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# PAYOFFS IN NONDIFFERENTIABLE PERFECTLY COMPETITIVE TU ECONOMIES

By

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Payoffs in Nondifferentiable Perfectly Competitive TU Economies*

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**Abstract**

We prove that a single-valued solution of perfectly competitive TU economies underling nonatomic vector measure market games is uniquely determined as the Mertens [18] value by four plausible value-related axioms. Since the Mertens value is always in the core of an economy, this result provides an axiomatization of the Mertens value as a core-selection. Previous works on this matter assumed the economies to be either differentiable (e.g., Dubey and Neyman [7]) or of uniform finite type (e.g., Haimanko [13]). This work does not assume that, thus it contributes to the axiomatic study of payoffs in perfectly competitive economies in general.

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# 1 Introduction

One of the most striking results in economics is the equivalence of the core and the set of competitive (Walras) payoffs in "perfectly competitive" economies. This was already conjectured by Edgeworth ([9]), and has been shown to hold in limit economies (e.g., [4, 16, 21]) and in nonatomic economies (e.g., [2]). It was later on observed by Aumann ([3]) that, in nonatomic economies with a smoothness assumption on the preferences, the "value allocations"[1] also coincide with the above two sets. This equivalence is known in the literature as the *equivalence phenomenon* (see [7]).

If the economies are assumed to have smooth, *transferable utilities* (TU), then the equivalence phenomenon turns out to be truly astonishing (see [1]): the core in this case consists of a single competi-

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[1] Allocations whose definition is based on the Shapley [20, 1] value.

tive payoff - the Aumann-Shapley ([1]) value. The coincidence and uniqueness[2] of these seemingly unrelated sets clearly demands an explanation. This was supplied by Dubey and Neyman [7] who proved a "meta-equivalence" theorem for perfectly competitive (nonatomic) TU economies with smooth preferences. Namely, they characterized the equivalence phenomenon axiomatically, by a list of four simple and "natural" axioms.

Dubey and Neyman meta-equivalence result holds, nonetheless, only (see [7, p. 1146]) under the smoothness assumption, namely - when the sets are single-valued. In general, however, the core of a perfectly competitive (nonatomic) economy is usually not single-valued. If, for example, we adopt the production interpretation for economies (as we usually shall), then in most applications the agents' production functions will have "kinks" that prevent the core from being single-valued. Furthermore, if the core is not single-valued, it is still equivalent to the set of competitive payoffs, but the Aumann-Shapley value may, however, not exist. These facts withhold any straightforward extension of Dubey and Neyman [7] "meta-equivalence" theorem to the more general setting.

Nevertheless, we may adopt a slightly different approach; the punch-line of the equivalence phenomenon is, essentially, that if the core is single-valued then a selection of an element of the core becomes trivial, and that this element is a competitive payoff that enjoys the nice properties of the (Aumann-Shapley) value. For general perfectly competitive TU economies we might thus wish to *select* an element of the core in a consistent and economically meaningful way, which extends the Aumann-Shapley value whenever it exists. Such a selection exists, and was first constructed by Mertens [17, 18], and it is thus known as the *Mertens value*. This selection avoids discontinuities, discrimination between agents, and inconsistencies in the selection of payoffs. It also coincides with the asymptotic Shapley value of economies, whenever the later exists. Thus it is natural to replace, in the more general setting, the Aumann-Shapley value with the Mertens value, and try to axiomatically characterize it as a payoff selection. We obviously wish to assume as less as we can to achieve this goal. Thus the list of axioms should be short and the axioms should not, individually, convey the information that the payoff is competitive nor an element of the core; this should be an *outcome* of the "meta-selection" theorem. We also wish that for a given solution, the axioms may be easily verified, that they will have a clear economic meaning, and that the list of axioms will be akin to the one in [7].

Such a list of axioms was suggested by Haimanko [13], who proved a "meta-selection" theorem for TU economies of *uniform finite-type*[3]. Haimanko's list consists of four axioms; two of them, *anonymity* and *separability*, were borrowed directly from Dubey and Neyman [7], and a third axiom, *contraction*, is similar to the continuity axiom of Dubey and Neyman [7].

Haimanko's [13] result, however, does not bring us much closer to obtaining a general "meta-selection" theorem for economies in general. First, unlike the case handled by Dubey and Neyman, economies cannot be approximated by economies of uniform finite-type in general; if the core is not single-valued then, typically, an economy will be "far away" from any economy of uniform finite-type, thus Haimanko's contraction axiom cannot be invoked to deduce a "meta-selection" theorem for economies that are not

[2] Namely, that each set consists of one element.

[3] Namely, economies with finitely many types of preferences and endowments.

of uniform finite-type, as it was done in [7]. Moreover, if an economy is of uniform finite-type, then its *derived market game*[4] is a *a vector measure game*, namely, it is a function of finitely many, mutually singular, nonatomic probability measures. Haimanko [13] observed that, in this case, a payoff obeying his axioms is expressible as a linear combination in the derived market game's measures, whose coefficients depend solely on the agents' production functions marginals. Haimanko then used the value axioms to prove that this observation gives rise to a simple representation formula of the payoff as a barycenter of the economy's core. This representation played a crucial role in the proof of his "meta-selection" theorem, and its derivation is no longer valid, for example, in economies with general endowments. Thus Haimanko's methods cannot be applied to prove a "meta-selection" theorem on more general domains than the one he had considered, and there does not seem to be any straightforward way to adapt his methods to the more general case. For this reason[5] proving a "meta-selection" theorem on the entire domain of perfectly competitive TU economies remained a formidable task, extremely susceptible to all previously known methods of analysis.

Nevertheless, we have recently introduced [11, 12] several new methods that may advance our understanding of the problem, and help us expand its analysis to more general subdomains of perfectly competitive TU economies than those that were previously considered. We obviously do not wish to consider any subdomain to which these methods are applicable, but rather to consider a subdomain with a clear importance in economic applications. Such an approach paid off in [10], where we proved a "meta-selection" theorem for the domain of *perfectly competitive exact TU economies*, namely, for economies whose derived market games are exact. This domain was considered for three main reasons. First, it contains many economies of interest, especially if we adopt the production interpretation. Second, continuous payoffs of perfectly competitive TU economy tend to have the *conic property* - they are completely determined by the behavior of the derived market game "near" its diagonal (see [7, Lemma 5.5] and [11]). However, exact market games are determined by the behavior of their derived market games "near" the diagonal, so we could dismiss matters that might be related to the continuity of payoffs for the time being. Third, the Mertens value has the following, quite miraculous, property - if two economies have the same core then they have the same Mertens value, so it seems that the nature of a payoff selection might be already determined, in some cases, on the domain of exact economies. These reasons indicated that exact economies are a good place to start our investigation.

Most economies, however, are not exact. Nevertheless, as the behavior of a payoff selection may be already determined on the domain of exact economies, we might wish do reduce the general problem to the case of exact economies. This may be a little far-fetched, for the time being, for perfectly competitive TU economies in general, as the conic property is only known to hold on certain (large) subdomains of economies. The largest subdomain on which the conic property is known to hold is $\mathfrak{E}^*$, containing the economies with vector measure[6] derived market games (see [11]). This domain is, nonetheless, of great importance in many economic applications; it contains, for example, all the economies of finite-type,

[4] A game in characteristic function form that assigns to every coalition of agents the maximum production output that it could achieve by reallocating its resources among its members.

[5] And also other, more technical, reasons.

[6] Namely, functions of finitely many probability measures

namely, the economies with finitely many types of production functions and general endowments. This domain is thus much larger than those considered in [13] and [10], and is of great economic interest. Thus, a "meta-selection" theorem for the domain $\mathfrak{E}^*$ is of substantial economic interest of its own. Furthermore, a proof of such a theorem for $\mathfrak{E}^*$ may supply a feasible road map for a proof of a "meta-selection" theorem for the entire domain of perfectly competitive TU economies.

In this work we prove a "meta-selection" theorem for economies in $\mathfrak{E}^*$, namely, we prove that the axioms offered by Haimanko [13] uniquely determine a payoff selection on $\mathfrak{E}^*$ as the Mertens [18] value. Table 1 may be found helpful in placing our work in the context of the main results on the value equivalence phenomenon.

| | |
|---|---|
| **Value Equivalence** | Aumann and Shapley [1] |
| **Value "Meta-Equivalence"** | Dubey and Neyman [7] |
| **Value Selection** | Mertens [18] |
| **Value "Meta-Selection"**<br>uniform finite-type economies | Haimanko [13] |
| **Value "Meta-Selection"**<br>general exact economies | Edhan [10] |
| **Value "Meta-Selection"**<br>derived market game is vector measure<br>(contains all economies of finite-type) | Edhan (this work) |

Table 1: Summary of results
for perfectly competitive (nonatomic) TU economies

The axioms will be spelled out precisely in Section 2, but let us present them at an intuitive level now. For the time being we shall spare the reader with most of the technical details. Denote the set of agents by $T$. An economy $E$ is essentially set by the agents' endowments $a$ and production functions $u$, so $(u_t, a_t)$ are the production function and endowment of agent $t \in T$. Recall that the *derived market game* $v_E$ of $E$ is the game in characteristic function form that assigns to every coalition of agents the maximum output that it could produce by reallocating its resources among its members. Payoffs of the economy $E$ may be identified with members of the set $FA_+$ of positive finitely additive and bounded measures. Thus, a payoff selection on $\mathfrak{E}^*$ is a mapping

$$\psi : \mathfrak{E}^* \to FA_+.$$

We will impose four axioms on $\psi$: "efficiency," "anonymity," "separability," and "positivity." Our main result is that these axioms uniquely determine $\psi$ as the Mertens value.

The *efficiency* axiom says that $\psi$ is Pareto optimal. The *anonymity* axiom asserts that the labels of the agents do not matter; their relabeling will only result in relabeling their payoffs, accordingly. These axioms hold for many solutions, not only on $\mathfrak{E}^*$ but also for more general domains and for finite economies.

The *separability* axiom considers an economy made up of two separate, noninteracting parts; suppose that, given economies $E', E''$, producing the same kind of output, we construct an economy in which each agent has and can access its endowments and production abilities in $E'$ and $E''$, but cannot use his endowment in $E'$ to produce output in $E''$ and vice versa. In this case, the output of every coalition of agents in the economy $E$ is just the sum of their outputs in $E'$ and $E''$. So, essentially, production in $E$ is equivalent to production in $E'$ and in $E''$ independently of each other. Thus, the payoff in $E$ should be *at least* as high as the separately obtained payoff of $E'$ and $E''$, namely for every coalition $S$ $\psi(E)(S) \geq \psi(E')(S) + \psi(E'')(S)$. This axiom is related to the additivity axiom for the value.

The *contraction* axiom asserts that "nearby" economies have "nearby" payoffs. This obviously depends on the interpretation of "nearby" in both cases. The metric that we employ on $\mathfrak{E}^*$ depends on the derived games, so if $v_E = v_{E'}$ then $\psi(E) = \psi(E')$. Our metric is, however, "large", and hence we obtain a quite weak continuity requirement. It is important to remark that our notion of distance between economies has an explicit economic meaning: the larger the total variation of the difference of the economies' outputs, the larger their distance is. The metric on payoffs is similarly induced from the total variation norm.

This axiomatization, offered in [13], slightly differs from the one offered in [7]. There are two main difference. The first is that we consider payoff selections, as the example in [7, p. 1146] shows that otherwise our chosen axiomatization will not work. The second is the replacement of the *inessential economy* axiom with the efficiency axiom. In [7], efficiency was implied by the axioms, and here we assumed it. This small sacrifice in the generality of our axioms is quite marginal as most solutions satisfy efficiency, while we obtain a substantially stronger result. A rather minor difference is our choice of a different notion of continuity; this is done as our contraction axiom implies a weak form of positivity that is needed in our analysis. Interestingly, contraction is implied by combining this weak form of positivity with the continuity axiom from [7].

Finally, let us mention that in the smooth case axioms of a similar nature can characterize the competitive payoff correspondence for perfectly competitive economies with nontransferable utilities (NTU) [8]. We hope that our approach can be translated into the setting of perfectly competitive NTU economies as well.

## 2 Definitions, Axioms, and The Main Result

Let $(T, \mathcal{C})$ be a standard measurable space. $T$ is the set of *agents*, and $\mathcal{C}$ is the $\sigma$-algebra of *coalitions*. The set of all bounded and finitely additive measures on $(T, \mathcal{C})$ is denoted by $FA$, and its subset of all nonatomic measures on $(T, \mathcal{C})$ is denoted by $NA$. The set of nonatomic probability measures on $(T, \mathcal{C})$ is denoted by $NA^1$. An *economy* $E$ is a triple $(u, a, \nu)$, where $u : T \times \mathbb{R}^k_+ \to \mathbb{R}$ and $a : T \to R^k_+$ for some $k$, and $\nu \in NA^1$. For each $t \in T$, $a_t = a(t)$ is agent $t$'s initial endowment of commodities 1, 2, ..., $k$, $u_t(\cdot) = u(t, \cdot)$ is his production function on the space of commodity bundles $R^k_+$, and $\nu$ is a *population measure* s.t. the following conditions hold:

1. $a$ is measurable;

2. $u$ is $\mathcal{C} \times \mathcal{B}_k$ measurable, where $\mathcal{B}_k$ denotes the Borel $\sigma$-algebra on $\mathbb{R}_+^k$; and,

3. $\int_T a_t d\nu(t) \in (0, \infty)^k$.

We further assume that for every $t \in T$:

4. $u_t$ is monotonically nondecreasing;

5. $u_t(x) = o(\|x\|)$ as $\|x\| \to \infty$;

6. $u_t$ is continuous; and

7. $u_t(0_k) = 0$.

A *game* is a function $v : \mathcal{C} \to \mathbb{R}$ with $v(\emptyset) = 0$. Given an economy $E = (u, a, \nu)$, the *derived market game* $v_E$ corresponding to $E$, assigns to each coalition the maximum output that it could achieve[7] by a reallocation of the resources of its own members, i.e., for every $S \in \mathcal{C}$

$$v_E(S) = \max \left\{ \int_S u_t(x_t) d\nu(t) \Big| x : T \to \mathbb{R}_+^k, \int_S x d\nu = \int_S a d\nu \right\}. \tag{2.1}$$

The *core* of an economy $E = (u, a, \nu)$ is the set of Pareto optimal payoffs of $E$ which no coalition can improve on, namely

$$Core(E) = \{\mu \in FA : \forall S \in \mathcal{C} \; \mu(S) \geq v_E(S), \mu(T) = v_E(T)\}. \tag{2.2}$$

The core of an economy that satisfies (1)-(7) above is nonempty and finite-dimensional (see [15, Equation (2.15), Corollary (2.16)]). In this work we shall be interested in economies $E$ s.t. $v_E$ is a *vector measure game*, namely, that there is some $k \geq 2$, $\mu \in \left(NA^1\right)^k$ and a continuous, concave, monotonically nondecreasing, and positively homogeneous function $f : \mathbb{R}_+^k \to \mathbb{R}$ with $v_E(S) = f(\mu(S))$ for each $S \in \mathcal{C}$. We denote in this case $v_E = f \circ \mu$. Given $\nu \in NA^1$ we denote by $\mathfrak{E}^*(\nu)$ the set of all economies $E = (u, a, \nu)$ s.t. $v_E$ is a vector measure game, and further denote $\mathfrak{E}^* = \bigcup_{\nu \in NA^1} \mathfrak{E}^*(\nu)$. A *payoff selection* is a mapping $\Psi : \mathfrak{E}^* \to FA_+$ ($FA_+$ is the set of bounded, finitely additive, and nonnegative measures on $(T, \mathcal{C})$), which satisfies axioms (1)-(4) that we state below.

**Axiom 1 (Efficiency).** *For any* $E \in \mathfrak{E}^*$

$$\Psi(E)(T) = v_E(T). \tag{2.3}$$

Denote by $\Theta$ the set of all measurable automorphisms of $(T, \mathcal{C})$, namely, the set of bi-measurable bijections $\theta : T \to T$. For $\mu \in FA$ and $\theta \in \Theta$ define[8] $\theta\mu$ by $(\theta\mu)(S) = \mu(\theta S)$. For any economy $E = (u, a, \nu) \in \mathfrak{E}^*$ let $\theta E = (\theta u, \theta a, \theta \nu) \in \mathfrak{E}^*$, where $(\theta u)_t(x) = u_{\theta(t)}(x)$ and $(\theta a)(t) = a_{\theta(t)}$. Notice that $v_{\theta E}(S) = v_E(\theta S) \doteq (\theta v_E)(S)$ for every $S \in \mathcal{C}$. This paves the path for the following axiom:

[7] See [1, Proposition 36.1].

[8] By abuse of notation.

**Axiom 2 (Anonimity).** *For every $E \in \mathfrak{E}^*$ and every $\theta \in \Theta$*

$$\Psi(\theta E) = \theta \Psi(E). \tag{2.4}$$

If $E = (u, a, \nu), E' = (u', a', \nu) \in \mathfrak{E}^*(\nu)$, where $a : T \to R_+^k$, $a' : T \to R_+^{k'}$, define the disjoint sum of the economies $E \oplus E' \in \mathfrak{E}^*$ by $(u \oplus u', a \oplus a', \nu)$, where $u \oplus u' : T \times R_+^{k+k'} \to \mathbb{R}$ and $a \oplus a' : T \to R_+^{k+k'}$ are given by $(u \oplus u')_t(x, y) = u_t(x) + u'_t(y)$ and $(a \oplus a')_t = (a_t, a'_t)$.

**Axiom 3 (Separability).** *For every $\nu \in NA^1$ and every $E, E' \in \mathfrak{E}^*(\nu)$*

$$\Psi(E \oplus E') \geq \Psi(E) + \Psi(E'). \tag{2.5}$$

*Remark* 2.1. Notice that by combining the efficiency and the separability axioms we obtain that for every $\nu \in NA^1$, $E, E' \in \mathfrak{E}^*(\nu)$, and $S \in \mathcal{C}$

$$\Psi(E \oplus E')(S) = \Psi(E)(S) + \Psi(E')(S)$$

A game is of *bounded variation* if it can be represented as a difference of two monotonic games. For a game $v$ of bounded variation we define its norm, $\|v\|$, by $\|v\| = inf(u(T) + w(T))$, where the infimum is taken over all monotonic games $u$ and $w$ with $v = u - w$. Differences of two derived market games or measures in $FA_+$ are clearly of bounded variation. A payoff selection $\Psi$ on $\mathfrak{E}^*$ is a *contraction* if it satisfies the following axiom

**Axiom 4 (Contraction).** *For every $E, E' \in \mathfrak{E}^*$*

$$\left\|\Psi(E) - \Psi(E')\right\| \leq \|v_E - v_{E'}\|. \tag{2.6}$$

*Remark* 2.2. The contraction and efficiency axioms entail the following weak form of *positivity* for $\Psi$: If $v_E - v_{E'}$ is monotonic then so is $\Psi(E) - \Psi(E')$.

The set of all payoff selections on $\mathfrak{E}^*$ is denoted $PS(\mathfrak{E}^*)$. Axioms 1-4 are among the basic properties of the Mertens [18] value, $\Psi_M$. Additionally, $\Psi_M$ is entirely determined by the core of the economy, and is itself an element of the core[9]. According to our result, this selection is uniquely determined by the axioms:

**Theorem 2.3.** $PS(\mathfrak{E}^*) = \{\Psi_M\}$.

## 3 Preparations

This paper utilizes several results that were obtained elsewhere. In this section we shall introduce some of these results, together with other results that are needed for the proof of Theorem 2.3.

[9]A brief description of this construction is given in Appendix B.

### 3.1 Presentations of Derived Market Games

Let $k \geq 2$. Denote by $M_+^k$ the cone of *market functions* on $\mathbb{R}_+^k$, i.e., those which are concave, continuous, nondecreasing, and homogenous of degree 1. Let $M^k$ be the vector space of differences of functions in $M_+^k$. Denote by $LM_+^k$ the cone of Lipschitz functions in $M_+^k$, and let $LM^k$ the vector space of differences of functions in $LM_+^k$.

For $x \in \mathbb{R}^k$ denote $\overline{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$, and for $\mu \in \left(NA^1\right)^k$ denote $\overline{\mu} = \frac{1}{k}\sum_{i=1}^{k} \mu_i$. Let $\mathcal{M}_+^k(\mu) = \left\{f \circ \mu : f \in M_+^k\right\}$ and $\mathcal{M}^k(\mu) = \left\{f \circ \mu : f \in M^k\right\}$. In the same manner, define $\mathcal{LM}_+^k(\mu)$ and $\mathcal{LM}^k(\mu)$ respectively. Let $\mathcal{LM}$ be the linear space spanned by $\bigcup_{k \geq 2, \mu \in (NA^1)^k} \mathcal{LM}^k(\mu)$. Denote by $\mathcal{M}$ the linear space spanned by $\mathcal{M}_+ = \{v_E : E \in \mathfrak{E}^*\}$.

**Lemma 3.1.** *$\mathcal{M}$ is the linear span of* $\bigcup_{k \geq 2, \mu \in (NA^1)^k} \mathcal{M}^k(\mu)$.

*Proof.* By definition $\mathcal{M}_+ \subset \bigcup_{k \geq 2, \mu \in (NA^1)^k} \mathcal{M}_+^k(\mu)$. To prove the inverse inclusion, notice that if $k \geq 2$ and $f \circ \mu \in \mathcal{M}_+^k(\mu)$ then the economy $E = (u, a, \overline{\mu}) \in \mathfrak{E}^*$, in which $u_t(x) \leq f(x)$ for every $t \in T$ and $x \in \mathbb{R}_+^k$, and s.t. equality holds instead of inequality for every $x \in [0,1]^k$, and $a = \frac{d\mu}{d\overline{\mu}}$, satisfies $v_E = f \circ \mu$. $\square$

*Remark* 3.2. Notice that if $E \in \mathfrak{E}^*(\nu)$ and $\nu \ll \mu$ then there is $E' \in \mathfrak{E}^*(\mu)$ with $v_E = v_{E'}$. Indeed, let $E = (u, a, \nu)$, and choose a measurable function $h : T \to \mathbb{R}$ with $h = \frac{d\mu}{d\nu}$ $\nu$-a.e.. define $a' = ah$ and $u'(t,x) = u(t,x)h(t)$. The choice $E' = (u', a', \nu)$ proves the claim.

*Remark* 3.3. An economy $E = (u, a, \nu)$ is of finite-type iff there is a finite measurable partition $\{T_i\}_{i=1}^n$ of $T$ s.t. the function $t \mapsto u_t$ is constant on each partition element. If $E$ is of finite type then $v_E \in \mathcal{M}_+$ (see [14, p. 33]).

### 3.2 Directional Derivatives of $M^k$ Functions

Given $f \in M_+^k$, $x \in \mathbb{R}_{++}^k$, and $y \in \mathbb{R}^k$, the directional derivative $df(x,y)$ of $f$ at $x$ in the direction $y$ is given by

$$df(x,y) = \lim_{\varepsilon \searrow 0} \frac{f(x + \varepsilon y) - f(x)}{\varepsilon}. \tag{3.1}$$

The limit exists for every concave function $f$ and hence for every $f \in M_+^k$.

If $f \in M_+^k$ then for every $x \in \mathbb{R}_{++}^k$ the function $df(x,\cdot) : \mathbb{R}^k \to \mathbb{R}$ is concave. Thus the directional derivative of $df(x,\cdot)$ at $y \in \mathbb{R}^k$ in the direction $z \in \mathbb{R}^k$ which is given by

$$df(x,y,z) = \lim_{\varepsilon \searrow 0} \frac{df(x, y + \varepsilon z) - df(x,y)}{\varepsilon}, \tag{3.2}$$

exists.

Obviously, the directional derivatives defined by equations (3.1)-(3.2) also exist for every $f \in M^k$.

Denote by $\mathbf{1}_k \in \mathbb{R}^k$ the vector with coordinates $(\mathbf{1}_k)_i = 1$ for every $1 \leq i \leq k$, and let $D^k = \{t\mathbf{1}_k : t \in \mathbb{R}_+\}$. Notice that if $f \in M^k$ then for every $x \in D^k \setminus \{0_k\}$, every $y, z \in \mathbb{R}^k$, every $a \in \mathbb{R}$, and every $b > 0$,

$$df(x, a\mathbf{1}_k + by, z) = df(\mathbf{1}_k, y, z). \tag{3.3}$$

Let $\chi : T \to \mathbb{R}^k$. For every $x \in \mathbb{R}^k_{++}$ and $y \in \mathbb{R}^k$ denote by $df(x, y, \chi)$ the function from $T$ to $\mathbb{R}$ given by

$$\forall t \in T,\ df(x, y, \chi)(t) = df(x, y, \chi(t)). \tag{3.4}$$

### 3.3 The Direction Space $\Omega_\lambda$

For every $k \geq 1$ denote[10] $\mathbb{S}^k_\perp = \left\{\frac{x}{\|x\|_2} : x \in \mathbb{R}^k, \overline{x} = 0\right\}$, and $\Delta^k = \left\{x \in \mathbb{R}^k_+ : k\overline{x} = 1\right\}$, and for $x \in \mathbb{R}^k$ let $\Upsilon^k(x) = \frac{x - \overline{x}\mathbf{1}_k}{\|x - \overline{x}\mathbf{1}_k\|_2}$. For every $\mu \in \left(NA^1\right)^k$ endow the set[11] $\Lambda_\mu = \left(\mathcal{R}(\mu) \cap \Delta^k \setminus D^k\right) \sqcup \mathbb{S}^k_\perp$ with the topology $\mathfrak{T}_\mu$ whose restriction to either $\mathcal{R}(\mu) \cap \Delta^k \setminus D^k$ or $\mathbb{S}^k_\perp$ is equivalent to the topology induced on these sets by the Euclidean topology on $\mathbb{R}^k$, and if a sequence $(x^n)_{n=1}^\infty \subseteq \mathcal{R}(\mu) \cap \Delta^k \setminus D^k$ converges, in the Euclidean topology, to some point in $D^k$ and satisfies $\Upsilon^k(x^n) \underset{n\to\infty}{\longrightarrow} y \in \mathbb{S}^k_\perp$ then $x^n \underset{n\to\infty}{\longrightarrow} y$ in $\mathfrak{T}_\mu$. The topological space $(\Lambda_\mu, \mathfrak{T}_\mu)$ is thus a compact metrizable space[12].

For every $\lambda \in NA^1$ and $k \geq 1$ denote $\mathcal{Z}^k_\lambda = \left\{\mu \in \left(NA^1\right)^k : \overline{\mu} \ll \lambda,\ \frac{d\overline{\mu}}{d\lambda} \in L^\infty(\lambda)\right\}$, and $\mathcal{Z}^*_\lambda = \bigcup_{k=1}^{\infty} \mathcal{Z}^k_\lambda$. Let $B^1_+(T, \mathcal{C})$ be the set of bounded measurable functions $\chi : T \to \mathbb{R}$ with $0 \leq \chi \leq 1$. The *direction space with perspective* $\lambda$, $\Omega_\lambda$, is the closure of the image of $B^1_+(T, \mathcal{C})$ in $\prod_{\mu \in \mathcal{Z}^*_\lambda} \Lambda_\mu$, under the mapping

$$y \mapsto (y(\mu))_{\mu \in \mathcal{Z}^*_\lambda}, \tag{3.5}$$

where for every $y \in B^1_+(T, \mathcal{C})$ and $\mu \in \mathcal{Z}^k_\lambda$, $y(\mu) = \begin{cases} \frac{\mu(y)}{k\overline{\mu}(y)}, & \mu(y) \notin D^k \\ 0_k, & \mu(y) \in D^k. \end{cases}$. $\Omega_\lambda$ is thus compact and Hausdorff, and every $x \in \Omega_\lambda$ has the form $x = (x(\mu))_{\mu \in \mathcal{Z}^*_\lambda}$ with $x(\mu) \in \Lambda_\mu$ for every $\mu \in \mathcal{Z}^*_\lambda$.

### 3.4 Values and their Representations

Given a subspace $Q \subseteq \mathcal{M}$, denote by $Q_+$ its subset of monotonic games. A *value* on $Q$ is a linear map $\phi : Q \to FA$ satisfying

i. efficiency - $\forall v \in Q\ \psi(v)(T) = v(T)$;

ii. symmetry - $\forall v \in Q, \forall \theta \in \Theta\ \psi(\theta v) = \theta(\psi v)$; and

[10] The convention $\frac{0_k}{0} = 0_k$ is used.

[11] $\mathcal{R}(\mu)$ stands for the range of the vector measure $\mu$.

[12] E.g., it is homeomorphic to $\left\{x \in \Delta^k : \left\|x - \frac{1}{k}\mathbf{1}_k\right\|_2 \geq \delta\right\}$ for a sufficiently small $\delta > 0$.

iii. positivity - $\forall v \in Q_+ \ \psi(v) \in FA_+$.

Given $\lambda \in NA^1$, $k \geq 2$, $f \in LM^k$, and $\mu \in \mathcal{Z}^k_\lambda$, define $\partial(f,\mu) : \Omega_\lambda \to L^\infty(\lambda)$ by

$$\partial(f,\mu)(x) = \begin{cases} df\left(x(\mu), \frac{d\mu}{d\lambda}\right), & x(\mu) \notin \mathbb{S}^k_\perp \\ df\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\lambda}\right), & x(\mu) \in \mathbb{S}^k_\perp. \end{cases} \tag{3.6}$$

Notice that indeed $\partial(f,\mu)(x) \in L^\infty(\lambda)$ for every $x \in \Omega_\lambda$ (see [12, Remark 4.3]).

The following Theorem is a direct consequence of [12, Theorem 2.6]. It is proved in Appendix A:

**Theorem 3.4.** *Let $\phi$ be a value on $\mathcal{LM}$. For every $\lambda \in NA^1$ there is a finitely additive, positive vector measure $P_\lambda$ of bounded semi-variation (i.e., $|P_\lambda|(\Omega_\lambda) < \infty$. See Appendix C for details.) on the Borel sets of $\Omega_\lambda$ with values in $\mathcal{L}\left(L^\infty(\lambda), L^2(\lambda)\right)$ s.t. for every coalition $S \in \mathcal{C}$ the vector measure $P^S_\lambda = \langle P_\lambda, \chi_S\rangle$ is positive, regular, and countably additive of bounded variation, and for every $f \in LM^k$ and $\mu \in \mathcal{Z}^k_\lambda$ we have for every $S \in \mathcal{C}$*

$$\phi(f \circ \mu)(S) = \int_{\Omega_\lambda} \partial(f,\mu)(x) dP^S_\lambda(x). \tag{3.7}$$

The following Lemma is exactly the content of [12, Remark 5.3]:

**Lemma 3.5.** *If $\phi$ and $P_\lambda$ are as in Theorem 3.4 then for every $\chi \in L^\infty(\lambda)$ and every Borel set $E \subseteq \Omega_\lambda$*

$$\langle \chi, P_\lambda\rangle(E) = \chi\langle 1, P_\lambda\rangle(E). \tag{3.8}$$

### 3.5 Values of Exact Market Games

Recall that for every $k \geq 2$, $\Delta^k$ is the $k-1$ dimensional simplex in $\mathbb{R}^k_+$. Denote by $EM^k_+$ the positive cone spanned by the functions $f_C : \mathbb{R}^k_+ \to \mathbb{R}$ given by $f_C(x) = \min_{c\in C} c \cdot x$, where $C \subseteq \Delta^k$ is compact and convex. Let $\mathcal{EM}_+$ be positive cone spanned by games of the form $f \circ \mu$ where $f \in EM^k_+$ and $\mu \in \left(NA^1\right)^k$ for some $k \geq 2$. The following Theorem follows from our main result in [10]:

**Theorem 3.6.** *The unique value on $\mathcal{EM}_+$ is the Mertens value $\Phi_M$, given by Equation (B.3).*

*Remark* 3.7. As $\mathcal{EM}_+ \subseteq \mathcal{M}$ the Mertens value $\Phi_M$ is well defined on $\mathcal{EM}$ by equation (B.3) (in the Appendix).

## 4 The Proof

This section is dedicated to the proof of Theorem 2.3. Before we begin, let us briefly describe the structure of the proof. We start by constructing the derived value $\Phi$ of $\Psi$, defined on $\mathcal{M}$, and proving that $\Phi$ uniquely determines $\Psi$. We prove that $\Phi$ is uniquely determined by its restriction to $\mathcal{LM}$. This is accomplished by a result obtained in [11]. Utilizing once again the results from [11] together with Theorem 3.4 and Lemma 3.5 we prove that the restriction of $\Phi$ to $\mathcal{LM}$ is completely determined by its values on $\mathcal{EM}_+$. We then deduce from Theorem 3.6 that $\Phi = \Phi_M$ on $\mathcal{EM}_+$, where $\Phi_M$ denotes the value derived from the Mertens value $\Psi_M$ (see Appendix B), and Theorem 2.3 follows.

Given a payoff selection $\Psi : \mathfrak{E}^* \to FA$, define its *derived value* $\Phi : \mathcal{M} \to FA$ as follows: If $v \in \mathcal{M}$, represent it as $v_E - v_{E'}$ for $E, E' \in \mathfrak{E}^*$, and let

$$\Phi(v) = \Psi(E) - \Psi(E') \tag{4.1}$$

**Lemma 4.1.** *The derived value $\Phi$ is a well defined linear map. Furthermore, it satisfies the axioms of efficiency, anonymity, and contraction, i.e., it is a value in the sense of Section 3.4, and it uniquely determines $\Psi$.*

*Proof.* Suppose $v = v_{E_0} - v_{E_0'} = v_{E_1} - v_{E_1'}$. By Remark 3.2 we may assume, w.l.o.g., that $E_0, E_0', E_1, E_1' \in \mathfrak{E}^*(\nu)$ for some $\nu \in NA^1$. Thus $v_{E_0 \oplus E_1'} = v_{E_0' \oplus E_1}$, and by first applying the contraction axiom and then Remark 2.1 we obtain

$$\begin{gathered}\Psi(v_{E_0 \oplus E_1'}) = \Psi(v_{E_0' \oplus E_1}) \Rightarrow \\ \Psi(E_0) - \Psi(E_0') = \Psi(E_1) - \Psi(E_1'),\end{gathered}$$

and hence $\Phi$ is well defined. Efficiency and symmetry of $\Phi$ follow easily from the efficiency and anonymity axioms for $\Psi$, respectively. If $v \in \mathcal{M}$ is monotonic and $v = v_E - v_{E'}$ for some $E, E' \in \mathfrak{E}^*$, then by Remark 2.2 we obtain $\Phi(v) = \Psi(E) - \Psi(E') \in FA_+$, hence $\Phi$ is also positive. For the linearity, first notice that by Remark 2.1 we have $\Phi(w + rv) = \Phi(w) + r\Phi(v)$ for every $v, w \in \mathcal{M}$ and $r \in \mathbb{Q}$, and the linearity now follows from the contraction axiom for $\Psi$. Furthermore, $\Phi$ uniquely determines $\Psi$ as $\Psi(E) = \Phi(v_E)$. □

Let $\epsilon > 0$ and $k \geq 2$. The set

$$U_\epsilon^k = \left\{ x \in \mathbb{R}^k : \forall 1 \leq i, j \leq k,\ |x_i - x_j| \leq \epsilon k \overline{x} \right\} \tag{4.2}$$

is a *conical diagonal neighborhood*. The following proposition is a direct consequence of Lemma 4.1 and [11, Theorem 2.2]:

**Proposition 4.2.** *Let $\Phi$ be a value on $\mathcal{M}$. If $h \in M^k$ vanishes on a conical diagonal neighborhood then for every $\mu \in \left(NA^1\right)^k$ we have*

$$\Phi(h \circ \mu) = 0. \tag{4.3}$$

As a result we have the following corollary:

**Corollary 4.3.** *Every value $\Phi$ on $\mathcal{M}$ is uniquely determined by its restriction to $\mathcal{LM}$. Namely, if $\Phi'$ is a continuous value on $\mathcal{M}$ with $\Phi'\big|_{\mathcal{LM}} = \Phi\big|_{\mathcal{LM}}$, then $\Phi' = \Phi$.*

*Proof.* This immediately follows from [11, Corollary 2.3]. □

The restriction of $\Phi$ to $\mathcal{LM}$ is a value. By abuse of notation we denote this mapping by $\Phi$. We are thus interested in continuous values[13] $\Phi$ on $\mathcal{LM}$. Recall that $\Phi_M$ denotes the value derived from the Mertens payoff selection $\Psi_M$.

**Proposition 4.4.** *If $\Phi$ is a continuous value on $\mathcal{LM}$, then $\Phi = \Phi_M$.*

We shall need the following Lemmata:

**Lemma 4.5.** *For every $w \in \mathbb{S}^k_\perp \setminus \{0_k\}$ and every sufficiently small $\epsilon > 0$ there is a continuous and positively homogeneous of degree 1 function $h^w_\epsilon : \mathbb{R}^k_+ \to \mathbb{R}$ which is twice continuously differentiable on $\mathbb{R}^k_+ \setminus \{0_k\}$, vanishes on the conical diagonal neighborhood $N_\epsilon = \{x : \sum_{\ell=1}^{k} |x_\ell - \overline{x}| < \epsilon k \overline{x}\}$, and for every $z \in \mathbb{R}^k_+$ with $w \cdot z \neq 0$ and every $1 \leq \ell \leq k$ we have $\frac{\partial h^w_\epsilon}{\partial x_\ell}(z) \underset{\epsilon \to 0^+}{\longrightarrow} w_\ell$, where the convergence is uniformly bounded in $\epsilon$.*

*Proof.* Choose a twice continuously differentiable function $g_\epsilon : \mathbb{R} \to \mathbb{R}$ satisfying $g_\epsilon = 0$ on $[-\epsilon, \epsilon]$, $g'_\epsilon = 1$ on $[-2\epsilon, 2\epsilon]^c$, and $0 \leq g'_\epsilon \leq 1$ on $\mathbb{R}$. Define $h^w_\epsilon : \mathbb{R}^k_+ \to \mathbb{R}$ by

$$h^w_\epsilon(x) = \begin{cases} k\overline{x} g_\epsilon\left(\frac{w \cdot x}{k\overline{x}}\right), & x \neq 0_k \\ 0, & x = 0_k. \end{cases} \tag{4.4}$$

The function $h^w_\epsilon$ is continuous and positively homogeneous of degree 1 on $\mathbb{R}^k_+$, and it is twice continuously differentiable on $\mathbb{R}^k_+ \setminus \{0_k\}$. It also vanishes on $N_\epsilon$; indeed if $x \in N_\epsilon \setminus \{0_k\}$ then

$$\left|\frac{w \cdot x}{k\overline{x}}\right| = \left|w \cdot \left(\frac{x}{k\overline{x}} - \frac{1}{k}\mathbf{1}_k\right)\right| \leq \sum_{i=1} \left|\frac{x_i}{k\overline{x}} - \frac{1}{k}\right| < \epsilon, \tag{4.5}$$

hence $h^w_\epsilon(x) = g_\epsilon\left(\frac{w \cdot x}{k\overline{x}}\right) = 0$.

As to the convergence of the partial derivative for $z \in \mathbb{R}^k_+$ with $w \cdot z \neq 0$, notice that

$$\frac{\partial h^w_\epsilon}{\partial x_\ell}(z) = g_\epsilon\left(\frac{w \cdot z}{k\overline{z}}\right) + \frac{k\overline{z} w_\ell - w \cdot z}{k\overline{z}} g'_\epsilon\left(\frac{w \cdot z}{k\overline{z}}\right) \underset{\epsilon \to 0^+}{\longrightarrow} w_\ell. \tag{4.6}$$

Also notice that the convergence is indeed uniformly bounded w.r.t. $\epsilon$. □

Given $\mu \in \left(NA^1\right)^k$ we denote by $AF(\mu)$ the affine space generated by the range $\mathcal{R}(\mu)$ of $\mu$.

[13] By abuse of terms, values are linear maps on $\mathcal{LM}$ satisfying the efficiency, anonymity, and positivity axioms.

**Lemma 4.6.** *Let $P_\lambda$ be the vector measure given in Theorem 3.4, let $\mu \in \mathcal{Z}^k_\lambda$ with $dim(AF(\mu)) \geq 2$, and denote $S_\mu = \{s \in I : \frac{d\mu}{d\lambda}(s) \notin D^k\}$. Then (in the vector lattice $L^2(\lambda)$)*

$$\langle \chi_{S_\mu}, P_\lambda\rangle(\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp\} = 0. \tag{4.7}$$

*Proof.* There is a Borel set $W \subseteq \mathbb{S}^k_\perp$ of Haar measure 1 s.t. for every $w \in W$ we have

$$\langle 1, P^I_\lambda\rangle(\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp, w \cdot x(\mu) = 0\}) = 0.$$

Hence for every $S \in \mathcal{C}$ and every $w \in W$ we have $\langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp, w \cdot x(\mu) = 0\}) = 0$. Choose $w \in W$ and given $\epsilon > 0$ choose a function $h^w_\epsilon$ as in Lemma 4.5. Then for any sufficiently small $\epsilon > 0$, $\|\partial(h^w_\epsilon, \mu)\|_\infty$ is uniformly bounded w.r.t. $\epsilon$, and for every $x \in \Omega_\lambda$ with $x(\mu) \notin \mathbb{S}^k_\perp$ and $w \cdot x(\mu) \neq 0$ we have $\partial(h^w_\epsilon, \mu)(x) \underset{\epsilon \to 0^+}{\longrightarrow} \frac{d(w\cdot\mu)}{d\lambda}$ in the $L^\infty(\lambda)$ norm. As by [13, Lemma 3.3], $h^w_\epsilon \in M^k$, we obtain

$$\Phi(h^w_\epsilon \circ \mu)(S) = \int_{\Omega_\lambda} \partial(h^w_\epsilon, \mu)(x) dP^S_\lambda(x) = \tag{4.8}$$

$$\int_{\{x\in\Omega_\lambda : x(\mu)\notin\mathbb{S}^k_\perp\}} dh^w_\epsilon\left(x(\mu), \frac{d\mu}{d\lambda}\right) dP^S_\lambda(x) \underset{\epsilon \to 0^+}{\longrightarrow} \langle \frac{d(w\cdot\mu)}{d\lambda}, dP^S_\lambda\rangle \left(\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp\}\right), \tag{4.9}$$

where the convergence in line (4.9) follows from the bounded convergence theorem C.2 (in the Appendix). By Proposition 4.2 we have $\Phi(h^w_\epsilon \circ \mu)(S) = 0$, and by combining that with Equations (4.8)-(4.9) we obtain for every $w \in W$

$$\forall S \in \mathcal{C}, \quad 0 = \langle \frac{d(w\cdot\mu)}{d\lambda}, P^S_\lambda\rangle \left(\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp\}\right) \Rightarrow \tag{4.10}$$

$$0 = \langle \frac{d(w\cdot\mu)}{d\lambda}, P_\lambda\rangle \left(\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp\}\right), \tag{4.11}$$

where the equality in line (4.11) holds in $L^2(\lambda)$ and follows by taking the Radon-Nikodym derivative in line (4.10). By combining Lemma 3.5 together with Equation (4.11) we obtain that for every $w \in W$ and $\lambda$-a.e. $s \in I$ with $\frac{d(w\cdot\mu)}{d\lambda}(s) \neq 0$

$$0 = \langle 1, P_\lambda\rangle \left(\left\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp\right\}\right)(s).$$

As $W$ is of full Lebesgue measure in $\mathbb{S}^k_\perp$ we thus conclude that for $\lambda$-a.e. $s \in I$ with $\frac{d\mu}{d\lambda}(s) \notin D^k$

$$0 = \langle 1, P_\lambda\rangle \left(\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp\}\right)(s),$$

hence, by Lemma 3.5 (in the vector lattice $L^2(\lambda)$)

$$0 = \langle \chi_{S_\mu}, P_\lambda\rangle \left(\{x \in \Omega_\lambda : x(\mu) \notin \mathbb{S}^k_\perp\}\right).$$

□

It is sufficient to verify Proposition 4.4 for games $f \circ \mu$ with $f \in LM_+^k$ and $\mu \in \left(NA^1\right)^k$ with $dim(AF(\mu)) \geq 2$. For such $f$ let $h(f) : \mathbb{R}_+^k \to \mathbb{R}$ be given by $h(f)(x) = df(\mathbf{1}_k, x)$. Then $h(f) \in EM_+^k$. Therefore, for every $S \in \mathcal{C}$

$$\Phi(f \circ \mu)(S) = \int\limits_{X_{\overline{\mu}}} \partial(f, \mu)(x) dP_{\overline{\mu}}^S(x) = \tag{4.12}$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \notin \mathbb{S}_\perp^k\}} df\left(x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP_{\overline{\mu}}^S(x) + \int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \in \mathbb{S}_\perp^k\}} df\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP_{\overline{\mu}}^S(x) \geq \tag{4.13}$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \notin \mathbb{S}_\perp^k\}} df\left(x(\mu), \frac{d\mu}{d\overline{\mu}} \chi_{S_\mu^c}\right) dP_{\overline{\mu}}^S(x) +$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \notin \mathbb{S}_\perp^k\}} df\left(x(\mu), \frac{d\mu}{d\overline{\mu}} \chi_{S_\mu}\right) dP_{\overline{\mu}}^S(x) +$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \in \mathbb{S}_\perp^k\}} df\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP_{\overline{\mu}}^S(x) = \tag{4.14}$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \notin \mathbb{S}_\perp^k\}} df\left(x(\mu), \chi_{S_\mu^c} \mathbf{1}_k\right) dP_{\overline{\mu}}^S(x) +$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \in \mathbb{S}_\perp^k\}} dh(f)\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP_{\overline{\mu}}^S(x) \geq \tag{4.15}$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \notin \mathbb{S}_\perp^k\}} dh(f)\left(x(\mu), \chi_{S_\mu^c} \mathbf{1}_k\right) dP_{\overline{\mu}}^S(x) +$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \in \mathbb{S}_\perp^k\}} dh(f)\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP_{\overline{\mu}}^S(x) = \tag{4.16}$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \notin \mathbb{S}_\perp^k\}} dh(f)\left(x(\mu), \frac{d\mu}{d\overline{\mu}} \chi_{S_\mu}\right) dP_{\overline{\mu}}^S(x) +$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \notin \mathbb{S}_\perp^k\}} dh(f)\left(x(\mu), \chi_{S_\mu^c} \mathbf{1}_k\right) dP_{\overline{\mu}}^S(x) +$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \in \mathbb{S}_\perp^k\}} dh(f)\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP_{\overline{\mu}}^S(x) = \tag{4.17}$$

$$\int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \notin \mathbb{S}_\perp^k\}} dh(f)\left(x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP_{\overline{\mu}}^S(x) + \int\limits_{\{x \in X_{\overline{\mu}} : x(\mu) \in \mathbb{S}_\perp^k\}} dh(f)\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP_{\overline{\mu}}^S(x) =$$

$$\int_{X_{\overline{\mu}}} dh(f)\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP^S_{\overline{\mu}}(x) = \Phi(h(f)\circ\mu)(S) = \tag{4.18}$$

$$\Phi_M(h(f)\circ\mu)(S) = \Phi_M(f\circ\mu)(S),$$

where the first equality in line (4.12) follows from Theorem 3.4, the inequality in line (4.13) follows from the concavity of the function $df(z,\cdot)$ for every $z \in \mathbb{R}^k_+ \setminus D^k$, the equality in line (4.14) follows by combining Lemma 4.6 with the definition of $h(f)$, the inequality in line (4.15) follows as $df(z,\mathbf{1}_k) \geq f(\mathbf{1}_k) = h(f)(\mathbf{1}_k) = dh(f)(z,\mathbf{1}_k)$ for every $z \in \mathbb{R}^k_+ \setminus D^k$, the equality in line (4.16) follows from Lemma 4.6, the equality in line (4.17) follows as $dh(f)(z,x) + dh(f)(z,a\mathbf{1}_k) = dh(f)(z,x+a\mathbf{1}_k)$ for every $z \in \mathbb{R}^k_+ \setminus D^k$, $x \in \mathbb{R}^k_+$, and $a \in \mathbb{R}_+$, the first equality in line (4.18) follows from Theorem 3.4, and the second equality in that line follows from Theorem 3.6.

Now, by the efficiency axiom we obtain for every $S \in \mathcal{C}$

$$\Phi(f\circ\mu)(S) = \Phi_M(f\circ\mu)(S), \tag{4.19}$$

and we are done.

## References

[1] R.J. Aumann, L.S. Shapley, *Values of Non-Atomic Games, Princton University Press* (1974).

[2] R.J. Aumann, *Markets with a Continuum of Traders, Econometrica 32 (1964),* 39–50.

[3] R.J. Aumann, *Values of Markets with a Continuum of Traders, Econometrica 43 (1975),* 611–646.

[4] G. Debru, H.E. Scarf, *A Limit Theorem on the Core of an Economy, Economic Review 4 (1963),* 235–269.

[5] J. Diestel, J.J. Uhl Jr., *Vector Measures, AMS Mathematical Surveys 15,* 1977.

[6] I. Dobrakov, *On integration in Banach spaces I, Czechoslovak Mathematical Journal 20 (1970),* 511–536.

[7] P. Dubey, A. Neyman, *Payoffs in Nonatomic Economies: An Axiomatic Approach, Econometrica 52 (1984),* 1129–1150.

[8] P. Dubey, A. Neyman, *An Equivalence Principle for Perfectly Competitive Economies, J. Econ. Theory 75 (1997),* 314–344.

[9] F.Y. Edgeworth, *Mathematical Physics, Kegan Paul, London,* 1881.

[10] O. Edhan, *Values of Exact Market Games, Discussion Paper 627, The Centre for the Study of Rationality, Hebrew University of Jerusalem,* 2012.

[11] O. Edhan, *Continuous Values of Market Games are Conic, Discussion Paper 623 , The Centre for the Study of Rationality, Hebrew University of Jerusalem,* 2012.

[12] O. Edhan, *Representations of Positive Projections on Lipschitz Vector Measure Games, Discussion Paper 624, The Centre for the Study of Rationality, Hebrew University of Jerusalem,* 2012.

[13] O. Haimanko, *Payoffs in non-differentiable perfectly competitive TU economies, Journal of Economics Theory 106 (2002),* 17–39.

[14] S. Hart, A. Neyman, *Values of non-atomic vector measure games: Are they linear combinations of the measures? Journal of Mathematical Economics 17 (1988),* 31–40.

[15] S. Hart, *Values of Non-Differentiable Markets with a Continuum of Traders, Journal of Mathematical Economics 4 (1977),* 103–116.

[16] Y. Kannai, *Continuity Properties of the Core of a Market, Econometrica 38 (1970),* 791–815.

[17] J.F. Mertens, *The Shapley value in the non-differentiable case, International Journal of Game Theory 17 (1988),* 1–65.

[18] J.F. Mertens, *Nondifferentiable TU markets: The value, in: Roth, A.E. (Ed.), The Shapley Value: Essays in Honor of Lloyd S. Shapley, Cambridge Univ. Press, Cambridge, UK, 1988,* pp. 235–264.

[19] A. Neyman, *Values of non-atomic vector measure games. Israel J. Math. 124 (2001),* 1-27.

[20] L.S. Shapley, *A value for n-person games, in: Kuhn, H.W, and Tucker, A.W. (Eds.), Contributions to the Theory of Games, Vol. II, Princeton University Press, Princeton NJ, 1953,* pp. 307–317.

[21] M. Shubik, *Edgeworth Market Games, Annals of Mathematics Studies 40 (1959),* 267–278.

# A Proofs

**Proof of Lemma 3.4.** *Set $Q = \mathcal{LM}$ in [12, Theorem 2.6], and let $\widehat{\mathcal{LM}}_\lambda$ be generated by pairs $(f,\mu)$ with $f \in LM_+^k$ and $\mu \in \mathcal{Z}_\lambda^k$ for some $k \geq 2$. Let $\left\{\widehat{P}_\lambda\right\}_{\lambda \in NA^1}$ be the representing measures of $\Psi$ w.r.t. $\mathcal{D}^{\widehat{\mathcal{EM}}}$, whose existence is proved in [12, Theorem 2.6]. For $x \in \prod_{\mu \in \mathcal{Z}_\lambda^*} \left[(\mathcal{R}(\mu) \setminus D^{k(\mu)}) \sqcup ([0,1]\boldsymbol{1}_k + \mathbb{S}_\perp^{k(\mu)})\right]$ define $\Pi(x) = \left(\Pi^{k(\mu)}(x_\mu)\right)_{\mu \in \mathcal{Z}_\lambda^*}$, where, for $z \in (\mathcal{R}(\mu) \setminus D^{k(\mu)}) \sqcup ([0,1]\boldsymbol{1}_k + \mathbb{S}_\perp^{k(\mu)})$*

$$\Pi^{k(\mu)}(z) = \begin{cases} \Upsilon^{k(\mu)}(z), & z \in [0,1]\boldsymbol{1}_k + \mathbb{S}_\perp^{k(\mu)} \\ \frac{z}{k\bar{z}}, & \mathcal{R}(\mu) \setminus D^{k(\mu)} \end{cases}.$$

*Choosing $P_\lambda = \Pi \circ \widehat{P}_\lambda$ for every $\lambda \in NA^1$ and recalling Equation (3.3) proves Equation (3.7).* $\square$

## B The Mertens Value on $\mathfrak{E}^*$

The construction of the Mertens value $\Psi_M$ for markets is carried out in details in [18]. Here we briefly describe the form of the Mertens value on $\mathfrak{E}^*$.

The Cauchy distribution with parameter $\alpha > 0$ is the distribution on $\mathbb{R}$ with density $\frac{\alpha}{\pi(\alpha^2+x^2)}$. If $X$ and $Y$ are independent Cauchy random variables with parameters $\alpha$ and $\beta$ respectively and $a, b \in \mathbb{R}$ s.t. $a^2 + b^2 \neq 0$ then $aX + bY$ is a Cauchy random variable with parameter $|a|\alpha + |b|\beta$. The characteristic function of the Cauchy distribution with parameter $\alpha$ is $\psi(t) = \exp(-\alpha|t|)$.

Recall that given a vector measure $\mu \in \left(NA^1\right)^k$ we defined $\overline{\mu} = \frac{1}{k}\sum_{i=1}^{k} \mu_i$. The $\mu$ *semi-norm* of $y \in \mathbb{R}^k$ is given by

$$\|y\|_\mu = \int |\sum_{i=1}^{k}(d\mu_i/d\overline{\mu})y_i|d\bar{\mu}. \tag{B.1}$$

By Lemma 1 in [19] the function $\phi_\mu : \mathbb{R}^k \longrightarrow \mathbb{R}$ given by $\phi_\mu(y) = exp(-\|y\|_\mu)$ is the characteristic function of a probability distribution $P_\mu$ on $AF(\mu)$ - *the affine space generated by* $\mathcal{R}(\mu)$. By [19, Lemma 1], $P_\mu$ is absolutely continuous w.r.t. the Lebesgue measure on $AF(\mu)$, and its density $\xi_\mu$ is a $C_0(AF(\mu))$ function.

For any $k \geq 2$ and $\mu \in \left(NA^1\right)^k$, consider $P_\mu$ as a measure on $\mathbb{R}^k$, and define a new measure $Q_\mu$ on $\mathbb{S}^k_\perp$ by

$$Q_\mu = \Upsilon^k \circ P_\mu. \tag{B.2}$$

The Mertens value on $\mathcal{M}$ can now be given using a simple formula: If $f \in M^k$, and $\mu \in \left(NA^1\right)^k$ then for any $S \in \mathcal{C}$ (see [18])

$$\Phi_M(f \circ \mu)(S) = \int df(\mathbf{1}_k, x, \mu(S))dQ_\mu(x). \tag{B.3}$$

If $E \in \mathfrak{E}^*$ then we define

$$\Psi_M(E) = \Phi_M(v_E). \tag{B.4}$$

## C Rudiments of Functional Analysis

Here we give some functional analysis background that is needed for the understanding of some of our results. For further reading, one is advised to use the reference.

## C.1 Vector measures

A function $F$ from an algebra $\mathfrak{F}$ of subsets of a set $\Omega$ to a Banach space $Z$ is called *finitely additive vector measure* or simply a *vector measure* if whenever $E_1, E_2 \in \mathfrak{F}$ are disjoint then $F(E_1 \cup E_2) = F(E_1) + F(E_2)$. If, in addition, $F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} F(E_n)$ in the norm topology of $Z$ for all sequences $(E_n)_{n=1}^{\infty}$ of pairwise disjoint members of $\mathfrak{F}$ s.t. $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{F}$ then $F$ is termed a *countably additive vector measure* or simply *countably additive.*

The *strong variation of* $F$ is the function $\|F\| : \mathfrak{F} \to \mathbb{R}$ defined by

$$\|F\|(E) = \sup_{\pi} \sum_{A \in \pi} \|F(A)\|,$$

where the supremum is taken over all finite partitions of $E$ into disjoint members of $\mathfrak{F}$. One may easily check that $\|F\|$ is a monotonic finitely additive measure. A measure $F$ is of *bounded variation* if $\|F\|(\Omega) < \infty$. Furthermore,

**Proposition C.1.** *[5, Proposition I.1.9] A vector measure of bounded variation is countably additive iff its variation is also countably additive.*

## C.2 Operator Valued Integration

Let $F$ be a vector measure on an algebra $\mathfrak{F}$ of subsets of $\Omega$ with values in the Banach space $\mathcal{L}(Y, Z)$ of bounded linear operators from $Y$ to $Z$, where $Y, Z$ are Banach lattices. Denote by $\mathcal{S}_{\Omega,\mathfrak{F}}(Y)$ the set of simple functions on $\Omega$ w.r.t. $\mathfrak{F}$ taking values in $Y$, i.e. the set of functions of the form $\sum_{i=1}^{n} a_i \chi_{E_i}$ where $E_i \in \mathfrak{F}$ and $a_i \in Y$ for every $1 \le i \le n$. The (Bartle) integral of the simple function $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ w.r.t. $F$ is given by

$$\int f dF = \sum_{i=1}^{n} F(E_i)(a_i).$$

A measurable function $f : \Omega \to Y$ is *strongly $F$-integrable*, or *integrable* for short, iff for every increasing sequence $(f_n)_{n=1}^{\infty}$ of simple functions $f_n : \Omega \to Y$ with $f_n \underset{n\to\infty}{\longrightarrow} f$ pointwise $\|F\|$-a.e., the limit $\nu(E) = \lim\limits_{n\to\infty} \int f_n \chi_E dF$ exists in the strong topology of $Z$ for every $E \in \mathfrak{F}$ and is independent of the choice of $(f_n)_{n=1}^{\infty}$. In that case we denote

$$\int_E f dF = \lim_{n\to\infty} \int_E f_n dF.$$

The following theorem is a version of the well-known Bartle bounded convergence theorem:

**Theorem C.2** (Bartle Bounded Convergence Theorem)**.** *Let $(f_n)_{n=1}^{\infty}$ be a uniformly bounded sequence of integrable functions $f_n : \Omega \to Y$, and suppose that $F$ above is countably additive of bounded variation. If*

*$(f_n)$ converges $\|F\|$-a.e. to $f$ then $f$ is integrable and*

$$\lim_{n\to\infty} \int f_n dF = \int f dF$$

*in the strong topology of $Z$.*

*Proof.* By Egorof-Lusin's theorem [6, p. 520] for every $\epsilon > 0$ there is a measurable subset $E = E(\epsilon) \subseteq \Omega$ s.t. $\|F(E^c)\| < \epsilon$ and $(f_n)$ converges uniformly to $f$ on $E$. Let $C > 0$ be s.t. $\sup_{x\in\Omega} \|f_n(x)\| \leq C$ for every $n \in \mathbb{N}$. Note that

$$\left\| \int_E f_n dF \right\| \leq C \left\|F\right\|(E)$$

for every $E \in \mathfrak{F}$, where $\|F\|$ denotes the variation of $F$. Let $N \in \mathbb{N}$ be s.t. for every $m, n > N$ and every $x \in E$, $\|f_m(x) - f_n(x)\| < \epsilon$. Then for every $m, n > N$ we have

$$\left\| \int f_m dF - \int f_n dF \right\| \leq \left\| \int_E (f_m - f_n) dF \right\| + \left\| \int_{E^c} (f_m - f_n) dF \right\| <$$
$$\epsilon \left\|F\right\|(E) + 2C \left\|F\right\|(E^c).$$

As $F$ is countably additive of finite variation we have $\left\|F\right\|(E(\epsilon)^c) \to 0$ as $\epsilon \to 0^+$, hence

$$\lim_{m,n\to\infty} \left\| \int f_m dF - \int f_n dF \right\| = 0, \tag{C.1}$$

proving that the integrals form a Cauchy sequence in $Z$ and hence convergence in its strong topology. As for every sequence of increasing functions $(g_n)_{n=1}^{\infty}$ converging pointwise to $f$ and $\epsilon > 0$ there is measurable subset $E$ and $N \in \mathbb{N}$ s.t. $|f_n(x) - g_n(x)| < \epsilon$ for every $x \in E$ and $n \geq N$, and as $\|g_n(x)\| \leq \|f(x)\| \leq C$ for every $x$, we deduce in a similar manner that $\lim_{n\to\infty} \int f_n dF = \lim_{n\to\infty} \int g_n dF$, hence $f$ is integrable, and the rest of the theorem now easily follows. $\square$