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BAYESIAN GAMES WITH A CONTINUUM OF STATES

By

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BAYESIAN GAMES WITH A CONTINUUM OF STATES

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ABSTRACT. Negative results on the the existence of Bayesian equilibria when state spaces have the cardinality of the continuum have been attained in recent years. This has led to the natural question: are there conditions that characterise when Bayesian games over continuum state spaces have measurable Bayesian equilibria? We answer this in the affirmative. Assuming that each type has finite or countable support, measurable Bayesian equilibria may fail to exist if and only if the underlying common knowledge σ-algebra is non-separable. Furthermore, anomalous examples with continuum state spaces have been presented in the literature in which common priors exist over entire state spaces but not over common knowledge components. There are also spaces over which players can have no disagreement, but when restricting attention to common knowledge components disagreements can exist. We show that when the common knowledge σ-algebra is separable all these anomalies disappear.

1. INTRODUCTION

What if we lived in a world in which Bayesian games were not guaranteed always to have Bayesian equilibria?

The effects might be felt widely throughout the literature, as it is difficult to exaggerate the importance which the concept of Bayesian games has attained in a wide range of subfields in economics and game theory, with subjects such as incomplete and asymmetric information models, signalling theory, principal-agent models, adverse selection and the provisions of public goods forming only a very

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partial list. Many papers in these fields start off by assuming the existence of an equilibrium and continuing their analyses from there. It would be challenging to gain significant theoretical traction, for example, in Bayesian truthful implementation and the related concepts of the revelation principle, the revenue equivalence theorem and optimal Bayesian methods, without first assuming that at least one Bayesian equilibrium exists in particular models being studied.

This isn't usually a concern at all, of course, since Harsányi (1967) proved (along with introducing the very concept of a Bayesian game) that every finite Bayesian game has an equilibrium. This positive result was extended to Bayesian games over countably many states in Simon (2003).

Over continuum state spaces, however, negative results have been shown in recent years. Simon (2003) presented an example of a three-player Bayesian game over a continuum state space with no Bayesian equilibrium.[1] Any hopes that positive results could be restored by considering approximate equilibria instead of exact equilibria were dashed when Hellman (2012b) showed an example of a two-player Bayesian game over a continuum state space with no measurable Bayesian ε-equilibrium for $\varepsilon \geq 0$.

These negative results are perturbing. One on occasion hears it said that it is sufficient to concentrate on finite games alone because the world itself is finite. However, as pointed out in Cotter (1991), since there are an infinity of continuous random variables, a more accurate statement would be that decision makers observe only a finite number of variables, each to a finite degree of accuracy. To model this as a finite space, however, requires that the modeller know *a priori* the set of variables actually observed and the degree of accuracy of each observed variable. Given this, the use of infinite games reflects the modeller's ignorance of the decision-making environment, just as infinite horizon models are routinely used to reflect ignorance of the life-span of the decision maker.

Furthermore, limiting attention to finite Bayesian games is far from being sufficient for capturing the full range of possible models that need to be studied. Many models in the literaure make use of uncountably many states. Examples include models in which prices (as in models of auctions or bargaining, such as that of Chatterjee and Samuelson (1983) for example) are the main state variables, or in which the main variables are profits and outputs in market models (for example Radner (1980)), continuous time points, accumulated wealth, accumulated resources, population percentages, share percentages and so forth.

[1] By the existence of an equilibrium we mean the existence of a measurable equilibrium. There are several reasons for restricting attention to measurable strategies (and hence measurable equilibria); to consider just two reasons, if a strategy is not measurable it cannot be constructed explicitly, and the payoffs of non-measurable strategies haven't got well-defined expected values. Measurability has in fact been included as a basic requirement in the definition of an equilibrium over uncountable spaces since the earliest literature on the subject (see Schmeidler (1973) for one such example). Throughout this paper we will therefore often use the term 'existence of an equilibrium' as synonymous with 'existence of a measurable equilibrium' without further qualification.

Furthermore, an extensively-used approach to dealing with a Bayesian game with a finite but large number of states is to analyse instead a similar game with a continuum of states. Myerson (1997), for example, informs readers of Chapter 2 of his textbook on game theory, when referring to Bayesian games, that 'it is often easier to analyze examples with infinite type sets than those with large finite type sets'.

Given the negative results mentioned earlier, however, modellers working with continuum state spaces face the perhaps uncomfortable situation in which they may need to check, in each separate model with which they are working, whether or not an equilibrium exists. This motivates our main result here, which is exhibiting conditions that guarantee the existence of Bayesian equilibria in Bayesian games over a continuum of states, restoring the confidence in the existence of equilibria in the class of games satisfying these conditions.

In most of the paper we assume that in the Bayesian games under consideration every atom of each player's posterior contains only a finite number of elements.[2] Note that every known example of a Bayesian game with no Bayesian equilibria satisfies this property, hence characterising conditions for the existence of equilibria in games with this property is of importance. Making this assumption is also concordant with some intuitions that although decision makers may *a priori* consider in their minds a continuum of possible states, for the purposes of observing a definite signal and moving to their posterior probabilities in most realistic cases they can only distinguish a finite number of posterior states to which they assign positive probability.

Part (I) of Theorem 2 then shows that if a Bayesian game satisfies the condition that the common knowledge σ-algebra of the underlying knowledge space is separable then there exists a measurable Bayesian equilibrium. Furthermore, this condition is not only sufficient, it is also necessary in the following sense, as shown in part (II) of Theorem 2: if Ω is a standard Borel space and $\mathcal{F}$ is a sub-σ-algebra of the Borel σ-algebra that is not separable and in which each atom is countable (and which can be generated by some beliefs), then there exists a Bayesian game Γ with state space Ω, a common prior, and common knowledge σ-algebra $\mathcal{F}$ that does not possess an ε-MBE for small enough $\varepsilon > 0$.

The condition of separability of the common knowledge σ-algebra also turns out to be the crucial factor in resolving a series of disturbing 'paradoxes' in models over continuum state spaces. These are detailed in Section 3: there are Bayesian games over continuum state spaces that have no Bayesian equilibria, yet if these games are restricted to being played over any common knowledge component of the players, Bayesian equilibria do exist; there are type spaces with common priors such that when restricting to any common knowledge component the resulting type space has no common prior; there are type spaces that exclude any possibility of disagreement between the players, but again when restricting to any common knowledge component the resulting type space does admit disagreements.

[2] This condition can be weakened to countably many elements; see Section 8.

These sorts of paradoxes are disturbing because they introduce instability in moving between the *ex ante* stage and the interim stage of analyses. Depending on the stage, one can get different answers to the questions of whether or not there exist equilibria, common priors or disagreements. As shown in this paper, however, all of these paradoxes disappear if the underlying knowledge spaces satisfy the condition of separability of the common knowledge σ-algebra.

Finally, we note that our results are attained mainly using results from descriptive set theory. In fact, we have found that there are parallels between concepts used in game theory and descriptive set theory concepts that are surprisingly useful for arriving at conclusions in game theoretic models. At several points in the body of the paper we strive to make these parallels explicit. Hopefully, these sorts of parallels can be deepened in future research, leading to more new results.

2. Mathematical Preliminaries

2.1. **Descriptive Set Theory Preliminaries.**

A *standard Borel* space is a topological space that is homeomorphic to a Borel subset of a Polish space.[3] For a standard Borel space X, let $\Delta(X)$ denote the space of regular Borel probability distributions on X, with the topology of weak convergence of probability measures, and let $\Delta_f(X) \subseteq \Delta(X)$ (resp. $\Delta_a(X) \subseteq \Delta(X)$) denote the subspace of finitely supported (resp. purely atomic) measures. $\Delta_f(X), \Delta_a(X)$ are Borel subsets[4] of $\Delta(X)$.

If $(\Omega, \mathcal{B})$ is a measurable space and $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{B}$, then a regular conditional distribution (henceforth, RCD) given $\mathcal{F}$ is a mapping $t : X \times \mathcal{F} \to [0,1]$ such that:

- For each $\omega \in \Omega$, $t(\omega)(\cdot)$ is a regular Borel probability measure on $(\Omega, \mathcal{B})$.
- For each $B \in \mathcal{B}$, the mapping $t(\cdot)(B)$ is $\mathcal{F}$-measurable.

If μ is a probability measure on $(\Omega, \mathcal{B})$, then an RCD t is an *RCD of μ given $\mathcal{F}$* if for each $A \in \mathcal{F}$, $B \in \mathcal{B}$,

$$\mu(A \cap B) = \int_A t(x)(B) d\mu(x) \tag{2.1}$$

Equivalently, an RCD t is an RCD of μ given $\mathcal{F}$ iff for all Borel $T \subseteq \Omega$,

$$t(\omega)(T) = E_\mu[1_T \mid \mathcal{F}](\omega),\ \mu\text{-a.e. } \omega \in \Omega$$

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[3] Equivalently, a measurable space $(X, \mathcal{B})$ is standard Borel if there exists a metric on X that makes it a complete separable metric space in such a way that $\mathcal{B}$ is then the Borel sigma-algebra, i.e., the smallest σ-algebra containing the open sets.

[4] $\Delta_f(X)$ can be viewed as $\cup_{n\in\mathbb{N}}\Delta_n(X)$, where $\Delta_n(X)$ consists of the probability measures supported on at most n points. $\Delta_n(X)$ can be viewed as the image in $\Delta(X)$ of $X^n \times \Delta_n$, where Δ_n is the n-simplex, under a finite-to-one map. Similarly, $\Delta_a(X)$ can be viewed as the image of $\{X \in X^{\mathbb{N}} \mid \forall n \neq m, x_n \neq x_m\} \times \{x \in \mathbb{R}^{\mathbb{N}} \mid x \geq 0, \sum_{n=1}^{\infty} x_n = 1\}$ under a countable-to-one map.

An RCD t is *proper at* ω (a notion introduced by Blackwell and Ryll-Nardzewski (1963)) if

$$t(\omega)(A) = 1, \text{ if } \omega \in A \in \mathcal{F}$$

An RCD is *proper* (resp. *proper* μ*-a.e.*) without qualification at any particular ω if it is proper at every point (resp. at μ-a.e. point) of Ω. Note that if an RCD is proper then in order to verify (2.1) it suffices to check that

$$\mu(B) = \int_{\Omega} t(x)(B) d\mu(x), \text{ for all } B \in \mathcal{B}$$

In terms that may be more familiar for game theorists, a proper RCD t of a probability measure μ may be thought of as the posterior t of a prior μ.

A sub σ-algebra of the Borel σ-algebra $\mathcal{F}$ on a space Ω is *separable* if there is a countable[5] collection of subsets $\{B_n\}_{n\in\mathbb{N}}$ of Ω that generates $\mathcal{F}$; that is, $\mathcal{F}$ is the smallest σ-algebra such that $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$.

A relation $\mathcal{E}$ on a standard Borel space Ω is said to be *Borel* if it is Borel as a subset of $\Omega \times \Omega$; i.e., if the set $\{(x, y) \in \Omega \mid x\mathcal{E}y\}$ is Borel. A Borel equivalence relationship is said to be *countable* if each equivalence class is countable. If $\mathcal{E}$ is a Borel equivalence relationship on a space Ω, then for each $\omega \in \Omega$, $[\omega]_{\mathcal{E}}$ (or just $[\omega]$ when it is clear to which relationship we are referring) denotes the equivalence class of ω.

Given a σ-algebra $\mathcal{F}$, there is an *induced equivalence relation*, denoted $\mathcal{E}_{\mathcal{F}}$, defined by

$$[\omega]_{\mathcal{E}_{\mathcal{F}}} := [\omega]_{\mathcal{F}} := \cap_{\omega\in A\in\mathcal{F}} A$$

Given a Borel equivalence relationship $\mathcal{E}$ on Ω and a set $T \subseteq \Omega$, the *saturation* $[T]_{\mathcal{E}}$ of T w.r.t. $\mathcal{E}$ is the smallest subset of Ω in $\mathcal{E}$ containing T; explicitly, $[T]_{\mathcal{E}} = \cup_{\omega\in T}[\omega]_{\mathcal{E}}$. We will sometimes write $[T]_{\mathcal{F}}$ instead of $[T]_{\mathcal{E}_{\mathcal{F}}}$ when $\mathcal{E}_{\mathcal{F}}$ is induced from $\mathcal{F}$. Conversely, if $\mathcal{E}$ is a Borel equivalence relationship, the induced σ-algebra $\mathcal{F}_{\mathcal{E}}$ is the collection of saturated Borel sets.

In terms that may be more familiar to game theorists used to working with finite atomic partitions as bases for σ-algebras, $[\omega]_{\mathcal{F}}$ is the atom containing ω and $[T]_{\mathcal{F}}$, for an event T, is the union of the atoms intersecting T.

If $\mathcal{F}$ is a σ-algebra such that each atom is countable, then it follows from the Lusin-Novikov theorem (see, for example, Theorem 18.10 of Kechris and Miller (2004)) that the induced equivalence relationship $\mathcal{E}_{\mathcal{F}}$ is Borel and that the saturation of each Borel set is Borel.

A *transversal* of an equivalence relationship is a set that intersects each equivalence class in exactly one point. A Borel equivalence relationship $\mathcal{E}$ is said to be *smooth* if there is a Borel mapping $\psi : \Omega \to X$, where X is some standard Borel space, such that $\psi(x) = \psi(y) \leftrightarrow x \sim y$.

Given a standard Borel space Ω and a sub-σ-algebra $\mathcal{F}$ of the Borel σ-algebra, we let $\Omega/\mathcal{F}$ denote the quotient space whose elements are the equivalence classes

[5] In this paper, 'countable' refers both to finite cardinalities and to countably infinite cardinalities.

induced by $\mathcal{F}$ and the induced σ-algebra consists of precisely the images of the sets in $\mathcal{F}$ under the quotient map.

We will make repeated use of the following proposition:

Proposition 1. *The following conditions are equivalent for a countable Borel equivalence relationship $\mathcal{E}_{\mathcal{F}}$ induced on Ω by a sub σ-algebra $\mathcal{F}$ of the Borel σ-algebra:*

(a) $\mathcal{F}$ is separable.
(b) There is a Borel transversal for $\mathcal{E}_{\mathcal{F}}$.
(c) The quotient space $\Omega/\mathcal{F}$ is standard Borel.
(d) The equivalence relationship $\mathcal{E}_{\mathcal{F}}$ is smooth.

Proof. The equivalence ((b))$\Longleftrightarrow$((c))$\Longleftrightarrow$((d)) is stated in Propositions 6.3 and 6.4 of Kechris and Miller (2004). If ((c)) holds and Λ is a countable collection of Borel sets generating the Borel structure on $\Omega/\mathcal{F}$, the collection $\{q^{-1}(U) \mid U \in \Lambda\}$, where $q : \Omega \to \Omega/\mathcal{F}$ is the quotient map, generates $\mathcal{F}$, and hence ((a)) holds.

Now, suppose ((a)) holds; let $B_1, B_2, \ldots \in \mathcal{F}$ generate $\mathcal{F}$. The map $p : \Omega \to 2^{\mathbb{N}}$ defined coordinate-wise by $p_n(\omega) = 1_{B_n}(\omega)$ is Borel and satisfies $p(x) = p(y)$ iff $x\mathcal{E}_{\mathcal{F}}y$, and hence $\mathcal{E}_{\mathcal{F}}$ is smooth. $\square$

2.2. **Knowledge Spaces.**

A *knowledge space* for a nonempty, finite set of players $\mathcal{P}$ is a given by a triple $(\Omega, \mathcal{P}, (\mathcal{F}^p)_{p\in\mathcal{P}})$, where Ω is a standard Borel space of *states*, and $\mathcal{F}^p$ for each $p \in \mathcal{P}$ is a σ-algebra over Ω, called p's *knowledge σ-algebra*. Intuitively, the elements in $\mathcal{F}^p$ represent the events that player p can identify, hence the name knowledge σ-algebra.

Let $\mathcal{F} := \cap_{p\in\mathcal{P}}\mathcal{F}^p$; that is, $\mathcal{F}$ is the finest σ-algebra contained in all the players' knowledge σ-algebras. $\mathcal{F}$ is called the *common knowledge σ-algebra* of the knowledge space. The elements of $\mathcal{F}$ intuitively represent the events of which all the players can have common knowledge.

For each $p \in \mathcal{P}$ and each set $N \subseteq \Omega$, let $K^p(N)$ denote the saturation of N w.r.t. $\mathcal{F}^p$, i.e., $K^p(N) = [N]_{\mathcal{F}^p}$. If $\omega \in \Omega$, write for short $K^p(\omega) = K^p(\{\omega\})$. Since the saturation of Borel sets under a Borel equivalence relationship is also Borel, we have:

Lemma 2. *If N is Borel, then so is $K^p(N)$.*

For each finite sequence $\hat{p} = (p_1, \ldots, p_k) \in \mathcal{P}^* := \cup_{n\geq 0}\mathcal{P}^n$ and $N \subseteq \Omega$, let

$$K^{\hat{p}}(N) = K^{p_k}\Big(K^{p_{k-1}}\big(\cdots(K^{p_1}(N))\cdots\big)\Big)$$

and $K^{\hat{p}}(\omega) = K^{\hat{p}}(\{\omega\})$. Then, define

$$K^{\infty}(N) = \cap_{\hat{p}\in\mathcal{P}^*}K^{\hat{p}}(N)$$

2.3. **Type Spaces.**

Fix a knowledge space $(\Omega, \mathcal{P}, (\mathcal{F}^p)_{p\in\mathcal{P}})$. For each $p \in \mathcal{P}$, a *type function* t^p is a proper RCD given $\mathcal{F}^p$. A triple $(\Omega, \mathcal{P}, (t^p)_{p\in\mathcal{P}})$ (with $(\mathcal{F}^p)_{p\in\mathcal{P}}$ understood) is called a *type space*.

Through most of this paper we will assume that $t^p(\omega)(\cdot) \in \Delta_f(\Omega)$,[6] for all $p \in \mathcal{P}$ and all $\omega \in \Omega$. This assumption will only be relaxed in Section 8. We note here that from the assumption that $t^p(\omega)(\cdot)$ is a measure with finite support for all $p \in \mathcal{P}$ and all $\omega \in \Omega$ it follows that each atom of the common knowledge σ-algebra $\mathcal{F}$ is countable.

Unless otherwise specified, we will henceforth assume[7] that type spaces satisfy *positivity*, i.e., that $t^p(\omega)[\omega] > 0$ for all $p \in \mathcal{P}$ and $\omega \in \Omega$. Type spaces that do not satisfy this condition are *non-positive*. We will offer justification below as to why we restrict attention to positive type spaces.

A measure $\mu^p \in \Delta(X)$ such that t^p is a proper RCD for μ^p given $\mathcal{F}^p$ is a *prior* for t^p. A *common prior* is a measure μ that is a prior for the type functions of all the players $p \in \mathcal{P}$.

Most game theory models[8] work with a special case of type spaces that are *partitionally generated*. In such models, each player p has a partition Π^p of Ω. That player's knowledge σ-algebra $\mathcal{F}^p$ is the σ-algebra generated by Π^p. A type function t^p is then defined by a Borel mapping $t^p : \Omega \times \Omega \to \mathbb{R}$ such that

(a) for each $\omega \in \Omega$, $t^p(\omega)(\cdot) \in \Delta(\Omega)$,
(b) if $\omega' \in \Pi^p(\omega)$ then $t^p(\omega')(\cdot) = t^p(\omega)(\cdot)$.

Intuitively, a type function t^p represents the probability distribution that player p ascribes to the states conditional on receiving a signal that ω is a possible true state.

In a type space generated by a partition, each state $\omega \in \Omega$ is contained in an atom[9] of the common knowledge σ-algebra $\mathcal{F}$ that is called the *common knowledge component* containing ω. We will denote the common knowledge component containing ω by $K(\omega)$.

We will say that a countable Borel equivalence $\mathcal{E}$ (or, equivalently, the σ-algebra it induces) is *belief induced* if there are finitely many equivalence relationships $\mathcal{E}_1, \ldots, \mathcal{E}_n$ refining it ($\mathcal{E}_k \subseteq \mathcal{E}$ for $k = 1, \ldots, n$) such that each $\mathcal{E}_k$ has finite equivalence classes and $\mathcal{E}$ is the finest common coarsening of $\mathcal{E}_1, \ldots, \mathcal{E}_n$, i.e., $\mathcal{E}$ is the transitive closure of $\cup_{k=1}^n \mathcal{E}_k$. This is equivalent to saying that $\mathcal{E}$ is the common knowledge equivalence relationship induced by some type space (with finitely supported types). Not all countable Borel equivalence relationships are belief induced; we elaborate in Appendix A.[10]

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[6] Recall that $\Delta_f(\Omega)$ is the set of finitely supported measures over Ω.

[7] This assumption also appears in Samet (1998).

[8] This can be broadened to: nearly all models in the economics, game theory and decision theory literature.

[9] An *atom* of $\mathcal{F}$ is an element of $\mathcal{F}$ that does not properly contain any non-empty element of $\mathcal{F}$.

[10] We are grateful to Benjamin Weiss for pointing this out to us.

Lemma 3. *Let Ω be a standard Borel space, let $\mathcal{G}$ be a sub σ-algebra of the Borel σ-algebra with finite atoms, let $\mu \in \Delta(\Omega)$ and let t be an RCD for μ given $\mathcal{G}$ that satisfies $t(\omega)[\omega] > 0$ for all $\omega \in \Omega$. Let $N \subseteq \Omega$ be a μ-measurable set satisfying $\mu(N) = 0$. Then there is $K \in \mathcal{G}$ with $N \subseteq K$ and $\mu(K) = 0$.*

Proof. For each $n \in \mathbb{N}$, define

$$N_n = \{\omega \in N \mid t(\omega)[\omega] \geq \frac{1}{n}\}$$

Let $K_n = [N_n]_{\mathcal{G}} \in \mathcal{G}$; in other words, $K_n = \cup_{\omega \in N_n}[\omega]$. For all $\omega \in \Omega$, if $\omega \notin K_n$ then

$$t(\omega)(K_n) = 0,$$

while if $\omega \in K_n$ then

$$t(\omega)(K_n) = t(\omega)[\omega] \leq n \cdot t(\omega)[\omega] \leq n \cdot t(\omega)(N_n),$$

Therefore

$$\mu(K_n) = \int_{\Omega} t(\omega)(K_n) d\mu(\omega) \leq n \cdot \int_{\Omega} t(\omega)(N_n) d\mu(\omega) = n \cdot \mu(N_n) = 0$$

and we can take $K = \cup_{n \in \mathbb{N}} K_n$. $\square$

Corollary 4. *Let τ be a positive type space with a common prior μ. Let $N \subseteq \Omega$ be a μ-measurable set satisfying $\mu(N) = 0$. Then there is $K \in \mathcal{F}$, the common-knowledge σ-algebra, with $N \subseteq K$ and $\mu(K) = 0$.*

To prove Corollary 4, one applies Lemma 3 inductively to show that for each $n \in \mathbb{N}$ and each $\hat{p} \in \mathcal{P}^n$, $K^{\hat{p}}(N)$ is Borel. Corollary 4 will often be used implicitly; in many proofs, when useful, we will automatically assume that some null set we are discarding is common knowledge – or, equivalently, and more to the point, that its complement is common knowledge.

Finally, we justify our concentration on positive type spaces. Proposition 5 essentially states that if a space has a common prior (whether or not positive) then under that prior the event containing the states to which any player p assigns zero probability in the posterior is a null event:

Proposition 5. *Let τ be a type space (not necessarily positive) with a common prior μ. Denote, for each $p \in \mathcal{P}$,*

$$N^p := \{\omega \in \Omega \mid t^p(\omega)[\omega] = 0\}$$

Then $\mu(N^p) = 0$ for all $p \in \mathcal{P}$.

Proof. Recall that each type function is $\mathcal{F}^p$-measurable. Let $x \in N^p$ and $\omega \in \Omega$. If ω is not in the same atom of x then $t^p(x)[\omega] = 0$ since ω is not in the support of $t^p(x)$. Otherwise, since $t^p(x)[x] = 0$ and $t^p(x)[\omega] = t^p(x)[x]$, we again conclude that $t^p(x)[\omega] = 0$. The proposition follows from the definition of an RCD. $\square$

2.4. **Bayesian Games.**

A Bayesian game $\Gamma = (\Omega, \mathcal{P}, (t^p)_{p\in\mathcal{P}}, (I^p)_{p\in\mathcal{P}}, (r^p)_{p\in\mathcal{P}})$ consists of the following components:

- $(\Omega, \mathcal{P}, (t^p)_{p\in\mathcal{P}})$ forms a type space.
- I^p is a finite action set for each Player $p \in \mathcal{P}$.
- $r : \Omega \times \prod_{p\in\mathcal{P}} I^p \to \mathbb{R}^{\mathcal{P}}$ is a bounded measurable payoff function.

As usual, we extend r multi-linearly to $r : \Omega \times \prod_{p\in\mathcal{P}} \Delta(I^p) \to \mathbb{R}^{\mathcal{P}}$.

A *strategy* of a player $p \in \mathcal{P}$ is a mapping $\Omega \to \Delta(I^p)$ that is $\mathcal{F}^p$-measurable. A *measurable Bayesian ε-equilibrium* (ε-MBE), with $\varepsilon \geq 0$, is a profile of strategies $\sigma = (\sigma^p)_{p\in\mathcal{P}}$ such that for each $p \in \mathcal{P}$, each atom A of $\mathcal{F}^p$, and each $x \in \Delta(I^p)$,

$$\int_A r^p(\omega, \sigma(\omega))dt^p(\omega) + \varepsilon \geq \int_A r^p(\omega, x, \sigma^{-p}(\omega))dt^p(\omega)$$

When $\varepsilon = 0$ we will refer simply to an MBE instead of a 0-MBE.

3. Three Paradoxes

The main motivation for the results of this paper is exhibiting conditions equivalent to the existence of measurable Bayesian equilibria in games over continuum many states. As further motivation, in this section we present 'three paradoxes' related to games and type spaces over continuum many states. The results in this paper characterise when these paradoxes hold and when they are guaranteed not to exist.

The first two paradoxes, on Bayesian games and common priors in spaces over continuum many states, have been well-known in the literature for about a decade. The third paradox, on no betting, is new.

The "Now You See It, Now You Don't" Bayesian Equilibrium.

Simon (2003) and Hellman (2012b) present examples of Bayesian games that have no measurable Bayesian equilibria. In greater detail, let $\mathbf{\Gamma}$ be one of these Bayesian games, with Ω the state space over which $\mathbf{\Gamma}$ is defined. Then there exists no vector of measurable strategies $(\varphi_1, \ldots, \varphi_n)$, one per player, that forms a Bayesian equilibrium.

However, in both cases, one can choose any $\omega \in \Omega$ and consider the common knowledge component of $K(\omega)$ (as determined by the partitions of the players). Let $\mathbf{\Gamma}|_{K(\omega)}$ be the Bayesian game derived by restricting $\mathbf{\Gamma}$ to the states in $K(\omega)$. Then there *is* a measurable Bayesian equilibrium of $\mathbf{\Gamma}|_{K(\omega)}$.

The "Now You See It, Now You Don't" Common Prior.

This paradox was first noted in Simon (2000). We present here a slight variation of a version appearing in Lehrer and Samet (2011).

Consider the following type space over a state space Ω, as depicted in Figure 3. Ω is constructed out of four disjoint subsets of $\mathbb{R}^2$, labelled A_j for $j \in \{1, 2, 3, 4\}$:

- $A_1 = \{(x, x + 1) \mid -1 \leq x < 0\}$

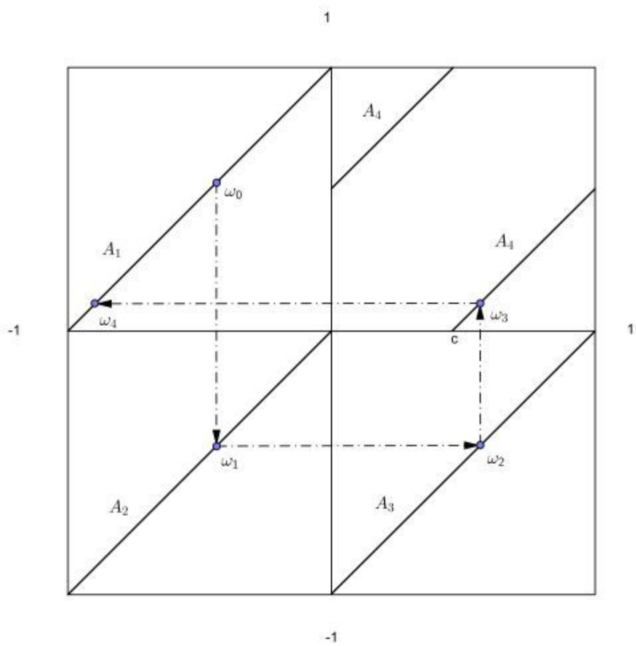

FIGURE 1. The state space consists of the three diagonals A_1, A_2, A_3 and of A_4. The latter is obtained by a rightward shift of the top-right diagonal by an irrational number c.

- $A_2 = \{(x, x) \mid -1 \leq x < 0\}$
- $A_3 = \{(x, x-1) \mid 0 \leq x \leq 1\}$
- $A_4 = \{(x, \psi(x)) \mid 0 \leq x \leq 1\}$, where $\psi(x) = x - c(\text{mod } 1)$ for a fixed irrational c in $(0, 1)$.

The knowledge space is partitionally generated, with Π_1 and Π_2 respectively the partitions of the two players. Player 1 is informed of the first coordinate of the state and player 2 is informed of the second coordinate. Thus, each element of $\Pi_1(\omega)$ is composed of the two points on the vertical line that contains the state ω. Similarly, $\Pi_2(\omega)$ contains the two points on the horizontal line that includes the state ω.

The posterior μ_i^ω for each of the two points in $\Pi_i(\omega)$ is $\frac{1}{2}$. Furthermore, let μ be the probability measure $\frac{1}{4}\sum_{j=1}^{4}\psi_j$, where ψ_j is the Lebesgue measure over A_i. Lehrer and Samet (2011) show that measurability conditions are satisfied by the posteriors and that μ is a common prior for μ_i^ω.

However, although the entire space Ω has a well-defined common prior, if we again concentrate on the common knowledge component $K(\omega_0)$ of any arbitrary state ω_0 (fixing the posteriors) then there is *no* common prior[11] over $K(\omega_0)$. The

[11] There may, however, be a *common improper prior* over $K(\omega_0)$. An improper prior allows for the possibility that the total measure it defines over a space diverges.

reason for this is that $K(\omega_0)$ is a doubly infinite countable sequence

$$\ldots, \omega_{-(k+1)}, \omega_{-k}, \ldots, \omega_{-1}, \omega_0, \omega_1, \ldots, \omega_k, \omega_{k+1}, \ldots$$

such that $(\{\omega_k, \omega_{k+1}\}) \subseteq \Pi_1$ for all odd $k \geq 1$, $(\{\omega_k, \omega_{k-1}\}) \subseteq \Pi_1$ for all even $k \leq 0$, $(\{\omega_k, \omega_{k+1}\}) \subseteq \Pi_2$ for all odd $k \geq 0$, and $(\{\omega_k, \omega_{k-1}\}) \subseteq \Pi_2$ for all even $k \leq -1$. Any common prior ν over $K(\omega_0)$ must satisfy the condition that $\mu(\omega_k) = \mu(\omega_k + 1)$ for all k. Thus all the countably many states in $K(\omega_0)$ must have the same probability, which is impossible.

The "Now You See It, Now You Don't" Agreement.

For a two-player type space with posteriors $\{\mu_i(\cdot)\}_{i=1,2}$ a *bet* is an integrable random variable $f : \Omega \to \mathbb{R}$. An *acceptable bet* is a bet that satisfies the condition that

$$\int_\Omega f(\cdot) d\mu_1(\omega)(\cdot) < 0 < \int_\Omega f(\cdot) d\mu_2(\omega)(\cdot) \text{ for all } \omega \in \Omega.$$

In greater generality, suppose that τ is a type space with a set of players $\mathcal{P}$ consisting of n players and posteriors $(\mu^p)_{p \in \mathcal{P}}$. Then a bet is an n-tuple of random variables $(f^1, \ldots, f^n)$ such that $\sum_{p \in \mathcal{P}} f^p(\omega) = 0$ for all states ω. It is an acceptable bet if $0 < \int_\Omega f^p(\cdot) d\mu^p(\omega)(\cdot)$ for all $\omega \in \Omega$.

By a result in Feinberg (2000) (see also Heifetz (2006)), if Ω is a compact space then there exists a common prior if and only if there is no acceptable bet. Since there is a common prior over the entire compact space in the example depicted in Figure 3, there can be no acceptable bet over the entire space.

Once again we concentrate on a particular state ω_0 and the common knowledge component $K(\omega_0)$ containing it. We make use of a variation of a construction from Hellman (2012a) to define the following function $f : K(\omega_0) :\to \mathbb{R}$ (with $K(\omega_0) = \ldots, \omega_{-(k+1)}, \omega_{-k}, \ldots, \omega_{-1}, \omega_0, \omega_1, \ldots, \omega_k, \omega_{k+1}, \ldots$):

$$f(\omega_n) = \begin{cases} 1 & \text{if } n = 0 \\ 1 + \sum_{i=1}^n \frac{1}{2^i} & \text{if } n > 0 \text{ is even, or } n < 0 \text{ is odd} \\ -(1 + \sum_{i=1}^n \frac{1}{2^i}) & \text{if } n > 0 \text{ is odd, or } n < 0 \text{ is even} \end{cases}$$

It is easy to check that f is an acceptable bet over $K(\omega_0)$, even though there is no globally acceptable bet over the entire space Ω.

Several researchers have noted that these sorts of paradoxes strike at the heart of major assumptions underpinning research in contemporary economics and game theory, namely those related to the distinction between the *ex ante* stage and the interim stage of analysis. The full state space, over which priors are defined, is usually taken to be the *ex ante* stage while the common knowledge component represents the interim stage after each player receives a signal.

In many presentations, the Bayesian games and type spaces are considered 'auxiliary constructions' for the sake of analysis. According to this view, in reality there is no chance move that selects a player's type; the knowledge and belief of each player determines his or her type. Type spaces and Bayesian games are merely ways to model the incomplete information each player has about the other players' types. The true situation the players face is the interim stage after the vector of types has been selected. However, incomplete information requires us to consider the *ex ante* stage in order to understand how the players make their choices in the interim stage. Furthermore, in this view, in dynamic situations in which several signals may be received in succession over time, with each such signal refining the information known to the players, what is the interim stage after the receipt of one signal may also be considered the *ex ante* stage with respect to subsequent signals that have not yet been received.

This very standard view is challenged by the paradoxes detailed in this section. They show that when the state space has the cardinality of the continuum there may be a disturbing instability as we move from *ex ante* to interim stages: is there or is there not a common prior? Is there or is there not a Bayesian equilibrium? Is there or is there not a possible disagreement?

Taken together, Theorem 1, Corollary 10 and Theorem 2 in this paper show that all three paradoxes essentially disappear when the underlying common knowledge σ-algebra is separable. In fact, the solutions to two of these paradoxes lead to full characterisations of when these pathologies occur.

4. Preliminary Notions and Results

The following is a slight strengthening of the Lusin-Novikov theorem (see, for example, Theorem 18.10 of Kechris (1995)):

Proposition 6. *Given a σ-algebra that induces a countable Borel equivalence relationship on a standard Borel space Ω, there exist partial*[12] *Borel mappings from Ω to Ω, $f_1, f_2, \dots$ such that for each $\omega \in \Omega$, $[\omega]_{\mathcal{F}} = \{f_n(\omega) \mid \omega \in Dom(f_n)\}$ and such that $f_n(\omega) \neq f_m(\omega)$ for $m \neq n$ and all $\omega \in Dom(f_n) \cap Dom(f_m)$.*

If $(\Omega, \mathcal{E})$, $(\Lambda, \mathcal{D})$ are standard Borel spaces with Borel equivalence relations $\mathcal{E}$ and $\mathcal{D}$ induced on them, $(\Omega, \mathcal{E})$ is said to *embeddable* into $(\Lambda, \mathcal{D})$ if there is an injective Borel mapping $\psi : \Omega \to \Lambda$ such that for all $\omega, \eta \in \Omega$, $\omega \mathcal{E} \eta \iff \psi(\omega) \mathcal{D} \psi(\eta)$; in this case, we denote $(\Omega, \mathcal{E}) \sqsubset (\Lambda, \mathcal{D})$.

A countable Borel equivalence relationship (henceforth CBER) is said to be *hyperfinite* (Dougherty et al. (1994)) if it is induced by the action of a Borel $\mathbb{Z}$-action on Ω; i.e., if there is a bijective[13] Borel mapping $T : \Omega \to \Omega$ such that $x \mathcal{E} y \iff \exists n \in \mathbb{Z}, T^n(x) = y$.

Proposition 7. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be non-smooth countable Borel equivalence relationships on standard Borel spaces Ω_1, Ω_2, with $\mathcal{E}_1$ being hyperfinite. Then $(\Omega_1, \mathcal{E}_1) \sqsubset (\Omega_2, \mathcal{E}_2)$.*

[12] That is, defined only on a subset of the domain; it follows that the domain, as in the inverse image of the entire range space, is Borel.

[13] If a Borel mapping is injective, a theorem by Kuratowski states that its image is standard Borel and that its inverse is Borel.

Proof. Let $\mathcal{E}_t$ be the tail equivalence relationship on $C = 2^{\mathbb{N}}$; i.e., if $S : C \to C$ is defined by $(Sx)_n = x_{n+1}$, then $x\mathcal{E}_t y$ iff $\exists k, m \in \mathbb{N}$ such that $S^k(x) = S^m(y)$; $\mathcal{E}_t$ is non-smooth and hyperfinite, see (Dougherty et al., 1994, Sec. 6). By the Glimm-Effros dichotomy for Borel equivalence relationships, Harrington et al. (1990), since $\mathcal{E}_2$ is not smooth,[14] $(C, \mathcal{E}_t) \sqsubset (\Omega_2, \mathcal{E}_2)$; denote such an embedding by τ. By Theorem 7.1 of Dougherty et al. (1994), any two non-smooth hyperfinite equivalence relationships can be embedded into each other, and $(C, \mathcal{E}_t)$ is known to be hyperfinite (Dougherty et al., 1994, Ch.6), hence $(\Omega_1, \mathcal{E}_1) \sqsubset (C, \mathcal{E}_t)$; denote such an embedding σ. This yields $\psi = \sigma \circ \tau$ as the required embedding. □

5. Common Priors over Components

Given a type space $\tau = (\Omega, \mathcal{P}, (t^p)_{p\in\mathcal{P}})$, if $K \in \mathcal{F}$ then $\tau_K := (K, \mathcal{P}, (t^p|_K)_{p\in\mathcal{P}})$, consisting of the state space K and the type functions restricted to K, is a well-defined type space. This is true in particular if K is an atom of $\mathcal{F}$.

Theorem 1 essentially states that given a type space τ with a common prior, the type space τ_K for any common knowledge component K is guaranteed also to have a common prior if and only if the underlying common knowledge σ-algebra is separable almost everywhere.

Theorem 1. *Let τ be a type space with a common prior μ. The following conditions are equivalent:*

(1) There exists $X \in \mathcal{F}$ with $\mu(X) = 1$ such that $\mathcal{F}|_X$ is separable.
(2) There exists $X \in \mathcal{F}$ with $\mu(X) = 1$ such that for each atom K of $\mathcal{F}$ satisfying $K \subseteq X$, the type space τ_K has a common prior.
(3) There exists $X \in \mathcal{F}$ with $\mu(X) = 1$ such that there is a proper regular conditional probability t of μ given $\mathcal{F}$, such that for each atom of $K \in \mathcal{F}$ and each $x \in K$, $t(x)$ is a common prior for τ_K.

Remark 8. In particular, it follows that the common knowledge σ-algebra $\mathcal{F}$ generated in Figure 3 is not separable. This, however, could be seen by more elementary means: the restriction of $\mathcal{F}$ to any one of the sets A_1, A_2, A_3, A_4 is easily seen to be induced by the equivalence relation induced by an rotational of the circle - i.e., $x \to x - c \ mod\ 1$, c being irrational – and this σ-algebra is well-known to be non-separable.

Lemma 9. *The correspondence*

$$\Psi(\omega) = \{\nu \in \Delta_a(\Omega) \mid \nu([\omega]_{\mathcal{F}}) = 1 \textit{ and } \nu|_{[\omega]_{\mathcal{F}}} \textit{ is a common prior for } \tau_{[\omega]_{\mathcal{F}}}\}$$

has a Borel graph, and $|\Psi(\omega)| \le 1$ for all $\omega \in \Omega$.

Proof. The fact that $|\Psi(\omega)| \le 1$ (i.e., that on a countable space in which no proper non-empty set is common knowledge there exists at most one common prior) follows from Proposition 3 of Hellman and Samet (2011).

[14] The Glimm-Effros dichotomy is usually stated for a state space Ω that is Polish; however, a Borel space can always be endowed with a Polish topology inducing the same Borel structure, since all standard Borel spaces are Borel isomorphic.

Let $(f_n)_{n\in\mathbb{N}}$ be as in Proposition 6. Define for each $n \in \mathbb{N}$, $g_n : \Delta_a(\Omega) \times \Omega \to [0, 1]$ by:

$$g_n(\nu, \omega) = \begin{cases} \nu(\{f_n(\omega)\}) & \text{if } \omega \in Dom(f_n) \\ 0 & \text{if } \omega \notin Dom(f_n) \end{cases}$$

and for each $m, n \in \mathbb{N}$ and $p \in \mathcal{P}$, define

$$h^p_{n,m}(\nu, \omega) = \begin{cases} 1 & \text{if } \omega \notin Dom(f_n) \text{ or } \omega \notin Dom(f_m) \\ 1 & \text{o.w. if } t^p(f_n(\omega))[f_m(\omega)] = 0 \\ 1 & \text{o.w. if } t^p(f_n(\omega))[f_m(\omega)] = \frac{\nu(\{f_m(\omega)\})}{\nu([f_n(\omega)]_p)} \\ 0 & \text{o.w.} \end{cases}$$

where $o.w.$ denotes 'otherwise'. g_n is Borel and hence so is h_n, and

$$\Psi(\omega) = \{\nu \in \Delta_a(\Omega) \mid \Big(\sum_{n=1}^{\infty} g_n(\nu, \omega) \in \{0, 1\}\Big) \wedge_{p\in\mathcal{P}} \wedge_{n,m\in\mathbb{N}} h^p_{n,m}(\nu, \omega) = 1\}$$

□

Proof. (of Theorem 1.) Clearly, property (3) implies property (2). Suppose (2) holds; then, for Ψ as in Lemma 9, $\Psi(\omega) = 1$ for μ-a.e. $\omega \in \Omega$. Hence, after restricting Ψ to some $X \in \mathcal{F}$ of full μ-measure, the graph of Ψ defines a Borel function $\psi : X \to \Delta_a(\Omega)$, which clearly satisfies $[x]_{\mathcal{F}} = [y]_{\mathcal{F}} \iff \psi(x) = \psi(y)$; hence $\mathcal{F}|_X$ is smooth.

Finally, assume property (1) holds, and assume w.l.o.g., $\Omega = X$. By[15] Theorem 1 of Blackwell and Ryll-Nardzewski (1963), there is a μ-a.e. proper RCD t for μ given $\mathcal{F}$. The claim that t is a common prior on μ-a.e. component follows now from Proposition 11 below. □

Corollary 10. *Let τ be a type space with a common prior μ satisfying the condition that there exists $X \in \mathcal{F}$ with $\mu(X) = 1$ such that $\mathcal{F}|_X$ is separable. Then there does not exist an acceptable bet over τ_K for each atom K of $\mathcal{F}$ satisfying $K \subseteq X$.*

Proof. As noted above, by the assumption that the support of each type $t^p(\omega)(\cdot)$ is finite for all p and ω, each atom K of $\mathcal{F}$ is countable. By Theorem 1, if $K \subseteq X$ is an atom of $\mathcal{F}$, then τ_K has a common prior.

This is sufficient, by Theorem 1.a. in Hellman (2012a), to conclude that there can be no acceptable bet over τ_K. □

Proposition 11. *Let $\mathcal{E}, \mathcal{E}'$ be smooth countable Borel equivalence relationships on a standard Borel space Ω, with $\mathcal{E}'$ refining $\mathcal{E}$ (that is, $\mathcal{E}' \subseteq \mathcal{E}$) let μ be a regular Borel probability measure on Ω, and let t, t' be proper RCD's of μ w.r.t. the σ-algebras $\mathcal{F}, \mathcal{F}'$ induced by $\mathcal{E}, \mathcal{E}'$, respectively. Then for μ-a.e. $\omega \in \Omega$ and $\mathcal{E}'$-equivalence class C' with $\omega \in C'$,*

$$t(\omega)(\cdot \mid C') = t'(\omega)(\cdot) \tag{5.1}$$

[15] The condition given there for the existence of proper RCD's is easily seen to follow from the existence of a Borel transversal, which – by Proposition 1 – follows from separability.

Proof. It suffices to show that for μ-a.e. $\omega \in \Omega$ and each $\mathcal{E}'$-equivalence class C' such that $\omega \in C'$,

$$t(\omega)(\{\omega\} \mid C') = t'(\omega)[\omega]$$

Indeed, this suffices since both t, t' are constant in each $\mathcal{E}'$-equivalence class, and both sides of (5.1) vanish for sets supported outside of C'. Since $\omega \in C'$, this is equivalent to showing that for μ-a.e. $\omega \in \Omega$ and such C',

$$t'(\omega)[\omega] \cdot t(\omega)(C') = t(\omega)[\omega] \tag{5.2}$$

In fact, throughout this proof, it will be convenient to denote $t(C)$ for $\omega \in C$ instead of $t(\omega)$ – i.e., to view the RCD as a function of the equivalence class, not of its elements. Note that since $\mathcal{E}, \mathcal{E}'$ are smooth, the induced quotient spaces $\Omega/\mathcal{F}, \Omega/\mathcal{F}'$ are standard Borel by Proposition 1 and μ-induced measures on these quotient spaces.

Lemma 12. *For any bounded real-valued random variable X on (Ω, μ),*

$$\int_{\Omega} X(\omega) d\mu(\omega) = \int_{\Omega/\mathcal{F}} \left(\sum_{\omega \in C} X(\omega) \cdot t(C)[\omega] \right) d\mu(C) \tag{5.3}$$

Proof. It suffices to verify (5.3) in the case $X = 1_A$, A being Borel, and then to use an approximation argument. In this case, the left-hand side of Equation (5.3) is just $\mu(A)$, while the other side is

$$\begin{aligned} \int_{\Omega/\mathcal{F}} \left(\sum_{\omega \in C} 1_A(\omega) \cdot t(C)[\omega] \right) d\mu(C) &= \int_{\Omega/\mathcal{F}} t(C)(A \cap C) d\mu(C) \\ &= \int_{\Omega/\mathcal{F}} t(C)(A) d\mu(C) \end{aligned}$$

In general, for an $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$ – which induces a measurable function $f : \Omega/\mathcal{F} \to \mathbb{R}$ – we have

$$\int_{\Omega/\mathcal{F}} f(C) d\mu(C) = \int_{\Omega} f(\omega) d\mu(\omega)$$

(again, one checks it first for simple $\mathcal{F}$-measurable functions) and in particular for $f(\cdot) = t(\cdot)(A)$. Hence,

$$\int_{\Omega/\mathcal{F}} t(C)(A) d\mu(C) = \int_{\Omega} t(\omega)(A) d\mu(\omega) = \mu(A)$$

as required. □

Now, note that on $\Omega/\mathcal{F}'$ there is the equivalence relationship $\mathcal{E}^*$ induced by $\mathcal{E}$; that is, two elements of $\Omega/\mathcal{F}'$ are $\mathcal{E}^*$ equivalent if they are subsets of the same equivalence class of $\mathcal{E}$. $\mathcal{E}^*$ is easily seen to be Borel and smooth as well; denote its induced σ-algebra as $\mathcal{F}^*$. Let t^* denote the proper RCD of μ (as a measure on $\Omega/\mathcal{F}$) w.r.t $\mathcal{F}^*$, which exists by[16] Theorem 1 of Blackwell and Ryll-Nardzewski (1963).

[16] See explanation and footnote when this result is used in the proof of Theorem 1.

Lemma 13. *For μ-a.e. $C \in \Omega/\mathcal{F}$ and each $\mathcal{E}'$-equivalence class $C' \subseteq C$,*

$$t^*(C)[C'] = t(C)(C')$$

Proof. For any bounded real-valued random variable X on $(\Omega/\mathcal{F}', \mu)$ (by abuse of notation, we let X also denote the induced $\mathcal{F}'$-measurable random variable defined on Ω), by repeated use of Lemma 12,

$$\begin{aligned}\int_{\Omega/\mathcal{F}} \left(\sum_{\omega \in C} X(\omega) \cdot t(C)[\omega] \right) d\mu(C) = \int_{\Omega} X(\omega) d\mu(\omega) &= \int_{\Omega/\mathcal{F}'} X(C') d\mu(C') \\ &= \int_{\Omega/\mathcal{F}} \left(\sum_{C' \in C} X(C') \cdot t^*(C)[C'] \right) d\mu(C)\end{aligned}$$

where the sum over $C' \in C$ is taken over $\mathcal{E}'$-equivalence classes. However, for μ-a.e. $\omega \in \Omega$,

$$\sum_{\omega \in C} X(\omega) \cdot t(C)[\omega] = \sum_{C' \in C} X(C') \cdot t(C)(C')$$

Hence,

$$\int_{\Omega/\mathcal{F}} \left(\sum_{C' \in C} X(C') \cdot t(C)(C') \right) d\mu(C) = \int_{\Omega/\mathcal{F}} \left(\sum_{C' \in C} X(C') \cdot t^*(C)[C'] \right) d\mu(C)$$

and this holds for any bounded real-valued random variable X. □

We now complete the proof. For any bounded real-valued random variable X on Ω, by Lemma 12 (applied first to the equivalence relationship $\mathcal{E}'$ on Ω, and then to the equivalence relationship $\mathcal{E}^*$ on $\Omega/\mathcal{F}'$),

$$\begin{aligned}\int_{\Omega} X(\omega) d\mu(\omega) &= \int_{\Omega/\mathcal{F}'} \left(\sum_{\omega \in C'} X(\omega) \cdot t'(C')[\omega] \right) d\mu(C') \\ &= \int_{\Omega/\mathcal{F}} \sum_{C' \subseteq C} t^*(C)[C'] \left(\sum_{\omega \in C'} X(\omega) \cdot t'(C')[\omega] \right) d\mu(C) \\ &= \int_{\Omega/\mathcal{F}} \left(\sum_{C' \subseteq C} \sum_{\omega \in C'} X(\omega) \cdot t(C)(C') \cdot t'(C')[\omega] \right) d\mu(C) \\ &= \int_{\Omega/\mathcal{F}} \left(\sum_{\omega \in C} X(\omega) \cdot t(C)([\omega]_{\mathcal{E}'}) \cdot t'([\omega]_{\mathcal{E}'})[\omega] \right) d\mu(C)\end{aligned}$$

Comparing this to Equation (5.3), we see that for μ-a.e. $\omega \in \Omega$,

$$t([\omega]_{\mathcal{E}})([\omega]_{\mathcal{E}'}) \cdot t'([\omega]_{\mathcal{E}'})[\omega] = t([\omega]_{\mathcal{E}})[\omega]$$

or, denoting $C' = [\omega]_{\mathcal{E}'}$ and recalling $t([\omega]_{\mathcal{E}}) = t(\omega)$, and similarly for $\mathcal{E}', t'$, we deduce Equation (5.2). □

6. MEASURABLE BAYESIAN EQUILIBRIA

This section is devoted to proving both parts of the following theorem:

Theorem 2.

I. Let Γ be a Bayesian game in which the common knowledge σ-algebra is separable. Then there exists an MBE for Γ.

II. Let Ω be a standard Borel space, and let $\mathcal{F}$ be sub-σ-algebra of the Borel σ-algebra that is not separable and which is belief induced. Then there exists a Bayesian game Γ with state space Ω, a common prior, and common knowledge σ-algebra $\mathcal{F}$ that does not possess an ε-MBE for small enough $\varepsilon > 0$, and in particular does not possess an MBE.

To prove Theorem 1.I, we proceed in three steps. First, we will develop a notion of the space of (not necessarily positive) Bayesian games with countably many states S, player set $\mathcal{P}$ and action sets $(I^p)_{p\in\mathcal{P}}$, which we will denote by $\mathfrak{B}(S, \mathcal{P}, (I^p)_{p\in\mathcal{P}})$ (or just $\mathfrak{B}$ for short). Afterwards, we will prove the existence of a measurable Bayesian equilibrium selection for this class of games. Then we will show how one can measurably map the games induced on each common knowledge component of a general game into the space of games on countably many states S; the composition of this mapping and the measurable Bayesian equilibrium selection from the second step will give us the required global Bayesian equilibrium.

Fix a countable set S and an element $s_0 \in S$. Let $\mathfrak{B}$ denote the collection of all $\mathcal{P}$-tuples $(s^p, g^p)_{p\in\mathcal{P}}$ for which $(S, \mathcal{P}, (I^p)_{p\in\mathcal{P}}, (s^p, g^p)_{p\in\mathcal{P}})$ constitutes a Bayesian game. $\mathfrak{B}$ is endowed with the topology of point-wise convergence[17]: $(s^p_\alpha, g^p_\alpha)_{p\in\mathcal{P}} = \Upsilon_\alpha \to \Upsilon = (s^p, g^p)_{p\in\mathcal{P}}$ in $\mathfrak{B}$ if for every player $p \in \mathcal{P}$, every $x \in S$, and every pure action profile $a \in \prod_{p\in\mathcal{P}} I^p$ in $g^p_\alpha(x, a) \to g^p(x, a)$:

Proposition 14. *$\mathfrak{B}$ is homeomorphic to a Borel subset $\Xi := \big((S \times [0,1])^* \times \mathbb{R}^{\prod_{p\in\mathcal{P}} I^p}\big)^{S\times\mathcal{P}}$ and hence is standard Borel (where for a set A, $A^* = \cup_{n=0}^\infty A^n$ with each A^n being both closed and open).*

The simple intuition is that for each player and state pair $(\omega, p) \in S \times \mathcal{P}$, we need to specify both an element in $(S \times [0,1])^*$ – a finite list of states that are in the same element of the knowledge partition as ω, and the probabilities themselves to these states – as well as an element of $\mathbb{R}^{\prod_{p\in\mathcal{P}} I^p}$, which specifies what payoff that player will receive as a result of each possible action profile.

Although we will not need it, the proof shows this mapping can be chosen to be natural up to a choice of a well-ordering on S. Henceforth, we will identify $\mathfrak{B}$ with some such fixed subset of Ξ.

Proof. Write $\mathfrak{B} = \prod_{p\in\mathcal{P}}(\mathfrak{B}^p_s \times \mathfrak{B}^p_g)$, where $\mathfrak{B}^p_s$ (resp. $\mathfrak{B}^p_g$) denotes the projection of $\mathfrak{B}$ to the space of types (resp. payoffs) for Player p, with the induced topologies.

[17] Defined in terms of nets.

It's enough to show that $\mathfrak{B}_s^p$ is homeomorphic to a Borel subset of $\left((S \times [0,1])^*\right)^S$ and that $\mathfrak{B}_g^p$ is homeomorphic to Borel subseteq of $\mathbb{R}^{S \times \prod_{p \in \mathcal{P}} I^p}$.

The latter claim is trivial once one notices that for any countable set C, the set of bounded functions in $\mathbb{R}^C$ is Borel, as it can be written

$$\cup_{n \in \mathbb{N}} \cap_{c \in C} \{a \in \mathbb{R}^C \mid |a_c| \leq n\}.$$

and that the Tychonoff topology is the topology of point-wise convergence. We turn to the former claim. Fix some well-ordering $<$ on S. As mentioned above, the intuition describing the map from $\mathfrak{B}_s^p$ to $\left((S \times [0,1])^*\right)^S$ is the following: for each $\omega \in S$, the player has to specify the finite list of states he believes he could be in and the weight each one receives. Finite lists of states are ordered by $>$. Hence, the image of $\mathfrak{B}_s^p$ under such a map is given by the subset of Ξ defined by:

$$\bigcap \cap_{\omega \in S} \cup_{F \subseteq S,\, |F| < \infty} \cap_{x \notin F} \{s^p \in \left((S \times [0,1])^*\right)^S \mid s^p(\omega)(x) = 0\}$$
$$\bigcap \cap_{\omega \in S} \{s^p \in \left((S \times [0,1])^*\right)^S \mid \sum_{x \in S} s^p(\omega)[x] = 1\}$$
$$\bigcap \cap_{\omega, \eta, \zeta \in S} \{s^p \in \left((S \times [0,1])^*\right)^S \mid s^p(\omega)[\eta] > 0 \to s^p(\omega)(\zeta) = s^p(\eta)[\zeta]\}$$

and, again the topology is the topology of point-wise convergence. □

The space Σ^p of strategies for Player p on a countable space is clearly a compact subspace of $(\Delta(I^p))^S$, hence the space of strategy profiles $\Sigma = \prod_{p \in \mathcal{P}} \Sigma^p$ is a compact space.

Proposition 15. *The Bayesian equilibrium correspondence $BE : \mathfrak{B} \to \Sigma$ has a Borel graph and takes on compact non-empty values.*

Proof. The fact that every Bayesian game with a countable state space has at least one Bayesian equilibrium follows from standard fixed point arguments; see, e.g., Simon (2003). The fact that the set of Bayesian equilibrium is compact also follows by standard arguments. To show that the graph G of the BE correspondence is Borel, note that

$$G = \{((s^p, g^p)_{p \in \mathcal{P}}, \sigma) \in \mathfrak{B} \times \Sigma \mid \forall \omega \in S, \forall p \in \mathcal{P}, \forall x \in \Delta_{\mathbb{Q}}(I^p),$$
$$\sum_{v \in S} 1_{v \in [\omega]_p} \cdot g^p(v, \sigma(w)) s^p(\omega)[v] \geq \sum_{v \in S} 1_{v \in [\omega]_p} \cdot g^p(v, x, \sigma^{-p}(w)) s^p(\omega)[v]\}$$

where for a finite set A, $\Delta_{\mathbb{Q}}(A)$ denotes the probability distributions on a A which give rational weights to all points. □

The following corollary then results from Proposition 15 and the selection theorem of Kuratowski and Ryll-Nardzewski (1965) (see also Himmelberg (1975)):

Corollary 16. *There exists a Borel mapping $\psi : \mathfrak{B} \to \Sigma$ such that for all $\Lambda \in \mathfrak{B}$, $\psi(\Lambda)$ is a Bayesian equilibrium of Λ.*

Given two Bayesian games

$$(S, \mathcal{P}, (I^p)_{p\in\mathcal{P}}, (s^p_S)_{p\in\mathcal{P}}, (g^p_S)_{p\in\mathcal{P}})$$

and

$$(T, \mathcal{P}, (I^p)_{p\in\mathcal{P}}, (s^p_T)_{p\in\mathcal{P}}, (g^p_T)_{p\in\mathcal{P}})$$

with countable state spaces and the same player and action sets, an *isomorphism* from S to T is a mapping $\phi : S \to T$ such that:

- For all $\omega \in S$ and pure action profile x, $g_S(\omega, x) = g_T(\phi(\omega), x)$.
- For all $\omega, \eta \in S$ and $p \in \mathcal{P}$, $s^p_S(\omega)[\eta] = s^p_T(\phi(\omega))[\phi(\eta)]$.

Proposition 17. *Let $\Gamma = (\Omega, \mathcal{P}, (I^p)_{p\in\mathcal{P}}, (t^p_S)_{p\in\mathcal{P}}, (r^p_S)_{p\in\mathcal{P}})$ be a Bayesian game such that the common knowledge equivalence relationship $\mathcal{E}$ is separable and aperiodic,*[18] *and $\mathfrak{B} = \mathfrak{B}(S, \mathcal{P}, (I^p)_{p\in\mathcal{P}})$ the set of Bayesian games with countable state space S with the same player and action space as Γ. Then $\Omega/\mathcal{E}$ is standard Borel and there is a Borel map $\Phi : \Omega \to S$ which is $\mathcal{F} = \mathcal{F}_{\mathcal{E}}$ measurable and a Borel map $\Lambda : \Omega/\mathcal{F} \to \mathfrak{B}$ such that for each $\omega \in \Omega$, if we denote*

$$\Gamma_\omega = ([\omega]_{\mathcal{F}}, \mathcal{P}, (I^p)_{p\in\mathcal{P}}, (t^p|_{[\omega]_{\mathcal{F}}})_{p\in\mathcal{P}}, (r^p|_{[\omega]_{\mathcal{F}}})_{p\in\mathcal{P}})$$

then $\Theta|_{[\omega]_{\mathcal{F}}}$ is an isomorphism from Γ_ω to $\Lambda([\omega]_{\mathcal{F}})$.

Proof. The space $Q := \Omega/\mathcal{F}$ is standard Borel by Proposition 1; hence, by choosing a Borel transversal, one uses Proposition 6 to see that there exists partial Borel mappings from Q to Ω, $f_1, f_2, \ldots$ such that for each $q \in Q$, $q = \{f_n(q) \mid n \in \mathbb{N} \text{ s.t. } q \in Dom(f_n)\}$ and such that $f_n(q) \neq f_m(q)$ for $m \neq n$.

Let $\sigma_1, \sigma_2, \ldots$ be an enumeration of S. Define $\Phi : \Omega \to S$ by $\Phi(\omega) = \sigma_{n(\omega)}$, where $n(\omega)$ is the unique element of $\mathbb{N}$ such that $f_{n(\omega)}([\omega]_{\mathcal{F}}) = \omega$. We can then define $\Lambda(q) = (g^p_q, s^p_q)_{p\in\mathcal{P}}$ by

$$g^p_q(\Phi(\omega), x) = r^p(\omega, x)$$

and

$$s^p_q(\Phi(\omega))[\Phi(\eta)] = t^p(\omega)[\eta]$$

It is straightforward to check that Φ and Λ so defined satisfy the requirements. $\square$

Proof. (of Theorem 1.I) For simplicity, take the case that the common knowledge equivalence relationship is aperiodic. Otherwise, partition the space into the common knowledge components of each size, and on each use a modified version of Proposition 17 with S being of a fixed countable or finite size.

Let $\psi : \mathfrak{B} \to (\prod_{p\in\mathcal{P}} \Delta(I^p))$ be a Bayesian equilibrium selection as in Corollary 16. Let Φ, Λ be as in Proposition 17 for some countable set S. For each $\omega \in \Omega$, define

$$\sigma(\omega) = \psi(\Lambda([\omega]_{\mathcal{F}}))(\Phi(\omega))$$

Such σ is then an MBE. $\square$

[18] I.e., each equivalence class is infinite.

Proof. (of Theorem 1.II) Let $C = 2^{\mathbb{N}}$ denote the Cantor space and let $\mathcal{E}_t$ be the tail equivalence relationship; i.e., if $S : C \to C$ is defined by $(Sx)_n = x_{n+1}$, then by $x\mathcal{E}_t y$ iff $\exists k, m \in \mathbb{N}, S^k(x) = S^m(x)$. This is a countable Borel equivalence relationship which is non-smooth and hyperfinite, see (Dougherty et al., 1994, Sec. 6). Now, let $X = \{-1, 1\} \times C$, and define $S_X : X \to X$ by $S_X(x_0, x_1, x_2, \ldots) = (-x_0, S(x_1, x_2, \ldots))$. Let $\mathcal{E}$ be the equivalence relationship on X given by

$$\mathcal{E} = \{(x, y) \mid \exists k, m \geq 0, S_X^k(x) = S_X^m(y)\}$$

This relationship is hyperfinite as the product of hyperfinite relationships (see Proposition 5.2 of Dougherty et al. (1994)) and hence by Proposition 7, $(X, \mathcal{E}) \sqsubset (C, \mathcal{E}_t)$; let ψ denote such an embedding.

Let $\Gamma_X = (X, \{1, 2\}, \{L, R\} \times \{L, R\}, t_X^1, t_X^2, r_X^1, r_X^2)$ be the two-player game presented in Hellman (2012b). Let $\mathcal{H}^1, \mathcal{H}^2$ be the image under ψ of the knowledge σ-algebras for players $1, 2$ on X in Γ_X, and let t^1, t^2 be the induced type spaces on Ω, which can be defined as having perfect knowledge outside of Ω - i.e., for $p = 1, 2$,

$$t^p(\omega)[x] \begin{cases} 0 & \text{if } \omega \neq x \text{ and } x \notin \psi(X) \\ 1 & \text{if } \omega = x \notin \psi(X) \\ t_X^p(\psi^{-1}(\omega))[\psi^{-1}(x)] & \text{if } \omega, x \in \psi(X) \end{cases}$$

Now, let $t^3, \ldots, t^n$ be types for players $3, \ldots, n$ for some $n > 3$ such that the common knowledge equivalence relationship induced by $t^3, \ldots, t^n$ is precisely $\mathcal{E}$; such exist, since $\mathcal{E}$ is belief induced. Define the payoffs for these players, each with actions $\{L, R\}$, in an arbitrary (but bounded and measurable) way.

Hellman (2012b) shows that Γ_X does not possess an ε-MBE for $\varepsilon > 0$ small enough. Hence, for appropriate $\varepsilon > 0$, the game

$$\Gamma = (\Omega, \{L, R\}^n, \{1, 2, 3, \ldots, n\}, t^1, t^2, t^3, \ldots, t^n, r^1, r^2, r^3, \ldots, r^n)$$

does not possess ε-MBE for ε-small enough. Furthermore, $\psi_*\mu$, defined by $\psi_*\mu(\cdot) = \phi(\psi^{-1}(\cdot))$ is a common prior for Γ. □

7. Agreeing to Agree

In this section we assume that there are only two players. The following definitions are taken from Lehrer and Samet (2011):

Definition 18. *Let E be an event in the state space $(\Omega, \mathcal{B})$ with information structure (Π^1, Π^2) and type functions (t^1, t^2). An agreement on E is an event of the form*

$$\{\omega \in \Omega \mid t^1(\omega)(E) = t^2(\omega)(E) = p\}$$

for some $0 < p < 1$. We say that agreeing to agree is possible for E (with μ) if there is a common prior μ for the type functions t^1, t^2 and an agreement A on E such that $\mu(K^\infty(A)) > 0$.

We also define an *ignorance operator* as in Lehrer and Samet (2011):

Definition 19. *The event that Player p is ignorant of event E is*

$$I^p(E) = (\Omega \backslash K^i(E)) \cap (\Omega \backslash K^i(\Omega \backslash E))$$

and $I(E) := I^1(E) \cap I^2(E)$.

In addition, for event F, we define the knowledge operator K^p_F, which is the knowledge operator induced by the partition generated Π^p and $\{F, \Omega\backslash F\}$, and we define the associated higher-order knowledge operators, the common knowledge operator K^∞_F, as well as the operator I_F.

Theorems 1 and 2 from Lehrer and Samet (2011) can be summarized as follows:

Theorem 3. *Assume the state space is countable.*[19] *The following conditions are equivalent for an event E:*

(i) Agreeing to agree is possible for E (with some common prior).
(ii) There exists a non-empty finite event F such that $F \subseteq K^\infty(I(E))$ and $F \subseteq K^\infty_F(I_F(E))$.
(iii) Agreeing to agree is possible for E with a common prior with finite support.

As remarked in Lehrer and Samet (2011), the implication (ii) $\to$ (i) holds even if the state space is uncountable; but, by example, the converse direction does not hold. We wish to prove the following, answering an open problem raised in Section 5.1 of Lehrer and Samet (2011).

Theorem 4. *Assume that the state space $(\Omega, \mathcal{B})$ is standard Borel.*

I. Assume the player's knowledge structure is such that the common knowledge σ-algebra $\mathcal{F}$ is separable. Let μ be a common prior. The following conditions are equivalent for an event E:
  (i) Agreeing to agree is possible for E with μ.
  (ii) There exists an event F with $\mu(F) > 0$ such that $F \subseteq K^\infty(I(E))$, $F \subseteq K^\infty_F(I_F(E))$, and such that the intersection of F with any equivalence class is finite.
  (iii) Agreeing to agree is possible for E with a common prior ν, which is absolutely continuous w.r.t. μ, for which there exists a Borel set G, intersecting each equivalence class in finitely many points, such that $\nu(G) = 1$.

II. If $\mathcal{F}$ is not separable but is belief induced, then there exists an information structure $(\Pi^1, \dots, \Pi^n)$ with common knowledge σ-algebra $\mathcal{F}$, type functions $(t^1, \dots, t^n)$ with common prior μ, and an event E, such that agreeing to agree is possible for E with μ, but conditions I.ii and I.iii do not hold.

Proof. Regardless of whether $\mathcal{F}$ is separable or not, by taking $\nu(\cdot) = \mu(\cdot \mid F)$, we see that I.ii implies I.iii; also, that I.iii implies I.i is immediate.

[19] Endowed with the discrete σ-algebra.

To prove that I.i implies I.ii, we rely on the countable case. For each common knowledge component C, let $I(\cdot, C)$, $K(\cdot, C)$ and, for each $H \subseteq C$ let $I_H(\cdot, C), K_H(\cdot, C)$ denote the versions of the operators $I(\cdot), K(\cdot), I_H(\cdot), K_H(\cdot)$ restricted to C. Note that if $\mathfrak{C}$ is a collection of common knowledge components and $A, H \subseteq \cup_{C\in\mathfrak{C}} C$ are any sets, then

$$K(A) = \cup_{C\in\mathfrak{C}} K(A \cap C, C)$$

and similarly for I, while

$$K_H(A) = \cup_{C\in\mathfrak{C}} K_{H\cap C}(A \cap C, C)$$

and similarly for I_H. (Intuitively, these operators apply independently on each common knowledge component.) Define a correspondence from the standard Borel space $\Omega/\mathcal{F}$ to the standard Borel space Z of non-empty finite subsets of Ω (identified with Ω^* modulo the appropriate permutations):

$$\Theta(C) = \{F \in Z \mid F \subseteq K^{\infty}(I(E \cap C, C)) \text{ and } F \subseteq K_F^{\infty}(I_F(E \cap C, C), C)\}$$

We note that since the common knowledge σ-algebra is separable, there is by Theorem 1 a mapping ρ from $\Omega/\mathcal{F}$ to distributions, assigning to each common knowledge component a common prior on it. We contend that there is $\mathfrak{C} \subseteq \Omega/\mathcal{F}$ satisfying $\mu(\mathfrak{C}) > 0$ (where μ also denotes the measure induced on $\Omega/\mathcal{F}$) such that, for all $C \in \mathfrak{C}$, agreeing to agree is possible for $E \cap C$ with $\rho(C)$. Indeed,

$$0 < \mu(K^{\infty}(E)) = \int_{\Omega} \rho([\omega])(E \cap C) d\mu(\omega) = \int_{\Omega/\mathcal{F}} \rho(C)(E \cap C) d\mu(C)$$

Hence, by Theorem 3, $\Theta(C) \neq \emptyset$ for all $C \in \mathfrak{C}$. By the Aumann selection theorem (e.g., Himmelberg (1975)), up to the need discard a set of measure zero, there is a Borel mapping $\theta : \mathfrak{C} \to Z$ such that $\theta(C) \in \Theta(C)$ for all $C \in \mathfrak{C}$. Since this map is clearly injective ($\theta(C) \cap \theta(C') = \emptyset$ in fact if $C \neq C'$) its image is Borel, and hence it's easy to see that so is $F = \cup_{C\in\mathfrak{C}} \theta(C)$. This F is the required set, since

$$F = \cup_{C\in\mathfrak{C}} \theta(C) \subseteq \cup_{C\in\mathfrak{C}} K^{\infty}(I(E \cap C, C)) = K^{\infty}(I(E))$$

and similarly

$$F = \cup_{C\in\mathfrak{C}} \theta(C) \subseteq \cup_{C\in\mathfrak{C}} K_{F\cap C}^{\infty}(I_{F\cap C}(E \cap C, C)) = K_F^{\infty}(I_F(E))$$

Finally, to prove (II), it suffices to show that the same holds in the example of Lehrer and Samet (2011); then, using an embedding similar to the one used in proving Theorem 1.I (except there it was used to embed the game of Hellman (2012b)) completes the proof. In fact, in the example of Lehrer and Samet (2011) discussed in Section 3, we show that there does not exist at all a Borel set G of non-null measure intersecting each equivalence class finitely; this will show that (iii) cannot hold, and hence neither does (ii). Suppose G were such a set. W.l.o.g., $\mu(G \cap A_4) > 0$; by Remark 8, it's enough to show that there does not exist a set G of positive measure in $[0, 1)$ intersecting finitely each equivalence class $\mathcal{E}$ of the operation $T(x) = x - c \mod 1$ for fixed irrational α. Let

$$H = \{x \in G \mid \forall y \in G, x\mathcal{E}y \to x \leq y\}$$

i.e., H is the minimal element of G in each equivalence class it intersects. We contend $\mu(H) > 0$; indeed, if not, $T^n(H) = 0$ for all $n \in \mathbb{Z}$, and yet $G \subseteq \cup_{n\in\mathbb{Z}} T^n(H)$, a contradiction. However, on the other hand, $\mu(H) = 0$, since

$$1 \geq \mu(\cup_{n\in\mathbb{Z}} T^n(H)) = \sum_{n\in\mathbb{Z}} \mu(T^n(H)) = \sum_{n\in\mathbb{Z}} \mu(H)$$

a contradiction. □

8. Extensions and Variations

8.1. **Countable Partitions.** Consider the following natural model. Let the continuum states be represented by the real numbers in the interval [0,1] and suppose that following receipt of a signal each player gives positive support to a sub-interval of [0,1]. Further assume that players are limited to some finite accuracy in their measurements and therefore the end-points of the sub-intervals in their posteriors are limited to rational numbers. In that case there can only be a countable number of distinct of partition elements in the posteriors.

Limiting the partitions to countable cardinalities is sufficient to guarantee the existence of Bayesian equilibria, even when the cardinality of the support of every posterior element is the continuum. This follows from Theorem 1 of Milgrom and Weber (1985), because the countable cardinality of the partition elements guarantees that the game has absolutely continuous information, as defined in that paper.

8.2. **Types with Countable Support.** None of the results of this paper would change if we allow for type spaces with countable support; that is, for each $\omega \in \Omega$ and each Player p, $t^p(\omega)$ is a purely atomic (but not necessarily finitely supported) Borel measure. The proofs all remain largely the same, with only minor alterations. The condition of belief induced also remains unaltered:

Proposition 20. *A countable Borel equivalence relationship $\mathcal{E}$ is belief induced iff there are smooth countable Borel equivalence relationships $\mathcal{E}_1, \ldots, \mathcal{E}_n$ which generate $\mathcal{E}$.*

Indeed, clearly the countable Borel equivalence relationship induced by a single player's type is smooth. To prove this proposition, we use the following easy consequence of Proposition 6:

Lemma 21. *Let $\mathcal{E}$ be a smooth countable Borel equivalence relationship. Then there exists Borel transversals $T_1, T_2, T_3, \ldots$ such that $\Omega = \cup_{n\in\mathbb{N}} T_n$, and such that if $k \in \mathbb{N}$ and F is an equivalence class such that $T_k \cap F = \emptyset$, then $T_n \cap F = \emptyset$ for all $n > k$.*

Proof. (of Proposition 20). It suffices to show that if $\mathcal{E}$ is a smooth countable Borel equivalence relationship, then there are Borel equivalence classes $\mathcal{E}_1, \mathcal{E}_2$ with finite equivalence classes which generate $\mathcal{E}$. Let $T_1, T_2, \ldots$ correspond to $\mathcal{E}$ as in Lemma 21. Then set,

$$\mathcal{E}_1 = \{(x, y) \in \Omega \times \Omega \mid x \sim_{\mathcal{E}} y \text{ and } \exists k \in \mathbb{N}, \{x, y\} \subseteq T_{2k-1} \cup T_{2k}\}$$

$$\mathcal{E}_2 = \{(x,y) \in \Omega \times \Omega \mid x \sim_{\mathcal{E}} y \text{ and } \exists k \in \mathbb{N}, \{x,y\} \subseteq T_{2k} \cup T_{2k+1}\}$$

It is easy to see that the equivalence classes of $\mathcal{E}_1, \mathcal{E}_2$ are all of size at most 2, and that $\mathcal{E}$ is generated by $\mathcal{E}_1, \mathcal{E}_2$. $\square$

9. Appendix: On Belief Induced Relationships

As we have mentioned, not all countable Borel equivalence relationships are belief induced. This can be shown using the concept of the *cost* of a countable Borel equivalence relationship $\mathcal{E}$ with an invariant[20] measure μ. We briefly recall this concept; for a more comprehensive treatment, see Kechris and Miller (2004).

A Borel graph G on a standard Borel space Ω is a Borel relation on Ω (i.e., a Borel subset of $\Omega \times \Omega$) that is irreflexive and symmetric. A Borel graph G induces a Borel equivalence relationship $\mathcal{E}$ on Ω: $\mathcal{E}$ is the reflexive and transitive closure of G. We say that G is a *graphing* of $\mathcal{E}$. Given such a graph, for each $v \in \Omega$, let $d_G(v)$ denote the cardinality of the set $\{w \in \Omega \mid (v,w) \in G\}$. Clearly, if $d_G(v)$ is countable for all $v \in \Omega$, then so is the induced equivalence relationship $\mathcal{E}$; conversely, if $\mathcal{E}$ is a countable Borel equivalence relationship, then it is induced by some Borel graph with vertices of countable degree.

The cost of a countable Borel equivalence relationship $\mathcal{E}$ (with respect to an invariant measure μ) is defined as:

$$C_\mu(\mathcal{E}) := \inf\{\frac{1}{2}\int_\Omega d_G(\omega)d\mu(\omega) \mid G \text{ spans } \mathcal{E}\}$$

A result of Levitt, e.g. (Kechris and Miller, 2004, Ch. 20), is that if T is a Borel transversal for a countable Borel equivalence relationship $\mathcal{E}$ with an $\mathcal{E}$-invariant measure μ, then $C_\mu(\mathcal{E}) = \mu(\Omega \backslash T)$; in particular, if μ is finite, then so is $C_\mu(\mathcal{E})$.

Suppose that $\mathcal{E}$ is a countable Borel equivalence relationship, and is the equivalence relationship generated by $\mathcal{E}_1, \ldots, \mathcal{E}_n$ (that is, the coarsest equivalence relationship that each $\mathcal{E}_k$ refines), and suppose that μ is $\mathcal{E}$-invariant. It is then clearly also $\mathcal{E}_k$ invariant for each $k = 1, \ldots, n$, and it's easy to see that[21]

$$C_\mu(\mathcal{E}) \leq \sum_{k=1}^{n} C_\mu(\mathcal{E}_k)$$

Combining this observation with the result of Levitt – and the fact that finite[22] Borel equivalence relationships are clearly always smooth[23] – we see that if $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are finite, $C_\mu(\mathcal{E})$ is finite.

[20] A measure is $\mathcal{E}$-invariant if for every Borel bijection $f : \Omega \to \Omega$ satisfying $f(\omega) \sim_{\mathcal{E}} \omega$ for all $\omega \in \mathcal{E}$, it holds that for all Borel $A \subseteq \Omega$, $\mu(f^{-1}(A)) = \mu(A)$.

[21] Note that μ is $\mathcal{E}_k$-invariant for each $k = 1, \ldots, n$; hence, for any graphings $G_1, \ldots, G_n$ of $\mathcal{E}_1, \ldots, \mathcal{E}_n$, respectively, $G = \cup_{k=1}^{n} G_n$ spans $\mathcal{E}$.

[22] I.e., with finite equivalence classes.

[23] E.g., take a Borel ordering $<$ on Ω, and choose the $<$-minimal element in each equivalence class to get a transversal

Hence, to show a non-belief induced countable Borel equivalence relationship, it suffices to find one with infinite cost w.r.t. some invariant measure $\mathcal{E}$ on it. A result of Gaboriau, e.g. (Kechris and Miller, 2004, Cor 27.10), states that if $\mathcal{E}$ is a countable Borel equivalence relationship with finite invariant measure μ, and $\mathcal{T}$ is a Borel tree[24] that is a graphing of $\mathcal{E}$, then $C_\mu(\mathcal{E}) = \frac{1}{2}\int_\Omega d_{\mathcal{T}}(\omega)d\mu(\omega)$.

Now, let F_∞ denote the free (non-abelian) group with countably many generations. This group acts on 2^{F_∞} via $(f(x))(g) = x(f \cdot g)$ for $x \in 2^{F_\infty}$, $f, g \in F_\infty$, and induces a countable Borel equivalence relationship by $x \sim y$ iff $\exists g \in F_\infty$ with $g \cdot x = y$. From this, one deduces easily that if $\mu = \prod_{f \in F_\infty}(\frac{1}{2}, \frac{1}{2})$ (which is clearly $\mathcal{E}$-invariant) it holds by Gaboriau's result that $C_\mu(\mathcal{E}) = \infty$.

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[24] I.e., a Borel graph with no cycles.

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