \text{ and } Q^{N \setminus S}\ \varepsilon\ L^{N \setminus S}\}.$$

(The minimum over the empty set is defined as $n + 1$.) Also $b(A) = n + 1$ and $b(\emptyset) = 0$. *H* is *independently blocking* if *b(.)* is additive; that is, if B_1 and B_2 are disjoint subsets of *A*, and *B* is the union of B_1 and B_2, then $b(B) = b(B_1) + b(B_2)$. *H* is *not imposed* if for every $x\ \varepsilon\ A$ there exists $R^N\ \varepsilon\ L^N$ such that $H(R^N) = \{x\}$. If *H* satisfies non-imposition then the effectivity function of *H*, E^H, is defined as follows. Let $S\ \varepsilon\ P_0(N)$ and let $B\ \varepsilon\ P_0(A)$. Then $B\ \varepsilon\ E^H(S)$ if there exists an *S*-profile R^S such that $H(R^S, Q^{N \setminus S})$ is contained in *B* for every $N \setminus S$-profile $Q^{N \setminus S}$. Finally, we need the following definition. Let $E: P(N) \rightarrow P(P_0(A))$ be an EF, let $R^N\ \varepsilon\ L^N$, let $S\ \varepsilon\ P_0(N)$, let $B\ \varepsilon\ E(S)$, and let $x\ \varepsilon\ A \setminus B$. *x is dominated by B via S at* R^N if $y\, R^i\, x$ for all $i\ \varepsilon\ S$ and $y\ \varepsilon\ B$. *x is dominated at* R^N if there exist *B* and *S* as above such that *B* dominates *x* via *S* at R^N . The *core of E at* R^N, $C(E, R^N)$, is the set of all undominated alternatives at R^N.

We have already mentioned that M is anonymous and Maskin monotonic. Also, it is obvious that M satisfies non-imposition and is independently blocking as $E^M = E_b$ (see Peleg and Peters 2010, Lemma 9.3.2). We shall now prove the converse result.

Theorem 5. *Let* $H: L^N \rightarrow P_0(A)$ *be an anonymous, non-imposed, Maskin monotonic, and independently blocking SCC. Furthermore, let* $b(x)$, $x\ \varepsilon\ A$, *be the blocking coefficients of H. Then H coincides with the SCC M, which is determined by feasible elimination procedures with respect to the blocking coefficients* $b(x)$, $x\ \varepsilon\ A$.

Proof. By assumption *H* is independently blocking. Hence, $b(A) = \Sigma_{x\, \varepsilon\, A}\, b(x) = n + 1$. Also, $b(x) \geq 1$ for all $x\ \varepsilon\ A$ as *H* is not imposed. Thus, *M* is well defined. We shall now prove that for every profile of preferences R^N, $M(R^N)$ is a subset of $H(R^N)$. Indeed, let $R^N\ \varepsilon\ L^N$ and let $x\ \varepsilon\ M(R^N)$. Then there exists an f.e.p. $(x_1, C_1; \ldots ;x_{m-1}, C_{m-1} ; x)$ with respect to R^N. Let now Q^N be the profile that is obtained from R^N by lowering x_j to the bottom of R^i for all $i\ \varepsilon\ C_j$ and for $j = 1, \ldots , m - 1$, and leaving everything else intact. By the definition of blocking coefficients $H(Q^N) = \{x\}$. Finally, as *H* is Maskin monotonic $x\ \varepsilon\ H(R^N)$.For the reverse inclusion we observe that *M* is equal to the core of E^M (Peleg and Peters 2010, Theorem 9.3.6). Also, *H* is contained in the core of E^H (see Lemma 6.5.6 in Peleg 1984; the reader should notice that since *H* is independently blocking, its first effectivity function (ibid., Definition 4.1.21) coincides with its alpha effectivity function, which we use exclusively in this paper). Furthermore, $E^H = E_b$ (Peleg 1984, Remark 6.2.35). We obtain that *H* is a sub-correspondence of *M*. Thus $H = M$.
Q.E.D.

We shall now prove that the four properties of Theorem 5 are logically independent. We have already seen that there exist anonymous selections from *M*. Such selections will satisfy non-imposition and will be independently blocking by Theorem 3. However, they cannot be Maskin monotonic (if $m \geq 3$) because of Muller and Satterthwaite (1977). Also, the Pareto correspondence is non-imposed, anonymous, and Maskin monotonic, but it is not independently blocking. Furthermore, a constant SCC (that is, an SCC *H* such that $H(R^N) = \{c\}$ for some $c \,\varepsilon\, A$ and all $R^N \varepsilon\, L^N$) satisfies all properties except non-imposition. For the independence of anonymity we consider the following example.

Example 1. *Let* $A = \{x, y\}$ *and let* $N = \{1, 2, 3, 4\}$. *Define an EF E by the following rules. Let* $E(\emptyset) = \emptyset$; $E(S) = \{y, A\}$ *if* $S = \{2, 3, 4\}$; $E(S) = \{x, A\}$ *if* $S \,\varepsilon\, \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$;$E(N) = P_0(A)$;*and* $E(S) = \{A\}$ *otherwise. Let further* $H(R^N) = C(E, R^N)$ *for all* $R^N \varepsilon\, L^N$. *Then H is Maskin monotonic, non-imposed, and independently blocking (* $b(x) + b(y) = 3 + 2 = 5 = b(A)$*). However, H is not anonymous.*

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