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VALUES OF EXACT MARKET GAMES

By

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Payoffs in Exact TU Economies

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Abstract

We prove that a single-valued solution of perfectly competitive TU economies underling nonatomic exact market games is uniquely determined as the Mertens value by four plausible value-related axioms. Since the Mertens value is always a core element, this result provides an axiomatization of the Mertens value as a core-selection. Previous works in this direction assumed the economies to be either differentiable (e.g., Dubey and Neyman [9]) or of uniform finite-type (e.g., Haimanko [14]). Our work does not assume that, thus it contributes to the axiomatic study of payoffs in perfectly competitive economies (or values of their derived market games) in general. In fact, this is the first contribution in this direction.

1 Introduction

The equivalence of the core and the set of competitive (Walras) payoffs in "perfectly competitive" economies is one of the most striking results in economics. It was already conjectured by Edgeworth[1] [11], and has been shown to hold in limit economies (e.g., [6, 17, 23]), nonstandard economies (e.g., [4]) and in nonatomic economies (e.g., [2]). It was later on observed by Aumann ([3]) that, in nonatomic economies with a smoothness assumption on the preferences, the set of "value allocations"[2] also coincides with the above two. This equivalence is known in the literature as the *equivalence phenomena* (see [9]).

The equivalence phenomena is even more astounding in economies with smooth, *transferable utilities* (TU): these have a single-valued core whose unique element, a competitive payoff, is the Aumann-Shapley [1] value. The coincidence and uniqueness[3] of these sets demanded an explanation. This was supplied by Dubey and Neyman [9] in the form of a "meta-equivalence" theorem for perfectly competitive (nonatomic) TU economies with smooth preferences. Namely, they characterized the equivalence phenomena axiomatically, by a list of four simple and plausible axioms.

Dubey and Neyman meta-equivalence result holds, nonetheless, only (see [9, p. 1146]) under the smoothness assumption, namely - when the sets are singletons. In general, however, the core of a perfectly

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[1] In his terminology the core was dubbed *the contract curve*.

[2] Allocations whose definition is based on the Shapley [22, 1] value.

[3] Namely, that each set consists of one element.

competitive (nonatomic) economy is usually not single-valued. If, for example, we adopt the production interpretation for economies (as we usually shall), then in most applications the agents' production functions will usually have "kinks" that will prevent the core from being single-valued. If the core is not single-valued, it is still equivalent to the set of competitive payoffs, but the Aumann-Shapley value may, however, not exist. Thus, the task of extending Dubey and Neyman [9] "meta-equivalence" theorem in its original form to the more general setting may seem to be impossible.

Nevertheless, a small divergence might help; we may wish to *select* an element of the core in a consistent and economically meaningful way, that extends the Aumann-Shapley value whenever it exists. Such a selection exists, and was first constructed by Mertens [18, 19], thus it is known as the *Mertens value*. This selection avoids discontinuities, discrimination between agents, and inconsistencies in the selection of payoffs. It also coincides with the asymptotic Shapley value of economies, whenever the later exists, and thus it is natural to replace, in the more general setting, the Aumann-Shapley value with the Mertens value, and to try to extend Dubey and Neyman [9] result into a "meta-selection" theorem for general economies.

The first breakthrough on this matter is due to Haimanko [14]. He proved that four plausible axioms, closely related to those set in [9], determine a unique core selection on the domain of perfectly competitive TU economies of *uniform finite-type*[4], and that this core selection is indeed the Mertens [19] value. However, Haimanko's work did not settle the matter on the entire domain of perfectly competitive TU economies, and not even on its subdomain consisting of *finite-type economies*[5]. Haimanko's methods relied heavily on the uniform finite-type assumption, with no apparent way to apply them in more general settings.

To be more specific, if an economy is of uniform finite-type, then its *derived market game*[6] is a function of finitely many, mutually singular, nonatomic probability measures. Haimanko [14] observed that the value axioms imply, in this case, that the a payoff obeying his axioms can be written as a linear combination of the derived market game's measures, whose coefficients depend solely on the agents' production functions marginals. Haimanko then used the value axioms to prove that this observation gives rise to a simple representation formula of the payoff as a barycenter of the economy's core. His axiomatization also yielded, in this case, a line of properties, geometric in nature, of the barycenter representation that lead in turn to the characterization of the payoff as the Mertens [19] value, which is known[7] to be a member of the core.

In general, a nonatomic market game derived from a perfectly competitive TU economy may not be a vector measure game. In fact, even if we restrict our attention to economies whose derived market games are vector measure games[8], it is unknown if a payoff satisfying Haimanko's axioms is indeed a linear combination of the derived game's measures. In fact, there are some examples, constructed on other domains, to contradict that (e.g., Hart and Neyman [15]). This fact limits any straightforward attempt to represent the payoff as a barycenter of the core and study the relationship between the different

[4]Namely, economies with finitely many types of production functions and endowments.

[5]Namely, those with finitely many type of production functions and general endowments.

[6]A game in characteristic function form that assigns to every coalition of agents the maximum production output that it could achieve by reallocating its resources among its members.

[7]In the special case of perfectly competitive TU economies, the Mertens value is given by an explicit formula as a barycenter of the core. See Mertens [19].

[8]E.g., in the case of finite-type economies, i.e., an economy with finitely many types of production functions and a general endowments.

representations, as it was done by Haimanko [14]. Thus, renouncing the uniform finite-type assumption makes the problem by far more immune to analysis.

To make the problem a bit more tractable to analysis, we might wish to consider a subdomain of "simple" perfectly competitive TU economies. We obviously do not wish to consider any subdomain, but one with a clear importance in economic applications and one that will advance our understanding of the problem. The payoffs studied in both Dubey and Neyman [9] and Haimanko [14] are assumed to follow some[9] continuity axiom. It is well known (e.g., Dubey and Neyman [9, Lemma 5.5], and recently also our [12] more general result) that continuous payoffs in a perfectly competitive TU economy are completely determined by the behavior of the derived market game in a small neighborhood of its diagonal. It may thus be appropriate to consider only economies whose derived market games are completely determined by their behavior in an "infinitesimal" neighborhood of the diagonal. In fact, we shall restrict our attention to the domain generated by *exact economies*, namely, perfectly competitive TU economies whose derived market games are exact. This domain contains many economies of interest, especially when we adopt the production interpretation for the economies. Furthermore, these economies have another helpful advantage; it is well known that perfectly competitive TU economies have finite dimensional cores (e.g., Hart [16]), and therefore exact market games have a simple representation as vector measure games.

In this work we prove a "meta-selection" theorem for exact economies in general. Namely, we prove that a single-valued solution of exact perfectly competitive TU economies is uniquely determined as the Mertens value by the axioms of "efficiency", "anonymity", "separability", and "positivity". Out axiomatization is akin to that of [14]. Table 1 may be found helpful in placing our work in the context of the main results on the value equivalence phenomenon.

Value Equivalence	Aumann and Shapley [1]
Value "Meta-Equivalence"	Dubey and Neyman [9]
Value Selection	Mertens [19]
Value "Meta-Selection" uniform finite-type economies	Haimanko [14]
Value "Meta-Selection" general exact economies	Edhan (this work)

Table 1: Summary of results
for perfectly competitive (nonatomic) TU economies

The axioms will be spelled out precisely in Section 2, but let us present them at an intuitive level now. For the time being we shall spare the reader with most of the technical details. Denote the set of agents by T. An economy E is essentially set by the agents' endowments a and production functions u, so (u_t, a_t) are the production function and endowment of agent $t \in T$. Recall that the *derived market game* v_E of E, is the game in characteristic function form that assigns to every coalition of agents the maximum output that it could produce by reallocating its resources among its members. Payoffs of the economy E may

[9] To be specific, it is assumed that the variation of the difference between the payoffs in two economies is sufficiently small whenever the variation of the difference of the market games derived from these economies is sufficiently small.

be identified with members of the set FA_+ of positive finitely additive and bounded measures. Thus, a payoff selection on the domain of exact economies (economies with an exact derived market game) $\mathfrak{E}_{ex}$ is a mapping

$$\psi : \mathfrak{E}_{ex} \to FA_+.$$

We will impose four axioms on ψ: "efficiency," "anonymity," "separability," and "positivity." Our main result is that these axioms uniquely determine ψ as the Mertens value.

The *efficiency* axiom says that ψ is Pareto optimal. The *anonymity* axiom asserts that the labels of the agents do not matter; their relabeling will only result in relabeling their payoffs, accordingly. These axioms hold for many solutions, not only on $\mathfrak{E}_{ex}$ but also for more general domains and also for finite economies.

The *separability* axiom considers an economy made up of two separate, noninteracting parts; suppose that, given economies E', E'', producing the same kind of output, we construct an economy in which each agent has and can access its endowments and production abilities in E' and E'', but cannot use his endowment in E' to produce output in E'' and vice versa. In this case, the output of every coalition of agents in the economy E is just the sum of their outputs in E' and E''. So, essentially, production in E is equivalent to production in E' and in E'' independently of each other. Thus, the payoff in E should be *at least* as high as the separately obtained payoff of E' and E'', namely for every coalition S $\psi(E)(S) \geq \psi(E')(S) + \psi(E'')(S)$. This axiom is related to the additivity axiom for the value.

The *positivity* axioms asserts that if the economy E has higher marginals than the economy E' then $\psi(E) - \psi(E') \in FA_+$. This is a weaker form of the positivity axiom for the value (see [1]).

The unique payoff selection satisfying these axioms turns out to be the Mertens [19] value[10]. It should be noted that, with the exception of the axiom of efficiency, the axioms do not involve any assumption on the range of the payoff selection. In particular, inclusion in the core is not implied by any of them individually. It is thus surprising that a core selection is determined by these axioms. Even more astounding is the fact that the payoff of an economy is completely determined by its core; indeed, economies sharing the same core also have the same Mertens value.

Due to their proximity, it is worth comparing our axiomatization with the one offered in [14]. The two axiomatizations differ in one axiom. We have replaced the *contraction* axiom, Haimanko's version of continuity, by our positivity axiom. The two axioms are in fact related, as the contraction and efficiency axioms implies the positivity of the payoff selection (Proposition 4.6 of Aumann and Shapley [1]). The continuity of the payoff selection is obtained, in this case, as an *outcome* of our axiomatization. It seems that the continuity is needed as a separate axiom only if more general production functions are to be considered.

Finally, let us mention that in the smooth utilities case axioms of a similar nature can characterize the competitive payoff correspondence for perfectly competitive economies with nontransferable utilities (see [10]). We hope that our approach can be translated into the setting of NTU economies as well.

[10]The Mertens value is constructed for economies in [19]. We briefly describe its formula in appendix B.

2 Definitions, Axioms, and The Main Result

Let $(T, \mathcal{C})$ be a standard measurable space. T is the set of *agents*, and $\mathcal{C}$ is the σ-algebra of *coalitions*. The set of all bounded and finitely additive measures on $(T, \mathcal{C})$ is denoted by FA, its cone of bounded, finitely additive, and nonnegative measures on $(T, \mathcal{C})$ is denoted by FA_+, and the set of all bounded nonatomic measures on $(T, \mathcal{C})$ is denoted by NA. The set of nonatomic probability measures on $(T, \mathcal{C})$ is denoted by NA^1. An *economy* E is a triple (u, a, ν), where $u : T \times \mathbb{R}^k_+ \to \mathbb{R}$ and $a : T \to R^k_+$ for some k, and $\nu \in NA^1$. For each $t \in T$, $a_t = a(t)$ is agent t initial endowment of commodities 1, ..., k, $u_t(\cdot) = u(t, \cdot)$ is its utility function on the space of commodity bundles R^k_+, and ν is a *population measure* s.t. the following conditions hold:

1. a is measurable;
2. u is $\mathcal{C} \times \mathcal{B}_k$ measurable, where $\mathcal{B}_k$ denotes the Borel σ-algebra on $\mathbb{R}^k_+$; and,
3. $\int_T a_t d\nu(t) \in (0, \infty)^k$.

We further assume that for every $t \in T$:

4. u_t is monotonically nondecreasing and continuous;
5. $u_t(x) = o(\|x\|)$ as $\|x\| \to \infty$; and
6. $u_t(0_k) = 0$.

Given $\nu \in NA^1$ we denote by $\mathfrak{E}(\nu)$ the set of all economies $E = (u, a, \nu)$, and further denote $\mathfrak{E} = \bigcup_{\nu \in NA^1} \mathfrak{E}(\nu)$.

A *game* is a function $v : \mathcal{C} \to \mathbb{R}$ with $v(\emptyset) = 0$. Given $E = (u, a, \nu) \in \mathfrak{E}(\nu)$, the *derived market game* v_E corresponding to E is given for every $S \in \mathcal{C}$ by

$$v_E(S) = \max \left\{ \int_T u_t(x_t) d\nu(t) \middle| x : T \to \mathbb{R}^k_+, x_\nu(S) = a_\nu(S) \right\},$$

where $y_\nu(S)$ abbreviates $\int_S y_t d\nu(t)$. The maximum is attained by [1, Proposition 36.1].

The core of an economy E is the set

$$Core(E) = \{\nu \in NA : \forall S \in \mathcal{C}\ \nu(S) \geq v_E(S), \nu(T) = v_E(T)\},$$

and this is a finite-dimensional set by [16, Eqaution (2.15), Corollary (2.16)]. An economy $E = (u, a, \nu) \in \mathfrak{E}(\nu)$ is *exact* if[11] $v_E(S) = \min_{\mu \in Core(E)} \mu(S)$. Let $\mathfrak{E}_{ex}(\nu)$ be the subset of $\mathfrak{E}(\nu)$ consisting of exact economies, and let $\mathfrak{E}_{ex} = \bigcup_{\nu \in NA^1} \mathfrak{E}_{ex}(\nu)$. By abuse of terminology we shall refer to a game v as an exact market game iff there is some $E \in \mathfrak{E}_{ex}$ with $v = v_E$. Denote by $\mathcal{EM}_+$ the set of exact market games.

A payoff selection on $\mathfrak{E}_{ex}$ is a mapping $\Psi : \mathfrak{E}_{ex} \to FA_+$, that satisfies axioms (1)-(4) stated below.

[11] The minimum exists as $Core(E)$ is compact.

Axiom 1 (Efficiency). *For every* $E \in \mathfrak{E}_{ex}$

$$\Psi(E)(T) = v_E(T).$$

Denote by Θ the set of all measurable automorphisms of $(T, \mathcal{C})$, namely, the set of bi-measurable bijections $\theta : T \to T$. For every economy $(u, a, \nu) \in \mathfrak{E}_{ex}$ and every $\theta \in \Theta$ define the economy $\theta E = (\theta u, \theta a, \theta \nu)$ where $(\theta u)_t = u_{\theta(t)}$ and $(\theta a)_t = a_{\theta(t)}$ for each $t \in T$. Notice that for every $S \in \mathcal{C}$ $v_{\theta E}(S) = v_E(\theta S) \doteq (\theta v_E)(S)$.

Axiom 2 (Anonimity). *For every* $v \in \mathfrak{E}_{ex}$ *and every* $\theta \in \Theta$

$$\Psi(\theta E) = \theta \Psi(E).$$

If $E = (u, a, \nu), E' = (u', a', \nu) \in \mathfrak{E}_{ex}(\nu)$, where $a : T \to R^k_+$, $a' : T \to R^{k'}_+$, and $\nu \in NA^1$ define an economy $E \oplus E' \in \mathfrak{E}$ by $(u \oplus u', a \oplus a', \nu)$, where $u \oplus u' : T \times R^{k+k'}_+ \to \mathbb{R}$ and $a \oplus a' : T \to R^{k+k'}_+$ are given by $(a \oplus a')_t = (a_t, a'_t)$ and $(u \oplus u')_t(x, y) = u_t(x) + u'_t(y)$. Observe that $v_{E \oplus E'} = v_E + v_{E'}$ and hence also that $v_{E \oplus E'} = \min_{\mu \in Core(v_{E \oplus E'})} \mu$. Hence $E \oplus E' \in \mathfrak{E}_{ex}$ and $\mathcal{EM}_+$ is a positive cone. The following axiom connects the payoff of the economies $E \oplus E'$ with that of the payoffs of $E, E' \in \mathfrak{E}_{ex}(\nu)$:

Axiom 3 (Separability). *For every* $\nu \in NA^1$, $E, E' \in \mathfrak{E}_{ex}(\nu)$, *and* $S \in \mathcal{C}$

$$\Psi(E \oplus E')(S) \geq \Psi(E)(S) + \Psi(E')(S).$$

Remark 2.1. Notice that by combining the efficiency and the separability axioms we obtain that for every $\nu \in NA^1$, $E, E' \in \mathfrak{E}_{ex}(\nu)$, and $S \in \mathcal{C}$

$$\Psi(E \oplus E')(S) = \Psi(E)(S) + \Psi(E')(S)$$

A game v is *monotonic* if for every $S \subseteq S' \in \mathcal{C}$, $v(S) \leq v(S')$.

Axiom 4 (Positivity). *If* $E, E' \in \mathfrak{E}_{ex}$ *and* $v_E - v_{E'}$ *is monotonic then*

$$\Psi(E) - \Psi(E') \in FA_+.$$

The set of all payoff selections on $\mathfrak{E}_{ex}$ is denoted by $PS(\mathfrak{E}_{ex})$. The existence of a payoff selection on $\mathfrak{E}_{ex}$ that satisfies axioms (1)-(4) is a direct corollary of Mertens [19] (see also Appendix B). The payoff selection constructed by Mertens [19] will be denoted by Ψ_M. Our main result is:

Theorem 2.2. $PS(\mathfrak{E}_{ex}) = \{\Psi_M\}$.

Remark 2.3. It is important to note that a payoff selection that satisfies axioms (1)-(4) may be viewed as a mapping on the space of exact market games, and that axioms (1)-(4) can be restated in the new setting in a straightforward way. Such a mapping is usually referred to as a *value* in the relevant literature (e.g., see [14]), and is closely related to the theory of values of nonatomic games (see [1]).

3 Preparations

3.1 Vector measure representation of games in $\mathcal{EM}_+$

For $k \geq 2$ denote by Δ^k the $k-1$ dimensional simplex in $\mathbb{R}^k_+$. Denote by EM^k_+ the positive cone generated by the functions $f_C : \mathbb{R}^k_+ \to \mathbb{R}$ given by $f_C(x) = \min_{c \in C} c \cdot x$ where $C \subseteq \Delta^k$ is compact and convex. Denote by HM^k_+ the positive cone generated by the functions $f_C : \mathbb{R}^k_+ \to \mathbb{R}$ where $C \subseteq \Delta^k$ is compact and strictly convex. Denote by EM^k and HM^k the linear spaces spanned by EM^k_+ and HM^k_+ respectively.

Lemma 3.1. *If $v \in \mathcal{EM}_+$ then $v = f \circ \mu$ with $f \in EM^k_+$ and $\mu \in \left(NA^1\right)^k$ for some $k \geq 2$.*

Proof. Suppose $v \in \mathcal{EM}_+$ and $v \neq 0$. As $Core(v)$ is finite-dimensional then $Core(v) = C \cdot \mu$ for some $\mu \in \left(NA^1\right)^k$, $C \subset \mathbb{R}^k_+$ is compact and convex, and $k \geq 2$. Notice that if $c \in C$ then $\sum_{i=1}^{k} c_i = v(I)$, hence

$$v(S) = \min_{\nu \in Core(v)} \nu(S) = v(I) \min_{c \in \frac{C}{v(I)}} c \cdot \mu \in \mathcal{EM}_+.$$

□

Denote by $\mathcal{HM}_+$ the positive cone generated by games of the form $f \circ \mu$ with $f \in HM^k_+$ and $\mu \in \left(NA^1\right)^k$ for some $k \geq 2$. Denote by $\mathcal{EM}$ and $\mathcal{HM}$ the linear space spanned by $\mathcal{EM}_+$ and $\mathcal{HM}_+$ respectively.

3.2 Directional derivatives of EM^k functions.

Given $f \in EM^k_+$, $x \in \mathbb{R}^k_{++}$, and $y \in \mathbb{R}^k$, the directional derivative $df(x, y)$ of f at x in the direction y is given by

$$df(x, y) = \lim_{\varepsilon \searrow 0} \frac{f(x + \varepsilon y) - f(x)}{\varepsilon}. \tag{3.1}$$

The limit exists for every $f \in EM^k_+$ by concavity. By linearity the definition may be extended to functions $f \in EM^k$.

If $f \in EM^k_+$ then for every $x \in \mathbb{R}^k_{++}$ the function $df(x, \cdot) : \mathbb{R}^k \to \mathbb{R}$ is concave. Thus the directional derivative of $df(x, \cdot)$ at $y \in \mathbb{R}^k$ in the direction $z \in \mathbb{R}^k$ that is given by

$$df(x, y, z) = \lim_{\varepsilon \searrow 0} \frac{df(x, y + \varepsilon z) - df(x, y)}{\varepsilon}, \tag{3.2}$$

exists, and by linearity the definition may be extended to functions $f \in EM^k$.

Denote by $\mathbf{1}_k \in \mathbb{R}^k$ the vector with coordinates $(\mathbf{1}_k)_i = 1$ for every $1 \leq i \leq k$, and let $D^k = \{t\mathbf{1}_k : t \in \mathbb{R}_+\}$. Notice that if $f \in EM^k$ then for every $x \in D^k \setminus \{0_k\}$, every $y, z \in \mathbb{R}^k$, every $a \in \mathbb{R}$, and every $b > 0$,

$$df(x, a\mathbf{1}_k + by, z) = df(\mathbf{1}_k, y, z). \tag{3.3}$$

Let $\chi : T \to \mathbb{R}^k$. For every $f \in EM^k$, $x \in \mathbb{R}^k_{++}$, and $y \in \mathbb{R}^k$ define $df(x, y, \chi) : T \to \mathbb{R}$ by

$$\forall s \in T,\ df(x, y, \chi)(s) = df(x, y, \chi(s)). \tag{3.4}$$

3.3 The direction space Ω_λ.

For every $\mu \in \left(NA^1\right)^k$ denote $\overline{\mu} = \frac{1}{k}\sum_{i=1}^{k} \mu_i$. For any $\lambda \in NA^1$ and $k \geq 1$ denote

$$\mathcal{Z}^k_\lambda = \left\{\mu \in \left(NA^1\right)^k : \overline{\mu} \ll \lambda,\ \frac{d\overline{\mu}}{d\lambda} \in L^\infty(\lambda)\right\}.$$

Denote $NA^* = \bigcup_{k=1}^{\infty} \left(NA^1\right)^k$, and $\mathcal{Z}^k_\lambda = \bigcup_{k=1}^{\infty} \mathcal{Z}^k_\lambda$. For $\mu \in NA^*$ with $\mu = (\mu_1, ..., \mu_m)$ denote $k(\mu) = m$. For any $k \geq 1$ and $x \in \mathbb{R}^k$ denote $\overline{x} = \frac{1}{k}\sum_{i=1}^{n} x_i$ and[12] $\Upsilon^k(x) = \frac{x - \overline{x}\mathbf{1}_k}{\|x - \overline{x}\mathbf{1}_k\|_2}$. Also denote $\mathbb{S}^k_\perp = \left\{\Upsilon^k(x) : x \in \mathbb{R}^k\right\}$. Let $B^1_+(T, \mathcal{C})$ be the set of bounded measurable functions $\chi : T \to \mathbb{R}$ with $0 \leq \chi \leq 1$. For any $y \in B^1_+(T, \mathcal{C})$ and $\mu \in NA^*$ denote $y(\mu) = \Upsilon^{k(\mu)}(\mu(y))$.

The mapping $y \mapsto (y(\mu))_{\mu \in \mathcal{Z}^*_\lambda}$ maps $B^1_+(T, \mathcal{C})$ onto $Y_\lambda \subseteq \prod_{\mu \in \mathcal{Z}^*_\lambda} \mathbb{S}^{k(\mu)}_\perp$. Denote the closure of Y_λ in $\prod_{\mu \in \mathcal{Z}^*_\lambda} \mathbb{S}^{k(\mu)}_\perp$ (w.r.t. the product topology) by Ω_λ. We refer to Ω_λ as the *direction space with perspective* λ. The following Lemmata are proved in Appendix A:

Lemma 3.2. *Let $\mu \in \mathcal{Z}^k_\lambda$ and $U : \mathbb{R}^k \to \mathbb{R}^m$ be a linear map with $U \circ \mu \in \mathcal{Z}^m_\lambda$. If $x \in \Omega_\lambda$ satisfies $U(x(\mu)) \notin D^m$ then*

$$x(U \circ \mu) = \left(\Upsilon^m \circ U\right)(x(\mu)). \tag{3.5}$$

Lemma 3.3. *Let $\mu \in \mathcal{Z}^{k+1}_\lambda$. For every $\alpha \in [0,1]$ and $1 \leq j \leq k$ denote $\mu^\alpha_j = (1-\alpha)\sum_{i=1}^{k} \mu_i e_i + \alpha(\mu_{k+1} - \mu_j)e_j \in \mathcal{Z}^k_\lambda$. If $x \in \Omega_\lambda$ satisfies $(x(\mu)_1, ..., x(\mu)_k) \notin D^k$, then*

$$x(\mu^\alpha_j) \underset{\alpha \to 0^+}{\longrightarrow} x(\mu^0_j). \tag{3.6}$$

3.4 The derived value on $\mathcal{EM}$

Lemma 3.4. *Every $\Psi \in PS(\mathfrak{E}_{ex})$ determines a value (see [1]) $\Phi : \mathcal{EM} \to FA$.*

[12] The convention $\frac{0_k}{0} = 0_k$ is used.

Proof. For $v \in \mathcal{EM}$ with $v = v_E - v_{E'}$ for some $E, E' \in \mathfrak{E}_{ex}$, let

$$\Phi(v) = \Psi(E) - \Psi(E').$$

Notice that for a choice of $E_0, E_0', E_1, E_1' \in \mathfrak{E}_{ex}$ with $v = v_{E_0} - v_{E_0'} = v_{E_1} - v_{E_1'}$ there is some $\nu \in NA^1$ with $E_0, E_0', E_1, E_1' \in \mathfrak{E}_{ex}(\nu)$. Therefore the mapping Φ is well defined by Remark 2.1. Linearity of Φ also follows from Remark 2.1, and efficiency and anonymity of Φ follow easily from the efficiency and anonymity axioms for Ψ, respectively. If $v \in \mathcal{EM}$ is monotonic and $v = v_E - v_{E'}$ for some $E, E' \in \mathfrak{E}_{ex}$, then by the positivity axiom for Ψ we obtain $\Phi(v) = \Psi(E) - \Psi(E') \in FA_+$, hence Φ is also positive, and therefore it is a value on $\mathcal{EM}$. □

We refer to Φ as the *derived value of* Ψ. We shall also consider, from now on, values on $\mathcal{EM}$. As we shall always mention the domain, and use suitable notation, no confusion should result.

3.5 Representations of values on $\mathcal{EM}$

Here we summarize some of the results in Edhan [13] that are needed for the proof of Theorem 2.2. The statement of these results require a somewhat deeper knowledge in functional analysis. The reader may refer to the functional analysis background appendix (Appendix D).

The following Theorem is a consequence of [13, Theorem 2.6]. It is proved in Appendix A:

Theorem 3.5. *Let Φ be a value on $\mathcal{EM}$. For every $\lambda \in NA^1$ there is a finitely additive and positive vector measure P_λ of bounded semi-variation (i.e., $|P_\lambda|(\Omega_\lambda) < \infty$. See Appendix D for details.) on the Borel sets of Ω_λ with values in $\mathcal{L}\left(L^\infty(\lambda), L^2(\lambda)\right)$ s.t. for every coalition $S \in \mathcal{C}$ the vector measure $P_\lambda^S = \langle P_\lambda, \chi_S \rangle$ is positive, regular, and countably additive of bounded variation, and for every $f \in EM^k$ and $\mu \in \mathcal{Z}_\lambda^k$ we have for every $S \in \mathcal{C}$*

$$\Phi(f \circ \mu)(S) = \int_{\Omega_\lambda} df\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\lambda}\right) dP_\lambda^S(x). \tag{3.7}$$

Furthermore, if $f \in HM^k$ then

$$\Phi(f \circ \mu)(S) = \int_S \left(\int_{\Omega_\lambda} df\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\lambda}\right) dP_\lambda(x)\right)(s) d\lambda(s). \tag{3.8}$$

For every $k \geq 1$, every nonempty $J \subseteq \{1, ..., k\}$, and every $x \in \mathbb{R}^k$, denote by $\pi_J^k(x) \in \mathbb{R}^{|J|}$ the projection of x onto the set of indices J. The following proposition summarizes some of the results of [13]:

Proposition 3.6. *If Φ is a value on $\mathcal{EM}$, then for every $\lambda \in NA^1$ the vector measure P_λ in Theorem 3.5 can be chosen s.t. the following properties hold:*

1. *If $\mu \in \mathcal{Z}_\lambda^k$ satisfies $\frac{d\mu}{d\lambda}(s) \notin D^k$ for λ-a.e. $s \in T$ then for every Borel set $E \subseteq \mathbb{S}_\perp^k$ we have (in the*

vector lattice $L^2(\lambda)$)

$$\langle 1, P_\lambda\rangle\left(\{x \in \Omega_\lambda : x(\mu) \in E\}\right) = \langle 1, P_\lambda\rangle\left(\{x \in \Omega_\lambda : x(\mu) \in -E\}\right); \tag{3.9}$$

2. *If $J \subseteq \{1, ..., k\}$ is nonempty, and $\mu \in \mathcal{Z}^k_\lambda$ satisfies $\frac{d(\pi^k_J \circ \mu)}{d\lambda}(s) \notin D^{|J|}$ for λ-a.e. $s \in T$ then (in the vector lattice $L^2(\lambda)$)*

$$\langle 1, P_\lambda\rangle\left(\left\{x \in \Omega_\lambda : \pi^k_J\left(x(\mu)\right) \in D^{|J|}\right\}\right) = 0; \tag{3.10}$$

3. *If $\mu \in \mathcal{Z}^k_\lambda$ satisfies $dim(AF(\mu)) \geq 2$ then (in the vector lattice $L^2(\lambda)$)*

$$\langle 1, P_\lambda\rangle\left(\{x \in \Omega_\lambda : x(\mu) = 0_k\}\right) = 0; \tag{3.11}$$

4. *If $\chi \in L^\infty(\lambda)$ then for every Borel set $E \subseteq \Omega_\lambda$ we have (in the vector lattice $L^2(\lambda)$)*

$$\langle \chi, P_\lambda\rangle\left(E\right) = \chi\langle 1, P_\lambda\rangle\left(E\right). \tag{3.12}$$

Proof. Given the choice of P_λ, property (1) is a consequence of [13, Lemma 5.5], property (2) is a consequence of [13, Corollary 5.6], property (3) is a consequence of [13, Remark 5.1], and property (4) is a consequence of [13, Remark 5.3]. □

4 The Proof of Theorem 2.2

This section is dedicated to the proof of Theorem 2.2. Before we begin, let us briefly describe the structure of the proof. For every payoff selection $\Psi \in PS(\mathfrak{E}_{ex})$ we consider its derived value Φ, which is a value on $\mathcal{EM}$ given by Lemma 3.4. We apply Theorem 3.5 and Proposition 3.6 to Φ and consider for every $\lambda \in NA^1$ the vector measure P_λ. We then study the Borel measures $\eta^S_{\lambda,\mu}$ on $\mathbb{S}^k_\perp$ given by

$$\eta^S_{\lambda,\mu}(E) = \langle 1, P^S_\lambda\rangle\left(\{x \in \Omega_\lambda : x(\mu) \in E\}\right) \tag{4.1}$$

for every $\lambda \in NA^1$, every $\mu \in \mathcal{Z}^k_\lambda$, every $S \in \mathcal{C}$, and every Borel set $E \subseteq \mathbb{S}^k_\perp$. We show that if $\lambda(S) > 0$ then the family $\mathfrak{M}^S_\lambda = \left\{\eta^S_{\lambda,\mu} : \mu \in \mathcal{Z}^*_\lambda\right\}$ induces an auxiliary family $\mathcal{Q}^S_\lambda = \left\{\kappa^S_{\lambda,\mu} : \mu \in \mathcal{Z}^*_\lambda\right\}$ of Borel probability measures on Euclidian spaces that in turn induces a conical set measure (see Appendix C for details) on $L^1(\lambda)$ with a certain invariance property w.r.t. members of the group $\Theta(\lambda)$, the subgroup of Θ of λ-preserving automorphisms. Theorem 2.2 is then deduced by the aid of Proposition C.2 (in the Appendix) and some of the properties described in Proposition 3.6.

4.1 The Measures $\kappa^S_{\lambda,\mu}$ and their properties.

From now on fix some $\lambda \in NA^1$ and $S \in \mathcal{C}$ with $\lambda(S) > 0$. For every $\mu \in \mathcal{Z}^k_\lambda$ the measure $\eta^S_{\lambda,\mu}$ given by Equation (4.1) may be considered as a Borel measure on $\mathbb{R}^k$ with total mass $\lambda(S)$ which is supported on $\mathbb{S}^k_\perp$. We shall, from now on, treat these measures as such and no confusion should thus result.

Denote[13] $NA^\lambda_0 = \left\{\mu \in NA : |\mu| \ll \lambda,\ \frac{d|\mu|}{d\lambda} \in L^\infty(\lambda),\ \mu(T) = 0\right\}$ and let $\mu \in \left(NA^\lambda_0\right)^k$. For each $1 \leq i \leq k$ with $\mu_i \neq 0$ let $\mu_i = \mu_i^+ - \mu_i^-$ be the Jordan decomposition of μ_i. This decomposition is uniquely determined. If $\mu_i = 0$ we arbitrarily define $\mu_i^+ = \mu_i^- = \lambda$. Let

$$\mu^* = \left(\frac{\mu_1^+}{\mu_1^+(T)}, ..., \frac{\mu_k^+}{\mu_k^+(T)}, \frac{\mu_1^-}{\mu_1^+(T)}, ..., \frac{\mu_k^-}{\mu_k^+(T)}, \lambda\right) \in \mathcal{Z}^{2k+1}_\lambda,$$

and define a linear mapping $T_\mu : \mathbb{R}^{2k+1} \to \mathbb{R}^k$ by $T_\mu(x) = \left(\mu_i^+(T)(x_i - x_{k+i})\right)_{i=1}^k$. Define a Borel probability measure $\kappa^S_{\lambda,\mu}$ on $\mathbb{R}^k$ by[14] $\kappa^S_{\lambda,\mu} = \frac{1}{\lambda(S)} T_\mu \circ \eta^S_{\lambda,\mu^*}$. For any measure P on $\mathbb{R}^k$ denote by ϕ_P its Fourier transform. Denote[15] by $\mathcal{B}^{con}(\mathbb{R}^k)$ the σ-algebra generated by open cones $C \subseteq \mathbb{R}^k$ based at 0_k. Denote by $\overline{\kappa}^S_{\lambda,\mu}$ the restriction of $\kappa^S_{\lambda,\mu}$ to $\mathcal{B}^{con}(\mathbb{R}^k)$.

Lemma 4.1. *The measure $\overline{\kappa}^S_{\lambda,\mu}$ is invariant under reflections, i.e., $\kappa^S_{\lambda,\mu}(E) = \kappa^S_{\lambda,\mu}(-E)$ for every $E \in \mathcal{B}^{con}(\mathbb{R}^k)$.*

Proof. If $dim(AF(\mu^*)) = 1$ then $\eta^S_{\lambda,\mu^*} = \lambda(S)\delta_{0_{2k+1}}$ and hence $\overline{\kappa}^S_{\lambda,\mu} = \delta_{0_k}$ which is invariant under reflections.

Suppose $dim(AF(\mu^*)) \geq 2$. Then by property (1) in Proposition 3.6 the measure η^S_{λ,μ^*} is invariant under reflections as for λ-a.e. $s \in T$ we have $\frac{d\mu^*}{d\lambda}(s) \notin D^{2k+1}$, and the Lemma follows by the definition of $\overline{\kappa}^S_{\lambda,\mu}$. □

Recall that for every $k \geq 1$, every nonempty $J \subseteq \{1, ..., k\}$, and every $x \in \mathbb{R}^k$ we denoted by $\pi^k_J(x)$ the projection of x onto the set of indices J, and let $\Upsilon^k_J(x) = \Upsilon^{|J|} \circ \pi^k_J(x)$.

Lemma 4.2. *For every $k \geq 1$, $\mu \in \left(NA^\lambda_0\right)^k$, and nonempty $J \subseteq \{1, ..., k\}$*

$$\overline{\kappa}^S_{\lambda,\pi^k_J \circ \mu} = \pi^k_J \circ \overline{\kappa}^S_{\lambda,\mu}.$$

Proof. First notice that $\pi^k_J \circ \mu \in \left(NA^\lambda_0\right)^{|J|}$, so $\overline{\kappa}^S_{\lambda,\pi^k_J \circ \mu}$ is well defined.

If $\pi^k_J \circ \mu = 0_{|J|}$ the Lemma is straightforward. Suppose otherwise. We claim that the Lemma follows if for every $m \geq 2$, $J' \subseteq \{1, ..., m\}$ with $|J'| \geq 2$, and $\nu \in \mathcal{Z}^m_\lambda$ satisfying $\frac{d(\pi^m_{J'} \circ \nu)}{d\lambda}(s) \notin D^{|J'|}$ for λ-a.e. $s \in T$, we have

$$\eta^S_{\lambda,\pi^m_{J'} \circ \nu} = \Upsilon^m_{J'} \circ \eta^S_{\lambda,\nu}. \tag{4.2}$$

[13] See appendix C for more details.
[14] Recall that by assumption $\lambda(S) > 0$.
[15] See appendix C for more details.

Indeed, in this case take $J' = \{1 \le j \le 2k : j \mod k \in J\} \cup \{2k+1\}$. Then $2k+1 \ge 2$, $|J'| \ge 2$, and $\frac{d(\pi^{2k+1}_{J'} \circ \mu^*)}{d\lambda}(s) \notin D^{|J'|}$ for λ-a.e. $s \in T$. Hence for every $E \in \mathcal{B}^{con}(\mathbb{R}^{|J|})$

$$\lambda(S)\kappa^S_{\lambda,\pi^k_J \circ \mu}(E) = \left(T_{\pi^k_J \circ \mu} \circ \eta^S_{\lambda,(\pi^k_J \circ \mu)^*}\right)(E) = \left(T_{\pi^k_J \circ \mu} \circ \eta^S_{\lambda,\pi^{2k+1}_{J'} \circ \mu^*}\right)(E) = \tag{4.3}$$

$$\left(T_{\pi^k_J \circ \mu} \circ \Upsilon^{2k+1}_{J'} \circ \eta^S_{\lambda,\mu^*}\right)(E) = \eta^S_{\lambda,\mu^*}\left((T_{\pi^k_J \circ \mu} \circ \Upsilon^{2k+1}_{J'})^{-1}(E)\right) = \tag{4.4}$$

$$\eta^S_{\lambda,\mu^*}\left((\pi^k_J \circ T_\mu)^{-1}(E)\right) = \left(\pi^k_J \circ T_\mu \circ \eta^S_{\lambda,\mu^*}\right)(E) = \lambda(S)\left(\pi^k_J \circ \kappa^S_{\lambda,\mu}\right)(E), \tag{4.5}$$

where the last equality in line (4.3) above follows from Equation (4.2), the last equality in line (4.4) follows from the fact that[16] $\left(T_{\pi^k_J \circ \mu} \circ \Upsilon^{2k+1}_{J'}\right)^{-1}(E) = \left(\pi^k_J \circ T_\mu\right)^{-1}(E)$ for every $E \in \mathcal{B}^{con}(\mathbb{R}^k)$, and the last equality in line (4.5) follows by the definition of $\kappa^S_{\lambda,\mu}$, and the Lemma follows.

We thus turn to prove Equation (4.2). Let $\nu \in \mathcal{Z}^m_\lambda$, and suppose $J' \subseteq \{1,...,m\}$ satisfies $|J'| \ge 2$. If $dim(AF(\pi^m_{J'} \circ \nu)) = 1$ then for every $x \in \Omega_\lambda$ we have $x(\pi^m_{J'} \circ \nu) = 0_{|J'|}$ and $\Upsilon^m_{J'}(x(\nu)) = 0_{|J'|}$. Thus

$$\eta^S_{\lambda,\pi^m_{J'} \circ \nu}(\{0_{|J'|}\}) = \langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : x(\pi^m_{J'} \circ \nu) = 0_{|J'|}\}) = \langle 1, P^S_\lambda\rangle(\Omega_\lambda) =$$
$$\langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : \Upsilon^m_{J'}(x(\nu)) = 0_{|J'|}\}) = \Upsilon^m_{J'} \circ \eta^S_{\lambda,\nu}(\{0_{|J'|}\}),$$

and therefore $\eta^S_{\lambda,\pi^m_{J'} \circ \nu} = \Upsilon^m_{J'} \circ \eta^S_{\lambda,\nu} = \lambda(S)\delta_{0_{|J'|}}$, and we are done.

Suppose $dim(AF(\pi^m_{J'} \circ \nu)) \ge 2$. By property (2) in Proposition 3.6 we have (as by our assumption $\frac{d(\pi^m_{J'} \circ \nu)}{d\lambda}(s) \notin D^{|J'|}$ for λ-a.e. $s \in T$)

$$\langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : \Upsilon^m_{J'}(x(\nu)) = 0_{|J'|}\}) = 0. \tag{4.6}$$

Thus for every Borel set $E \subseteq \mathbb{S}^{|J'|}_\perp$

$$\Upsilon^m_{J'} \circ \eta^S_{\lambda,\nu}(E) = \langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : \Upsilon^m_{J'}(x(\nu)) \in E\}) = \tag{4.7}$$
$$\langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : \Upsilon^m_{J'}(x(\nu)) \in E \setminus \{0_{|J'|}\}\}) = \tag{4.8}$$
$$\langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : x(\pi^m_{J'} \circ \nu) \in E \setminus \{0_{|J'|}\}\}) = \tag{4.9}$$
$$\langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : x(\pi^m_{J'} \circ \nu) \in E\}) = \eta^S_{\lambda,\pi^m_{J'} \circ \mu},$$

where the last equality in line (4.7) follows from Equation (4.6), the equality in line (4.8) follows from Lemma 3.2, and the equality in line (4.9) follows as[17] $\langle 1, P^S_\lambda\rangle(\{x \in \Omega_\lambda : x(\pi^m_{J'} \circ \nu) = 0_{|J'|}\}) = 0$. □

[16]Indeed if we order the elements of J as $i_1 < ... < i_{|J|}$ then

$$T_{\pi^k_J \circ \mu} \circ \Upsilon^{2k+1}_{J'}(x) = \frac{\left(\mu^+_{i_m}(T)(x_{i_m} - x_{i_m+k})\right)^{|J|}_{m=1}}{\left\|\pi^{2k+1}_{J'}(x) - \overline{\pi^{2k+1}_{J'}(x)}\mathbf{1}_{|J'|}\right\|_2} = \frac{\pi^k_J \circ T_\mu(x)}{\left\|\pi^{2k+1}_{J'}(x) - \overline{\pi^{2k+1}_{J'}(x)}\mathbf{1}_{|J'|}\right\|_2},$$

hence for $E \in \mathcal{B}^{con}(\mathbb{R}^{|J|})$ we have $T_{\pi^k_J \circ \mu} \circ \Upsilon^{2k}_{J'}(x) \in E \iff \pi^k_J \circ T_\mu(x) \in E$.

[17]since $dim(AF(\pi^m_{J'} \circ \mu)) \ge 2$. See property (3) in Proposition 3.6.

Lemma 4.3. *Suppose $\mu \in \left(NA_0^\lambda\right)^k$ is a vector of mutually singular measures and $\theta \in \Theta(\lambda)$. Then $\overline{\kappa}^S_{\lambda,\theta^*\mu} = \overline{\kappa}_{\lambda,\mu}$.*

Proof. It is sufficient to prove that if $\nu \in \mathcal{Z}^m_\lambda$ is a vector of mutually singular measures for some $m \geq 2$, then for every $\theta \in \Theta(\lambda)$

$$\eta^S_{\lambda,\theta\nu} = \eta^S_{\lambda,\nu}. \tag{4.10}$$

Indeed, denote $\pi_k = \pi^{2k+1}_{\{1,...,2k\}}$, and given a vector of mutually singular measures $\mu \in \left(NA_0^\lambda\right)^k$ let $A_\mu : \mathbb{R}^{2k} \to \mathbb{R}^k$ be given by $A_\mu(x) = \left(\mu_i^+(T)(x_i - x_{i+k})\right)_{i=1}^k$. Then $T_\mu = A_\mu \circ \pi_k$ and for every and every $t \in \mathbb{R}^k$

$$\lambda(S)\phi_{\kappa^S_{\lambda,\theta\mu}}(t) = \phi_{\eta^S_{\lambda,\theta\mu^*}}(T^*_{\theta\mu}(t)) = \phi_{\eta^S_{\lambda,\theta\mu^*}}(T^*_\mu(t)) = \tag{4.11}$$

$$\phi_{\eta^S_{\lambda,\theta(\pi_k\circ\mu^*)}}(A^*_\mu(t)) = \phi_{\eta^S_{\lambda,\pi_k\circ\mu^*}}(A^*_\mu(t)) = \phi_{\eta^S_{\lambda,\mu^*}}(T^*_\mu(t)) = \lambda(S)\phi_{\kappa^S_{\lambda,\mu}}(t), \tag{4.12}$$

where the second equality in line (4.11) follows as $T^*_{\theta\mu} = T^*_\mu$, the last equality in line (4.11) and the second equality in line (4.12) follow from Lemma 4.2, and the first equality in line (4.12) follows from Equation (4.10).

We turn to prove Equation (4.10). Let $m \geq 2$, let $f \in HM^m$, and let $\nu \in \mathcal{Z}^m_\lambda$ be a vector of mutually singular measures. Choose a function g in the equivalence class of $\frac{d\nu}{d\lambda}$. Let $S_{m+1} = S_{m+1}(g) = \{s \in T : g(s) = 0_m\}$, and let $\nu_{m+1} \in NA$ be the measure whose Radon-Nykodim derivative w.r.t. λ is the extended function $\frac{\chi_{S_{m+1}}}{\lambda(S_{m+1})}$. For $\alpha \in [0,1]$ denote $\nu^\alpha = \nu + \alpha \cdot sign(\lambda(S_{m+1}))(\nu_{m+1} - \nu_1)e_1$. By combining Theorem B.1 (in the Appendix) with Theorem 3.5 we obtain for every $R \in \mathcal{C}$ and $\alpha \in (0,1]$

$$\int_{\mathbb{S}^m_\perp} df(\mathbf{1}_m, z, \nu^\alpha(R))dQ^m(x) = \int_{\Omega_\lambda} df\left(\mathbf{1}_m, x(\nu^\alpha), \frac{d\nu^\alpha}{d\lambda}\right)dP^R_\lambda(x), \tag{4.13}$$

where Q^m is the measure given by Equation (B.2). Choose a measurable partition $S_1, ..., S_m$ of $T \setminus S_{m+1}$ s.t. for every $1 \leq i \leq m$ $\frac{d\nu_i}{d\lambda}$ is supported on S_i. For every $1 \leq i \leq m+1$ and $R \in \mathcal{C}$ with $R \subseteq S_i$ we thus obtain by combining Equation (4.13) with property (4) in Proposition 3.6

$$\int_{\mathbb{S}^m_\perp} df(\mathbf{1}_m, z, e_{i \bmod m})dQ_\nu(x)\nu_{i \bmod m}(R) = \tag{4.14}$$

$$\int_{\Omega_\lambda} df\left(\mathbf{1}_m, x(\nu^\alpha), e_{i \bmod m}\right) d\langle\frac{d\nu_{i \bmod m}}{d\lambda}, P^R_\lambda\rangle(x).$$

Taking $\alpha \to 0^+$ in Equation (4.14) we obtain[18] by the bounded convergence theorem D.2 (in the Appendix)

$$\int_{\mathbb{S}_\perp^m} df(\mathbf{1}_m, z, e_{i \bmod m}) dQ^m(x) \nu_{i \bmod m}(R) = \tag{4.15}$$
$$\int_{\{x:(x((\nu,\nu_{m+1}))_1,\ldots,x((\nu,\nu_{m+1}))_m)\notin D^m\}} df\left(\mathbf{1}_m, x(\nu), e_{i \bmod m}\right) d\langle \frac{d\nu_{i \bmod m}}{d\lambda}, P_\lambda^R\rangle(x)$$

for every $1 \le i \le m+1$ and every coalition $R \subseteq S_i$.

By [14, Proposition 3.8], for every $1 \le j \le m$ the set of functions $\{df(\mathbf{1}_m, \cdot, e_j) : f \in HM^m\}$ is dense in $C(\mathbb{S}_\perp^m \setminus \{0_m\})$. Hence for every $1 \le i \le m+1$ and every coalition $R \subseteq S_i$ we obtain from Equation (4.15)

$$\langle \frac{d\nu_{i \bmod m}}{d\lambda}, P_\lambda^R\rangle = \nu_{i \bmod m}(R) Q^m. \tag{4.16}$$

By passing to the Radon-Nykodim derivatives in Equation (4.16) and recalling the definition of the partition $S_1, ..., S_m, S_{m+1}$ of T we deduce that for every Borel set $E \subseteq \mathbb{S}_\perp^m$ we have for λ-a.e. $s \in T$

$$\langle 1, P_\lambda\rangle(\{x \in \Omega_\lambda : x(\nu) \in E\})(s) = Q^m(E). \tag{4.17}$$

Thus, integrating Equation (4.17) over $S \in \mathcal{C}$ yields

$$\langle 1, P_\lambda^S\rangle(\{x \in \Omega_\lambda : x(\nu) \in E\}) = Q^m(E)\lambda(S) \tag{4.18}$$

for every Borel set $E \subseteq \mathbb{S}_\perp^m$. Equation (4.10) now follows directly from the definition of the measure $\eta_{\lambda,\nu}^S$, and the Lemma follows. □

Lemma 4.4. *For every $\mu \in \mathcal{Z}_\lambda^k$ with $dim(AF(\mu)) \ge 2$ and every $E \in \mathcal{B}(\mathbb{S}_\perp^k)$*

$$\lambda(S)\Upsilon^k \circ \kappa_{\lambda,\mu-\overline{\mu}\mathbf{1}_k}^S(E) = \eta_{\lambda,\mu}^S(E).$$

Proof. Denote $\xi = \mu - \overline{\mu}\mathbf{1}_k$. For every Borel set $E \subseteq \mathbb{S}_\perp^k$

$$\lambda(S)\Upsilon^k \circ \kappa_{\lambda,\xi}^S(E) = \Upsilon^k \circ T_\xi \circ \eta_{\lambda,\xi^*}(E) = \tag{4.19}$$
$$\langle 1, P_\lambda^S\rangle\left(\left\{x :\in \Omega_\lambda : \Upsilon^k \circ T_\xi(x(\xi^*)) \in E\right\}\right) = \langle 1, P_\lambda^S\rangle\left(\{x :\in \Omega_\lambda : x(\mu) \in E\}\right) = \eta_{\lambda,\mu}^S(E), \tag{4.20}$$

where the equalities in line (4.19) follow by the definitions of the measures $\kappa_{\lambda,\xi}^S$ and $\eta_{\lambda,\mu}^S$ respectively, the first equality in line (4.20) follows by combining the fact that for each $x \in \Omega_\lambda$ with $T_\xi(x(\xi^*)) \ne 0_k$ we have $\Upsilon^k \circ T_\xi(x(\xi^*)) = x(\mu)$ with the fact that[19] $\langle 1, P_\lambda^S\rangle(\{x \in \Omega_\lambda : T_\xi(x(\xi^*)) = 0_k\} = 0$, and the last equality in line (4.20) follows by the definition of the measure $\eta_{\lambda,\mu}^S$. □

[18] Notice that as $m \ge 2$ we have $\langle 1, P_\lambda^R\rangle(\{x : (x(\nu,\nu_{m+1})_1, ..., x(\nu,\nu_{m+1})_m) \in D^m\}) = 0$ by property (2) in Proposition 3.6. Also notice that by combining Lemma 3.3 with the continuity, for every $1 \le i \le m$, of the function $df(\mathbf{1}_m, \cdot, e_i)$ on $\mathbb{S}_\perp^m$, we obtain $df(\mathbf{1}_k, x(\nu^\alpha), e_i) \underset{\alpha \to 0^+}{\longrightarrow} df(\mathbf{1}_m, x(\nu), e_i)$ for every $1 \le i \le m$ and $x \in \Omega_\lambda$ with $(x(\nu,\nu_{m+1})_1, ..., x(\nu,\nu_{m+1})_m) \in D^m$.

[19] As $dim(AF(\xi^*)) \ge 2$ we have by Lemma 4.2 $\eta_{\lambda,\xi^*}^S\left(\{x \in \mathbb{R}^{2k+1} : \forall 1 \le i \le k,\ x_i = x_{k+i}\}\right) = 0$ and hence $\langle 1, P_\lambda^S\rangle\left(\{x \in \Omega_\lambda : T_\xi(x(\xi^*)) = 0\}\right) = \eta_{\lambda,\xi^*}^S\left(\{x \in \mathbb{R}^{2k+1} : \forall 1 \le i \le k,\ x_i = x_{k+i}\}\right) = 0$.

The following Proposition summarizes the results of this subsection:

Proposition 4.5. *For every $S \in \mathcal{C}$ with $\lambda(S) > 0$ and $\mu \in \left(NA_0^\lambda\right)^k$ the probability measure $\overline{\kappa}^S_{\lambda,\mu}$ induced by restricting the probability measure $\kappa^S_{\lambda,\mu}$ to $(\mathbb{R}^k, \mathcal{B}^{con}(\mathbb{R}^k))$ satisfies the following conditions:*

1. *if $\mu \neq 0_k$ then $\overline{\kappa}^S_{\lambda,\mu}(\{0_k\}) = 0$ and if $\mu = 0_k$ then $\overline{\kappa}^S_{\lambda,\mu} = \delta_{0_k}$;*
2. *$\forall E \in \mathcal{B}^{con}(\mathbb{R}^k)$, $\overline{\kappa}^S_{\lambda,\mu}(E) = \overline{\kappa}^S_{\lambda,\mu}(-E)$;*
3. *$\forall J \subseteq \{1, ..., k\}$ with $J \neq \emptyset$, $\overline{\kappa}^S_{\lambda,\pi^k_J \circ \mu} = \pi^k_J \circ \overline{\kappa}^S_{\lambda,\mu}$;*
4. *if μ is a vector of mutually singular measures then for every $\theta \in \Theta(\lambda)$, $\overline{\kappa}^S_{\lambda,\theta\mu} = \overline{\kappa}^S_{\lambda,\mu}$; and,*
5. *if $\mu \in \mathcal{Z}^k_\lambda$ then for every $E \in \mathcal{B}(\mathbb{S}^k_\perp)$, $\lambda(S)\Upsilon^k \circ \kappa^S_{\lambda,\mu-\overline{\mu}\mathbf{1}_k}(E) = \eta^S_{\lambda,\mu}(E)$.*

4.2 The proof of Theorem 2.2

By combining properties (1)-(3) in Proposition 4.5 with Lemma C.1 (in the Appendix) we deduce that for every $S \in \mathcal{C}$ with $\lambda(S) > 0$ the set $\mathcal{Q}^S_\lambda = \left\{\overline{\kappa}^S_{\lambda,\mu} : \exists k \in \mathbb{N},\ \mu \in \left(NA_0^\lambda\right)^k\right\}$ induces a conical measure Q^S_λ on $L^1(\lambda)$, and for every $\mu \in \left(NA_0^\lambda\right)^k$ we have $Q^S_\lambda \circ \mu^{-1} = \overline{\kappa}^S_{\lambda,\mu}$ on $(\mathbb{R}^k, \mathcal{B}^{con}(\mathbb{R}^k))$. By combining that with property (4) in Proposition 4.5 and Proposition C.2 (in the Appendix), we deduce that the conical measure Q^S_λ is unique, and for every $\mu \in \left(NA_0^\lambda\right)^k$ we have $Q^S_\lambda \circ \mu^{-1} = \Upsilon^k \circ Q_\mu$ where Q_μ is the measure given by Equation B.1. Therefore by property (5) in Proposition 4.5 we obtain for every $S \in \mathcal{C}$ with $\lambda(S) > 0$

$$\eta^S_{\lambda,\mu} = \lambda(S) Q_\mu.$$

Therefore for every Borel set $E \subseteq \mathbb{S}^k_\perp$ we have for λ-a.e. $s \in T$

$$\langle 1, P_\lambda \rangle(\{x \in \Omega_\lambda : x(\mu) \in E\})(s) = Q_\mu(E). \tag{4.21}$$

To prove Theorem 2.2 it is now sufficient to verify that if $\Psi \in PS(\mathfrak{E}_{ex})$ then for every $k \geq 2$, every compact and convex $C \subseteq \Delta^k$, every $\mu \in \left(NA^1\right)^k$ with $dim(AF(\mu)) \geq 2$, every economy $E \in \mathfrak{E}_{ex}$ with $Core(E) = C \cdot \mu$ and every $S \in \mathcal{C}$, $\Psi(E)(S) = \Psi_M(E)(S)$. We first consider compact and strictly convex C. In this case

$$\Psi(E)(S) = \int_{\Omega_\lambda} df_C\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\lambda}\right) dP^S_\lambda(x) = \tag{4.22}$$

$$\int_S \left(\int_{\Omega_\lambda} df_C\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\lambda}\right) dP_\lambda(x)\right)(s) d\lambda(s) = \tag{4.23}$$

$$\sum_{i=1}^{k} \int_S \left(\int_{\Omega_\lambda} df_C\left(\mathbf{1}_k, x(\mu), e_i\right) \frac{d\mu_i}{d\lambda}(s) d\langle 1, P_\lambda\rangle(x)\right)(s) d\lambda(s) = \tag{4.24}$$

$$\sum_{i=1}^{k} \int_{\mathbb{S}^k_\perp} \left(\int_S df_C\left(\mathbf{1}_k, x, e_i\right) \frac{d\mu_i}{d\lambda}(s) d\lambda(s)\right) dQ_\mu(x) = \int_{\mathbb{S}^k_\perp} df_C\left(\mathbf{1}_k, x, \mu(S)\right) dQ_\mu(x) = \Psi_M(E)(S),$$

where the first equality in line (4.22) follows from Theorem 3.5, the last equality in line (4.22) follows by combining Equation (3.8) with property (4) in Proposition 3.6, the equality in line (4.23) follows as f_C is continuously differentiable on $\mathbb{R}^k_+ \setminus D^k$ and $\langle 1, P_\lambda\rangle(\{x : x(\mu) = 0_k\}) = 0$, and the equality in line (4.24) follows from Fubini's theorem. We thus also deduce that $\Psi = \Psi_M$ on $\mathcal{HM}$.

Let $C \subseteq \Delta^k$ be compact and convex. Let Φ_M the value derived from Ψ_M. For each $n \geq 1$ choose a compact and strictly convex $C_n \subseteq \Delta^k$ s.t.[20] $f_{C_n} \underset{n\to\infty}{\longrightarrow} f$ pointwise. Then for every $S \in \mathcal{C}$

$$\Psi(E)(S) = \int_{\Omega_{\overline{\mu}}} df_C\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP^S_{\overline{\mu}}(x) \leq \tag{4.25}$$

$$\int_{\Omega_{\overline{\mu}}} \liminf_{n\to\infty} df_{C_n}\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP^S_{\overline{\mu}}(x) \leq \liminf_{n\to\infty} \int_{\Omega_{\overline{\mu}}} df_{C_n}\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\overline{\mu}}\right) dP^S_{\overline{\mu}}(x) = \tag{4.26}$$

$$\liminf_{n\to\infty} \Phi(f_{C_n} \circ \mu)(S) = \liminf_{n\to\infty} \Phi_M(f_{C_n} \circ \mu)(S) = \Phi_M(f_C \circ \mu)(S) = \Psi_M(E)(S), \tag{4.27}$$

where the first equality in line (4.25) follows by Theorem 3.5, the last inequality in line (4.25) follows from [21, Theorem 24.5], the inequality in line (4.26) follows from Fatou's lemma D.3 (in the Appendix), the first equality in line (4.27) follows as $\Phi = \Phi_M$ on $\mathcal{HM}$, and the last equality in line (4.27) follows by combining formula (B.3) (in the Appendix) with the fact that the measure Q_μ is absolutely continuous w.r.t. the Haar measure on $\mathbb{S}^k_\perp$ and with the fact that by [21, Theorem 24.5] we have $df_{C_n}(\mathbf{1}_k, x, \mu(S)) \underset{n\to\infty}{\longrightarrow} df_C(\mathbf{1}_k, x, \mu(S))$ at every differentiability point $x \in \mathbb{S}^k_\perp$, i.e., almost everywhere. By the efficiency axiom we thus obtain for every $S \in \mathcal{C}$

$$\Psi(E)(S) = \Psi_M(E)(S),$$

and we are done.

[20]E.g., we may choose the sequence $(C_n)_{n=1}^\infty$ to converge to C in the Hausdorff metric.

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References

[1] Aumann, R.J., Shapley, L.S., 1974. *Values of Non-Atomic Games, Princton University Press.*

[2] Aumann, R.J., 1964. *Markets with a Continuum of Traders. Econometrica 32,* 39–50.

[3] Aumann, R.J., 1975. *Values of Markets with a Continuum of Traders. Econometrica 43,* 611–646.

[4] Brown, D.J. Robinson, A., 1975. *Nonstandard Exchange Economies. Econometrica 43,* 41–55.

[5] Bukhvalov, A.V, Veksler, A.I, Lozanovskii, G.Ya., 1979. *Banach Lattices - some Banach aspects of their theory. Russian Math. Surveys 34:2,* 159–212.

[6] Debru, G., Scarf, H.E., 1963. *A Limit Theorem on the Core of an Economy. Economic Review 4,* 235–269.

[7] Diestel, J., Uhl Jr., J.J., 1977. *Vector Measures, AMS Mathematical Surveys 15.*

[8] Dobrakov, I. 1970. *On integration in Banach spaces I. Czechoslovak Mathematical Journal, 20,* 511–536.

[9] Dubey, P., Neyman, A., 1984. *Payoffs in Nonatomic Economies: An Axiomatic Approach. Econometrica 52,* 1129–1150.

[10] Dubey, P., Neyman, A., 1997. *An Equivalence Principle for Perfectly Competitive Economies. J. Econ. Theory 75,* 314–344.

[11] Edgeworth, F.Y., 1881. *Mathematical Physics. London, Kegan Paul.*

[12] O. Edhan, *Continuous Values of Market Games are Conic. Discussion Paper 623 , The Centre for the Study of Rationality, Hebrew University of Jerusalem.*

[13] O. Edhan, *Representations of Positive Projections on Lipschitz Vector Measure Games. Discussion Paper 624, The Centre for the Study of Rationality, Hebrew University of Jerusalem.*

[14] Haimanko, O., 2002. *Payoffs in non-differentiable perfectly competitive TU economies. Journal of Economics Theory 106,* 17–39.

[15] Hart, S., Neyman, A., 1988. *Values of non-atomic vector measure games: Are they linear combinations of the measures? Journal of Mathematical Economics 17,* 31–40.

[16] Hart, S., 1977. *Values of Non-Differentiable Markets with a Continuum of Traders. Journal of Mathematical Economics 4,* 103-116.

[17] Kannai, Y., 1970. *Continuity Properties of the Core of a Market. Econometrica 38,* 791–815.

[18] Mertens, J.F., 1988. *The Shapley value in the non-differentiable case. International Journal of Game Theory 17,* 1–65.

[19] Mertens, J.F., 1988. *Nondifferentiable TU markets: The value, in: Roth, A.E. (Ed.), The Shapley Value: Essays in Honor of Lloyd S. Shapley, Cambridge Univ. Press, Cambridge, UK, Chap. 15.*

[20] Neyman, A., 2001 *Values of non-atomic vector measure games. Israel J. Math. 124,* 1-27.

[21] Rockafellar, R.T., 1970. *Convex Analysis, Princton University Press.*

[22] L.S. Shapley, 1953. *A value for n-person games, in: Kuhn, H.W, and Tucker, A.W. (Eds.), Contributions to the Theory of Games, Vol. II, Princeton University Press, Princeton NJ,* pp. 307–317.

[23] Shubik, M., 1959. *Edgeworth Market Games. Annals of Mathematics Studies 40,* 267–278.

A Proofs

Proof of Lemma 3.1. *For $x \in \Omega_\lambda$ with $U(x(\mu)) \notin D^m$, choose a net $\left(y^\beta\right)_{\beta\in B} \subseteq Y_\lambda$ converging to x. Notice that for any sufficiently large $\beta \in B$, $U(\mu(y^\beta)) \notin D^m$. Thus*

$$x(U \circ \mu) = \lim_{\beta\in B} \Upsilon^m \left((U \circ \mu)(y^\beta)\right) = \lim_{\beta\in B} \Upsilon^m \left(U(\mu(y^\beta)) - U(\overline{\mu}(y^\beta)\boldsymbol{1}_k)\right) = \tag{A.1}$$

$$\lim_{\beta\in B} \Upsilon^m \left(U(\mu(y^\beta) - \overline{\mu}(y^\beta)\boldsymbol{1}_k)\right) = \lim_{\beta\in B} \Upsilon^m \left(U \circ \Upsilon^k(\mu(y^\beta))\right) = \tag{A.2}$$

$$\Upsilon^m \left(U\left(\lim_{\beta\in B} \Upsilon^k(\mu(y^\beta))\right)\right) = \left(\Upsilon^m \circ U\right)(x(\mu)),$$

where the second equality in line (A.1) follows from the fact that $U(\boldsymbol{1}_k) = \boldsymbol{1}_m$ and the last equality in line (A.2) follows by combining the continuity of Υ^k on $\mathbb{S}^k_\perp \setminus \{0_k\}$ with the assumption $U(x(\mu)) \notin D^m$. □

Proof of Lemma 3.2. *Let $A^\alpha_j, A : \mathbb{R}^{k+1} \to \mathbb{R}^k$ be given by $A^\alpha_j(x) = (1-\alpha)(x_1, ..., x_k) + \alpha(x_{k+1} - x_j)e_j$, and $A(x) = \frac{k}{k+1}(x_1, ..., x_k) + \frac{1}{k+1}x_{k+1}\boldsymbol{1}_k$. Let $x \in U_\eta$.*

If $x \in \Omega_\lambda$ satisfies $(x(\mu))^0_j \notin D^k$ then by Lemma 3.2 for every small enough $\alpha \in (0,1)$ we have

$$x(\mu_j^\alpha) = \left(\Upsilon^k \circ A_j^\alpha\right)(x(\mu)), \tag{A.3}$$

and also

$$\left(\Upsilon^k \circ A\right)(x(\mu)) = x(A \circ \mu) \tag{A.4}$$

Thus,

$$\lim_{\alpha \to 0^+} x(\mu_j^\alpha) \overset{1}{=} \Upsilon^k \left(\lim_{\alpha \to 0^+} A_j^\alpha (x(\mu)) \right) = \Upsilon^k \left((x(\mu))^0_j \right) = \tag{A.5}$$

$$\Upsilon^k \left((x(\mu))^0_j - \overline{(x(\mu))^0_j} \boldsymbol{1}_k \right) = \Upsilon^k \left((x(\mu))^0_j + \frac{1}{k} x(\mu)_{k+1} \boldsymbol{1}_k \right) \overset{2}{=} \tag{A.6}$$

$$= x(A \circ \mu) = x(\mu_j^0),$$

where the first equality in line (A.5) follows by combining Equation (A.3) with the continuity of Υ^k outside of D^k, and the last equality in line (A.6) follows from Equation (A.4). □

Proof of Lemma 3.4. *Set $Q = \mathcal{EM}$ in [13, Theorem 2.6], and let $\widehat{\mathcal{EM}}_\lambda$ be generated by pairs (f,μ) with $f \in EM^k_+$ and $\mu \in \mathcal{Z}^k_\lambda$ for some $k \geq 2$. Let $\left\{\widehat{P}_\lambda\right\}_{\lambda \in NA^1}$ be the representing measures of Ψ w.r.t. $\mathcal{D}^{\widehat{\mathcal{EM}}}$, whose existence is proved in [13, Theorem 2.6]. For $x \in \prod_{\mu \in \mathcal{Z}^*_\lambda} \mathbb{R}^{k(\mu)}$ define $\Upsilon(x) = \left(\Upsilon^{k(\mu)}(x_\mu)\right)_{\mu \in \mathcal{Z}^*_\lambda}$. Choosing $P_\lambda = \Upsilon \circ \widehat{P}_\lambda$ for every $\lambda \in NA^1$ and recalling Equation (3.3) proves Equation (3.7). Equation (3.8) follows by combining [13, Theorem 2.1] with [13, Equation (2.2)].* □

B The Mertens Value on $\mathcal{EM}$

The construction of the Mertens value Ψ_M for market games is carried out in details in [19]. Here we briefly describe the form of the Mertens value on $\mathcal{EM}$.

The Cauchy distribution with parameter $\alpha > 0$ is the distribution on $\mathbb{R}$ with density $\frac{\alpha}{\pi(\alpha^2+x^2)}$. If X and Y are independent Cauchy random variables with parameters α and β respectively and $a, b \in \mathbb{R}$ s.t. $a^2 + b^2 \neq 0$ then $aX + bY$ is a Cauchy random variable with parameter $|a|\alpha + |b|\beta$. The characteristic function of the Cauchy distribution with parameter α is $\phi(t) = \exp(-\alpha|t|)$.

Recall that given a vector measure $\mu \in \left(NA^1\right)^k$ we defined $\overline{\mu} = \frac{1}{k}\sum_{i=1}^k \mu_i$. The μ *semi-norm* of $y \in \mathbb{R}^k$ is given by

$$\|y\|_\mu = \int |\sum_{i=1}^k (d\mu_i/d\overline{\mu}) y_i| d\overline{\mu}.$$

By [20, Lemma 1] the function $\phi_\mu : \mathbb{R}^k \longrightarrow \mathbb{R}$ given by $\phi_\mu(y) = exp(-\|y\|_\mu)$ is the characteristic function of a probability measure P_μ on $AF(\mu)$ - *the affine space generated by* $\mathcal{R}(\mu)$. By [20, Lemma 1], P_μ is absolutely continuous w.r.t. the Lebesgue measure on $AF(\mu)$, and its density ξ_μ is a $C_0(\mathbb{R}^k)$ function.

Recall that for every $k \geq 1$ and $x \in \mathbb{R}^k$ we denote[21] $\Upsilon^k(x) = \frac{x - \overline{x}\mathbf{1}_k}{\|x - \overline{x}\mathbf{1}_k\|_2}$, and $\mathbb{S}^k_\perp = \left\{\Upsilon^k(x) : x \in \mathbb{R}^k\right\}$. Given $\mu \in \left(NA^1\right)^k$ and the measure P_μ on $\mathbb{R}^k$, define a the probability measure Q_μ on $\mathbb{S}^k_\perp$ by

$$Q_\mu = \Upsilon^k \circ P_\mu. \tag{B.1}$$

If $\mu \in \left(NA^1\right)^k$ is a vector of mutually singular measures then the measure Q_μ is independent of the specific choice of the vector measure μ, and we shall denote

$$Q_\mu = Q^k. \tag{B.2}$$

The Mertens value on $\mathcal{EM}$ can now be given using a simple formula: If $f \in EM^k$, $\mu \in \left(NA^1\right)^k$, and $S \in \mathcal{C}$ then (See Mertens [19])

$$\Phi_M(f \circ \mu)(S) = \int df(\mathbf{1}_k, x, \mu(S))dQ_\mu(x). \tag{B.3}$$

For any economy $E \in \mathfrak{E}_{ex}$ there is $k \geq 2$, $C \subset \mathbb{R}^k_+$, and $\mu \in \left(NA^1\right)^k$ with $Core(E) = C \cdot \mu$. In this case $f_C \in EM^k_+$ and the Mertens value of the economy E is given by

$$\Psi_M(E) = \Phi_M(f_C \circ \mu). \tag{B.4}$$

Denote by $\mathcal{EM}_F$ the space of games generated by the games $f \circ \mu$ with $f \in EM^k$ and $\mu \in \left(NA^1\right)^k$ for some $k \geq 2$ s.t. μ is a vector of mutually singular measures. A close examination of the proof of the main result in Haimanko [14] leads to the following Theorem:

Theorem B.1. *If Φ is a value on $\mathcal{EM}$ then its restriction to $\mathcal{EM}_F$ is the Mertens value.*

Proof. Here explain how the Theorem follows from the proof of the main result in Haimanko [14]. Let Φ be a value on $\mathcal{EM}_F$. As in [14, Remark 4], it follows from the anonymity axiom that if $f \in EM^k$ and $\mu \in \left(NA^1\right)^k$ is a vector of mutually singular measures then

$$\Psi(f \circ \mu) = c^k_i(f)\mu_i,$$

where the coefficients $c^k_1(f), ..., c^k_k(f)$ are independent, for every $k \geq 2$, from the specific choice of the vector measure μ. Define, as in [14, Equation (20)], $g(f) : \mathbb{S}^k_\perp \setminus \{0_k\} \to \mathbb{R}^k$ by

$$g(f)_i(x) = df(\mathbf{1}_k, x, e_i)$$

[21] We use the convention $\frac{0_k}{0} = 0_k$.

for every $1 \le i \le k$. Let $V = \left\{g(f) : f \in EM^k\right\}$, and for $g \in V$ let $f(g) \in EM^k$ be the unique element s.t. $g(f(g)) = g$. Define for every $g \in V$ and $1 \le i \le k$

$$\psi_i^k(g) = c_i^k(f(g)).$$

Following the same reasoning that follows [14, Equation (21)], we deduce that ψ_i^k is a positive projection and thus there is a positive linear functional on $B(\mathbb{S}_\perp^k \setminus \{0_k\})$ that we shall denote by $\overline{\psi}_i^k$ s.t. $\psi_i^k(g) = \overline{\psi}_i^k(g_i)$ for every $g \in V$. By the Riesz representation theorem we have for every $g \in C(\mathbb{S}_\perp^k \setminus \{0_k\})$

$$\overline{\psi}_i^k(g) = \int_{\mathbb{S}_\perp^k \setminus \{0_k\}} g(x) d\lambda_i^k(x),$$

where λ_i^k is a Borel probability measure on $\mathbb{S}_\perp^k \setminus \{0_k\}$. Thus, for every $f \in HM^k$ we obtain in the same manner as in [14, Equation (24)]

$$\overline{\psi}_i^k(g) = \int_{\mathbb{S}_\perp^k \setminus \{0_k\}} df(\mathbf{1}_k, x, e_i) d\lambda_i^k(x). \tag{B.5}$$

Notice that the proofs of [14, Lemma 3.5], [14, Corollary 3.9], [14, Remark 7], and [14, Lemmata 3.10-3.13] remain valid. Thus it follows from [14, Lemma 3.14] that $\lambda_i^k = Q^k$. Since Q^k is absolutely continuous with respect to the Haar measure on $\mathbb{S}_\perp^k \setminus \{0_k\}$, and for every $f \in EM^k$ the function $g(f)$ is a.e. continuous on $\mathbb{S}_\perp^k$, then by [14, Lemma 3.5] Equation (B.5) holds for every $f \in EM^k$, and we are done. □

C Conical Measures on $L^1(\lambda)$

For every $k \ge 1$ we denote by $\mathcal{B}^{con}(\mathbb{R}^k)$ the σ-algebra generated by open cones $C \subseteq \mathbb{R}^k$ based at 0_k. For every $\lambda \in NA^1$ denote $NA_0^\lambda = \left\{\mu \in NA : \mu(T) = 0, |\mu| \ll \lambda, \frac{d\mu}{d\lambda} \in L^\infty(\lambda)\right\}$. A conical measure Q on $L^1(\lambda)$ is a finitely additive measure on conical sets[22] of $L^1(\lambda)$ s.t. for every $\mu \in \left(NA_0^\lambda\right)^k$ the measure $Q_\mu = Q \circ \mu^{-1}$ on $\mathcal{B}^{con}(\mathbb{R}^k)$ induces a countably additive measure on k-dimensional projective space. Recall that for every $k \ge 1$, every $x \in \mathbb{R}^k$, and every nonempty $J \subseteq \{1, ..., k\}$ we denote by $\pi_J^k(x) \in \mathbb{R}^{|J|}$ the projection of x on the set of indices J.

Lemma C.1. *Let $\mathcal{Q} = \left\{Q_\mu \in M^1(\mathbb{R}^k, \mathcal{B}^{con}(\mathbb{R}^k)) : k \ge 1, \mu \in \left(NA_0^\lambda\right)^k\right\}$ be a set of probability measures s.t.*

1. *for every $k \ge 1$ and nonzero $\mu \in \left(NA_0^\lambda\right)^k$ the measure Q_μ is invariant under reflections and $Q_\mu(\{0_k\}) = 0$; and*
2. *for every $k \ge 1$, every $\mu \in \left(NA_0^\lambda\right)^k$, and every nonempty $J \subseteq \{1, ..., k\}$ we have $Q_{\pi_J^k \circ \mu} = \pi_J^k \circ Q_\mu$.*

For every basic conical set $E(\mu, A) = \mu^{-1}(A)$ (where $\mu \in \left(NA_0^\lambda\right)^k$ and $A \in \mathcal{B}^{con}(\mathbb{R}^k)$) let $Q(E(\mu, A)) = Q_\mu(A)$. Then Q induces a conical measure on $L^1(\lambda)$.

[22] I.e., cylinder sets C s.t. $\chi \in C, a, b \in \mathbb{R} \Rightarrow a + b\chi \in C$.

Proof. The algebra of conical sets is generated by $\bigcup_{k=1}^{\infty}\left\{E(\mu,A):\mu\in\left(NA_0^{\lambda}\right)^k, A\in\mathcal{B}^{con}(\mathbb{R}^k)\right\}$. Thus in order to prove that Q induces a finitely additive measure on the algebra of conical sets it is sufficient to prove the consistency of Q, i.e., if $E(\mu,A)=E(\nu,B)$ for $\mu\in\left(NA_0^{\lambda}\right)^k$, $\nu\in\left(NA_0^{\lambda}\right)^{\ell}$, $A\in\mathcal{B}^{con}(\mathbb{R}^k)$, and $B\in\mathcal{B}^{con}(\mathbb{R}^{\ell})$ then $Q(E(\mu,A))=Q(E(\nu,B))$. Notice that

$$E(\mu,A)=E((\mu,\nu),A\times\mathbb{R}^{\ell}))=E((\mu,\nu),A\times B)=E((\mu,\nu),\mathbb{R}^k\times B)=E(\nu,B). \tag{C.1}$$

Thus

$$Q(E(\mu,A))=Q_{\mu}(A)=\left(\pi^{k+\ell}_{\{k+1,\dots,k+\ell\}}\circ Q_{(\mu,\nu)}\right)(A)=Q_{(\mu,\nu)}(A\times\mathbb{R}^{\ell})= \tag{C.2}$$

$$Q_{(\mu,\nu)}(A\times B)=Q_{(\mu,\nu)}(\mathbb{R}^{\ell}\times B)=\left(\pi^{k+\ell}_{\{1,\dots,k\}}\circ Q_{(\mu,\nu)}\right)(B)=Q_{\nu}(B)=Q(E(\nu,B)), \tag{C.3}$$

where the first equality in line (C.2) and third equality in line (C.3) follow from condition (2) of the Lemma and the last equality in line (C.2) and the first equality in line (C.3) follow from Equation (C.1). Thus the consistency of Q follows. By the definition of Q it follows that $Q\circ\mu^{-1}=Q_{\mu}$ on $\mathcal{B}^{con}(\mathbb{R}^k)$ is countably additive for every $k\geq 1$ and $\mu\in\left(NA_0^{\lambda}\right)^k$, and by property (1) it follows that $Q\circ\mu^{-1}$ induces a measure on k-dimensional projective space. Therefore Q is conical measure on $L^1(\lambda)$. ☐

Proposition C.2. *There is a unique conical measure Q on $L^1(\lambda)$ s.t. for every vector of mutually singular measures $\mu\in\left(NA_0^{\lambda}\right)^k$ and every $\theta\in\Theta(\lambda)$*

$$Q\circ\mu^{-1}=Q\circ(\theta\mu)^{-1}.$$

Furthermore, for every $\mu\in\mathcal{Z}_{\lambda}^k$ we have

$$Q\circ(\mu-\overline{\mu}\boldsymbol{1}_k)^{-1}=\Upsilon^k\circ Q_{\mu},$$

where Q_{μ} is the Borel probability measure on $\mathbb{S}_{\perp}^k$ given by Equation (B.3).

Proof. Choose a sequence $(\nu_n)_{n=1}^{\infty}\subseteq\mathcal{Z}_{\lambda}^1$ of mutually singular measures. Notice that for $i\neq j$, $\nu_i-\nu_j\neq 0$ with probability one, otherwise it would not induce a distribution on the zero-dimensional projective space.

For $n\geq 4$ we may thus consider the sequence $(f_n)_{n=4}^{\infty}$ of random variables given by $f_n(\chi)=\frac{(\nu_n-\nu_1)(\chi)}{(\nu_3-\nu_2)(\chi)}$. For any permutation π on the integers that fixes $n=1,2,3$, there is an automorphism $\theta_{\pi}\in\Theta(\lambda)$ s.t. $(\theta_{\pi}f_n)_{n=4}^{\infty}=\left(f_{\pi(n)}\right)_{n=4}^{\infty}$. As Q is a conical measure, we deduce by Kolmogorov's extension theorem that there is a unique probability distribution $P=P_{\nu_1,\nu_2,\dots}$ on the Borel sets of $\prod_{n=1}^{\infty}\mathbb{R}$ that extends the distribution of the finite sequences $f_4,\dots,f_n$ for every $n\geq 4$. Notice that we have also proved that the sequence $(f_n)_{n=4}^{\infty}$ is finitely exchangeable, hence by de-Finneti's theorem $(f_n)_{n=4}^{\infty}$ is, conditionally on the σ-algebra of tail events $\mathcal{F}_{\infty}$, independent and identically distributed (i.i.d.), with some distribution F. Notice that, by exchangeability, for any two different sequences $(\nu_n)_{n=1}^{\infty},(\nu'_n)_{n=1}^{\infty}\subseteq\mathcal{Z}_{\lambda}^1$ of mutually singular measures with $\nu_i = \nu'_i$ for $i = 1, 2, 3$, we have[23] $P_{\nu_1,\nu_2,...} = P_{\nu'_1,\nu'_2,...}$, hence $P = P_{\nu_1,\nu_2,\nu_3}$. Thus for a.e. (hence every) $\alpha \in [0, 1]$ the sequence $(f_4, ..., (1-\alpha)f_k + \alpha f_{k+1}, ...)$ has the same distribution as $(f_n)_{n=4}^{\infty}$. It follows that for every $\alpha \in [0, 1]$, conditionally on $\mathcal{F}_\infty$, the random variables $(1-\alpha)f_4 + (1-\alpha)f_5$, f_4, and f_5 have the same distribution, and as f_4, f_5 are i.i.d. they have a stable distribution of index 1. Hence, conditionally on $\mathcal{F}_\infty$, $(f_n)_{n=4}^{\infty}$ is distributed as $(m + \sigma U_n)_{n=4}^{\infty}$ with m, σ measurable w.r.t. $\mathcal{F}_\infty$ and $(U_n)_{n=4}^{\infty}$ are i.i.d. Cauchy distributions. Notice that $\sigma \neq 0$ with probability 1 (otherwise $\nu_4 = \nu_5 = ... = \nu_k$ for any $k \geq 4$ with some positive probability, a contradiction). Thus, conditionally on $\mathcal{F}_\infty$, $\frac{f_n - f_4}{f_6 - f_5} = \frac{U_n - U_4}{U_6 - U_5}$ for any $n \geq 7$. Hence the sequence $\left(\frac{f_n - f_4}{f_6 - f_5}\right)_{n=7}^{\infty}$ is distributed as $\left(\frac{U_n - U_4}{U_6 - U_5}\right)_{n=7}^{\infty}$, proving that P_{ν_4,ν_5,ν_6} is independent of the choice of ν_4, ν_5, ν_6, and therefore for every choice of ν_1, ν_2, ν_3 we have $P_{\nu_1,\nu_2,\nu_3} = P$ for some constant distribution P.

Any conical set is determined by the ratios of such a finite sequence - where the ν_i's are not necessarily mutually singular - but can be taken as linearly independent. Thus we need the distribution of $\left(\frac{\nu_i}{\nu}\right)_{i=1}^{n}$, with $\nu_1, ..., \nu_n, \nu \in NA_0^\lambda$, s.t. the measures $\nu_1, ..., \nu_n$ are linearly independent, and ν may be chosen as mutually singular of $\nu_1, ..., \nu_n$. These distributions are completely determined, via Fourier transforms, by the distribution of all the linear combinations $\sum_{i=1}^{n} t_i \frac{\nu_i}{\nu}$ with $t_1, ..., t_n \in \mathbb{R}$. Thus we need only to determine the distribution of $\frac{\mu}{\nu}$ with $\mu, \nu \in NA_0^\lambda$ mutually singular and non-zero. But this is uniquely determined[24] by the distribution of $\frac{\mu_1 - \mu_2}{\nu_1 - \nu_2}$ where $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{Z}_\lambda^1$ are mutually singular. □

D Rudiments of Functional Analysis

Here we give some functional analysis background that is essential for the understanding of the statement and proof of some of our results. For further reading, one is advised to use the references.

D.1 Banach Lattices

A *Banach lattice* Z is a Banach space that is also a lattice, whose lattice structure is commensurable with its Banach space topology, i.e., if $0 \leq x \leq y$ then $\|x\| \leq \|y\|$. A Banach lattice Z is a *K-space* if it is order complete, i.e., if every nonempty and bounded from above (below) set $A \subseteq Z$ has a least (greatest) upper (lower) bound.

Example: For every $1 < p \leq \infty$, every standard measure space $(I, \mathcal{C})$, and every $\lambda \in NA^1$ the space $L^p(\lambda)$ is a K-space. In fact, if X is a Banach lattice then X^* with its positive cone

$$X_+^* = \{x^* \in X^* : \forall x \in X_+, x^*(x) \geq 0\}$$

is a K-space (see [5, p. 162]), and hence every reflexive Banach lattice is a K-space.

[23] Choose a third sequence $(\nu''_n)_{n=1}^{\infty} \subseteq \mathcal{Z}_\lambda^1$, s.t. the arrangement of $(\nu_n)_{n=1}^{\infty}$ and $(\nu''_n)_{n=1}^{\infty}$ into a sequence $(\mu_n)_{n=1}^{\infty}$, and the arrangement of $(\nu'_n)_{n=1}^{\infty}$ and $(\nu''_n)_{n=1}^{\infty}$ into a sequence $(\mu'_n)_{n=1}^{\infty}$ form sequences of mutually singular measures. By exchangeability we deduce $P_{\mu_1,\mu_2,....} = P_{\mu'_1,\mu'_2,...} = P_{\nu_1,\nu_2,...} = P_{\nu'_1,\nu'_2,...}$.

[24] taking the Jordan decompositions of μ, ν.

D.2 Vector Measures

A function F from an algebra $\mathfrak{F}$ of subsets of a set Ω to a Banach space Z is called *finitely additive vector measure* or simply a *vector measure* iff whenever $E_1, E_2 \in \mathfrak{F}$ are disjoint then $F(E_1 \cup E_2) = F(E_1) + F(E_2)$. If, in addition, $F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} F(E_n)$ in the norm topology of Z for all sequences $(E_n)_{n=1}^{\infty}$ of pairwise disjoint members of $\mathfrak{F}$ s.t. $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{F}$ then F is termed a *countably additive vector measure* or simply *countably additive.*

The *strong variation of* F is the function $\|F\| : \mathfrak{F} \to \mathbb{R}$ defined by

$$\|F\|(E) = \sup_{\pi} \sum_{A \in \pi} \|F(A)\|,$$

where the supremum is taken over all finite partitions of E into disjoint members of $\mathfrak{F}$. One may easily check that $\|F\|$ is a monotonic finitely additive measure. A measure F is of *bounded variation* if $\|F\|(\Omega) < \infty$. Furthermore,

Proposition D.1. *[7, Proposition I.1.9] A vector measure of bounded variation is countably additive iff its variation is also countably additive.*

D.3 Operator Valued Integration

Let F be a vector measure on an algebra $\mathfrak{F}$ of subsets of Ω with values in the Banach space $\mathcal{L}(Y, Z)$ of bounded linear operators from Y to Z, where Y, Z are Banach lattices. Denote by $\mathcal{S}_{\Omega,\mathfrak{F}}(Y)$ the set of simple functions on Ω w.r.t. $\mathfrak{F}$ taking values in Y, i.e. the set of functions of the form $\sum_{i=1}^{n} a_i \chi_{E_i}$ where $E_i \in \mathfrak{F}$ and $a_i \in Y$ for every $1 \le i \le n$. The (Bartle) integral of the simple function $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ w.r.t. F is given by

$$\int f dF = \sum_{i=1}^{n} F(E_i)(a_i).$$

A measurable function $f : \Omega \to Y$ is *strongly F-integrable*, or *integrable* for short, iff for every increasing sequence $(f_n)_{n=1}^{\infty}$ of simple functions $f_n : \Omega \to Y$ with $f_n \underset{n\to\infty}{\longrightarrow} f$ pointwise $\|F\|$-a.e., the limit $\nu(E) = \lim_{n\to\infty} \int f_n \chi_E dF$ exists in the strong topology of Z for every $E \in \mathfrak{F}$ and is independent of the choice of $(f_n)_{n=1}^{\infty}$. In that case we denote

$$\int_E f dF = \lim_{n\to\infty} \int_E f_n dF.$$

The following theorem is a version of the well-known Bartle bounded convergence theorem:

Theorem D.2 (Bartle Bounded Convergence Theorem)**.** *Let $(f_n)_{n=1}^{\infty}$ be a uniformly bounded sequence of integrable functions $f_n : \Omega \to Y$, and suppose that F above is countably additive of bounded variation. If*

(f_n) converges $\|F\|$-a.e. to f then f is integrable and

$$\lim_{n\to\infty}\int f_n dF = \int f dF$$

in the strong topology of Z.

Proof. By Egorof-Lusin's theorem [8, p. 520] for every $\epsilon > 0$ there is a measurable subset $E = E(\epsilon) \subseteq \Omega$ s.t. $\|F(E^c)\| < \epsilon$ and (f_n) converges uniformly to f on E. Let $C > 0$ be s.t. $\sup\limits_{x\in\Omega}\|f_n(x)\| \leq C$ for every $n \in \mathbb{N}$. Note that

$$\left\|\int_E f_n dF\right\| \leq C\,\|F\|\,(E)$$

for every $E \in \mathfrak{F}$, where $\|F\|$ denotes the variation of F. Let $N \in \mathbb{N}$ be s.t. for every $m, n > N$ and every $x \in E$, $\|f_m(x) - f_n(x)\| < \epsilon$. Then for every $m, n > N$ we have

$$\left\|\int f_m dF - \int f_n dF\right\| \leq \left\|\int_E (f_m - f_n) dF\right\| + \left\|\int_{E^c} (f_m - f_n) dF\right\| <$$
$$\epsilon\,\|F\|\,(E) + 2C\,\|F\|\,(E^c).$$

As F is countably additive of finite variation we have $\|F\|\,(E(\epsilon)^c) \to 0$ as $\epsilon \to 0^+$, hence

$$\lim_{m,n\to\infty}\left\|\int f_m dF - \int f_n dF\right\| = 0, \tag{D.1}$$

proving that the integrals form a Cauchy sequence in Z and hence convergence in its strong topology. As for every sequence of increasing functions $(g_n)_{n=1}^{\infty}$ converging pointwise to f and $\epsilon > 0$ there is measurable subset E and $N \in \mathbb{N}$ s.t. $|f_n(x) - g_n(x)| < \epsilon$ for every $x \in E$ and $n \geq N$, and as $\|g_n(x)\| \leq \|f(x)\| \leq C$ for every x, we deduce in a similar manner that $\lim\limits_{n\to\infty}\int f_n dF = \lim\limits_{n\to\infty}\int g_n dF$, hence f is integrable, and the rest of the theorem now easily follows. □

We also prove the following version of Fatou's Lemma

Lemma D.3. *Suppose Y is a K-space. Let $(f_n)_{n=1}^{\infty}$ be a uniformly bounded sequence of integrable functions $f_n : \Omega \to Y$. Suppose that the vector measure F is positive, countably additive of finite variation, and $Z = \mathbb{R}$. Then*

$$\liminf_{n\to\infty}\int f_n dF \geq \int \liminf_{n\to\infty} f_n dF.$$

Proof. Denote $h_n = \inf\limits_{k\geq n} f_n$ and $h = \liminf\limits_{n\to\infty} f_n$. Then $(h_n)_{n=1}^{\infty}$ is uniformly bounded sequence of integrable functions and $h_n \underset{n\to\infty}{\longrightarrow} h$ pointwise, hence h is integrable. Thus

$$\forall n \geq 1, \quad \int_\Omega f_n(w)dF(w) \geq \int_\Omega h_n(w)dF(w) = \int_\Omega h(w)dF(w) + \int_\Omega (h_n - h)(w)dF(w) \Rightarrow \tag{D.2}$$

$$\liminf_{n\to\infty} \int_\Omega f_n(w)dF(w) \geq \int_\Omega h(w)dF(w), \tag{D.3}$$

where the first inequality in line (D.2) follows by the positivity of F and line (D.3) follows from line (D.2) as by the bounded convergence theorem D.2 $\int_\Omega (h_n - h)(w)dF(w) \underset{n\to\infty}{\longrightarrow} 0$. □

D.4 Representation of Bounded Linear Operators

Let Z, Y be Banach spaces, Ω a compact and Hausdorff space. If G is a measure on the Borel σ-algebra $\mathcal{B}_\Omega$ of Ω taking values in $\mathcal{L}(Y, Z^{**})$ then for every $z^* \in Z^*$ we define the measure $G_{z^*} : \mathcal{B}_\Omega \to Y^*$ by $G_{z^*}(A)(y) = \langle G(A)(y), z^* \rangle$ where $\langle \cdot, \cdot \rangle$ is the usual pairing. The *semi-variation* $|G|(E)$ of G on $E \in \mathcal{B}_\Omega$ is given by $|G|(E) = \sup\{\|G_{z^*}\|(E) : ||z^*|| \leq 1\}$.

Let $T : C(\Omega, Y) \to Z$ be a bounded linear operator. The following theorem, due to Dinculeanu and Singer, is a fortification of the Riesz representation theorem:

Theorem D.4 (Dinculeanu-Singer)**.** *[7, p. 182] There exists a unique finitely additive measure G of bounded semi-variation (i.e.- $|G|(\Omega) < \infty$), defined on $\mathcal{B}_\Omega$ with values in $\mathcal{L}(Y, Z^{**})$ s.t. $T(f) = \int_\Omega f(\omega)dG(\omega)$ and,*

(i) G_{z^*} is a regular and countably additive Borel measure for each $z^* \in Z^*$;

(ii) the mapping $z^* \mapsto G_{z^*}$ of Z^* into[2] $C(\Omega, Y)^*$ is weak* to weak* continuous;

(iii) $\langle T(f), z^* \rangle = \int_\Omega f(\omega)dG_{z^*}(\omega)$, for every $f \in C(\Omega, Y)$ and every $z^* \in Z^*$.

Remark D.5. Notice that if T is positive then its representing measure G is also positive. Indeed, for every $E \in \mathcal{B}_\Omega$ choose a sequence of continuous functions $(f_n)_{n=1}^\infty \subset C(\Omega, [0, 1])$ with $f_n \underset{n\to\infty}{\longrightarrow} \chi_E$ pointwise. Thus for every two positive elements $y \in Y$ and $z^* \in Z^*$ we have

$$\langle G(E)(y), z^* \rangle = \lim_{n\to\infty} \int_\Omega (f_n(\omega)y)\, dG_{z^*}(\omega) = \lim_{n\to\infty} \langle T(f_n y), z^* \rangle \geq 0, \tag{D.4}$$

where the first equality in line (D.4) follows by combining property (i) of Theorem D.4 with the bounded convergence theorem D.2 and the last inequality in that line follows from the positivity of T. Hence $G(E) : Y \to Z^{**}$ is a positive operator for every $E \in \mathcal{B}_\Omega$.

[2] This space isomorphic to the space of regular countably additive vector measures of bounded variation on $\mathcal{B}_\Omega$ taking values in Y^*.