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**CONTINUOUS VALUES OF MARKET
GAMES ARE CONIC**

By

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Continuous Values of Market Games are Conic*

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Abstract

We prove that every continuous value on a space of vector measure market games Q , containing the space of nonatomic measures NA , has the *conic property*, i.e., if a game $v \in Q$ coincides with a nonatomic measure ν on a conical diagonal neighborhood then $\varphi(v) = \nu$. We deduce that every continuous value on the linear space \mathcal{M} , spanned by all vector measure market games, is determined by its values on \mathcal{LM} - the space of vector measure market games which are Lipschitz functions of the measures.

1 Introduction

The value of a nonatomic game was introduced by Aumann and Shapley [1]. One of the basic properties shared by all the values discussed by Aumann and Shapley [1] is the diagonal property, which states that if a game vanishes on an entire diagonal neighborhood then its value also vanishes. Whether or not this property is a consequence of the value axioms was left as an open problem by Aumann and Shapley [1]. Neyman and Tauman [6] proved the existence of a nondiagonal value. Nevertheless, Neyman [5] proved that every continuous value is diagonal.

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In many cases it is helpful to completely determine a value on a space of games by its restriction to a certain (nondense) subspace. This idea was first introduced by Dubey [2] for the Shapley value of finite games. Another example is Monderer's [4] proof of the uniqueness of the value on pNA_∞ . In general, however, the uniqueness of the value on a space of games does not entail the uniqueness of the value on any of its subspaces or superspaces. Following this line of thought, Neyman [5] in fact proved that every continuous value on a symmetric space spanned by a space of nonatomic games Q and a subspace of $DIAG$ (the space of games of bounded variation vanishing on an entire diagonal neighborhood) is uniquely determined by its restriction to Q .

However, the diagonal property may prove to be useless in some cases. One such case is the space generated by nonatomic market games. The diagonal property in this case is meaningless, since a market game vanishes on an entire neighborhood of the diagonal iff it is the zero game. Nevertheless, a weaker property, namely, the *conic property*, has been proved efficient in the study of the uniqueness of the value on these spaces. A long standing problem in the theory of nonatomic games concerned the uniqueness of the value on the linear space spanned by differentiable market games. Dubey and Neyman [3] proved the uniqueness of a continuous value, the Aumann-Shapley value, on this space. A crucial step in their proof was the reduction of the problem to the study of the uniqueness of the value on the linear space generated by market games which coincide with a linear function of finitely many mutually singular measures on some conical diagonal neighborhood. A crucial step in this reduction was obtained by Dubey and Neyman's [3] proof of the conic property for continuous values: If a market game is a function of finitely many mutually singular measures and it coincides with some additive game ν on a conical diagonal neighborhood then its (continuous) value is ν . This raises a natural question - is the assumption of mutual singularity of the measures crucial for Dubey and Neyman's [3] result? A solution to this problem is an important step towards a solution of the age-old open problem of characterizing the (continuous) value on the domain of nonatomic vector measure market games.

In this paper we prove that every continuous value on a symmetric linear space of nonatomic vector measure market games which contains all nonatomic measures is conic, i.e., the value of every nonatomic vector measure market game which coincides with some nonatomic measure ν on a conical diagonal neighborhood is ν . In particular, this result implies that every continuous value on the linear space spanned by nonatomic vector measure market games is completely determined

by its values on the linear space spanned by market games which are Lipschitz functions of finitely many nonatomic probability measures.

2 Definitions and Statement of Results

Let (I, \mathcal{C}) be a measurable space isomorphic to $([0, 1], \mathcal{B})$, where \mathcal{B} is the Borel σ -algebra on $[0, 1]$. We shall call the members of I *players* and the members of \mathcal{C} *coalitions*. A *game* is a real valued function $v : \mathcal{C} \rightarrow \mathbb{R}$ s.t. $v(\emptyset) = 0$. A game v is *monotonic* iff $v(S) \leq v(T)$ whenever $S \subset T$. If Q is a set of games then Q^+ denotes the subset of monotonic games in Q , and Q^1 denotes the subset $\{v \in Q^+ : v(I) = 1\}$. A game v is of *bounded variation* iff it is the difference between two monotonic games. The space of all games of bounded variation will be denoted by BV . The *variation* of a game $v \in BV$ is the supremum of the variation of v over all increasing chains $S_0 \subset S_1 \subset \dots \subset S_m$ in \mathcal{C} , or equivalently,

$$\|v\| = \inf \{u(I) + w(I) : u, w \text{ are monotonic games s.t. } v = u - w\}. \quad (2.1)$$

The variation defines a norm on BV (see [1]). FA denotes the subspace of BV consisting of all finitely additive games, and NA denotes its subspace consisting of nonatomic and countably additive measures.

Let Θ denote the group of measurable automorphisms of (I, \mathcal{C}) . Each $\theta \in \Theta$ induces a linear mapping of BV onto itself by¹ $(\theta v)(S) = v(\theta S)$. A linear subspace $Q \subseteq BV$ is *symmetric* iff $\theta Q = Q$ for each $\theta \in \Theta$.

Let $Q \subseteq BV$ be a symmetric space. A map $\varphi : Q \rightarrow BV$ is *positive* iff $\varphi(Q^+) \subset BV^+$, *symmetric* iff $\theta\varphi = \varphi\theta$ for every $\theta \in \Theta$, and *efficient* iff $\varphi(v)(I) = v(I)$ for every $v \in Q$.

Definition 2.1. Let $Q \subseteq BV$ be a symmetric linear space. A *value* on Q is a symmetric, positive and efficient linear map $\varphi : Q \rightarrow FA$.

¹By abuse of notation.

For every $x \in \mathbb{R}^k$, $k \geq 2$, denote $\bar{x} = \frac{1}{k} \sum_{i=1}^k x_i$. For every $\delta > 0$, call the set

$$U_\delta^k = \{x \in \mathbb{R}_+^k : \forall 1 \leq i, j \leq k \ |x_i - x_j| \leq \delta k \bar{x}\} \quad (2.2)$$

a *conical diagonal neighborhood*. Define *CONIC* as the set of all $v \in BV$ satisfying:

- i. *There exist an integer $k \geq 2$, a vector measure $\xi \in (NA^1)^k$, a measure $\nu \in NA$, and a conical diagonal neighborhood U s.t. $v(S) = \nu(S)$ whenever $\xi(S) \in U$.*

Note that *CONIC* is a symmetric subspace of BV . Let Q be a symmetric subspace of BV and φ a value on Q . The pair (Q, φ) has the *conic property* and we say that φ is *conic* iff $\varphi(v) = \nu$ whenever $v \in Q \cap \text{CONIC}$, where $\nu \in NA$ satisfies $v(S) = \nu(S)$ whenever $\xi(S) \in U$, for some $\xi \in (NA^1)^k$ and a conical diagonal neighborhood $U \subset \mathbb{R}_+^k$, for some $k \geq 2$.

Define *CONIC'* as the set of all $v \in BV$ satisfying:

- ii. *There exist an integer $k \geq 2$, a vector measure $\xi \in (NA^1)^k$, and a conical diagonal neighborhood U s.t. $v(S) = 0$ whenever $\xi(S) \in U$.*

A space Q is *massive* iff $NA \subseteq Q$. If φ is a value on a massive space Q then $\varphi(v) = \nu$ for every $\nu \in NA$. In this case the pair (Q, φ) has the conic property iff $\varphi(v) = 0$ whenever $v \in Q \cap \text{CONIC}'$. Indeed, if (Q, φ) has the conic property and $v \in Q \cap \text{CONIC}'$ then by the conic property $\varphi(v) = 0$. On the other hand, suppose $\varphi(v) = 0$ whenever $v \in Q \cap \text{CONIC}'$ and let $w \in Q \cap \text{CONIC}$. Then there exist a positive integer $k \geq 2$, a vector measure $\xi \in (NA^1)^k$, a measure $\nu \in NA$, and a conical diagonal neighborhood $U \subseteq \mathbb{R}_+^k$ s.t. $w(S) = \nu(S)$ whenever $\xi(S) \in U$. Define $v = w - \nu$. Then $v \in Q \cap \text{CONIC}'$ and therefore $\varphi(v) = 0$. Thus $\varphi(w) = \varphi(v) = \nu$, and φ has the conic property. As we are dealing only with massive subspaces $Q \subseteq \mathcal{M}$, we shall from now on consider (Q, φ) as having the conic property iff $\varphi(v) = 0$ whenever $v \in Q \cap \text{CONIC}'$.

We call a concave, continuous, nondecreasing, and homogeneous of degree 1 function on \mathbb{R}_+^k a *market function*. Denote by M_+^k the cone of market functions on \mathbb{R}_+^k , and let M^k be the vector space of differences of functions in M_+^k . Denote by LM_+^k the set of *Lipschitz market functions*, i.e., the subset of M_+^k consisting of Lipschitz functions. Denote by LM^k the vector space of differences

of Lipschitz market functions. Denote by \mathcal{M} the space of games of the form $f \circ \mu$ where $f \in M^k$ and $\mu \in (NA^1)^k$ for some $k \geq 2$. We refer to \mathcal{M} as the *space of market games*. Denote by \mathcal{LM} the linear subspace of \mathcal{M} consisting of games of the form $f \circ \mu$ where $f \in LM^k$ and $\mu \in (NA^1)^k$ for some $k \geq 2$.

The main result of the present paper is the following:

Theorem 2.2. *Let $Q \subseteq \mathcal{M}$ be a symmetric, linear, and massive space. Then for every continuous value φ on Q , (Q, φ) has the conic property.*

The following is an immediate Corollary:

Corollary 2.3. *Every continuous value φ on \mathcal{M} is uniquely determined by its values on \mathcal{LM} , i.e., if φ' is a continuous value on \mathcal{M} with $\varphi' = \varphi$ on \mathcal{LM} then $\varphi' = \varphi$ on \mathcal{M}*

Proof. For every $k \geq 2$, every concave and monotonically nondecreasing $f \in M^k$ and every $n \in \mathbb{N}$ let $C_n = \{x \in \mathbb{R}^k : x \cdot \mathbf{1}_k = 1 + f(\mathbf{1}_k), \|x\| \leq n\}$ and consider the homogeneous of degree 1 functions f^n given on Δ^k by

$$f^n(x) = \min \{f(x), f_{C_n}(x)\}. \quad (2.3)$$

Being concave, f is Lipschitz on any subset of \mathbb{R}_+^k whose boundary does not intersect the boundary of \mathbb{R}_+^k . For n large enough, f^n coincides with f in such a set and it is Lipschitz continuous on its complement in \mathbb{R}_+^k . Therefore f^n is Lipschitz continuous on \mathbb{R}_+^k , homogeneous of degree 1, monotonically nondecreasing, and concave. Hence $f^n \circ \mu \in \mathcal{LM}$ for every $\mu \in (NA^1)^k$. Furthermore, f^n coincides with f on a conical diagonal neighborhood. By Theorem 2.2 we have

$$\varphi(f \circ \mu - f^n \circ \mu) = \varphi((f - f^n) \circ \mu) = 0, \quad (2.4)$$

and the corollary follows. □

3 Proof of Theorem 2.2

Let $Q \subseteq \mathcal{M}$ be a symmetric, linear, and massive space of games. Let $v \in Q$ and φ be a value on Q . Neyman [5] proved that there is a constant $K > 0$ s.t. for every $\tau \in \Theta$ and every $\ell \geq 1$ there exists a selection of signs $\epsilon_i = \pm 1$, $1 \leq i \leq \ell$, with

$$\frac{1}{\sqrt{\ell}} \left\| \sum_{i=1}^{\ell} \epsilon_i \tau^i \varphi v \right\| \geq K \|\varphi v\|. \quad (3.1)$$

We shall prove that if $v \in Q \cap CONIC'$ there is some $\theta \in \Theta$ s.t. for every $\ell \geq 1$ and every choice of constants a_i , $1 \leq i \leq \ell$, with $|a_i| \leq 1$ we have

$$\frac{1}{\sqrt{\ell}} \left\| \sum_{i=1}^{\ell} a_i \tau^i v \right\| = o(1), \quad (3.2)$$

as $\ell \rightarrow \infty$. If φ is a continuous value on Q then by equation (3.2)

$$\lim_{\ell \rightarrow \infty} \varphi \left(\frac{1}{\sqrt{\ell}} \left\| \sum_{i=1}^{\ell} a_i \tau^i v \right\| \right) = 0. \quad (3.3)$$

By combining that with equation (3.1) we obtain

$$\|\varphi(v)\| = 0, \quad (3.4)$$

which proves Theorem 2.2.

Remark 3.1. Let² $f' \circ \eta' \in Q \cap CONIC'$ where $\eta' \in (NA^1)^{m'}$ and $f' \in M^{m'}$. Choose $\epsilon' > 0$, and $\eta \in (NA^1)^m$, for some $m \geq 2$, s.t. $\eta(S) \in U_{\epsilon'}^m \Rightarrow f' \circ \eta'(S) = 0$. Let $k = m + m'$, let $\mu = (\eta', \eta)$, and let $f : \mathbb{R}^{m'} \times \mathbb{R}^m \rightarrow \mathbb{R}$ be given by $f(x, y) = f'(x)$. As $f \in M^k$ and $f \circ \mu = f' \circ \eta'$ we have $f \circ \mu \in Q$, and there is some $\epsilon > 0$ s.t. $f \circ \mu(S) = 0$ whenever $\mu(S) \in U_{\epsilon}^k$.

A sequence of finite measurable partitions $(\Pi_n)_{n=1}^{\infty}$ of I is *admissible* iff it is increasing and the σ -algebra generated by $\bigcup_{n=1}^{\infty} \Pi_n$ is \mathcal{C} . For every $1 \leq i \leq k$ we have $\phi_i = \frac{d\mu_i}{d\bar{\mu}} \in L^{\infty}(\bar{\mu})$. Thus, there exists an admissible sequence of partitions $(\Pi_n)_{n=1}^{\infty}$ s.t. for every $1 \leq i \leq k$ and $n \geq 1$ the

²If $v \in Q$ then we may always write $v = f' \circ \eta'$ with $f' \in M^{m'}$ and $\eta' \in (NA^1)^{m'}$ for some $m' \geq 2$

functions $\phi_i^n \in L^\infty(\bar{\mu})$ given by $\phi_i^n(s) = \frac{\mu_i(a)}{\bar{\mu}(a)}$ whenever $s \in a \in \Pi_n$ and $\bar{\mu}(a) > 0$ converge to ϕ_i in the $L^\infty(\bar{\mu})$ norm as $n \rightarrow \infty$. Fix one such an admissible sequence of partitions $(\Pi_n)_{n=1}^\infty$, and let μ_i^n be the measure whose Radon-Nikodym derivative w.r.t. $\bar{\mu}$ is³ ϕ_i^n . Notice that $\mu^n \in (NA^1)^k$ for every $n \geq 1$.

Remark 3.2. Notice that we may write (in $L^\infty(\bar{\mu})$)

$$\phi_i^n = \sum_{a \in \Pi_n, \bar{\mu}(a) > 0} \frac{\mu_i(a)}{\bar{\mu}(a)} \chi_a. \quad (3.5)$$

Lemma 3.3. *For every $n \geq 1$, there is an integer $p = p(n) \geq 1$, a vector ξ^n of p mutually singular NA^1 measures, and an affine transformation $A_n : [0, 1]^p \rightarrow \mathcal{R}(\mu)$ s.t. $\mu^n = A_n \circ \xi^n$.*

Proof. Let $J_n = \{a \in \Pi_n : \bar{\mu}(a) > 0\}$. For every $a \in J_n$ let $\xi_a^n \in NA^1$ be the measure whose Radon-Nikodym derivative w.r.t. $\bar{\mu}$ is $\frac{\chi_a}{\bar{\mu}(a)}$. By Remark 3.2 we have for every $1 \leq j \leq k$

$$\mu_j^n(S) = \sum_{a \in J_n} \frac{\mu_j(a)}{\bar{\mu}(a)} \bar{\mu}(a \cap S) = \sum_{a \in J_n} \mu_j(a) \xi_a^n(S). \quad (3.6)$$

Thus setting $p(n) = |J_n|$, $\xi^n = (\xi_a^n)_{a \in J_n}$, and choosing A_n as the $k \times p(n)$ matrix whose rows are the vectors $(\mu_j(a))_{a \in J_n}$ for $1 \leq j \leq k$ completes the proof. \square

Remark 3.4. Notice that for every $n \geq 1$ we have $\bar{\mu}^n = \bar{\mu}$. Indeed, by Remark 3.2 for every $1 \leq j \leq k$ we have⁴ (in $L^\infty(\bar{\mu})$) $\phi_j^n = \sum_{a \in J_n} \frac{\mu_j(a)}{\bar{\mu}(a)} \chi_a$. Thus (in $L^\infty(\bar{\mu})$)

$$\bar{\phi}^n = \frac{1}{k} \sum_{j=1}^k \phi_j^n = \frac{1}{k} \sum_{j=1}^k \sum_{a \in J_n} \mu_j(a) \frac{\chi_a}{\bar{\mu}(a)} = \sum_{a \in J_n} \chi_a \left(\sum_{j=1}^k \frac{\mu_j(a)}{k \bar{\mu}(a)} \right) = \sum_{a \in J_n} \chi_a = 1. \quad (3.7)$$

We can summarize the choice of $(\mu^n)_{n=1}^\infty$, Lemma 3.3, and Remark 3.4 as follows:

1. $\forall \epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ s.t. for every $n \geq N_\epsilon$ $\max_{1 \leq i \leq k} \|\phi_i - \phi_i^n\|_\infty < \frac{k\epsilon}{4}$;
2. $\forall n \geq 1$ there is an integer $p(n) \geq 1$, a vector measure ξ^n of $p(n)$ mutually singular probability measures, and an affine transformation $A_n : [0, 1]^{p(n)} \rightarrow \mathcal{R}(\mu)$ s.t. $\mu^n = A_n \circ \xi^n$; and,

³Obviously, ϕ_i^n depends on our specific choice of Π_n for every $n \geq 1$.

⁴The set J_n is defined as in Lemma 3.3

3. $\forall n \geq 1 \bar{\mu}^n = \bar{\mu}$.

We shall further assume, w.l.o.g., that $p(n)$ above is minimal, i.e., if ξ' is a vector of p' mutually singular probability measures and $A' : [0, 1]^{p'} \rightarrow \mathcal{R}(\mu)$ is an affine transformation s.t. $\mu^n = A' \circ \xi'$ then $p' \geq p(n)$.

For every $\alpha > 0$ and $\eta \in (NA^1)^k$ denote

$$D_\alpha(\eta) = \{S \in \mathcal{C} : \eta(S) \in U_\alpha^k\}. \quad (3.8)$$

We chose $\epsilon > 0$ s.t. $x \in U_\epsilon^k \Rightarrow f(x) = 0$. Choose N_ϵ as in property (1) above. For convenience denote $\nu = \mu^{N_\epsilon}$, $\xi = \xi^{N_\epsilon}$, $A = A_{N_\epsilon}$, and $p = p(N_\epsilon)$.

Lemma 3.5. $D_{\epsilon/2}(\nu) \subset D_\epsilon(\mu)$.

Proof. Suppose $S \in D_{\epsilon/2}(\nu)$. Then for every $1 \leq i, j \leq k$

$$|\nu_i(S) - \nu_j(S)| < k \frac{\epsilon}{2} \bar{\nu}(S) = k \frac{\epsilon}{2} \bar{\mu}(S). \quad (3.9)$$

By applying first the definition of ν and then Hölder's inequality we obtain for every $1 \leq i \leq k$

$$|\mu_i(S) - \nu_i(S)| = \left| \int_S (\phi_i - \phi_i^{N_\epsilon})(s) d\bar{\mu}(s) \right| \leq \|\phi_i - \phi_i^{N_\epsilon}\|_\infty \|\chi_S\|_1 \leq \frac{k\epsilon}{4} \bar{\mu}(S), \quad (3.10)$$

Thus, for every $1 \leq i, j \leq k$

$$\begin{aligned} |(\mu_j - \mu_i)(S)| &\leq |(\nu_j - \nu_i)(S)| + |(\nu_i - \mu_i)(S)| + |(\mu_j - \nu_j)(S)| \\ &\leq \frac{k\epsilon}{2} \bar{\mu}(S) + \frac{k\epsilon}{4} \bar{\mu}(S) + \frac{k\epsilon}{4} \bar{\mu}(S) = \epsilon k \bar{\mu}(S), \end{aligned} \quad (3.11)$$

hence $S \in D_\epsilon(\mu)$ and the lemma follows. \square

Lemma 3.6. $D_{\gamma\epsilon/4}(\xi) \subset D_{\epsilon/2}(\nu)$ for some $\gamma > 0$.

Proof. The matrix A has the form $(a_{im} | 1 \leq i \leq k, 1 \leq m \leq p)$, where $(a_{im})_{m=1}^p \in \Delta^p$ for every $1 \leq i \leq k$. For every $1 \leq m \leq p$ denote $\gamma_m = \sum_{i=1}^k a_{im}$. As ξ was chosen s.t. its length p is

minimal we have $\gamma_m > 0$ for every $1 \leq m \leq p$; indeed, if w.l.o.g. $\gamma_p = 0$, define $\xi' = (\xi_1, \dots, \xi_{p-1})$ and consider $A' = (a_{im} : i = 1, \dots, k, m = 1, \dots, p-1)$. Then $\nu = A' \circ \xi'$ which contradicts the minimality of p . Choose $\gamma = \min_{1 \leq m \leq p} \gamma_m > 0$.

Thus for every $S \in D_{\gamma\epsilon/4}(\xi)$ we obtain

$$|\nu_i(S) - \nu_j(S)| = \left| \sum_{m=1}^p (a_{im} - a_{jm}) \xi_m(S) \right| = \quad (3.12)$$

$$\begin{aligned} & \left| \sum_{m=1}^p a_{im} (\xi_m(S) - \|\xi(S)\|_\infty) + \sum_{m=1}^p a_{jm} (\|\xi(S)\|_\infty - \xi_m(S)) \right| \leq \\ & \sum_{m=1}^p a_{im} |\xi_m(S) - \|\xi(S)\|_\infty| + \sum_{m=1}^p a_{jm} |\|\xi(S)\|_\infty - \xi_m(S)| \leq \quad (3.13) \\ & \sum_{m=1}^p a_{im} p \frac{\gamma\epsilon}{4} \bar{\xi}(S) + \sum_{m=1}^p a_{jm} p \frac{\gamma\epsilon}{4} \bar{\xi}(S) = \frac{\gamma\epsilon}{2} p \bar{\xi}(S) = \frac{\epsilon}{2} \sum_{m=1}^p \gamma \xi_m(S) \leq \\ & \frac{\epsilon}{2} \sum_{m=1}^p \gamma_m \xi_m(S) = \frac{\epsilon}{2} \sum_{m=1}^p \sum_{i=1}^k a_{im} \xi_m(S) = \frac{\epsilon}{2} \sum_{i=1}^k \sum_{m=1}^p a_{im} \xi_m(S) = k \frac{\epsilon}{2} \bar{\nu}(S), \end{aligned}$$

where the last equality in line (3.12) follows from the fact that $\sum_{m=1}^p a_{im} = \sum_{m=1}^p a_{jm} = 1$ and the inequality in line (3.13) follows as $S \in D_{\gamma\epsilon/4}(\xi)$. Hence $S \in D_{\epsilon/2}(\nu)$ and the lemma follows. \square

For $h : \mathbb{R}_+^k \rightarrow \mathbb{R}$ and $b \in \mathbb{R}_+^k$ write

$$\|h\|_b = \sup \sum_{j=1}^t |h(x^j) - h(x^{j-1})|, \quad (3.14)$$

where the supremum is over all finite sequences of points $0 = x^0 \leq x^1 \leq \dots \leq x^t = b$.

By Remark 3.1 it is sufficient to prove Theorem 2.2 for games of the form $v = f \circ \mu$ s.t. $f \in M^k$ and there is $\epsilon > 0$ with $f(x) = 0$ whenever $x \in U_\epsilon^k$. Assume, w.l.o.g., that $\epsilon < 1$. Set $\beta = (1 - \frac{\epsilon}{4})$ and let $N > 0$ be the smallest positive integer s.t. $N^3 > \ell$. Let Ω be a finite increasing chain of coalitions. As $\|\cdot\|_\Omega$ is monotonic in Ω , we may assume w.l.o.g. that for every $0 \leq t \leq N$ there is a coalition $S^t \in \Omega$ s.t. $\bar{\xi}(S^t) = \beta^t \bar{\xi}(I) = \beta^t$ and $S^0 = I$. For $0 \leq t < N$ denote by Ω_t the subchain of all $S \in \Omega$ with $S^{t+1} \subset S \subset S^t$, and by Ω_N the subchain of all $S \in \Omega$ with $S \subset S^N$. Neyman

and Dubey proved⁵ that for every choice of constants $(a_i)_{i=1}^\ell$ with $|a_i| \leq 1$ and every $\theta \in \Theta$

$$\left\| \sum_{i < \ell} a_i \theta^i (f \circ \mu) \right\|_{\Omega} \leq \sum_{i < \ell} k \bar{\mu}(\theta^i S^N) \|f\|_{\mu(I)} + \sum_{t < N} k \|f\|_{\mu(I)} \sum_{i < \ell} \bar{\mu}(\theta^i S^t) I(\theta^i \Omega_t \not\subseteq D_\epsilon(\mu)), \quad (3.15)$$

where $I(\theta^i \Omega_t \not\subseteq D_\epsilon(\mu)) = 1$ if $\theta^i \Omega_t \not\subseteq D_\epsilon$ and 0 otherwise. By Lemma 3.5

$$I(\theta^i \Omega_t \not\subseteq D_\epsilon(\mu)) \leq I(\theta^i \Omega_t \not\subseteq D_{\epsilon/2}(\nu)) \quad (3.16)$$

for every $t < N$ and $i < \ell$. Combining equation (3.16) and Lemma 3.6 we obtain

$$I(\theta^i \Omega_t \not\subseteq D_\epsilon(\mu)) \leq I(\theta^i \Omega_t \not\subseteq D_{\gamma\epsilon/4}(\xi)) \quad (3.17)$$

for every $t < N$ and $i < \ell$.

Now, combining equations (3.15) and (3.17) we obtain

$$\begin{aligned} \left\| \sum_{i < \ell} a_i \theta^i (f \circ \mu) \right\|_{\Omega} &\leq \sum_{i < \ell} k \bar{\mu}(\theta^i S^N) \|f\|_{\mu(I)} + \\ &\sum_{t < N} k \|f\|_{\mu(I)} \sum_{i < \ell} \bar{\mu}(\theta^i S^t) I(\theta^i \Omega_t \not\subseteq D_{\gamma\epsilon/4}(\xi)). \end{aligned} \quad (3.18)$$

By [3, Lemma 5.5, equations (5.9)-(5.10)] we may choose a $\bar{\xi}$ -preserving $\theta \in \Theta$ s.t. for every $t < N$

$$\sum_{i < \ell} I(\theta^i \Omega_t \not\subseteq D_{\gamma\epsilon/4}(\xi)) \leq \frac{64(p-1)}{\gamma^2 \epsilon^2 \bar{\xi}(S^t)}. \quad (3.19)$$

Notice that for every $m \leq N$ and $i < \ell$

$$\bar{\mu}(\theta^i S^t) \stackrel{1}{=} \bar{\nu}(\theta^i S^t) = \overline{A \circ \xi}(\theta^i S^t) \stackrel{2}{\leq} p \bar{\xi}(\theta^i S^t) \stackrel{3}{=} p \bar{\xi}(S^t), \quad (3.20)$$

where equality (1) above follows by combining Remark 3.4 and the choice of ν , inequality (2)

⁵See [3, Lemma 5.5, equation (5.6)] and the discussion prior to [3, equation (5.7)] and [3, equation (5.8)]

follows as for each $T \in \mathcal{C}$ we have $\sum_{t=1}^k \sum_{j=1}^p a_{tj} \xi_j(T) = \sum_{j=1}^p \left(\sum_{t=1}^k a_{tj} \right) \xi_j(T) \leq k \sum_{j=1}^p \xi_j(T)$, and equality (3) follows from the assumption that θ preserves $\bar{\xi}$. In particular for every $i < \ell$

$$\bar{\mu}(\theta^i S^N) \leq p \bar{\xi}(S^N) = p \beta^N \bar{\xi}(I) = p \beta^N. \quad (3.21)$$

By combining equations (3.18)-(3.21) we deduce that

$$\left\| \sum_{i < \ell} a_i \theta^i (f \circ \mu) \right\|_{\Omega} \leq \ell k p \beta^N \|f\|_{\mu(I)} + \sum_{t < N} \frac{64 k p (p-1)}{\gamma^2 \epsilon^2} \|f\|_{\mu(I)} = \left(\ell \beta^N + N \frac{64 (p-1)}{\gamma^2 \epsilon^2} \right) k p \|f\|_{\mu(I)}. \quad (3.22)$$

As $\frac{1}{\sqrt{\ell}} \left(\ell \beta^N + N \frac{64 (p-1)}{\gamma^2 \epsilon^2} \right) \xrightarrow{\ell \rightarrow \infty} 0$, equation (3.2) is proved and we are done.

References

- [1] R.J. Aumann and L.S. Shapley, *Values of Non-Atomic Games*, Princeton University Press, 1974.
- [2] P. Dubey, *On the uniqueness of the shapley value*, International Journal of Game Theory **4** (1975), 131–139.
- [3] P. Dubey and A. Neyman, *Payoffs in nonatomic economies: An axiomatic approach*, Econometrica **52** (1984), 1129–1150.
- [4] D. Monderer, *A Milnor condition for non-atomic lipschitz games and its applications*, Math. Oper. Res. **15** (1990), 714–723.
- [5] A. Neyman, *Continuous values are diagonal*, Math. Oper. Res. **2** (1977), 338–342.
- [6] A. Neyman and Y. Tauman, *The existence of a non-diagonal axiomatic values*, Math. Oper. Res. **1** (1976), 246–250.