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**THE INSTABILITY OF  
BACKWARD INDUCTION IN  
EVOLUTIONARY DYNAMICS**

**By**

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**Discussion Paper # 633**

**Jan 2013**

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# The instability of backward induction in evolutionary dynamics

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18th December 2012

## Abstract

This paper continues the work initiated in [19]. We adopt the same model as in [19]. We show that the non-backward-induction equilibrium component may be evolutionarily stable for any population size in a finite stopping game where the two equilibrium components are terminated by different players. A surprising result is that the backward induction equilibrium component may not be evolutionarily stable for large populations. Finally, we study the evolutionary stability result in a different limiting process where the expected number of mutations per generation is bounded away from both zero and infinity.

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<sup>1</sup>The author would like to acknowledge financial support by European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement No. 249159.

# 1 introduction

The *backward induction equilibrium* (the so-called ‘*subgame-perfect equilibrium*’) introduced by Selten in [17] and [18] is traditionally regarded as the unique outcome for finite extensive-form games with perfect information in the realm of full rationality and common knowledge of full rationality. That is, each player never plays anything which she knows is not optimal, and each player knows that everyone is rational, and also knows that everyone knows that everyone is rational, and so on (see Aumann [1]). However, there is wide disagreement between game theorists regarding the epistemic foundation of the solution concept of backward induction equilibria. In particular, Binmore in [3] argues that a rigorous model should incorporate what would happen if a rational player were to deviate from rational play. Ben-Porath in [2] shows that the backward induction outcome is not the only outcome consistent with common certainty of rationality. We are not going to substantiate one claim or another in this paper. Instead, we focus on an evolutionary approach proposed by Hart in [11] for equilibrium selection. In [11] and [10], the backward induction equilibrium of a perfect information game is shown to be the unique *evolutionarily stable* outcome for dynamic models consisting of selection and mutation, when the mutation rate is low and the populations are large. We show this is not always true in an extensive-form game with *multiple moves* from a player: regardless of population size, other Nash equilibria may also be evolutionarily stable.

A dynamic evolutionary process is essentially a mutation-selection process where mutation plays the role of perturbations. For finite extensive-form games of perfect information, the dynamic process approaches the backward induction equilibrium points, as the perturbation goes to zero. (Recall that in these games trembling-hand perfection is the same as subgame perfection.)

A natural question is whether the evolutionary models with the mutation rate decreasing to zero yield the analogous limit points, namely that most individuals in each population follow their own backward induction strategies most of the time. With notions in Hart's paper [11], this question can be rephrased as 'is backward induction equilibrium the unique evolutionarily stable outcome for every finite extensive-form game of perfect information?'. Loosely speaking, a state is evolutionarily stable if it is robust against persistent random perturbations, not just isolated random perturbations, as is assumed for evolutionarily stable strategies (see [15]). The notion evolutionarily stability is also called 'stochastically stable' by Foster and Young in [7], by Young in [21] and [22], 'long-run equilibrium' by Kandori et al. in [13], 'in the support of limit distribution' by Samuelson in [16] and by Fudenberg and Levine in [6].

This question has been extensively analysed under various selection mechanism. For the mechanism of replicator dynamics, Gale et al. [8], Cressman and Schlag [5], and Gintis et al. [9] have shown that, in many cases, Nash equilibria other than the backward induction equilibria are also stable. For the mechanism of best or better reply, Noldeke and Samuelson [12] consider a model in which each individual is characterised by its own action and its conjecture about the composition of the populations at all nodes. They give an example to show that, when the mutation rate goes to zero, the non-backward induction equilibrium-strategies may have positive probability in the invariant distribution of the Markov chain generated from the evolutionary dynamic process with finite population size. In contrast to this, Hart [11] proves that, in a basic and operational model with no conjectures or beliefs involved, the backward induction equilibrium is the unique evolutionarily stable outcome, when the populations go to infinity, the mutation rate

decreases to zero, and the product of population size and mutation rate is bounded away from zero. This boundary condition can be interpreted as requiring that the expected number of mutations per generation does not go to zero. Gorodeisky [10] relaxes this constraint on the product of population size and mutation rate. Kuzmics [14] extends Hart's result to the dynamic process proposed by Noldeke and Samuelson, although it requires both population size and the expected number of mutation per generation go to infinity. Note that in all these papers any player can never play at more than one node along any path of the game.

We adopt the model in [11] but allow multiple moves of a player along a path of the game. In [19], we present a two-player *finite stopping game* where the backward induction equilibrium component is not the only evolutionary stable result for large populations. A finite stopping game is a finite extensive-form game of perfect information where each node has at most one immediate succeeding node and all its other immediate children are terminal vertices. This is the starting point to study evolutionary stability in extensive-form games. We do not need to consider a (reduced-form) strategy consisting of moves in multiple branches. (See the second paragraph in Section 2 for the definition of reduced-form strategies.) We also do not need an assumption on whether a better/best strategy will change its moves at unaffected disconnected nodes. When a mutation happens, we do not need to compare the mutation rate of moves at different branches in a strategy. The example in [19] applies to the case in which both the backward induction equilibrium component (abbreviated as  $BC$ ) and the alternative Nash equilibrium component (abbreviated as  $NC$  in this section) are terminated by the same player. The dynamics of the population at the disconnected node is supported by a combinatorial lemma, Lemma 8.7 in [19]. In that example,

the strategy of player II in the alternative pure Nash equilibrium is a weakly dominated one.

For the case that  $BC$  and  $NC$  are terminated by different players in a finite stopping game, we show an example in Section 4 below that  $NC$  is also evolutionarily stable for any population size. Here, neither the strategy of player I nor the one of player II in the alternative pure Nash equilibrium is weakly dominated. Moreover, this example is a minimum one: it contains the minimum number of nodes in a game where  $BC$  and  $NC$  are terminated by different players and  $NC$  is evolutionarily stable for large populations. In section 7, we discuss why such a game needs more than three nodes, using a shortened version of this example. The conclusions hold for both the best reply and the better reply selection mechanisms. Here the dynamics of the population at the disconnected node are supported by the combinatorial lemma, Lemma 8.7.

Unlike in the case of normal-form games, the dynamic process applied to extensive-form games incorporates individuals' repeated assessment of the consequences of diverting play to unreached nodes. It is thus not straightforward to produce a general characterisation theorem. Each such game needs a supporting combinatorial result to describe the dynamics of the proportions of the strategies different from the backward induction ones. (Few general results exist for extensive-form games under other selection mechanisms or stability notions as well. cf. [4]) It is thus an open question whether a concise characterisation theorem exists for all finite extensive-form games of perfect information.

An unexpected result in this paper is that under the best-reply dynamics a game exists where  $BC$  is not evolutionarily stable when the population size approaches infinity. This is proved to be not true in either the model of Hart

[11] or the one of Nodelke and Samuelson [12], where players can never play at more than one node along each path in the game. When the mutation rate decreases to zero, we can approximate the Markov process by ‘blocks’ of time in best-reply dynamics (with no mutation) where every two consecutive blocks are separated by just one mutation. If from some equilibrium component which is evolutionarily stable for large populations, with probability bounded from below when populations increase to infinity, a mutation can trigger a transition to another component, then that arrival component can be shown to be evolutionarily stable for large populations. However, this approach does not work well for all extensive-form games. We present a two-player finite stopping game in Section 5 where such a transition triggered by one mutation does not exist in both directions between  $BC$  and  $NC$  with probability approaching 1 when the population size increasing to infinity. The relative frequency of occurrence of each component is thus determined by some event of small probability when populations are large. Note that in our model, when there are multiple best replies, each one is selected with equal probability. For the example in Section 5, it is the event, inconsistent with the strong law of large numbers, that the relative proportions of a population at a disconnected node can trigger a further transition to the other component. We compare such ‘inconsistency probability’ between two components in both directions. We can further show that the probability of transition from  $BC$  to  $NC$  is ‘much greater’ than the probability in the other direction, although the absolute values of both probabilities are very small. Hence, we can conclude that only  $NC$  is evolutionarily stable in that game for large populations.

We consider a different limiting process of the evolutionary dynamics in Section 6. As before, the populations are still going to infinity and the mu-

tation rate is decreasing to zero, but the product of population size and mutation rate is bounded away from both infinity and zero. We show that the backward induction equilibrium is not the unique evolutionarily stable state in this game for large populations. Although the result of this example is weaker than the one in Section 4, it is under a different boundary condition. In the example in Section 4, we need instead the mutation rate to be extremely small with respect to the population size. That is, the expected number of mutations per generation may not be bounded away from zero. Moreover, the techniques in the proofs are different. In the latter example, we compute the invariant distribution directly, while, in the former one, we compare the transition times in the induced Markov chain.

## 2 The Model

The model is the same as in [19]. For convenience, we copy it below.

We consider a generic finite  $N$ -player game  $\Gamma$  in extensive form with perfect information. In the generic assumption, no player obtains the same utility at any two terminal vertices. Thus, the backward induction equilibrium in  $\Gamma$  is unique. Without loss of generality, we assume that  $\Gamma$  does not contain any chance move. We denote the strategy set of player  $i$  to be  $A^i$ ,  $i = 1, \dots, N$ . Put  $A := \prod_{1 \leq i \leq N} A^i$ . For every  $a$  in  $A$  players receive payoffs determined by  $a$  in  $\Gamma$ . Without loss of generality, we consider *reduced-form strategies* in this paper. That is, we view two strategies of a player the same strategy if it is impossible to reach different payoff to this player from these two strategies. For instance, given the game in Figure 1, conventionally, for a strategy of player I where she plays the action  $a_1$  at the top node, that strategy must also specify the action she would play at the bottom node.



We, however, do not specify it, as it is impossible to reach the bottom node in that case. Hence, in this one-player game, there are only three strategies  $a_1$ ,  $a_2$  and  $a_3$  in our definition.

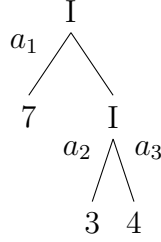


Figure 1: Example to the definition of strategies

For each player  $i$ , the set of mixed strategies of player  $i$  is a unit simplex on  $A^i$ , and we denote it by

$$X^i := \Delta(A^i) = \{x^i \in \mathbb{R}_+^{A^i} : \sum_{a^i \in A^i} x_{a^i}^i = 1\}.$$

We then denote the set of mixed strategy profiles by  $X := \prod_{1 \leq i \leq N} X^i$ . Let  $u^i : A \rightarrow \mathbb{R}$  be the payoff function of player  $i$ .

We consider the following *population game* associated to  $\Gamma$ . Assume that for each  $i$  with  $1 \leq i \leq N$  there is a non-empty population  $M(i)$  of individuals playing the game in the role of player  $i$ . For simplicity, we put  $|M(i)| = m$  for all  $i$ . We define a state  $w$  to be the collection of pure strategies of each individual in each population, i.e.,  $w = (w^i)_{1 \leq i \leq N}$  where  $w^i := (w^i(q))_{q \in M(i)}$  and  $w^i(q) \in A^i$  for each  $i$ . Let the state space  $W$  be  $\prod_{1 \leq i \leq N} (A^i)^{M(i)}$ . At a state  $w$ , for each  $i$ , let  $m_{a^i}(w)$  be the number of individuals playing strategy  $a^i \in A^i$ , and let  $x_{a^i}^i(w)$  be the proportion of population  $M(i)$  that plays the strategy  $a^i$ , i.e.,  $x_{a^i}^i(w) = m_{a^i}(w)/m$ . We put  $x^i(w) := (x_{a^i}^i(w))_{a^i \in A^i}$ , and thus we may view each state  $w$  as an  $N$ -tuple mixed strategy  $x(w) = (x^i(w))_i$  in  $X$ .

We define a discrete-time Markov chain  $(Z_t)_{t \geq 0}$  such that each  $Z_t$  is a random variable valued in  $W$ . Note that for a rigorous analysis, one needs to define first the sample space  $\Omega := W^{\mathbb{N}}$ , which is the necessity for the definition of this Markov chain and its transition probabilities. We shall not refer to it explicitly, when the context allows. Given the population size  $m$  and mutation rate  $\mu$ , the transition probabilities  $P_{\mu,m}$  of this Markov chain specify the probability that  $Z_{t+1}$  equals a state  $\tilde{w}$  given a history  $Z_1 = w_1, \dots, Z_t = w_t$ , i.e.,  $P_{\mu,m}(Z_{t+1} = \tilde{w} | Z_1 = w_1, \dots, Z_t = w_t)$ , for each  $t \geq 0$ . We further assume  $(Z_t)_{t \geq 0}$  to be a stationary Markov chain dictated by a one-step transition probability matrix  $Q_{\mu,m}$ , i.e.,

$$P_{\mu,m}(Z_{t+1} = \tilde{w} | Z_1 = w_1, \dots, Z_t = w_t) = Q_{\mu,m}(\tilde{w} | w_t),$$

for every  $w_1, \dots, w_t, \tilde{w}$  in  $W$  and  $t = 1, 2, \dots$ . If context allows, we drop the subscript  $\mu$  and  $m$  of  $P$  and  $Q$ ; for a strategy  $a$  of some player  $i$ , at stage  $t$ , we also abbreviate  $m_a(Z_t)$  by  $m_a(t)$  or simply  $A(t)$ , which is the random number of individuals in population  $M(i)$  playing the strategy  $a$  in the Markov chain at stage  $t$ . We view  $Q$  as a transition rule concerning ‘selection’ and ‘mutation’ described in the following basic model with a *best-reply selection mechanism*, which is adapted from the better-reply selection mechanism introduced by Hart in [11].

- The chosen  $q^i$  undergoes mutation, selection or no change, with probability  $\mu$ ,  $\sigma$  and  $1 - \mu - \sigma$ , respectively.
- Conditional on mutation, the individual  $q^i$  chooses a random strategy, i.e.,  $\tilde{w}^i(q^i) = a^i$  with probability  $1/|A^i|$  for each  $a^i$  in  $A^i$ .
- Conditional on selection, we define a set of ‘best strategies’, namely

$$BS^i(w) := \arg \max_{a^i \in A^i} u^i(a^i, w^{-i}). \quad (2.1)$$

where  $w^{-i}$  indicates the collection of pure strategies of each individuals in all populations other than  $M(i)$ . If no ambiguity, we abbreviate the notation  $BS^i(w)$  by  $BS^i$ . If  $w^i(q^i) \notin BS^i$ , then the new strategy  $\tilde{w}^i(q^i)$  of  $q^i$  is a randomly chosen best strategy, i.e.,  $\tilde{w}^i(q^i) = a^i$  with probability  $1/|BS^i|$  for each  $a^i \in BS^i$ . If  $w^i(q^i) \in BS^i$ , then there is no change in the strategy of  $q^i$ .

- For the case of no change, the strategy of  $q^i$  does not change:  $\tilde{w}^i(q^i) = w^i(q^i)$ .

Thus, the one-step transition probabilities  $Q(\tilde{w}|w)$  is well defined.

### 3 Preview of main results

To give the formal definition of evolutionary stability, we study the behaviour of the steady-state analysis and limiting distribution of the Markov chain  $(Z_t)_{t \geq 0}$  with the one-step transition probability matrix  $Q_{\mu,m}$  generated from the transition process in Section 2. We first consider the selection mechanism of better reply applied in the model. It is straightforward to check that, for every finite  $m$  and every  $\mu > 0$ , the induced Markov chain is irreducible and aperiodic. Hence, for every pair of population size  $m$  and mutation rate  $\mu$ , there exists a unique invariant distribution  $\pi_{\mu,m} \in \Delta(W)$  on  $W$  such that  $\pi_{\mu,m} = \pi_{\mu,m} Q_{\mu,m}$ , or

$$\pi_{\mu,m}[\tilde{w}] = \sum_{w \in W} \pi_{\mu,m}[w] Q_{\mu,m}(\tilde{w}|w)$$

for every  $\tilde{w} \in W$ .

**Lemma 3.1** *Given an one-step transition probability matrix  $Q_{\mu,m}$  defined in Section 2, for a fixed population size  $m$ , the limit distribution  $\lim_{\mu \rightarrow 0} \pi_{\mu,m}$  exists.*

**Proof.** It immediately follows from Theorem 3.1 in [22].  $\square$

For a fixed population size, a state  $w$  in  $W$  is called *m-evolutionarily stable* if its invariant probability  $\pi_{\mu,m}[w]$  does not go to zero as  $\mu$  decreases to zero, i.e.,  $\lim_{\mu \rightarrow 0} \pi_{\mu,m}[w] > 0$ . We call a state  $w$  *evolutionarily stable for large populations* if  $\liminf_{m \rightarrow \infty} \lim_{\mu \rightarrow 0} \pi_{\mu,m}[w] > 0$ . Given a limiting process that  $m$  increases to infinity,  $\mu$  decreases to zero, and that  $\mu m$  is always in a set  $R \subseteq \mathbb{R}^+$ , we call a state  $w$  *evolutionarily stable in this limiting process* if  $\pi_{\mu,m}[w]$  is bounded away from zero throughout this process. All definitions above can be naturally extended to subsets included in the state space. For every set of mixed strategy profiles  $Y \subseteq X$ , if no ambiguity, we abbreviate  $\pi_{\mu,m}[w \in W : x(w) \in Y]$  by  $\pi_{\mu,m}[Y]$ . Recall the game we consider is generic. Denote the unique backward induction equilibrium in  $\Gamma$  by  $BI = (s^i)_{1 \leq i \leq N}$ , and denote the  $\epsilon$ -neighbourhood of  $BI$  by

$$BI_\epsilon := \{x \in X : x_{s^i}^i \geq 1 - \epsilon \forall 1 \leq i \leq N\}.$$

(For convenience, we sometimes write  $BI$  as  $e_b$ .)

We have proved the following theorem in [19] that only Nash equilibria are evolutionarily stable regardless of population size.

**Theorem 3.2** *Under the best-reply or better-reply selection mechanism, for a finite generic  $N$ -player game in extensive form with perfect information, it has*

$$\lim_{\mu \rightarrow 0} \pi_{\mu,m}[NE] = 1$$

for all  $m$  in  $\mathbb{N}$ , and hence

$$\lim_{m \rightarrow \infty} \lim_{\mu \rightarrow 0} \pi_{\mu,m}[NE] = 1.$$

The combined results from Hart [11] and Gorodeisky [10] are

$$\forall \epsilon > 0, \lim_{\mu \rightarrow 0, m \rightarrow \infty} \pi_{\mu, m}[BI_\epsilon] = 1,$$

in the case that each player can only play at one node in a game. Thus, in the context of only two players, their conclusion can only be applied to a two-node game.

Recall that all equilibria in the same Nash equilibrium component give rise to an identical outcome over terminal vertices, i.e., they differ only off the equilibrium path. We denote the backward induction equilibrium component by  $BC$ . We then denote  $NC$  to be the set of Nash equilibria other than those included in the backward induction equilibrium component, and  $NE$  to be the set of Nash equilibria. It follows that  $NC = NE \setminus BC$ . We have proved the following theorem in [19] by a finite stopping game where  $BC$  and  $NC$  are terminated by the same player.

**Theorem 3.3** *For the best-reply or better-reply selection mechanism, there exists a two-player game of perfect information and a positive number  $c$  such that for all finite  $m > 1$   $NC$  is  $m$ -evolutionarily stable and*

$$\lim_{\mu \rightarrow 0} \pi_{\mu, m}[NC] \geq c.$$

*Thus,  $NC$  is evolutionarily stable for large populations and*

$$\liminf_{m \rightarrow \infty} \lim_{\mu \rightarrow 0} \pi_{\mu, m}[NC] \geq c. \tag{3.1}$$

In Section 4, we give another proof of Theorem 3.3 by the finite stopping game denoted by  $\Gamma_1$  in Figure 2. We follow a similar approach in [19] and

prove the theorem by studying its time-average asymptotic behaviour of the Markov chain:

$$\forall S \subseteq W, \lim_{T_2 - T_1 \rightarrow \infty} \frac{|\{t : T_1 < t \leq T_2, Z_t \in S\}|}{T_2 - T_1} = \pi[S].$$

We can think of the duration between two consecutive mutations as a block, and the average length of a block is  $1/(N\mu)$  where  $N$  is the number of players. If, in expectation, after  $n$  consecutive blocks of states in a component  $C_1$ , there is at least one block of states in a different component  $C_2$ , then  $\pi[C_2] \geq (\pi[C_1] + \pi[C_2])/(n + 1)$ . To prove Theorem 3.3, we need Lemma 8.6 and related preliminary results, which have been proved in [19]. For convenience, we attach a copy of the necessary lemmas in the Appendix. The dynamic in this example is different from the one in [19], and it is described in Lemma 8.7 in Appendix.

We show an example in Section 5 that the backward induction equilibrium component is not evolutionarily stable for large populations, and thus we arrive at the following theorem. In the proof, we compare the small probability of the event which is inconsistent with the strong law of large numbers in each direction of transition between  $BC$  and  $NC$ .

**Theorem 3.4** *Under the best-reply dynamic, there exists a two-player game of perfect information such that for any positive number  $\epsilon < 1$  there exists a population size  $m(\epsilon) > 1$  with the property that*

$$\lim_{\mu \rightarrow 0} \pi_{\mu, m(\epsilon)}[BC] < \epsilon.$$

In Section 6, we study a transition process that the expected number of mutations per generation, i.e.,  $m\mu$ , is bounded away from both zero and infinity, while the population size increases to infinity. By Example 3, we prove the following theorem.

**Theorem 3.5** *Under the best-reply or better-reply dynamic, there exists a two-player game of perfect information and a positive  $\epsilon$  such that, for any pair of numbers  $\delta$  and  $\eta$  with  $0 < \delta < \eta$ , we can find a triple positive numbers  $c(\delta, \eta)$ ,  $m(\delta, \eta)$  and  $c_{MU}$  with the property*

$$\pi_{\mu, m}[BI_\epsilon] < 1 - c(\delta, \eta),$$

for all  $\mu < c_{MU}$  and  $m > m(\delta, \eta)$  with  $\delta \leq \mu m \leq \eta$ .

Thus, for any limiting process with  $m \rightarrow \infty$ ,  $\mu \rightarrow 0$  and  $\delta \leq \mu m \leq \eta$ ,  $BI$  is not the only evolutionarily stable outcome in this game.

#### 4 Example of $0 < \pi[NC] < 1$

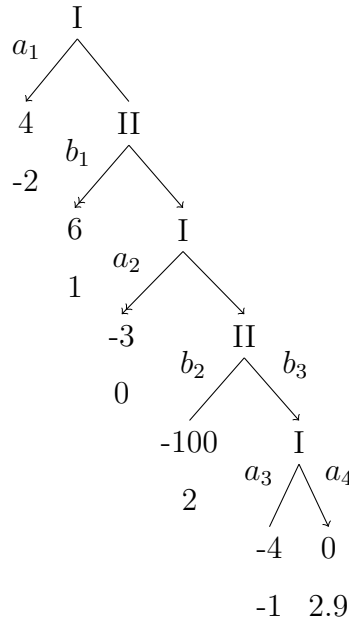


Figure 2:  $\Gamma_1$

In game  $\Gamma_1$  in Figure 2, the path of  $BC$  is terminated by player I, while the path of  $NC$  is terminated by player II. We mark the pure strategies of player

I by  $a_1, a_2, a_3$  and  $a_4$ , and the pure strategies of player II by  $b_1, b_2$  and  $b_3$ . There is only one pure Nash equilibrium  $(a_2, b_1)$  different from the backward induction equilibrium  $(a_1, b_3)$ . For convenience, all backward induction moves are arrowed and the terminating moves in the pure Nash equilibrium are double arrowed. Note that in this game,  $b_1$  is clearly not weakly dominated. For the strategy  $a_2$ , when player II plays  $b_1$ ,  $a_2$  is better than  $a_1$ ; when player II plays  $b_2$ ,  $a_2$  is better than  $a_3$  and  $a_4$ . So, no strategy in the profile  $(a_2, b_1)$  is weakly dominated in  $\Gamma_1$ . (In the game  $\Gamma_1$  in [19], strategy *no*, which is in the non-backward-induction pure Nash equilibrium, is weakly dominated by strategy *yes*.)

**Proposition 4.1** *Given the game  $\Gamma_1$  in Figure 2, for the best-reply selection mechanism described in Section 2, there exists a positive number  $c$  such that for all finite  $m > 1$ ,*

$$\lim_{\mu \rightarrow 0} \pi_{\mu, m}[NC] > c.$$

With a similar (but more subtle) approach as in the proof of Theorem 3.3 in [19], we consider a process with two mutations in population  $M(1)$  to strategy  $a_3$  and  $a_2$  in order. The proof of this theorem and discussion on better-reply selection mechanism are relegated to Appendix. In the proof, we apply Bertrand's ballot theorem for the dynamic of population  $M(2)$  on strategy  $b_2$  and  $b_3$ .

We consider a general five-node game in Figure 3 below with backward induction actions arrowed and the pure Nash equilibrium double arrowed. By Lemma 8.7 and analogous analysis as in Proposition 4.1, one can find the alternative Nash equilibrium component with payoff vector  $(u_2^1, u_2^2)$  evolutionarily stable regardless of population size if the following conditions are



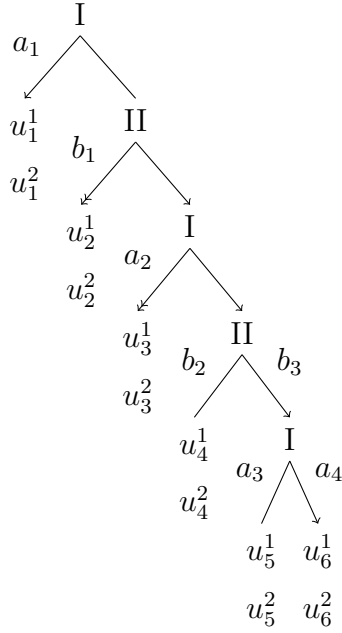


Figure 3: A general five-node game

satisfied:

$$u_4^1 < u_3^1 < u_6^1 < u_1^1 < u_2^1, \quad u_3^1 > \frac{u_6^1 + u_4^1}{2}, \quad u_6^1 > u_5^1$$

and

$$u_3^2 < u_2^2 < u_4^2 < u_6^2, \quad u_4^2 > u_5^2.$$

It is clear that payoff  $u_2^1$  is the best payoff to player I, and  $u_2^2$  is not the worst payoff to player II. This alternative Nash equilibrium component is Pareto-efficient.

We could also give a more general sufficient condition for  $NC$  being evolutionarily stable in a two-player finite extensive-form game of perfect information, similar to Theorem 6.3 in [19]. However, the conditions and the proofs would be more delicate. (Hint: there are three subgames with their own  $NC$  where the paths of  $NC$  and  $BC$  split at the root of each subgame. In  $\Gamma_1$  in Figure 2, these three subgames are rooted at the second last node, the

third node and the second node, respectively. More details available upon request.)

## 5 Backward induction may not be evolutionarily stable.

We study the game  $\Gamma_2$  below in Figure 4, and we show that the backward induction equilibrium component is not evolutionarily stable in  $\Gamma$  for large populations.

The backward induction equilibrium in  $\Gamma$  is  $e_b = (a_1, b_2)$ , and the alternative pure Nash equilibrium is  $e_n = (a_1, b_1)$ . We only give an informative sketch proof of all claims below.

### 5.1 A game with no transition between components triggered by a single mutation

The key idea of Lemma 8.6 (used to prove Proposition 4.1 above) is to consider as if each transition to a state in  $NE$  is triggered by one mutation only, and no more mutation until it reaches  $NE$ . From an initial state, if no mutation forever, the best-reply dynamic will drive the process into a Nash equilibrium. Back to the original dynamic process with mutation involved, since the mutation rate is low, we can assume that the relative proportion of time spent on an equilibrium is approximately 1. Furthermore, we can infer that if a state is disturbed by a mutation, then with asymptotic probability 1 there will be no more mutation until it reaches a Nash equilibrium again. As in Section 3.2 in [20], we can view the duration of two consecutive mutations as a ‘block’. The time spent on the non-Nash-equilibrium states can

be ignored, and we only need to count the relative visiting frequency of each component (or block). See the discussion in Section 5 in [19]. Unfortunately, in some games, not every two Nash equilibrium components are connected by such a transition triggered by a single mutation. (The formal definition is introduced in Section 3.2 in [20].) For the game  $\Gamma_2$  in Figure 4, the probability of such a transition between  $BC$  and  $NC$  goes to 0 in either direction when the population size goes to infinity.

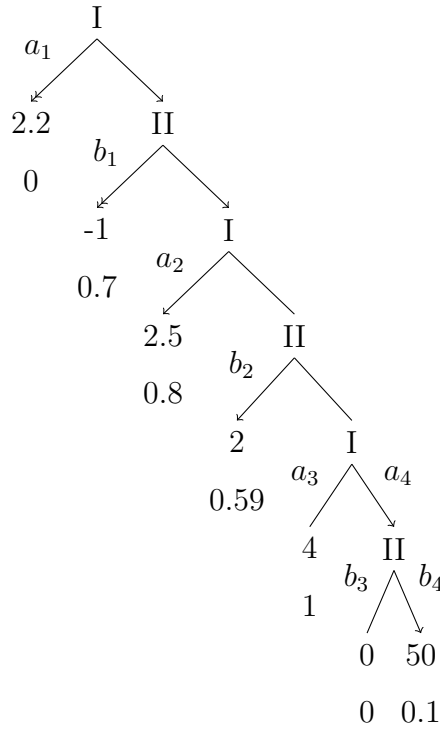


Figure 4:  $\Gamma_2$

For a more detailed analysis on the simplified version of the core transitions below in this game, please refer to Section 7.

[The transition process from  $BC$  to  $NC$ ] From the arguments later, we shall see that we can assume without loss of generality that the whole  $M(2)$  is playing the strategy  $b_2$  at stage 0. The mutation in  $M(1)$  from  $a_2$  to  $a_4$  or

$a_1$  cannot have any effect on  $M(2)$ . If a mutation happens in  $M(1)$  from  $a_2$  to  $a_3$ , then  $M(2)$  is under the selection pressure to strategy  $b_3$  and  $b_4$  with equal probability. It is possible that the mutant will stay there for sufficiently long time: for any positive number  $l$ , with probability  $1/(4^l)$ , the mutant will stay there for at least  $lm$  stages. The movement of  $M(2)$  will continue until a significant proportion of  $M(2)$ , say  $cm$  ( $c$  is a positive constant), moves away from  $b_2$ , which then attracts  $M(1)$  to deviate from  $a_2$ . Because  $m$  is very large, we can apply the strong law of large numbers, and it is an approximate uniform distribution of  $M(2)$  at  $b_3$  and  $b_4$  with probability close to 1. If  $M(1)$  is going to pick a best reply between  $a_3$  and  $a_4$ , it is  $a_4$ , due to this uniform distribution. Therefore, when  $cm$  of individuals in  $M(2)$  are accumulated at  $b_3$  and  $b_4$ ,  $M(1)$  is moving towards  $a_4$ . However, once there is one individual in  $M(1)$  playing  $a_4$ , the bottom node is connected. More importantly, the relative proportion of  $M(1)$  playing  $a_3$  and  $a_4$  (1:1 in this case), makes the best reply of  $M(2)$  changed to  $b_2$ . It will then just take several stages (with probability close to 1) that the number of individuals playing either  $b_3$  or  $b_4$  drops below  $cm$ . Then,  $M(1)$ 's best reply reverts to  $a_2$ .

There may be a repeat of this process, if the individual playing  $a_4$  converts back to  $a_2$  earlier than the one playing  $a_3$ . Nevertheless, once the one playing  $a_4$  leaves later, the process will convert back to  $BC$ . For  $M(2)$ , a small proportion of it is in an approximate uniform distribution of playing  $b_3$  and  $b_4$ , and the majority is playing  $b_2$ . When the next mutation comes, the analysis is exactly the same. Therefore, there is no one-mutation transition with positive probability from  $BC$  to  $NC$ , when populations increase to infinity. We can also see that for the argument above it is safe to assume that at stage 0 the whole  $M(2)$  is playing  $b_2$ .

[The transition process from  $NC$  to  $BC$ ] With analogous argument above,

we shall see that we can assume without loss of generality that the whole  $M(2)$  is playing the strategy  $b_1$  at stage 0. (A more general situation is an approximate uniform distribution of a minority of  $M(2)$  playing at  $b_3$  and  $b_4$ .) The mutation in  $M(1)$  from  $a_1$  to  $a_4$  cannot have any effect on  $M(2)$ . The transitions due to the mutation from  $a_1$  to  $a_2$  and from  $a_1$  to  $a_3$  are similar. We show the former case here. It is possible that the mutant will stay at  $a_2$  for sufficiently long time (with similar argument as in the above paragraph). While the mutant is playing  $a_2$ ,  $M(2)$  is under the selection pressure to strategy  $b_2$ ,  $b_3$  and  $b_4$  with equal probability. The movement of  $M(2)$  will continue until a significant proportion of  $M(2)$ , say  $cm$  ( $c$  is a positive constant), moves away from  $b_1$ , which then attracts  $M(1)$  to deviate from  $a_1$ . Because  $m$  is very large, we can apply the strong law of large numbers, and it is an approximate uniform distribution of  $M(2)$  at  $b_2$ ,  $b_3$  and  $b_4$  with probability close to 1. If  $M(1)$  is going to pick a best reply between  $a_2$ ,  $a_3$  and  $a_4$ , it is  $a_4$ , due to this uniform distribution. Therefore, when  $cm$  of individuals in  $M(2)$  are cumulated at  $b_2$ ,  $b_3$  and  $b_4$ ,  $M(1)$  is moving towards  $a_4$ . However, once there is one individual in  $M(1)$  playing  $a_4$ , the bottom node is connected. More importantly, the relative proportion of  $M(1)$  playing  $a_2$  and  $a_4$  (1:1 in this case), makes the best reply of  $M(2)$  changed to  $b_1$ . It will then just take several stages (with probability close to 1) that the number of individuals playing one strategy in  $\{b_2, b_3, b_4\}$  drops below  $cm$ . Then,  $M(1)$ 's best reply reverts to  $a_1$ .

There may be a repeat of this process, if the individual playing  $a_4$  converts back to  $a_1$  earlier than the one playing  $a_2$ . Nevertheless, once the one playing  $a_4$  leaves later, the process will convert back to  $NC$ . For  $M(2)$ , small proportion of it is in an approximate uniform distribution of playing  $b_2$ ,  $b_3$  and  $b_4$ , and the majority is playing  $b_1$ . When the next mutation comes, the

analysis is exactly the same. Therefore, there is no transition triggered by one mutation with positive probability from  $NC$  to  $BC$ , when populations increase to infinity. We can also see that for the argument above it is safe to assume that at stage 0 the whole  $M(2)$  is playing  $b_2$ .

## 5.2 The non-negligible small probability

From the arguments above, we find that the probability is approaching 0 for an one-mutation transition between  $BC$  and  $NC$ . However, we know the unique invariant distribution exists in the Markov chain of this dynamic process. So how can we find the exact probability of  $BC$  in the invariant distribution?

Even when population size is large, here we cannot ignore the small probability that a distribution is not consistent with the strong law of large numbers. Compared with the ever decreasing mutation rate, this is the main source of transition between two components.

To make the idea clear, we rewrite the game  $\Gamma_2$  as follows in Figure 5. The only changes are the payoff  $u_1 = 0.8 - 10^{-2k}$ ,  $u_2 = 2.5 \cdot 10^k - 1$ ,  $u_3 = 5 \cdot 10^k$  and  $u_4 = -5 \cdot 10^{2k}$  for a  $k \in \mathbb{N}$  with  $k > 2$ . The conclusion in the above section still holds.

In  $\Gamma_2$ , for a state  $z$  with

$$m_{a_2}(z) < \frac{u_1 - 0.5}{0.3} \cdot m, \quad (5.1)$$

$M(2)$  prefers  $b_1$  to  $b_2$ . Define  $\bar{m} := \frac{u_1 - 0.5}{0.3} \cdot m$ . For a state  $z$  with

$$\frac{m_{b_1}(z)}{m} > \frac{u_3 - 2.2}{u_3 - u_4} \quad (5.2)$$

$a_1$  is the best reply of  $M(1)$ .

We first consider the transition from  $BC$  to  $NC$ . Without loss of generality, suppose that at the initial state, the proportions of  $M(2)$  on  $b_3$  and  $b_4$

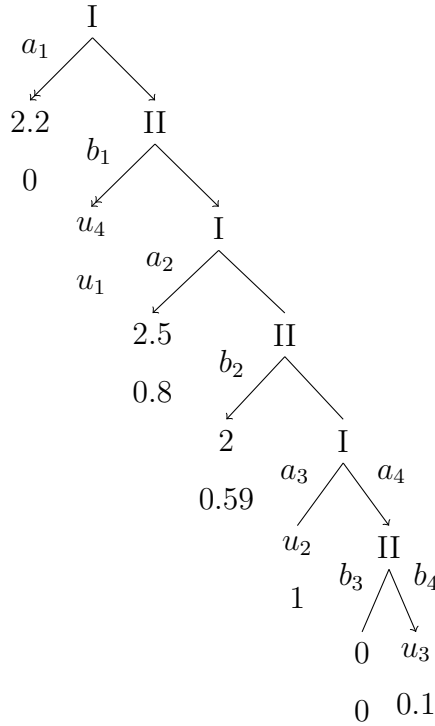


Figure 5: Example 3

are almost equal, and the state is in  $BC$ . (One can prove that, for sufficiently large  $t$ , if  $Z_t \in BC$ , then for any  $\epsilon > 0$ , it is with probability bounded from below that  $|m_{b_3}(t) - m_{b_2}(t)| < \epsilon m$  when  $m$  goes to infinity.)

**Step 1:** After a mutation from  $a_2$  to  $a_3$ ,  $M(2)$  is moving towards  $b_3$  and  $b_4$ . When a significant proportion of  $M(2)$ , say  $xM$  individuals, have left  $b_2$ ,  $M(1)$  is under the selection pressure to move away from  $a_2$ . The exact  $x$  depends on the distribution of  $M(2)$  on  $b_3$  and  $b_4$ , however there is a bound  $\bar{x}(k)$  such that  $x < \bar{x}(k)$ , where  $\bar{x}(k)$  is determined by the payoffs to player I for the path to  $b_2$  and the path to  $a_3$ . We denote the first stage where  $M(1)$ 's best reply is not  $a_2$  as stage  $t_0$

We now assume that at stage  $t_0$  the proportion rate of  $M(2)$  at  $b_3$  to  $b_4$  is greater than 5:3. We denote the event of this distribution at  $t_0$  as event

$F_{BC \rightarrow NC}$ . (Once again, note that the probability of  $F_{BC \rightarrow NC}$  is approaching 0 when the population size goes to infinity.)

**Step 2:** Because at stage  $t_0$ , the distribution on  $M(2)$  is advantageous to  $a_3$ .  $M(2)$  then continues to moves to  $b_3$  and  $b_4$ , at least until the ration of proportion of  $M(2)$  playing  $b_4$  and  $b_3$  exceeds  $(2.5 \cdot 10^k - 1) : (5 \cdot 10^k - 2.5 \cdot 10^k - 1)$ . So, there will be more than  $y$  stages when the best reply of  $M(1)$  keeps as  $a_3$  where  $y$  satisfies

$$2x/5 + y = \rho(y + x) \text{ and } \rho = \frac{2.5 \cdot 10^k - 1}{5 \cdot 10^k}.$$

(This is the extreme case that for all stages from stage  $t_0$  all individuals chosen in  $M(2)$  are playing  $b_2$  and they then move to  $b_4$ .)

**Step 3:** From Step 1 and Step 2, we know that  $M(1)$  is under the selection pressure to  $a_3$  between stages  $t_0$  and  $t_0 + y$ . It is possible that more than  $m - \bar{m}$  (introduced in (5.1)) individuals in  $M(1)$  have moved away from  $a_2$  to  $a_3$ . From Lemma 8.9, it follows that this probability is greater than  $(m - \bar{m})/(y\bar{m}\bar{\sigma})$ .

**Step 4:** If the event in Step 3 is true, then  $M(2)$  will prefer  $b_1$  to  $b_2$  from stage  $t_0 + y$  on. By the strong law of large numbers, the distribution of  $M(2)$  at  $t_3$  and  $t_4$  will become more even from any stage on. So with probability close to 1,  $a_4$  will become the best reply of  $M(1)$  finally. When not many individuals in  $M(1)$  are playing  $a_4$ , the best reply of  $M(2)$  can still be both  $b_3$  and  $b_4$ . However, when enough individuals in  $M(1)$  have accumulated at  $a_4$ , the best reply of  $M(2)$  changes to  $b_1$ . Meanwhile, the proportions of  $M(2)$  playing  $b_3$  and  $b_4$  are decreasing and their distribution towards more even, so  $M(1)$  will still move to  $a_4$ .

**Step 5:** Once the number of individuals in  $M(2)$  playing  $b_1$  is enough large (this threshold number cannot be greater than the  $m_{b_1}(z)$  in (5.2)),  $M(1)$  is under the selection pressure to  $a_1$ . Furthermore, the state is in the



basin of attraction of  $NC$ . During the process to  $NC$ , there may be some times when  $M(2)$  is leaving  $b_1$  due to the relative proportions of  $M(1)$  playing  $a_2$ ,  $a_3$  and  $a_4$ . However, as long as the populations are large, with similar arguments as in the above sub section, the probability that it will enter  $NC$  before the next mutation is close to 1.

In summary, conditional on that  $F_{BC \rightarrow NC}$  introduced in Step 1 happens, the state will transit from  $BC$  to  $NC$  with positive probability bounded from below, when the population size goes to infinity.

We now turn to the possible transition from  $NC$  to  $BC$ . Without loss of generality, suppose that at the initial state in  $NC$ , the proportions of  $M(2)$  on  $b_2$ ,  $b_3$  and  $b_4$  are almost equal, and the majority of  $M(2)$  is on  $b_1$ . (It can be supported by an analogous argument as in the transition from  $BC$  to  $NC$  above.) One can show that a transition triggered by a mutation in  $M(1)$  from  $a_1$  to  $a_3$  or  $a_4$  cannot lead to  $BC$ . If there is a mutation from  $a_1$  to  $a_2$ , then  $M(2)$  is under the selection pressure to  $b_2$ ,  $b_3$  and  $b_4$ . Firstly, from (5.1), it follows that, when

$$m_{b_1} < \frac{(u_3 - 2.2)m}{u_3 - u_4}, \quad (5.3)$$

$M(1)$  is possibly under the selection pressure to leave  $a_1$ . Secondly, if  $M(1)$ 's best reply is not  $a_1$ , then it needs

$$m_{b_2} > \frac{(u_2 - 2.5)m}{u_2 - 2} > \frac{246}{250} \quad (5.4)$$

to provide enough incentive such that  $a_2$  becomes the new best reply of  $M(1)$ . We denote by  $F_{NC \rightarrow BC}$  the event that  $m_{b_2}$  satisfies (5.4) when  $M(1)$ 's best reply diverts from  $a_1$ . Note this is a necessary condition for a transition from  $NC$  to  $BC$ . Otherwise the transition process is similar to the one triggered by a mutation from  $a_1$  to  $a_3$  or  $a_4$ . It is obvious that the probability of  $F_{NC \rightarrow BC}$  is approaching 0 when the population size  $m$  goes to infinity.

While both  $P(F_{BC \rightarrow NC})$  and  $P(F_{NC \rightarrow BC})$  are approaching 0 when  $m$  increases to infinity, we can show that for any fixed  $m$ ,  $P_m(F_{BC \rightarrow NC}) > P_m(F_{NC \rightarrow BC})$ . Furthermore, for any positive number  $c$  between 0 and 1, there is an  $\bar{m}(c) \in \mathbb{N}$  such that

$$c \cdot P(F_{BC \rightarrow NC}) > P(F_{NC \rightarrow BC}) \quad \forall m > \bar{m}(c).$$

Note that, conditional on  $F_{BC \rightarrow NC}$ , a transition from  $BC$  to  $NC$  has positive probability. For a transition from  $NC$  to  $BC$ ,  $F_{NC \rightarrow BC}$  is a necessary condition. Since the definition of  $m$ -evolutionary stability is concerning with a process that the mutation rate  $\mu$  decreases to 0 under a fixed  $m$ , we can conclude that in  $\Gamma$ , given any  $\epsilon > 0$ , there is a natural number  $\bar{m}(\epsilon)$  such that

$$\epsilon \cdot \lim_{\mu \rightarrow 0} \pi_{\mu, m}[NC] > \lim_{\mu \rightarrow 0} \pi_{\mu, m}[BC] \quad \forall m > \bar{m}(\epsilon),$$

and thus

$$\liminf_{m \rightarrow \infty} \lim_{\mu \rightarrow 0} \pi_{\mu, m}[NC] = 1.$$

## 6 Dynamics in other limiting processes

We consider a limiting process that populations go to infinity and mutation rate goes to zero under the condition that the product of population size and mutation rate (expected number of mutations per generation) is bounded away from both zero and infinity. Recall in the limiting process of the last two examples, the product of population size and mutation rate goes to zero. We show backward induction equilibrium is not the only evolutionarily stable result in the game  $\Gamma_3$  in Figure 6.

In this example, for a  $\mu > 0$ , we put  $\sigma = 1 - \mu$  as the probability that selection implementation (c.f. (2.1)) is chosen for the individual  $q^i$  in the

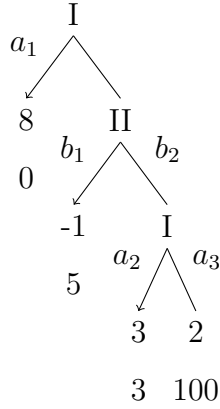


Figure 6:  $\Gamma_3$

population  $M(i)$  for  $i = 1, 2$  at any stage. In this game  $\Gamma_3$ , we mark the pure strategies of player I by  $a_1, a_2$  and  $a_3$ , and the pure strategies of player II by  $b_1$  and  $b_2$ . It is easy to check that the backward induction equilibrium in  $\Gamma_3$  is  $(a_1, b_1)$ . We consider here the case of better-reply selection mechanism, which is the one Hart applies in [11] for this particular limiting process. For this mechanism, the only change for the model in Section 2 is to replace  $BS^i(q^i, w)$  by a set of ‘better strategies’

$$BT^i(q^i, w) := \{a^i \in A^i : u^i(a^i, w^{-i}) > u^i(w^i(q^i), w^{-i})\},$$

for a chosen individual  $q^i$  in  $M(i)$  at the state  $w$ . Roughly speaking, the selected individual will choose one better strategy randomly. Our proof can be adapted to the case of best-reply selection mechanism easily as well. To reach Theorem 3.5, we prove the following proposition.

**Proposition 6.1** *Given the game  $\Gamma_3$  in Figure 6 and  $\delta, \eta$  with  $0 < \delta < \eta$ , there is a positive number  $c(\delta, \eta)$  such that for all pairs of  $(\mu, m)$  with  $0 < \mu < 2/9$ ,  $m \geq 2(3 + 4\eta)/3$  and  $\delta \leq \mu m \leq \eta$*

$$\pi_{\mu, m}[x_{b_2}^2(w) \geq 1/40] > c(\delta, \eta).$$

Note first that there is always selection pressure at population  $M(1)$  towards strategy  $a_1$  regardless of the distribution of population  $M(2)$ . The proportions of population  $M(1)$  playing strategy  $a_2$  and  $a_3$  are almost surely not always the same during an infinitely long dynamic process. In particular, at the state  $w$  where  $m_{a_3}(w) = 1$  but  $m_{a_2}(w) = 0$ , player II is reached and the selection is towards  $b_2$ . Before the proof of Proposition 6.1, we give four preliminary lemmas concerning the limiting behaviour of the generated Markov chain in  $\Gamma_3$  when  $m$  goes to infinity,  $\mu$  goes to zero and  $\mu m$  is bounded away from both zero and infinity. In Lemma 6.4 and 6.5 below, we shall show that the invariant probability of the whole population playing  $a_1$  except one mutant playing  $a_3$  is positive, if  $m\mu$  is bounded away from both infinity and zero. Lemma 6.2 estimates the probability that for no less than  $m/4$  consecutive stages population  $M(1)$  stays still with one mutant at  $a_3$ . Conditional on that, Lemma 6.3 shows that the probability is approaching 1 that at least  $1/40$  of population  $M(2)$  is attracted to strategy  $b_2$  within these  $m/4$  stages when  $m$  goes to infinity. We shall complete the proof by the theorem on Markov chain convergence to equilibrium.

The proofs of all four lemmas are relegated to Appendix.

Recall that, at any  $t$ , for a population  $M(i)$ , if  $Z_t = w \in W$ , then  $Z_t^i = w^i$ .

**Lemma 6.2** *For a Markov chain  $(Z_t)_{t \geq 0}$  induced in  $\Gamma_3$  with  $m \geq 2(3+2\eta)/3$  and  $0 < \mu m \leq \eta$ , for a state  $w$  with  $m_{a_3}(w) = 1$  and  $m_{a_2}(w) = 0$  (and hence  $m_{a_1}(w) = m - 1$ ), there is a constant  $c_1(\eta)$  such that*

$$P_{\mu,m}(Z_t^1 = Z_0^1 \ \forall 0 < t \leq m/4 | Z_0 = w) \geq c_1(\eta).$$

**Lemma 6.3** *Denote  $\{w \in W : m_{a_3}(w) = 1 \ \& \ m_{a_2}(w) = 0\}$  by  $S(m)$ . Then for a Markov chain  $(Z_t)_{t \geq 0}$  induced in  $\Gamma_3$  with population size  $m \geq 2(3+2\eta)/3$*

and mutation rate  $\mu > 0$ , there is a constant  $c_2(\eta)$  such that

$$P_{\mu,m}(B_2(m/4) \geq m/40 | Z_t \in S(m) \forall 0 \leq t \leq m/4) \geq c_2(\eta).$$

**Lemma 6.4** For a Markov chain  $(Z_t)_{t \geq 0}$  induced in  $\Gamma_3$  with  $m \geq 8\eta/3$ ,  $\mu < 2/9$  and  $0 < \mu m \leq \eta$ , there exists a constant  $c_3(\eta)$  such that

$$\pi_{\mu,m}[\{w : m_{a_1}(w) = m\}] \geq c_3(\eta).$$

**Lemma 6.5** For a Markov chain  $(Z_t)_{t \geq 0}$  induced in  $\Gamma_3$  with  $m \geq 8\eta/3$  and  $\delta \leq \mu m \leq \eta$ , there exists a constant  $c_4(\delta, \eta)$  such that

$$\pi_{\mu,m}[\{w : m_{a_3}(w) = 1, m_{a_2}(w) = 0\}] \geq c_4(\delta, \eta).$$

**Proof of Proposition 6.1:** Suppose  $(\bar{Z}_l)_{l \geq 0}$  is a Markov chain with the one-step transition probability matrix  $Q$  induced in Example 4 and with the initial condition  $m_{a_3}(Z_0) = 1$ ,  $m_{a_2}(Z_0) = 0$ . By Markov property, conditional on any  $\bar{Z}_t$  at some period  $t$  with  $m_{a_3}(\bar{Z}_t) = 1$  and  $m_{a_2}(\bar{Z}_t) = 0$ , the process  $(\bar{Z}_{t+i})_{i \geq 0}$  is still a Markov chain and has the same one-step transition probability matrix  $Q$  as in  $(\bar{Z}_l)_{l \geq 0}$ . Recall the definition of  $S(m)$  in the proof of Lemma 6.3. It follows from the Markov property that

$$P_m(m_{b_2}(\bar{Z}_{t+m/4}) \geq m/40 | \bar{Z}_t \in S(m)) = P_m(m_{b_2}(\bar{Z}_{m/4}) \geq m/40) \quad \forall t \geq 0.$$

So, for any  $t \geq 0$ ,

$$\begin{aligned} & P_m(m_{b_2}(\bar{Z}_{t+m/4}) \geq m/40) \\ & \geq P_m(\bar{Z}_t \in S(m)) P_m(m_{b_2}(\bar{Z}_{t+m/4}) \geq m/40 | \bar{Z}_t \in S(m)) \\ & = P_m(\bar{Z}_t \in S(m)) P_m(m_{b_2}(\bar{Z}_{m/4}) \geq m/40). \end{aligned} \tag{6.1}$$

From Lemma 6.2 and 6.3, it follows that, for the Markov chain  $(\bar{Z}_l)_{l \geq 0}$ ,

$$P_{\mu,m}(m_{b_2}(\bar{Z}_{m/4}) \geq m/40) \geq c_1(\eta)c_2(\eta).$$

For any  $t > 0$ , we find

$$P_{\mu,m}(m_{b_2}(\bar{Z}_{t+m/4}) \geq m/40) \geq c_1(\eta)c_2(\eta)P_{\mu,m}(\bar{Z}_t \in S(m)). \quad (6.2)$$

Recall the definition of the invariant distribution  $\pi$  in the original Markov chain  $(Z_t)_{t \geq 0}$ . By the theorem on Markov chain convergence to equilibrium, i.e.,

$$\lim_{t \rightarrow \infty} P(\bar{Z}_t = w) = \pi[w] \quad \forall w \in W,$$

we may infer that for a population size  $m$  and a mutation rate  $\mu$

$$\begin{aligned} & \pi_{\mu,m}[\{w : m_{a_3}(w) = 1 \ \& \ m_{a_2}(w) = 0\}] \\ &= \lim_{t \rightarrow \infty} P_{\mu,m}(m_{a_3}(\bar{Z}_t) = 1 \ \& \ m_{a_2}(\bar{Z}_t) = 0) \\ &= \lim_{t \rightarrow \infty} P_{\mu,m}(\bar{Z}_t \in S(m)), \end{aligned} \quad (6.3)$$

and

$$\pi_{\mu,m}[\{w : m_{b_2}(w) \geq m/40\}] = \lim_{t \rightarrow \infty} P_{\mu,m}(m_{b_2}(\bar{Z}_t) \geq m/40). \quad (6.4)$$

Let  $t$  increase to  $\infty$  in (6.2) and apply (6.3) and (6.4), we have

$$\begin{aligned} \pi_{\mu,m}[w : m_{b_2}(w) \geq m/40] &= \lim_{t \rightarrow \infty} P_{\mu,m}(m_{b_2}(\bar{Z}_t) \geq m/40) \\ &= \lim_{t \rightarrow \infty} P_{\mu,m}(m_{b_2}(\bar{Z}_{t+m/4}) \geq m/40) \\ &\geq c_1(\eta)c_2(\eta) \lim_{t \rightarrow \infty} P_{\mu,m}(\bar{Z}_t \in S(m)) \\ &= c_1(\eta)c_2(\eta)\pi_{\mu,m}[\{w : m_{a_3}(w) = 1, \ m_{a_2}(w) = 0\}]. \end{aligned}$$

Lemma 6.5 then completes the proof.  $\square$

We consider a general three-node game in Figure 7 below with backward induction actions arrowed.

By analogous analysis as in Proposition 6.1, one can find the set of alternative Nash equilibria other than  $(a_1, b_1)$  evolutionarily stable in the limiting

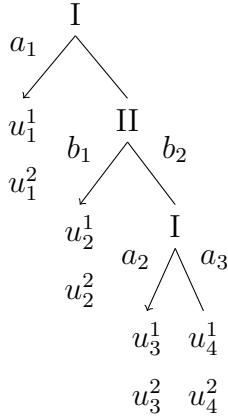


Figure 7: A general three-node game

process with the expected number of mutations per generation bounded away from both zero and infinity, if the following conditions are satisfied:

$$u_1^1 > \max\{u_2^1, u_3^1, u_4^1\}, \quad u_3^1 > u_4^1$$

and

$$u_3^2 < u_2^2 < u_4^2.$$

## 7 Further comments

We have shown the game in Example 1 and a game in [19] (the one with the paths of backward induction equilibrium and the alternative equilibrium terminated by the same player) where the all equilibrium components are evolutionarily stable outcome for large populations. It is not difficult to see that, for an example to (3.1) in Theorem 3.3 in the case that the paths of  $BC$  and  $NC$  are terminated by the same player, the corresponding game needs at least four nodes. Otherwise, one can find the game essentially reduces to the case that each player can only play at one node, which can only yield the

unique evolutionarily stable outcome of the backward induction equilibrium, as analysed in [11] and [10].

It is interesting and useful to see why, for an example to (3.1) in Theorem 3.3 with the paths of  $BC$  and  $NC$  terminated by different players, the example needs at least five nodes for large populations. We have follow the similar approach in Section 5.1. Consider an adapted version of  $\Gamma_1$  with three nodes as in Figure 8 below. The  $BI$  of this game is  $(a_1, b_2)$  and the only alternative

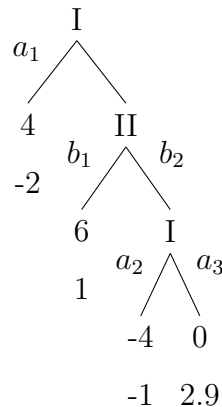


Figure 8: The shortened version of  $\Gamma_1$

pure Nash equilibrium is  $(a_2, b_1)$ . In this game,  $NC$  is evolutionarily stable for every finite population size, but not for the case that the population size goes to infinity.

Suppose the state is in  $BC$  at stage 0. We now take a look at an attempt to escape to  $NC$  by a mutant in population  $M(1)$  to strategy  $a_2$  at stage 1. We further suppose that  $\mu < g(m)$  and no mutation ever happens before the state back to NE, which has probability close to 1. (See Lemma 8.3.) Given  $A_2(1) > A_3(1) = 0$ , the mutant triggers the selection process towards  $b_1$  in population  $M(2)$ . Once the proportion of  $X_{b_1}$  exceeds  $2/3$ , population  $M(1)$  is under the selection pressure. Note the best reply of population  $M(1)$  in



the case of  $2/3 < X_{b_1} < 1$  cannot be  $a_2$ , as the bottom node is still connected by some individuals in population  $M(2)$ , and  $a_3$  is strictly better than  $a_2$  conditional on the bottom node is connected. Thus, population  $M(1)$  is under the selection pressure to  $a_3$  only, whenever  $2/3 < X_{b_1} < 1$ . It then follows that  $m_{a_2} \leq 1$  at every stage before the next mutation. We may then infer that the relative proportion of  $m_{a_3}/m_{a_2}$  increases rapidly once  $X_{b_1}$  exceeds  $2/3$ . Of course, one may argue that, with positive probability, the chosen individual in population  $M(1)$  is always the one playing  $a_3$  in very long times, e.g. in the next  $m$  stages. However, this probability goes to zero when the population size goes to infinity.

From the first stage with  $m_{a_3} > m_{a_2}$ , individuals in population II begin to revert back to  $b_2$ . After  $X_{b_1}$  drops below  $2/3$ , the strategy  $a_1$  becomes the favourite of population  $M(1)$  again. Therefore,  $a_1$  and  $b_2$  are the best reply of population  $M(1)$  and population  $M(2)$  thereafter, respectively, and the state goes back into  $BC$  before the next mutation with probability close to 1. During the process above, population  $M(2)$  swings back and forth between strategy  $b_1$  and  $b_2$ , similarly as shown in Example 3. However, at most a tiny proportion of population  $M(1)$  is attracted to strategy  $a_3$  while the majority still stay at strategy  $a_1$ , if  $M(1)$  is large. From a macro perspective, this is because player I's support in  $NC$  is strategy  $a_2$ , and  $a_2$  can never be the best reply of player I if  $X_{b_2} > 0$ . The analysis for the selection mechanism of better reply is slightly more complicated, but with the same flavour above. (The full details will be presented together with other positive results in the future.)

To construct a game such that a successful transition to  $NC$  from  $BC$  has probability bounded away from zero, we need to make player I's support strategy to  $NC$  as a best reply to some strategy of player II, and also need to

distract the majority of population  $M(2)$  away from its backward induction strategy during the process. This can only be achieved with no less than five nodes. In Example 1, each strategy taken below the path of  $BC$  is a best reply to the other player's one strategy taken below the path of  $BC$  as well. It should be clear so far why, for a transition from  $BC$  to  $NC$  in the proof of Proposition 4.1, we need a mutant to  $a_3$  before a mutant to  $a_2$  in population  $M(1)$ .

## 8 Appendix

For all the omitted proofs of the lemmas below, please refer to the Appendix in [19].

### 8.1 Results on transition time

The following corollary to Theorem 3.2 estimates the expected transition time when no mutation involved. Given a Markov chain  $(Z_t)_{t \geq 0}$  generated in the selection-mutation process with population size  $m$  and mutation rate  $\mu$ , for a state  $w \in W$  and a subset  $S \subseteq W$ , we define the random variable  $T_{\mu,m}(w, S) := \min\{t \geq 0 : Z_t \in S, Z_0 = w\}$  to be the transition time from  $w$  to  $S$ , and define the expected transition time from  $w$  to  $S$  to be

$$U_{\mu,m}(w, S) := E_{\mu,m}[T_{\mu,m}(w, S)].$$

For every two subsets  $S_1, S_2 \subseteq W$ , we put

$$U_{\mu,m}(S_1, S_2) := \mathbb{6}_{w \in S_1} U_{\mu,m}(w, S_2). \quad (8.1)$$

**Lemma 8.1** *In a finite  $N$ -player game in extensive form of perfect information, we suppose that each player has no more than  $S$  pure strategies. Given*

a Markov chain  $(Z_t)_{t \geq 0}$  with  $\mu = 0$  and the associated

$$f_\sigma(m) := \frac{S^{mN}}{\left(2 \left(\frac{\sigma}{2mS}\right)^N\right)^{S^{mN}}}.$$

for a  $\sigma$  with  $0 < \sigma < 1$ , we have  $U_{0,m}(W, NE) < f_\sigma(m)$ .

We need the following lemma to show the expected transition time to Nash equilibria when  $\mu$  is small.

**Lemma 8.2** *Given a number  $p$  in  $(0, 1/2)$ , we consider a random variable  $X_p : \Omega \rightarrow \mathbb{N}$  in geometric distribution with*

$$P(X_p = k) = (1 - p)^{k-1} p \quad \forall k \in \mathbb{N}.$$

*Then, we have  $P(X_p > E[X_p]) > 1/8$ .*

The next lemma shows that the expected transition time to Nash equilibria is bounded when mutation rate  $\mu$  is small.

**Lemma 8.3** *Consider a two-player extensive-form game of perfect information with the induced Markov chain. Without loss of generality, we suppose that there exists a number  $\bar{\sigma}$  with  $0 < \bar{\sigma} < 1$  such that  $\sigma \geq \bar{\sigma}$  in the limiting process of evolutionary dynamics. Denote  $c_{\bar{\sigma}}$  to be  $e^{-200/\bar{\sigma}}$ . Then, given  $f_{\bar{\sigma}}(m)$  defined in (8.1), for every  $\mu < c_{\bar{\sigma}}/(32f_{\bar{\sigma}}(m))$ , it satisfies  $U_{\mu,m}(W, NE) < c_{\bar{\sigma}}\mu^{-1}$ . Define  $g_{\bar{\sigma}}(m) := c_{\bar{\sigma}}/(32f_{\bar{\sigma}}(m))$ .*

The following lemma shows in expectation the transition conditional on passing some subset takes longer than the one without such condition.

**Lemma 8.4**

$$\forall W_3 \subseteq W, U(W_3, NC) \leq U(W_3, BC) + U(BC, NC).$$

It is also important to know in expectation how long the Markov chain stays at the same state each time. The next lemma answers this question for an arbitrary state in  $NE$ . To present it formally, we first introduce some terminology. Given the Markov chain  $(Z_t)_{t \geq 0}$ , for a state  $w$  in  $W$ , suppose that  $Z_0 \neq w$  and mark stage 0 as  $t_0^e(w)$ . For each  $i \geq 0$ , we mark the stage  $\min\{t > t_i^e(w) : Z_t = w\}$  as  $t_{i+1}^s(w)$  and then the stage  $\min\{t > t_{i+1}^s(w) : Z_t \neq w\}$  as  $t_{i+1}^e(w)$  in turn. We define  $\bar{U}(w)$  to be the expected length of the period during which the dynamic process stays at  $w$ , i.e.,

$$\bar{U}(w) := E \left[ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq i \leq N} (t_i^e(w) - t_i^s(w)) \right].$$

Note that the expected length of the period at  $w$  does not depend on the state at initial stage, so we can assume  $Z_0 \neq w$  in the condition above.

**Lemma 8.5** *Given a two-player extensive-form game of perfect information, for the induced Markov chain with the mutation rate  $\mu < g_a(m)$  where  $0 < a \leq 1$ , we have*

$$\bar{U}(w) \geq 1/(2\mu)$$

for all  $w$  in  $NE$ .

## 8.2 From time-average asymptotic to evolutionary stability

This subsection is included in [19].

We shall give a proofs of Theorem 3.3 by Proposition 4.1 in the following approach.

Recall that the long-run behaviour of a Markov chain  $(Z_t)$  is well described by its invariant distribution  $\pi$  in the following way. In any long enough

period of time, the relative frequency of visits at a subset  $S$  included in  $W$  is approximately  $\pi[S]$ . That is,

$$\forall S \subseteq W, \lim_{T_2 - T_1 \rightarrow \infty} \frac{|\{t : T_1 < t \leq T_2, Z_t \in S\}|}{T_2 - T_1} = \pi[S]. \quad (8.2)$$

This property holds regardless of the initial state.

Our approach is to analyse the long-term relative proportion of visiting time in  $NC$ . Most of the proof for Proposition 4.1 is devoted to the calculation of  $U(BC, NC)$ . (If no ambiguity, we drop the subscript  $\mu$  and  $m$  of  $U$ .) Consider the case of  $\mu < g(m)$ , where  $g(m)$  is defined in Lemma 8.3.

Recall that an event is a subset of the sample space  $\Omega = W^{\mathbb{N}}$  which is specified in Section 2. An element  $\omega$  in the sample space  $\Omega$  is an infinite sequence of states, i.e.,  $\omega = (\omega_0, \omega_1, \dots)$  with  $\omega_i$  in  $W$  for all  $i \geq 0$ . Denote the truncated sequence of  $\omega$  with length  $n$  by  $\omega|(n-1) = (\omega_0, \omega_1, \dots, \omega_{n-1})$ . We can view an event as a set of sequences of states. We denote  $\{\omega \in \Omega : \omega \in F\}$  by simply  $F$ . In the proof of Proposition 4.1, we shall define an event  $F$  as an intersection of a finite sequence of events on the dynamic process such that, if  $F$  happens, then  $Z_t$  is in  $NC$  at some stage  $t > 0$ . Note that we can in fact start from any stage  $\bar{t} > 0$ , ignore the partial history  $(Z_0, \dots, Z_{\bar{t}-1})$ , and check whether it will reach  $NC$  after  $\bar{t}$  according to a ‘similar’ transition pattern as in  $F$ . That is, by Markov property, we can shift the original Markov chain  $Z_t$  to  $Z_{t-\bar{t}}$  and then consider the event  $F$ . For instance, when we count from stage  $\bar{t}$ ,  $F$  includes that the first mutant in population  $M(1)$  picks strategy  $a_2$  and the second mutation happens  $2\mu^{-1}$  stages later... Here the exact time index is not important, only transition order and time interval matter.

Note that  $F$  may only be a sufficient condition but not a necessary condition. Denote the complement of  $F$  by  $F^c$ . For an initial state  $s$ , we define  $T_s(F)$  (or  $T_s(F^c)$ ) to be the expected time that we know  $F$  is true (or false),

respectively. That is,

$$T_s(F) = E[\min\{t : \forall \omega \in \Omega \text{ with } \omega|t = (Z_0 = s, Z_1, \dots, Z_t), \omega \in F\} | F], \quad (8.3)$$

and

$$T_s(F^c) = E[\min\{t : \forall \omega \in \Omega \text{ with } \omega|t = (Z_0 = s, Z_1, \dots, Z_t), \omega \notin F\} | F^c].$$

**Lemma 8.6** *Without loss of generality, we suppose that there exists a number  $\bar{\sigma}$  with  $0 < \bar{\sigma} < 1$  such that  $\sigma \geq \bar{\sigma}$  in the limiting process of evolutionary dynamics. Given the induced Markov chain with a finite population size  $m$ , suppose if  $F$  happens, then  $Z_t$  is in  $NC$  at some stage  $t > 0$ . We further suppose there exist two positive numbers  $k$  and  $p$  such that given any  $g' \leq g_{\bar{\sigma}}(m)$  and for any mutation rate  $\mu < g'$  it has the property*

$$\min_{w \in BC} P(F | Z_0 = w) \geq p$$

and

$$\max_{w \in BC} T_w(F) + \max_{w \in BC} T_w(F^c) \leq k\mu^{-1}.$$

Then there exists a positive number  $c(p, k)$  such that

$$\forall \mu < g', \pi_{\mu, m}[NC] \geq c(p, k).$$

### 8.3 A combinatorial lemma

We shall use the following lemma to analyse the ‘dynamic proportion’ of two strategies played at some distant node from the root, when we only know its initial distribution and that the sum of individuals playing at these two strategies is decreasing to zero.

**Lemma 8.7** *Suppose you have three boxes, A, B and C, and m balls. At stage 0, Box B contains  $\lceil(1 - k)m\rceil$  balls and C contains  $\lfloor km\rfloor$  balls, where  $0 < k < 1/2$ . At each stage, one ball is randomly chosen among those m balls. If it is in Box A, then nothing will happen. If it is in Box B or C, it will be moved to Box A or remain in the same box, both with probability 1/2. So, all balls will be moved to Box A almost surely. We define the minimum time of the situation of all balls in Box A by T. For each stage t with  $t < T$ , define  $B_t$  and  $C_t$  to be the number of balls in Box B and Box C at stage t, respectively.*

*Then,  $P(B_t > C_t \forall t < T) \geq 1 - 2k$ , independently to m.*

The proof below is inspired by the Ballot Problem. One may also prove it by Doob's Optional-Stopping Theorem.

**Proof.** From  $B_0/(B_0 + C_0) \geq 1 - k$ , we may infer the probability of the last ball drawn to Box A from Box B is no less than  $1 - k$ . This is because for any particular ball in Box B or Box C, the probability of being taken to Box A as the first, second,..., or the last ball is equal. (For example, when  $k=1/10$ , one can view Box B as divided into 9 sub-boxes  $B^1, B^2, \dots, B^9$  with  $B_0^i = \lfloor m/10 \rfloor$  for all  $1 \leq i < 9$  and  $B_0^9 = m - 9\lfloor m/10 \rfloor \geq \lfloor m/10 \rfloor$ . Since every ball is picked randomly from  $B^1, \dots, B^9$  and C to A, the probability that the last ball is drawn from C is no greater than from any sub-box  $B^i$ . Thus  $P(C_{T-1} = 1) \leq 1/10$ .

For a random trajectory  $V := (t, B_t - C_t)_{0 \leq t \leq T}$  in this dynamic process, denote the minimum time  $t$  with  $B_t - C_t = 0$  by  $t_0(V)$ . If  $t_0(V) = T$ , then  $B_t > C_t \forall 0 \leq t < T$ . We put

$$S_1 := \{v = (t, B_t - C_t)_{0 \leq t \leq T} : B_{T-1} - C_{T-1} = 1, t_0(v) < T\}$$

and

$$S_2 := \{v = (t, B_t - C_t)_{0 \leq t \leq T} : B_{T-1} - C_{T-1} = -1\}.$$

We deduce  $P(S_1)$  by a reflection method. Given a trajectory  $v = (t, y_t = B_t - C_t)$  with  $t_0(v) < T$  and  $y_{T-1} = 1$ , the probability that the dynamic process agrees with this trajectory can be determined as follows.

$$\begin{aligned}
& P((t, y_t)_{0 \leq t \leq T}) \\
&= P(B_t - C_t = y_t \forall 0 \leq t \leq T) \\
&= P(B_t - C_t = y_t \forall 0 \leq t \leq t_0(v)) \cdot \\
& P(B_t - C_t = y_t \forall t_0(v) < t \leq T | B_i - C_i = y_i \forall i \leq t_0(v)).
\end{aligned}$$

Now we reflect the trajectory  $v = (t, y_t)$  from stage  $t_0(v)$  with respect to the line  $y_t = B_t - C_t = 0$  for all  $t$  to form a new trajectory  $v' = (t, y'_t)$  with  $y'_t = y_t \forall t \leq t_0$  and  $y'_t = -y_t \forall t_0 < t < T$ . Note that  $B_{t_0(v)} = C_{t_0(v)}$ . We may then infer that

$$\begin{aligned}
& P(B_t - C_t = y_t \forall t_0(v) < t < T | B_t - C_t = y_t \forall t \leq t_0) \\
&= P(B_t - C_t = y'_t = -y_t \forall t_0(v) < t < T | B_t - C_t = y_t \forall t \leq t_0),
\end{aligned}$$

since the truncated stochastic process starting from stage  $t_0(v)$  is symmetric with respect to the line  $B_t - C_t = 0$  for all  $t$ . It follows that  $P((t, y_t)_{0 \leq t \leq T}) = P((t, y'_t)_{0 \leq t \leq T})$ . Moreover, for each trajectory  $v = (t, y_t)$  with  $t_0(v) < T$  and  $y_{T-1} = 1$ , there is a unique reflected trajectory  $v = (t, y'_t)$ , and vice versa. Thus, it is a bijection from  $S_1$  to  $S_2$  in the reflection relationship. From  $P(S_2) = P(C_{T-1} = 1) \leq k$ , it follows that  $P(B_t > C_t \forall t < T) = 1 - P(S_1) - P(S_2) \geq 1 - 2k$ .  $\square$

## 8.4 Preliminary lemmas for evolutionary dynamics

**Lemma 8.8** *Given the induced Markov chain  $(Z_t)$  of a finite extensive-form game of perfect information with population size  $|m| > 1$ , suppose an individual  $q$  is chosen in population  $M(i) \ni q$  at some stage  $t$ , then for any positive*



number  $k$ , the probability that  $q$  is not chosen in  $M(i)$  for all stages between  $t$  and  $t + \lceil km \rceil$  is greater than  $2^{-2k-1}$ .

**Lemma 8.9** *Assume that at each stage the probability that selection takes effect at a chosen individual is no less than  $\bar{\sigma}$ . Given the induced Markov chain  $(Z_t)$  of a finite extensive-form game of perfect information, suppose that population  $M(i)$  is under selection pressure moving away strategy  $s_i$  for all stages  $t > \bar{t}$ . Then, for any pair of positive numbers  $k$  and  $\lambda$ , it follows that*

$$P(m_{s_i}(\bar{t} + \lceil km \rceil) > \lambda m | G_{\bar{t}, \bar{t} + \lceil km \rceil}) < \frac{1 - \lambda}{k\lambda\bar{\sigma}},$$

where  $G_{t_1, t_2}$  requires no mutation in all populations between stage  $t_1$  and  $t_2$ .

## 8.5 Proof of Proposition 4.1

It is easy to check that the backward induction equilibrium in  $\Gamma_3$  is  $(a_1, b_3)$ , and the corresponding backward induction equilibrium component  $BC = \{(x^1, x^2)\}$  satisfies

$$x_{a_1}^1 = 1, \quad 106x_{b_2}^2 + 3x_{b_3}^2 \geq 2 \quad \text{and} \quad 9x_{b_2}^2 + 9x_{b_3}^2 \geq 2. \quad (8.4)$$

One alternative pure Nash equilibrium is  $(a_2, b_1)$ . Its corresponding Nash equilibrium component  $NC = \{(x^1, x^2)\}$  satisfies

$$x_{b_1}^2 = 1, \quad x_{a_2}^1 \geq x_{a_3}^1 + x_{a_4}^1 \quad \text{and} \quad x_{a_2}^1 + 2x_{a_3}^1 \geq 2x_{a_4}^1. \quad (8.5)$$

We now prove Theorem 3.3 by Proposition 4.1 below.

Without loss of generality, we assume  $\sigma > \bar{\sigma} > 0$  in the limiting process. Recall that the population size  $m$  is finite and we suppose  $\mu < g_{\bar{\sigma}}(m)$ . We define an event  $F$  below in a similar procedure as in the proof of the analogous proposition in [19]. We are going to define a finite sequence of

events  $F_1, F_2, \dots, F_9$  with respect to the Markov chain  $(Z_t)_{t \geq 0}$ , and put event  $F$  to be  $\bigcap_{i=1}^9 F_i$ . For simplicity, we denote  $\bigcap_{i=1}^l F_i$  by  $D_l$  for all  $1 \leq l \leq 9$ . Thus  $D_l = D_{l-1} \cap F_l$ . We would like to apply Lemma 8.6 after obtaining  $U(BC, NC)$ . To this end, we assume  $Z_0 \in BC$  at stage 0. We shall see the argument below is independent to the exact initial state in  $BC$ . We enumerate the stages that at least one mutation happens as  $u_1, u_2, \dots$

**Event**  $F_1 := \{Z_{u_1}^1(q^1) = a_3, m_{b_i}(Z_{u_1}) = m_{b_i}(Z_0) \forall i = 1, 2, 3\}$ . [The first mutation is in population  $M(1)$  only and that yields an  $a_3$  strategy.]

Similarly to  $F_1$  in [19], we see that, for all  $w$  in  $BC$ ,

$$P(F_1|Z_0 = w) = (1 - \mu)/6 > 1/7, \quad (8.6)$$

when  $\mu < 1/7$ .

**Event**  $F_2 := \{\mu^{-1} < u_2 - u_1 < 100\mu^{-1}\}$ . [The number of stages between the first and the second mutation is between  $\mu^{-1}$  and  $100\mu^{-1}$ .]

Similarly to  $F_2$  in [19], we see that, for all  $w$  in  $BC$ ,

$$P(F_2|Z_0 = w) > 1/16 - 1/200, \quad (8.7)$$

when  $\mu < 1/4$ .

Note that between stage  $u_1$  and  $u_2$ , population  $M(1)$  is not under selection pressure except the mutant playing  $a_3$ . To see this, note that for population  $M(2)$ , the best strategy to  $a_3$  is  $b_2$ . At stage 0,  $a_1$  is the best strategy to  $Z^2$ . ( $Z^2$  is the population distribution of  $M(2)$ .) The increase of  $Z_{b_2}^2$  can only consolidate the preference of population I to strategy  $a_1$ . Hence, the only possible change of  $Z^1$  between stage  $u_1$  and  $u_2$  is that the mutant of  $a_3$  reverts back to  $a_1$ . We denote the stage that the mutant  $q^1$  changes his strategy by  $t_1 := \min\{t > u_1 : Z_t^1(q^1) \neq Z_{u_1}^1(q^1)\}$ .

**Event**  $F_3 := \{t_1 - u_1 < 1/(2\mu)\}$ . [From stage  $u_1$ , it takes less than  $1/(2\mu)$  stages for the mutant to change his strategy from  $a_3$ .]

Similar to the deduction to  $F_3$  in [19], we see that, for all  $w$  in  $BC$ ,

$$P(F_3|D_2, Z_0 = w) \geq 1 - e^{-\frac{200}{\bar{\sigma}}}. \quad (8.8)$$

**Event**  $F_4 := \{t_1 - u_1 > \lceil 72m/\bar{\sigma} \rceil\}$ . [From stage  $u_1$ , it takes more than  $\lceil 72m/\bar{\sigma} \rceil$  stages for the mutant to change his strategy from  $a_3$ .]

**Claim:** when  $\mu < g_{\bar{\sigma}}(m)$ , for all  $w$  in  $BC$ ,

$$P(D_4, Z_0 = w) \geq P(D_2, Z_0 = w)(e^{-\frac{144}{\bar{\sigma}}-1} - e^{-\frac{200}{\bar{\sigma}}}) \geq c_4(\bar{\sigma}) \quad (8.9)$$

where  $c_4(\bar{\sigma})$  is denoted to be  $(\frac{1}{16} - \frac{1}{200})(e^{-\frac{144}{\bar{\sigma}}-1} - e^{-\frac{200}{\bar{\sigma}}})/7$ .

We take  $k = 72/\bar{\sigma}$  in Lemma 8.8. It follows that, for all  $w$  in  $BC$ ,

$$P(F_4|D_2, Z_0 = w) > 2^{-\frac{144}{\bar{\sigma}}-1} > e^{-\frac{144}{\bar{\sigma}}-1}, \quad (8.10)$$

for all  $m > 1$ . From (8.8), it follows that, for all  $w$  in  $BC$ ,

$$P(F_3 \cap F_4|D_2, Z_0 = w) \geq e^{-\frac{144}{\bar{\sigma}}-1} - e^{-\frac{200}{\bar{\sigma}}}.$$

Therefore, by (8.7) and (8.6), the claim is proved.

**Event**  $F_5 := \{m_{b_2}(u_2) \geq 9m/10\}$ . [At the stage of the second mutation, the proportion of population  $M(2)$  playing  $b_2$  is no less than  $9/10$ .]

**Claim:** for all  $w$  in  $BC$ ,

$$P(D_5, Z_0 = w) \geq \frac{7}{8}P(D_4, Z_0 = w), \quad (8.11)$$

when  $\mu < g_{\bar{\sigma}}(m)$ .

When  $m_{a_3} = 1 > m_{a_2} = m_{a_4} = 0$ , for population  $M(2)$ , the selection towards strategy  $b_2$  is effective. We regard  $s_i$  as the combined strategy  $b_1$  and  $b_3$ , take  $k = 72/\bar{\sigma}$  and  $\lambda = 1/10$ , and apply Lemma 8.9. It follows from the definition of  $D_4$  that, for all  $w$  in  $BC$ ,

$$P(m_{b_2}(\lceil 72m/\bar{\sigma} \rceil + u_1) \geq 9m/10 | D_4, Z_0 = w) \geq 7/8,$$

which completes the proof of the claim.

**Event**  $F_6 := \{Z_{u_2}^1(q^1) = a_2, m_{b_i}(Z_{u_2}) = m_{b_i}(Z_{u_2-1}), i = 1, 2, 3\}$ . [The second mutation is in population  $M(1)$  only and that yields an  $a_2$  strategy.]

Similarly to  $F_1$ , we find that, for all  $w$  in  $BC$ ,

$$P(F_6|D_5, Z_0 = w) > 1/7, \quad (8.12)$$

when  $\mu < g_{\bar{\sigma}}(m)$ .

**Event**  $F_7 := \{\mu^{-1} < u_3 - u_2 < 100/\mu\}$ . [The number of stages between the second and the third mutation is between  $\mu^{-1}$  and  $100\mu^{-1}$ .]

With similar analysis as in  $F_2$ , we find that, for all  $w$  in  $BC$ ,

$$P(F_7|D_6, Z_0 = w) > 1/16 - 1/200, \quad (8.13)$$

when  $\mu < g_{\bar{\sigma}}(m)$ .

We denote  $t_2$  to be the first stage after  $u_2$  when the state is in  $NE$ , i.e.,  $t_2 = \min\{t > u_2 : Z_t \in NE\}$ .

**Event**  $F_8 := \{t_2 - u_2 < 1/(2\mu)\}$ . [From stage  $u_2$ , it takes less than  $1/(2\mu)$  stages back to  $NE$ .]

With similar analysis as in  $F_3$ , we find that, for all  $w$  in  $BC$ ,

$$P(F_8|D_7, Z_0 = w) \geq 1 - e^{-\frac{200}{\sigma}}. \quad (8.14)$$

**Event**  $F_9 := \{\exists t : u_2 < t < u_3, m_{a_1}(Z_t) = 0, m_{a_3}(Z_t) + m_{a_4}(Z_t) \leq m/10, m_{b_1}(Z_t) = m\}$ . That means

$$F_9 \subseteq \{\exists t : u_2 < t < u_3, Z_t \in NC\}.$$

[Between the second and the third mutation in the process, it enters into  $NC$  at some stage.]

**Claim:** When  $\mu < g_{\bar{\sigma}}(m)$ , for all  $w$  in  $BC$ ,

$$P(F_8 \cap F_9|D_7, Z_0 = w) > 2^{-\frac{160}{\sigma}} (7/40) - e^{-\frac{200}{\sigma}} > 0. \quad (8.15)$$

Assume that, at stage  $u_2$ , a mutant appears at  $a_2$  and  $m_{b_2}(u_2)/m_{b_3}(u_2) \geq 9$ , which is consistent with  $F_5$ . We show below some movements with positive probability of two populations  $M(2)$  and  $M(1)$ , respectively.

$M(2)$ : denote the event of  $m_{a_2}(t) > m_{a_3}(t) = m_{a_4}(t) = 0$  at stage  $t$  by  $F_a(t)$ . At any stage  $t > u_2$ , population  $M(2)$  is under selection to  $b_1$ , when  $F_a(t)$  is true. Denote  $Y_t$  to be the event  $\{m_{b_2}(t') \geq m_{b_3}(t') \forall u_2 \leq t' \leq t\}$ . Denote the event  $G_{t_1, t_2}$  to be the one with no mutation in both populations between stage  $t_1$  and  $t_2$ . By Lemma 8.7 in the case of  $k = 1/10$ , we know that, given any  $t > u_2$ , (for example, take  $t = \min\{t' > u_2, m_{a_1}(t') = 0\}$ )

$$P(Y_t | G_{u_2, t}, F_a(t')) \forall t' \text{ with } u_2 \leq t' \leq t \geq 4/5. \quad (8.16)$$

Regarding  $s_i$  as the combined strategy of  $b_2$  and  $b_3$  and taking  $k = 8/\bar{\sigma}$  and  $\lambda = 1/5$  in Lemma 8.9, we find

$$P\left(m_{b_1}(u_1 + \lceil km \rceil) > \frac{4m}{5} \mid G_{u_1, u_1 + \lceil km \rceil}, F_a(t) \forall u_2 < t \leq u_2 + \lceil km \rceil\right) \geq 1/2. \quad (8.17)$$

$M(1)$ : we take  $k = 8/\bar{\sigma}$  in Lemma 8.8 and conclude that the probability is greater than  $2^{-1-2k}$  that the mutant will keep playing strategy  $a_2$  between stage  $u_2$  and  $u_2 + \lceil km \rceil$ . (\*\*)

If  $m_{b_2}(t) > m_{b_3}(t)$ , then for population  $M(1)$   $a_2$  is a better reply than  $a_3$  or  $a_4$  at stage  $t$ . Furthermore, if  $m_{b_2}(t) > m_{b_3}(t)$  and  $m_{b_1}(t) \geq 4m/5$  from some stage on after  $u_2$ , then population  $M(1)$  is under the selection pressure to  $a_2$ .

Now the question is which population completes the selection process first, conditional on  $t_2 < u_3$ . Denote  $\min\{t > u_2 : m_{b_1}(t) = m\}$  by  $t_3$ . Consider the following two cases which satisfy  $t_3 < u_3$ .

**Case A:**  $m_{a_1}(t_3) = 0$ , i.e.,  $t_2 = t_3$ . Then, between stage  $\min\{t > u_2 : m_{a_1}(t) = 0\}$  and  $t_3$ , population  $M(2)$  is under selection pressure to  $b_1$ , while

population  $M(1)$  stays still. If the events in (\*\*), (8.16), (8.17) and Case A are all true, then  $(z_t)_{u_2 \leq t < u_3}$  is in a trajectory to  $NC$ . Finally, all individuals of populations II plays  $b_1$  at stage  $t_2$ . Thus  $m_{a_2}(t) = m_{b_1}(t) = m$ , i.e.,  $Z_t \in NC$ , for all  $t_3 < t < u_3$ .

**Case B:**  $t_3 < t_2$ , which means some proportion of population I still plays  $a_1$  at stage  $t_3$ . If  $m_{a_2}(t_3) \geq 9m/10$ , then  $b_1$  is the best reply for population  $M(2)$  from stage  $t_3$  on. (★) If the events in (\*\*), (8.16), (8.17), Case B, and (★) are all true, then  $(z_t)_{u_2 \leq t < u_3}$  is in a trajectory to  $NC$ . To analyse (★), we first find a lower bound of the probability that  $t_3$  happens more than  $72m/\bar{\sigma}$  stages later than  $u_2$ , based on the last individual converting to strategy  $b_1$ :

$$\begin{aligned} & P\left(t_3 - u_2 > \frac{72m}{\bar{\sigma}} \mid F_a(t) \ \forall u_2 < t \leq t_3\right) \\ & \geq \left(1 - \frac{1}{m}\right)^{\frac{72m}{\bar{\sigma}}} \\ & \geq 2^{-\frac{144}{\bar{\sigma}}}, \end{aligned} \tag{8.18}$$

where  $m > 1$ . Taking  $k = 72/\bar{\sigma}$  and  $\lambda = 1/10$  in Lemma 8.9, we may infer that

$$P\left(m_{a_1}(c(\bar{\sigma}) + u_2) > \frac{m}{10} \mid Y_{t+c(\bar{\sigma})}, G_{u_2, u_2+c(\bar{\sigma})}\right) < \frac{1}{8}, \tag{8.19}$$

where  $c(\bar{\sigma}) := \lceil \frac{72m}{\bar{\sigma}} \rceil$ . Combining the results of (\*\*), (8.16), (8.17), (8.18) and (8.19), we find that

$$P(F_9 \mid F_8 \cap D_7, Z_0 = w) > 2^{-\frac{160}{\bar{\sigma}}}(7/40),$$

for all  $w$  in  $BC$ . (Note that the mutation is independent of the selection process.) We take (8.14) and  $\bar{\sigma} < 1$  into account, and complete the proof of the claim.

Finally, from (8.9), (8.11), (8.12) (8.13) and (8.15)

$$\min_{w \in BC} P(F|Z_0 = w) > (2^{\frac{-160}{\bar{\sigma}}} (7/40) - e^{\frac{-200}{\bar{\sigma}}})(1/16 - 1/200)(1/7)(7/8)c_4(\bar{\sigma}),$$

when  $\mu < g_{\bar{\sigma}}(m)$ .

Now we calculate the maximum expected transition time from BC to NC conditional on F: for each  $w$  in  $BC$ ,

$$T_w(F) \leq E[u_1|F, Z_0 = w] + E[u_2 - u_1|F, Z_0 = w] + E[t_2 - u_2|F, Z_0 = w].$$

For all  $w$  in  $BC$ , by the definition of  $F_2$ ,  $E[u_2 - u_1|F, Z_0 = w] < 100/\mu$ ; by the definition of  $F_8$ ,  $E[t_2 - u_2|F, Z_0 = w] < 1/(2\mu)$ . So,

$$\max_{w \in BC} T_w(F) < 1/(2\mu) + 100/\mu + 1/(2\mu) = 101/\mu.$$

It follows from the definition of  $F_2$  and  $F_7$  that, for all  $w$  in  $BC$ ,

$$T_w(F^c) \leq (E[u_1|F, Z_0 = w] + 100/\mu + 100/\mu) < 1/(2\mu) + 100/\mu + 100/\mu.$$

We complete the proof by Lemma 8.6.  $\square$

**Comment:** For the selection mechanism of better reply, one needs to note that, after  $u_1$ , both  $b_1$  and  $b_2$  are better strategy than  $b_3$ . So one has to consider, for instance,

$$P(m_{b_1}(t) \leq m/2 \forall u_1 < t < t_1 | t_1 < u_2).$$

If  $m_{b_1} \leq m/2$ , then  $a_1$  is still the best strategy of population  $M(1)$ . Hoeffding's inequality can be applied to obtain a lower bound of the probability above.

## 8.6 Proofs for Example 3

For the proofs of four lemmas in Section 6, given a state  $w$  in  $W$ , we denote  $\tilde{w}$  to be the next random state chosen with probability  $Q[\tilde{w}|w]$ , as denoted in Section 2.

**Proof of Lemma 6.2:** Conditional on a  $w$  with  $m_{a_3}(w) = 1$  and  $m_{a_2}(w) = 0$  at any stage, the probability is  $1/m$  for that the chosen individual  $q(1)$  is the one playing  $a_3$ , i.e.,  $P(w^1(q^1) = a_3|w) = 1/m$ . If  $w^1(q^1) = a_3$ , then  $BT^1(q^1, w) \neq \emptyset$ , regardless of whether  $m_{b_2}(w)$  is positive. So,

$$Q(\tilde{w}^1 \neq w^1 | w^1(q^1) = a_3) = 1 - \mu/3.$$

We also observe that  $P(w^1(q^1) = a_1|w) = (m-1)/m$ ,  $BT^1(q^1, w) = \emptyset$  when  $w^1(q^1) = a_1$ , and

$$Q(m_{a_1}(\tilde{w}) < m_{a_1}(w) | w^1(q^1) = a_1) = 2\mu/3.$$

We may then infer that

$$Q(\tilde{w}^1 \neq w^1 | w) = \frac{1 - \frac{\mu}{3}}{m} + \frac{m-1}{m} \frac{2\mu}{3} \leq \frac{1}{m} + \frac{2\mu}{3}.$$

Therefore, for a state  $w$  with  $m_{a_3}(w) = 1$  and  $m_{a_2}(w) = 0$ ,

$$\begin{aligned} P(Z_t^1 = Z_0^1 \forall 0 < t \leq m/4 | Z_0 = w) &= \prod_{t=1}^{m/4} P(Z_t^1 = Z_{t-1}^1 | Z_{t-1}^1 = w^1) \\ &\geq (1 - 1/m - 2\mu/3)^{m/4} \\ &\geq (1 - 1/m - 2\eta/3m)^{m/4} \\ &= \left(1 - \frac{1}{3m}\right)^{\frac{3m}{3+2\eta} \frac{3+2\eta}{12}}, \end{aligned}$$

with the condition  $\mu m \leq \eta$ . When  $m \geq 2(3+2\eta)/3$ , the probability above is strictly increasing with respect to  $m$ . So,

$$P_{\mu,m}(Z_t^1 = Z_0^1 \forall 0 < t \leq m/4 | Z_0 = w) \geq \left(1 - \frac{1}{2}\right)^{\frac{3+2\eta}{6}}.$$

Denote the right hand side above by  $c_1(\eta)$ .  $\square$

**Proof of Lemma 6.3:** We abbreviate the notation  $S(m)$  to  $S$ , if no ambiguity. Recall that at a state  $w$  with  $m_{a_3}(w) = 1$  and  $m_{a_2}(w) = 0$ ,



the selection at population  $M(2)$  is towards strategy  $b_2$ . Denote  $\{w \in W : m_{b_1}(w) = m\}$  by  $S_0$ . It is straightforward that, for the Markov chain  $(Z_t)_{t \geq 0}$ ,

$$\begin{aligned} & P(B_2(m/4) \geq m/40 | Z_t \in S \forall 0 \leq t \leq m/4) \\ & \geq P(B_2(m/4) \geq m/40 | Z_0 \in S_0, Z_t \in S \forall 0 \leq t \leq m/4). \end{aligned} \quad (8.20)$$

That is,  $Z_0$  in  $S_0$  is the ‘worst’ situation to reach  $B_2(m/4) \geq m/40$ . In the case of  $(Z_t)_{t \geq 0}$  with  $Z_0 \in S_0$ ,  $B_1(t) \geq 3m/4$  for all  $0 \leq t \leq m/4$ . Hence, for all  $t$  with  $0 \leq t < m/4$ ,

$$P(Z_t^2(q^2) = b_1 | Z_0 \in S_0) \geq 3/4. \quad (8.21)$$

It is straightforward that, at any  $t$  with  $0 \leq t < m/4$ , if  $Z_t \in S$ , then

$$BT^2(q^2, Z_t) = \{b_2\} \setminus \{Z_t^2(q^2)\} \quad (8.22)$$

since selection is always towards  $b_2$  when  $Z_t$  is in  $S$ . We also observe that

$$P(B_2(t+1) > B_2(t) | Z_t^2(q^2) = b_1, Z_t \in S) = 1 - \mu/2, \quad (8.23)$$

as the chosen individual will undergo mutation with probability  $\mu$ .

From (8.21), (8.22) and (8.23), we may infer that, given any  $\mu < 2/9$ , for all  $t$  with  $0 \leq t < m/4$ ,

$$\begin{aligned} & P(B_2(t+1) = B_2(t) + 1 | Z_t \in S, Z_0 \in S_0) > 2/3, \text{ and} \\ & P(B_1(t+1) \geq B_1(t) | Z_t \in S, Z_0 \in S_0) < 1/3. \end{aligned}$$

If among these  $m/4$  periods it occurs in at least  $\lceil \frac{11m}{80} \rceil$  periods that  $B_2(t+1) = B_2(t) + 1$ , then  $B_2(m/4) \geq m/40$ . So,

$$\begin{aligned} & P(B_2(m/4) \leq m/40 | Z_0 \in S, Z_t \in S \forall 0 \leq t \leq m/4) \\ & \leq \sum_{i=0}^{\lceil \frac{11m}{80} \rceil} \binom{\frac{m}{4}}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{\frac{m}{4}-i}. \end{aligned}$$

By Hoeffding's inequality, we find

$$\begin{aligned} & P(B_2(m/4) \leq m/40 | Z_0 \in S, Z_t \in S \forall 0 \leq t \leq m/4) \\ & \leq \frac{1}{2} \exp\left(-2 \frac{(\frac{m}{4} \frac{2}{3} - \frac{11m}{80})^2}{\frac{m}{4}}\right). \end{aligned}$$

It then follows from (8.20) that

$$\begin{aligned} & P(B_2(m/4) \geq m/40 | Z_t \in S(m) \forall 0 \leq t \leq m/4) \\ & \geq P(B_2(m/4) \geq m/40 | Z_0 \in S_0, Z_t \in S(m) \forall 0 \leq t \leq m/4) \\ & \geq 1 - \frac{1}{2} \exp\left(-2 \frac{(\frac{m}{4} \frac{2}{3} - \frac{11m}{80})^2}{\frac{m}{4}}\right). \end{aligned}$$

Note the last term in the inequality above is strictly increasing with respect to  $m$ . Denote  $c_2(\eta)$  to be its value when  $m = 2(3 + 2\eta)/3$ , and the result follows.  $\square$

**Proof of Lemma 6.4:** From the one step transition probability matrix, it follows that

$$\begin{aligned} Q(\tilde{w}^1(q^1) = a_1 | w^1(q^1) = a_2) &= 1 - 2\mu/3, \\ Q(\tilde{w}^1(q^1) \neq a_1 | w^1(q^1) = a_1) &= 2\mu/3, \\ Q(\tilde{w}^1(q^1) = a_1 | w^1(q^1) = a_3) &\geq (1 - \mu/3)/2. \end{aligned} \tag{8.24}$$

For the last formula, when  $m_{b_2}(w) > 0$ , the equality holds; otherwise, the probability is  $1 - 2\mu/3$ .

Let  $H_k := \{w : m_{a_2}(w) + m_{a_3}(w) = k\}$ , for  $k = 0, \dots, m$ . Then

$$Q(\tilde{w} \in H_{k+1} | w \in H_k) = \frac{m - k}{m} \frac{2\mu}{3},$$

and

$$Q(\tilde{w} \in H_k | w \in H_{k+1}) \geq \frac{m_{a_2}(w)}{m} (1 - 2\mu/3) + \frac{m_{a_3}(w)}{m} \frac{1 - \frac{\mu}{3}}{2} \geq \frac{k+1}{m} \frac{3 - \mu}{6}.$$

We know the invariant distribution property implies that

$$\pi[H_k]Q(\tilde{w} \in H_{k+1}|w \in H_k) = \pi[H_{k+1}]Q(\tilde{w} \in H_k|w \in H_{k+1}),$$

so

$$\pi[H_k] \frac{m-k}{m} \frac{2\mu}{3} \geq \pi[H_{k+1}] \frac{k+1}{m} \frac{3-\mu}{6}.$$

Then,

$$\pi[H_{k+1}] \leq \pi[H_k] \frac{4\mu}{3-\mu} \frac{m-k}{k+1}.$$

Therefore, we have

$$\pi[H_k] \leq \pi[H_0] \binom{m}{k} \left( \frac{4\mu}{3-\mu} \right)^k.$$

From  $1 = \sum_{k=0}^m \pi[H_k]$  and the Binomial Formular, it follows that

$$\pi[H_0] \geq \left( 1 + \frac{4\mu}{3-\mu} \right)^{-m} > \left( 1 - \frac{4\mu}{3} \right)^m.$$

Since  $m\mu \leq \eta$ , we find

$$\pi[H_0] > \left( \left( 1 - \frac{4\eta}{3m} \right)^{\frac{3m}{4\eta}} \right)^{\frac{4\eta}{3}}.$$

Note that the right hand side above is strictly increasing with  $m$  when  $m \geq 8\eta/3$ . Hence,

$$\pi_{\mu,m}[\{w : m_{a_1}(w) = m\}] \geq \left( 1 - \frac{1}{2} \right)^{\frac{8\eta}{3}}.$$

Denote the right hand side above by  $c_3(\eta)$ .  $\square$

**Comment:** For the selection mechanism of best reply, the result still holds, with the condition (8.24) replaced by the following inequality.

$$Q(\tilde{w}^1(q^1) = a_1 | w^1(q^1) = a_3) \geq 1 - (2\mu/3).$$

This is the only difference for two selection mechanisms in the proof of Proposition 6.1.

**Proof of Lemma 6.5:** From the property of invariant distribution, it follows that

$$\begin{aligned} & \pi[\{w : m_{a_1}(w) = m\}]Q(\tilde{w} : m_{a_3}(\tilde{w}) = 1 | m_{a_1}(w) = m) \\ & \leq \pi[\{w : m_{a_3}(w) = 1 \ \& \ m_{a_2}(w) = 0\}]Q(\tilde{w} \neq w | m_{a_3}(w) = 1 \ \& \ m_{a_2}(w) = 0). \end{aligned}$$

Hence, by the similar argument as in the proof of Lemma 6.4, we find

$$\pi[\{w : m_{a_3}(w) = 1 \ \& \ m_{a_2}(w) = 0\}] \geq \pi[\{w : m_{a_1}(w) = m\}] \frac{\mu}{\frac{3-\mu}{m} + \frac{m-1}{m} 2\mu}.$$

Note

$$\begin{aligned} \frac{\mu}{\frac{3-\mu}{m} + \frac{m-1}{m} 2\mu} &= \frac{m\mu}{3 + 2m\mu - 3\mu} \\ &> \frac{1}{\frac{3}{m\mu} + 2} \\ &\geq \frac{1}{\frac{3}{\delta} + 2}, \end{aligned}$$

as  $m\mu \geq \delta$ . We may then infer that

$$\begin{aligned} & \pi_{\mu,m}[\{w : m_{a_3}(w) = 1 \ \& \ m_{a_2}(w) = 0\}] \\ & \geq \pi_{\mu,m}[\{w : m_{a_1}(w) = m\}] \frac{\delta}{2\delta + 3} \\ & \geq \frac{c_3(\eta)\delta}{2\delta + 3}, \end{aligned}$$

by Lemma 6.4. Denote  $(c_3(\eta)\delta)/(2\delta + 3)$  by  $c_4(\delta, \eta)$ .  $\square$

## Acknowledgements

The author is grateful to Sergiu Hart for many suggestions and discussions. The author would also like to thank Katsuhiko Aiba, Tomas Rodriguez Baraquero, Yosef Rinott and Eyal Winter for their comments.

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