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COALITIONAL GAMES**

By

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SPECTRUM VALUE FOR COALITIONAL GAMES

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ABSTRACT. Assuming a ‘spectrum’ or ordering on the players of a coalitional game, as in a political spectrum in a parliamentary situation, we consider a variation of the Shapley value in which coalitions may only be formed if they are connected with respect to the spectrum. This results in a naturally asymmetric power index in which positioning along the spectrum is critical. We present both a characterisation of this value by means of properties and combinatoric formulae for calculating it. In simple majority games, the greatest power accrues to ‘moderate’ players who are located neither at the extremes of the spectrum nor in its centre. In supermajority games, power increasingly accrues towards the extremes, and in unanimity games all power is held by the players at the extreme of the spectrum.

1. INTRODUCTION

The Shapley value (Shapley, 1953) has for decades been one of the main indices used in the literature for measuring the relative power of players in voting situations. A voting situation is characterized by the set of agents participating in it and the subsets of this set that have enough power to pass a bill. This two elements together define a simple game. The Shapley-Shubik power index (Shapley and Shubik, 1954) is the restriction of the Shapley value to simple games. For a comprehensive overview on simple games and power indices see (Felsenthal and Machover, 1998). The Shapley-Shubik and related power indices are used to measure the power of the agents in main institutions around the world, for instance (Bilbao et al., 2002) study the implications of the enlargement of the European Union and (Alonso-Meijide and Bowles, 2005) study the power distribution in the International Monetary Fund. In these papers, however, the ideology of the agents is not taken into account which is of key importance in political situations.

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Consider for example a parliamentary situation in which there are n players with the same number of votes and a simple majority of them is required to form a government. A straightforward application of the Shapley value grants each player $1/n$, using symmetry considerations. In real-life parliaments, however, it is intuitively clear to all observers that not all members have equal power. It is highly unusual to see, for example an extreme right party joining an extreme left party in a coalition without any centre parties also included in the coalition to bridge political differences between them.

As the above discussion indicates, part of the problem is that the standard Shapley approach assumes that all possible permutations of the players be used in forming coalitions. That means that even highly unlikely coalitions, such as those formed by an extreme left party joining with an extreme right party while by-passing all the parties in between, including their most natural political allies, must be counted equally along every other coalition.

Different approaches have been proposed in the literature to study situation in which not all coalitions are feasible or equally likely. In many papers, the problem is tackled by considering some structure on the set of players to describe the way in which players can form coalitions. Coalitional games together with these kind of structures are usually denoted games with restricted cooperation. One of the most widely studied model of games with restricted cooperation is the one of games with restricted communication proposed by (Myerson, 1977). This approach considers besides the game itself an undirected graph that describes the communication possibilities among the players. A modification of the Shapley value is then proposed assuming that coalitions which are not connected by the graph are split into connected components. The main difference with respect to our approach is that our focus is on permutations, that is, on the way in which coalitions are formed instead of imposing the restrictions directly on the coalitions.

We propose here an intuitive way to modify the Shapley value by taking the political spectrum explicitly into account. The incorporation of the ideological positions of the agents for the study of the power distribution of a decision making body was first done in (Owen, 1972). In that work agents' political positions are given as points in a high dimensional Euclidean space and a probability distribution on the set of all permutations is inferred from them. Then, a modification of the Shapley value is proposed based on two properties, namely that an ordering and its reverse ordering should have the same probability and that the removal of a subset of agents should not affect the probabilities assigned to the relative orderings of the remaining agents.

As we show, our approach is much simpler, first because we can allow the positions to be one-dimensional and second, since we consider only a set of admissible permutations and assign the same probability to all of

them. With regard to the properties of the (Owen, 1972) value, the value introduced here shares the first of those properties but not the second one. (Shapley, 1977) also considered that the political positions of the agents should be taken into account and he proposed an asymmetric generalization of the Shapley value. This modification of the original Shapley value was also considered in (Owen and Shapley, 1989) to study the optimal ideological position of candidates. Our proposal is based on very similar ideas; however as is shown here, computing our value is much simpler and allows for a characterization of the value by means of a set of properties.

In this work we assume that there exists a spectrum, from ‘left to right’ according to which the players are ordered linearly. We then impose the condition that as coalitions are formed *a la* Shapley, they must be connected with respect to the spectrum. Hence, we propose a novel way to generalize the Shapley value to games with restricted cooperation in which the restrictions arise from the position of the agents in a one-dimensional spectrum. This leads to an interesting new value that may shed light on relative power values in situations in which there is a natural ordering of the players.

Perhaps the paper with the most similar general motivation to ours is (Gilles and Owen, 1999), in which an exogenously given hierarchy amongst the players is assumed (as opposed to the exogenously given spectrum as in our paper). A player in the (Gilles and Owen, 1999) model may join a coalition only if s/he received permission from one or more ‘supervisors’. As in this paper, this has the result of limiting the admissible coalitions that may be formed, thus affecting the value. The value in (Gilles and Owen, 1999), however, differs from the spectrum value because of the different assumptions regarding which coalitions are admissible.

Nowak and Tadeusz (1994) introduce a value called the solidarity value by adding a new postulate to the Shapley properties based on the average marginal contributions of the members of coalitions that may be formed. In its basic interpretation, if a coalition S is formed then the players who contribute more to S than the average marginal contribution members of S contribute to supporting the ‘weaker’ partners in S . Consideration of the solidarity value, however, does not involve any restriction on the admissible coalitions that can be formed, in contrast with the model used in this paper.

(Calvo and Gutierrez, 2011) start from similar suppositions to those in (Owen, 1977), namely that coalitional games are endowed with a coalitional structure, an exogenously given partition of the players. When coalitions are formed, the players interact at two levels: first, bargaining takes place among the unions and then bargaining takes place inside each union. (Calvo and Gutierrez, 2011) make use of the solidarity value in this model:

first, unions play a quotient game among themselves according to the Shapley value, and then the outcome obtained by the union is shared among its members by paying the solidarity value in the internal game. The coalitional value that they define is then applied in political situations to explain why government coalitions are sometimes larger than minimal winning coalitions.

2. SPECTRA AND VALUES

2.1. Definitions.

As usual, a *coalitional game* is a pair (N, v) where N is a finite set of players and v , the characteristic function, is a real valued function over the set of all coalitions, i.e., $v : 2^N = \{S : S \subseteq N\} \rightarrow \mathbb{R}$ with the convention that $v(\emptyset) = 0$. Denote the set of all coalitional games by \mathcal{G} . A *value on* \mathcal{G} , φ , is a map that associates to every coalitional game, (N, v) , a payoff vector, $\varphi(N, v) \in \mathbb{R}^N$. The cardinality of a finite set N will be denoted by $|N|$ or simply by a lower case n . We also denote the set of all permutations over a finite set N by Ω_N .

Definition 1. The *Shapley value*, ϕ^{Sh} , is the value on \mathcal{G} defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ by,

$$\phi_i^{Sh}(N, v) = \frac{1}{n!} \sum_{\omega \in \Omega_N} [v(P_i^\omega \cup \{i\}) - v(P_i^\omega)],$$

where P_i^ω stands for the set of predecessors of i at ω , i.e., $P_i^\omega = \{j \in N : \omega(j) < \omega(i)\}$.

For every bijection $\sigma : N \rightarrow \{1, \dots, |N|\}$, define a *spectrum on* N to be a strict total ordering \prec^σ of N as follows: for every $i, j \in N$, $i \prec^\sigma j$ if and only if $\sigma(i) < \sigma(j)$. The name spectrum is, of course, chosen because of the intention of modelling a political spectrum in a parliament, from left to right. For this reason, if $j \prec^\sigma i$, we will sometimes say that j comes before i in the spectrum ordering, or that j is to the left of i (and i is to the right of j).

A *coalitional game with a spectrum* will be formally denoted by (N, v, \prec^σ) , where $(N, v) \in \mathcal{G}$ and \prec^σ is a spectrum on N . Furthermore, we will usually assume that the relative positions of the players in N are compatible with the given spectrum, i.e., $N = \{1, \dots, n\}$ and for every $i \in N$, $\sigma(i) = i$. In this case for every $i, j \in N$, $i \prec^\sigma j$ if and only if $i < j$. Denote the set of all coalitional games with a spectrum by \mathcal{GS} . A *value on* \mathcal{GS} , φ , is a map that associates to every coalitional game with a spectrum, (N, v, \prec^σ) , a payoff vector, $\varphi(N, v, \prec^\sigma) \in \mathbb{R}^N$. A coalition $S \subseteq N$ is *connected* (with

respect to \prec^σ) if for all $i, j \in S$, $i \prec^\sigma k \prec^\sigma j$ implies that $k \in S$.¹ Denote the set of all connected coalitions with respect to \prec^σ by $C^{\prec^\sigma}(N)$ (or simply by $C(N)$ if \prec^σ is obvious from the context). Given a coalition $S \subseteq N$, we can always identify the player at the left end $\min(S)$, who is the player $i \in N$ such that $i \prec^\sigma k$ for all $k \in S \setminus \{i\}$, and similarly, the player at the right end $\max(S)$, who is the player $j \in N$ such that $k \prec^\sigma j$ for all $k \in S \setminus \{j\}$. Every connected coalition $S \in C(N)$ with $|S| > 1$ contains two distinguished ‘extreme’ players $i = \min(S)$ and $j = \max(S)$, and all the players in between. Hence, the two ‘extreme’ players characterize the coalition. We will therefore sometimes write $[i \dots j]$ to denote the connected coalition of all the players between i and j , inclusive, with the understanding that if $i > j$ then $[i \dots j] := \emptyset$.

Definition 2. For every coalitional game with an spectrum, $(N, v, \prec^\sigma) \in \mathcal{GS}$, the associated *connected-reduced game*, $(N, v^{\prec^\sigma}) \in \mathcal{G}$ is defined for every $S \subseteq N$ by

$$v^{\prec^\sigma}(S) = \begin{cases} v(S) & \text{if } S \text{ is connected with respect to } \prec^\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Associating a coalitional game to every coalitional game with a spectrum allows one to define a value on \mathcal{GS} *à la* Myerson, i.e., defining the value that is obtained by applying the Shapley value to the connected-reduced game. Note, however, that the connected-reduced game of a monotone game may not be monotone; this may lead to negative payoffs in monotone games. We follow a different approach here that does not suffer from this problem.

A permutation $\omega \in \Omega_N$ is *admissible* (with respect to a spectrum \prec^σ) if P_i^ω is connected (with respect to \prec^σ) for all $i \in N$. Denote the set of all admissible permutations with respect to \prec^σ by Ω_{\prec^σ} (or simply by Ω if \prec^σ is obvious by context).

We may interpret this in the spirit of one of the interpretations of the Shapley value. Regard each permutation as an ordered queue of the players, who enter a room one by one according to their number in the queue. Each permutation thus defines a dynamic way of forming a coalition, which grows by one player at a time, thus enabling us to measure the contribution of each player to the coalition formed by the players who preceded him or her in entering the room. A connected admissible permutation, as defined here, relates this queue to the spectrum ordering, in the sense that each player that enters the room must be the player who is immediately to the left or immediately to the right of the set of predecessors. The coalitions

¹ The empty set and singleton coalitions vacuously satisfy the condition of being connected.

that are thus dynamically formed are always connected, as defined above. Therefore, Ω_{\prec^σ} represents the set of all possible ways of dynamically forming the grand coalition while ensuring that the sub-coalitions formed in each step are connected.

Furthermore, for every $i \in N$ denote the set of coalitions which can be expressed as sets of predecessors of i at a connected admissible permutation by $C_i(N)$, i.e., $C_i(N) = \{P_i^\omega : \omega \in \Omega\}$. Note that we can write $C_i(N)$ as a disjoint union of the empty set, the set of connected coalitions ‘immediately to the left of’ player i , and the set of connected coalitions ‘immediately to the right of’ player i . Formally, $C_i(N) = L_i(N) \cup R_i(N) \cup \emptyset$ where $L_i(N) = \{[k \dots i - 1] : 1 \leq k < i\}$ and $R_i(N) = \{[i + 1 \dots k] : i < k \leq n\}$.

2.2. The Spectrum Value. We are now in the position to introduce the formal definition of the spectrum value for games with a spectrum

Definition 3. The *spectrum value*, ϕ , is the value on \mathcal{GS} defined for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and $i \in N$ by

$$\phi_i(N, v, \prec^\sigma) := \frac{1}{|\Omega_{\prec^\sigma}|} \sum_{\omega \in \Omega_{\prec^\sigma}} [v(P_i^\omega \cup i) - v(P_i^\omega)]$$

When N , \prec^σ and v are clear from context, we will sometimes save space by writing $\phi_i(v)$ or simply ϕ_i instead of $\phi_i(N, v, \prec^\sigma)$.

2.3. Combinatoric Analysis of the Spectrum Value.

The spectrum value lies within several families of values studied in the literature. Before proceeding, we look at some combinatorial calculations that are tedious but straightforward to calculate. It is instructive to compare them to their standard analogues.

$$(1) \quad |C(N)| = \frac{n(n+1)}{2} + 1$$

(in contrast, the cardinality of the set of all coalitions is 2^n);

$$(2) \quad |\Omega_{\prec^\sigma}| = 2^{n-1}$$

(in contrast, the cardinality of Ω_N is $n!$); finally, when k is the k -th player in the ordering given by \prec^σ ,

$$(3) \quad |\Omega_k| = \binom{n-1}{k-1}$$

where $\binom{n}{k}$ is the standard binomial coefficient. Hence, we may rewrite the spectrum value as follows

$$(4) \quad \phi_i(N, v, \prec^\sigma) := \frac{1}{2^{n-1}} \sum_{\omega \in \Omega_{\prec^\sigma}} [v(P_i^\omega \cup i) - v(P_i^\omega)]$$

For an arbitrary set A , denote by $\Delta(A)$ the set of all probability distributions on A . Denote by E_p the expectation operator with respect to $p \in \Delta(M)$. Then, a value φ is a *random order value* (Weber, 1988) if for every $(N, v) \in \mathcal{G}$ and $i \in N$ there is a probability distribution $b \in \Delta(\Omega_N)$ such that,

$$\varphi_i(N, v) = E_b(v(P_i^\omega \cup \{i\}) - v(P_i^\omega)) = \sum_{\omega \in \Omega_N} b(\omega) [v(P_i^\omega \cup \{i\}) - v(P_i^\omega)].$$

It is well known that the standard Shapley value is a random order value for the uniform distribution over all permutations of Ω_N . In analogy, from Definition 3, it is easy to check that the spectrum value is also a random order value for the uniform distribution over the set of connected admissible permutations Ω_{\prec^σ} (which gives zero probability to $\Omega_N \setminus \Omega_{\prec^\sigma}$).

Recall that for any finite set N , 2^N stands for the set of all subsets of N . Then, a value ϕ is known as a *probabilistic value* (Weber, 1988) if for every $(N, v) \in \mathcal{G}$ and $i \in N$ there is a probability distribution $p^i \in \Delta(2^{N \setminus \{i\}})$ such that,

$$\phi_i(N, v) = E_{p^i}(v(S \cup \{i\}) - v(S)) = \sum_{S \in 2^{N \setminus \{i\}}} p^i(S) [v(S \cup \{i\}) - v(S)].$$

The Shapely value is the probabilistic value defined by the probability distributions $p^i \in \Delta(2^{N \setminus \{i\}})$ such that for every $S \subseteq N \setminus \{i\}$, $p^i(S) = 1/(n \binom{n-1}{|S|})$. The spectrum value can similarly be expressed as a probabilistic value.

Claim 1.

$$(5) \quad \phi_i(N, v, \prec^\sigma) = \frac{1}{2^{n-1}} \left\{ \binom{n-1}{i-1} v(\{i\}) + \sum_{S \in L_i(N)} 2^{|S|-1} \binom{n - (|S| + 1)}{\min(S) - 1} [v(S \cup \{i\}) - v(S)] + \sum_{T \in R_i(N)} 2^{|T|-1} \binom{n - (|T| + 1)}{i-1} [v(T \cup \{i\}) - v(T)] \right\}$$

Proof. For each player i , the only relevant coalitions for calculating $\phi_i(N, \prec^\sigma, v)$ are the connected coalitions in $C_i(N)$. That is, coalitions immediately to i 's left and immediately to i 's right, along with the empty set. Hence, we only need to count how many times each coalition is in $L_i(N) \cup R_i(N) \cup \emptyset$ appearing in the expression of Eq. (4).

Let $S \in L_i(N)$. By Equation (2) the number of different ways in which the coalition S can be formed in a connected manner under an admissible permutation is $2^{|S|-1}$. Next, we only need to count the number of admissible permutations of $N \setminus S$ that start with i . By Equation (3) it is easy to check that this number is precisely $\binom{n-|S|-1}{\min(S)-1}$.

The proof for the connected coalitions immediately to i 's right and the empty set follows similar lines. ■

Claim 2.

$$\begin{aligned}
 \phi^i(N, v, \prec^\sigma) &= \frac{1}{2^{n-1}} \left\{ \binom{n-1}{i-1} v(\{i\}) \right. \\
 (6) \quad &+ \sum_{k < i} 2^{i-k-1} \binom{n-1-(i-k)}{k-1} [v([k \dots i]) - v([k \dots i-1])] \\
 &\left. + \sum_{k > i} 2^{k-i-1} \binom{n-1-(k-i)}{i-1} [v([i \dots k]) - v([i+1 \dots k])] \right\}
 \end{aligned}$$

Proof. This is the previous claim re-written in terms of the identities of the players as determined by the spectrum ordering. ■

3. MAJORITY GAMES

Next, we illustrate the Spectrum value analysing the payoffs it prescribes in simple voting situations.

3.1. Simple Majority Games. Consider a decision-making body in which all players have the same weight (1 for instance) and a simple majority of votes is needed to pass a bill. The situation may be described by a simple majority game $(N, v) \in \mathcal{G}$ where $N = \{1, \dots, n\}$ and $v(S) = 1$ if $|S| > n/2$ and $v(S) = 0$ otherwise. The standard Shapley value in the simple majority game is symmetric for all the players – it grants each player an equal payoff or power $1/n$. However, if we take the political spectrum into account the result is quite different. As usual, suppose that the spectrum is consistent with the relative positions of the agents in N .

Definition 4. A player is $i \in M$ is *left-moderate* with respect to a spectrum if $i \in \{LM_1, LM_2\}$ where

$$\begin{aligned} LM_1 &= \lceil \frac{n}{4} \rceil \text{ and } LM_2 = \lfloor \frac{n}{4} \rfloor + 1 && \text{if } n \text{ is even} \\ LM_1 &= \lceil \frac{n+1}{4} \rceil \text{ and } LM_2 = \lfloor \frac{n+1}{4} \rfloor + 1 && \text{if } n \text{ is odd} \end{aligned}$$

and is *right-moderate* with respect to a spectrum if $i \in \{RM_1, RM_2\}$ where

$$\begin{aligned} RM_1 &= \lceil \frac{3}{4}n \rceil \text{ and } RM_2 = \lfloor \frac{3}{4}n \rfloor + 1 && \text{if } n \text{ is even} \\ RM_1 &= \lceil \frac{3}{4}(n+1) \rceil \text{ and } RM_2 = \lfloor \frac{3}{4}(n+1) \rfloor + 1 && \text{if } n \text{ is odd} \end{aligned}$$

Observe that if $\frac{n+1}{2}$ is odd $LM_1 = LM_2$ and $RM_1 = RM_2$. Note also that moderate players are located virtually half way between the center and one of the most extreme players (when n is large enough).

Claim 3. In a simple majority game with a spectrum $LM_1, LM_2, RM_1,$ and RM_2 are the players earning the highest payoff. Furthermore, ϕ^i (as a function of i) is

- strictly increasing in the interval $[1, LM_1]$,
- strictly decreasing in the interval $[LM_2, \lceil n/2 \rceil]$,
- strictly increasing in the interval $[\lfloor n/2 \rfloor + 1, RM_1]$,
- strictly decreasing in the interval $[RM_2, n]$.

Proof.

If n is even, the vector of spectrum payoffs looks like

$$\frac{1}{2^{n-1}} \left(\binom{\frac{n}{2}-1}{0}, \binom{\frac{n}{2}-1}{1}, \dots, \binom{\frac{n}{2}-1}{\frac{n}{2}-1}, \binom{\frac{n}{2}-1}{\frac{n}{2}-1}, \dots, \binom{\frac{n}{2}-1}{1}, \binom{\frac{n}{2}-1}{0} \right)$$

or in other words like two copies of the corresponding row of Pascal's triangle concatenated one after another.

If n is odd, the vector of spectrum payoffs looks like

$$\frac{1}{2^{n-1}} \left(\binom{\frac{n-1}{2}}{0}, \binom{\frac{n-1}{2}}{1}, \dots, \binom{\frac{n-1}{2}}{\frac{n-1}{2}}, 2 \binom{\frac{n-1}{2}}{\frac{n-1}{2}}, \binom{\frac{n-1}{2}}{\frac{n-1}{2}-1}, \dots, \binom{\frac{n-1}{2}}{1}, \binom{\frac{n-1}{2}}{0} \right)$$

which almost looks like two copies of the corresponding row of Pascal's triangle concatenated one after another.

Given the well known fact that the maximum of each row of Pascal's triangle is attained at its central position (or central positions, depending on

whether the number of elements is odd or even), the claim follows immediately. ■

3.2. Supermajority Games. Let $1/2 < \alpha \leq 1$. A coalitional game (N, v) is an α -*supermajority game* if each player has one vote and a quota of $\lceil \alpha n \rceil$ is needed to pass a bill, i.e., if for every $S \subseteq N$, $v(S) = 1$ if $|S| \geq \lceil \alpha n \rceil$ and $v(S) = 0$ otherwise.

Let $0 \leq k \leq n - 1$. Denote by A_k^l the set of players whose distance to the left extreme is greater than or equal to k and by A_k^r the set of players whose distance to the right extreme is greater than or equal to k , i.e., $A_k^l = [k + 1 \dots n]$ and $A_k^r = [1 \dots n - k]$.

Claim 4. *Let (N, v) be an α -supermajority game with a spectrum and $k = \lceil \alpha n \rceil$. Then the spectrum value of each player i is*

$$\phi^i = \begin{cases} 2^{\frac{k-2}{n-1}} \binom{n-k}{i} & i \in A_{k-1}^r, \\ 2^{\frac{k-2}{n-1}} \binom{n-k}{n-i} & i \in A_{k-1}^l, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $i \notin A_{k-1}^r \cup A_{k-1}^l$, then there is no connected coalition of size $k - 1$ either to the right or the left of i in the spectrum, hence i can never be the k -th to join a coalition and therefore he receives a 0 payoff.

The calculations of the values for players $i \in A_{k-1}^r \cup A_{k-1}^l$ are straightforward combinatorial calculations. ■

Given the result of Claim 4, we may call the set $N \setminus A_{k-1}^r \cup A_{k-1}^l$ the *null zone* of a supermajority game, because players in the null zone all get spectrum value 0. The null zone, if non-empty, always includes the central players with respect to the spectrum. The maximum spectrum value goes to the players about halfway between the extremes of the null zone and the extremes of the spectrum, one on each side of the null zone.

In general, as α increases the null zone grows larger. When $\alpha = 1$, the game is the unanimity game of the grand coalition in which the worth 1 is attained only by the grand coalition N . In that case, the null zone includes all the players except for the leftmost extreme player and the rightmost extreme player; players 1 and n each receive $1/2$, and all other players receive zero. This fits with an intuition that when unanimity is required to pass a measure it is the most extreme elements who usually wield the strongest veto power.

3.3. Characterisation of the Spectrum Value by means of properties. Fix $(N, v, \prec^\sigma) \in \mathcal{GS}$. For each player $i \in N$, define i 's *reverse twin* (with

respect to \prec^σ ² to be the player $j = n + 1 - i$. For every $S \subseteq N$, the *reverse coalition* of S is the coalition S^{-1} composed by the reverse twins of S , i.e., $S^{-1} = \{j \in N : j = n + 1 - i \text{ for some } i \in S\}$. The reverse game (N, v^{-1}) of a coalitional game with a spectrum $(N, v, \prec^\sigma) \in \mathcal{GS}$ is given by $v^{-1}(S) = v(S^{-1})$. Two players $i, j \in N$ are *connected symmetric* to each other if i and j are reverse twins and $v(S \cup \{i\}) = v(S^{-1} \cup \{j\})$ for all $S \in C_i(N)$. A player $i \in N$ is a *veto player* if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$. A player $i \in N$ is a *connected null player* if $v(S \cup \{i\}) = v(S)$ for all $S \in C_i(N)$. Hence, a standard null player is in particular a connected null player. Finally, for every permutation $\sigma : N \rightarrow \{1, \dots, n\}$ and every $i \in N$ with $\sigma(i) < n$ we define $\sigma_{i \rightarrow} = \sigma_{\leftarrow(i+1)}$ as the permutation obtained from σ when player i swaps his position with player $(i + 1)$.

In what follows, we state properties that a value on \mathcal{GS} might be expected to satisfy.

EFF: A value on \mathcal{GS} , φ , satisfies *efficiency* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$,

$$\sum_{i \in N} \varphi_i(N, v, \prec^\sigma) = v(N).$$

ADD: A value on \mathcal{GS} , φ , satisfies *additivity* if for every $(N, v, \prec^\sigma), (N, w, \prec^\sigma) \in \mathcal{GS}$,

$$\varphi(N, v + w, \prec^\sigma) = \varphi(N, v, \prec^\sigma) + \varphi(N, w, \prec^\sigma).$$

CNP: A value on \mathcal{GS} , φ , satisfies the *connected null player property* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and every connected null player $i \in N$,

$$\varphi_i(N, v, \prec^\sigma) = 0.$$

CSP: A value on \mathcal{GS} , φ , satisfies the *connected symmetric player property* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and every pair of connected symmetric players $i, j \in N$,

$$\varphi_i(N, v, \prec^\sigma) = \varphi_j(N, v, \prec^\sigma).$$

ROE: A value on \mathcal{GS} , φ , satisfies *reverse ordering equivalence* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and $i \in N$,

$$\varphi_i(N, v, \prec^\sigma) = \varphi_{n+1-i}(N, v^{-1}, \prec^{\sigma^{-1}}).$$

BCVP: A value on \mathcal{GS} , φ , satisfies *balanced contributions for veto players* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and every pair of veto players $i, j \in N$ such that $1 < i < j < n$,

$$\varphi_i(N, v, \prec^\sigma) - \varphi_i(N, v, \prec^{\sigma_{j \rightarrow}}) = \varphi_j(N, v, \prec^\sigma) - \varphi_j(N, v, \prec^{\sigma_{i \leftarrow}}).$$

²A player i is his own reverse twin iff n is odd and $i = (n + 1)/2$.

EEVP: A value on \mathcal{GS} , φ , satisfies the *effect on extreme veto players property* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and every pair of veto players $i, j \in N$ such that $1 = i < j < n$, the effect on player i 's payoff of player j becoming one step more extreme depends only on the worth of the coalition $[i \dots j]$ and on the distance from j to n . That is,

$$\varphi_i(N, v, \prec^\sigma) - \varphi_i(N, v, \prec^{\sigma_{j \rightarrow}}) = f(v([i \dots j]), n + 1 - j),$$

where f is a function. Moreover f is increasing in the first argument and decreasing in the second argument.

The first two properties, EFF and ADD, are the standard efficiency and additivity properties stated in the framework of games with a spectrum. CNP and CSP are based on the standard null player and symmetric player properties. Note, however, that the definition of connected null player is less demanding than its standard counterpart and hence, CNP is a stronger property than the standard null player property. In the case of CSP the definition of a connected symmetric player is more demanding than the definition of a standard symmetric player. CSP is therefore a weaker property than the standard symmetry property.

Under ROE, the payoff of a player is equal to the payoff of his reverse twin when we consider the reverse spectrum and the reverse game. This is essentially stating that a player's payoff is only affected by his or her relative position in the spectrum. In particular, this implies there is no bias in favour of any political orientation (left or right).

Finally, both BCVP and EEVP only apply to veto players in a particular position and are therefore not overly demanding. On the one hand, BCVP is a reciprocity property in the spirit of various versions of the balanced contributions property that have been proposed in the literature. It states that when neither of the two veto players is at the extreme of the spectrum then the gain or loss that a veto player can inflict on the other veto player by moving one position towards the more extreme end of the spectrum is equal to the gain or loss that the second veto player can inflict to the first one by doing the same. On the other hand, EEVP only applies when one of the veto players is at the left extreme of the spectrum and describes the effect on this player's payoff by the other veto player becoming one step more extreme. Moreover, note that it asserts that for monotone games the more to the right j is, the greater the effect his move towards the extreme has.

Lemma 1. *The spectrum value, ϕ , satisfies EFF, ADD, CNP, CSP, ROE, BCVP, and EEVP.*

Proof. The proof for EFF and ADD follows directly from Definition 3. CNP follows from Claim 1 since the definition of the spectrum value depends only on the connected coalitions in $C_i(N)$.

To show that ϕ satisfies CSP and ROE note that:

- (1) $S^{-1} \in C^{\prec^\sigma}(N)$ if and only if $S \in C^{\prec^\sigma}(N)$.
- (2) $S \in C^{\prec^\sigma}(N)$ if and only if $S \in C^{\prec^{\sigma^{-1}}}(N)$.
- (3) Let $r : 2^N \rightarrow 2^N$ be given by $r : S \mapsto S^{-1}$. If i and j are reverse twins then $r(L_i(N)) = R_j(N)$, $r(R_i(N)) = L_j(N)$, $r(L_j(N)) = R_i(N)$, and $r(R_j(N)) = L_i(N)$. Indeed $r|_{L_i(N)}$, $r|_{R_i(N)}$, $r|_{L_j(N)}$, and $r|_{R_j(N)}$ are all bijective.

Both CSP and ROE then follow from Claim 1.

Both BCVP and EEVP follow from Equation (6). Indeed, let $i, j \in N$ be two veto players such that $1 < i < j < n$, then

$$\begin{aligned} \phi^i(N, v, \prec^\sigma) - \phi^i(N, v, \prec^{\sigma_{j \rightarrow}}) &= \frac{2^{j-i-1}}{2^{n-1}} \binom{n-j-i+1}{i-1} v([i \dots j]) \\ &= \phi^j(N, v, \prec^\sigma) - \phi^j(N, v, \prec^{\sigma_{\leftarrow i}}). \end{aligned}$$

The proof of case $i = 1$ (EEVP) is similarly a straightforward application of Equation (6).

Theorem 1. *The spectrum value is the unique value on \mathcal{GS} satisfying EFF, ADD, CNP, CSP, ROE, BCVP, and EEVP.*

Proof. Given the result of Lemma 1 we need only prove uniqueness. Let φ be a value satisfying the properties above. By the assumption of ADD we only need to check uniqueness in unanimity games. Let $S \subseteq N$ and let u_S be the unanimity game with carrier S .

First of all, note that by CNP $\varphi_k(N, v, \prec^\sigma) = 0$ for every $k \in N \setminus \{\min(S), \max(S)\}$. Define the *span* of S to be

$$s(S) := \max(S) + 1 - \min(S) \in \{1, \dots, n\}.$$

We prove uniqueness by backward induction on the size of S .

First step of the induction. If $s(S) = n$ then $1, n \in S$. Hence by CSP $\varphi_1(N, u_S, \prec^\sigma) = \varphi_n(N, u_S, \prec^\sigma)$. Uniqueness then follows by EFF.

Inductive hypothesis. Suppose that $\varphi(N, u_S, \prec^\sigma)$ is uniquely determined for every $S \subseteq N$ such that $s(S) > k$ with $k < n$.

Induction step. Let $S \subseteq N$ satisfy $s(S) = k$. If $1 \in S$ (that is, $\min(S) = 1$ and $\max(S) = k$), then by EEVP, $\varphi_1(N, u_S, \prec^\sigma) - \varphi_1(N, u_S, \prec^{\sigma_{k \rightarrow}})$ is uniquely determined. Note that $(N, u_S, \prec^{\sigma_{k \rightarrow}})$ is a unanimity game of

span $k + 1$. Hence, it follows that $\varphi_1(N, u_S, \prec^{\sigma_{k \rightarrow}})$ is unique by the inductive hypothesis. Finally, by EFF $\varphi_k(N, u_S, \prec^\sigma)$ is also unique. In case $n \in S$, uniqueness follows from ROE and the proof above. Thus, we may assume that $1 < \min(S) < \max(S) < n$. Let $i = \min(S)$ and $j = \max(S)$, then by BCVP, $\varphi_i(N, u_S, \prec^\sigma) - \varphi_j(N, u_S, \prec^\sigma) = \varphi_i(N, u_S, \prec^{\sigma_{j \rightarrow}}) - \varphi_j(N, u_S, \prec^{\sigma_{\leftarrow i}})$. Note that both $(N, u_S, \prec^{\sigma_{j \rightarrow}})$ and $(N, u_S, \prec^{\sigma_{\leftarrow i}})$ are unanimity games with carriers of span $k + 1$, hence the payoffs according to φ are unique by the induction hypothesis. Finally, $\varphi_i(N, u_S, \prec^\sigma) + \varphi_j(N, u_S, \prec^\sigma)$ is also unique by EFF. That is, both $\varphi_i - \varphi_j$ and $\varphi_i + \varphi_j$ are uniquely determined, which completes the proof. ■

4. CASE STUDY: THE ISRAELI GENERAL ELECTIONS OF 1981 AND 1984

4.1. Background.

We present here a case study analysis of the Israeli general elections for the Knesset in 1981 and 1984 using the spectrum value.

We chose to look at Knesset elections in the 1980s mainly because national political discourse in Israel in that decade was dominated by a single issue: the Arab-Israeli conflict and the future disposition of the West Bank and the Gaza Strip. This meant that there was general agreement on a one dimensional spectrum that could be applied to virtually all the political parties in the Knesset. The Labour-Alignment party was considered to the left of the Likud party because it advocated a more conciliatory negotiating position in peace talks. Parties that were more dovish than Labour-Alignment comprised the left wing of the Israeli political spectrum in a fairly linear ordering of positions with respect to the extent of peace negotiation concessions proposed by the parties. Similarly, parties that were more hawkish than Likud comprised the right wing. We will list here the political parties winning representation in the Knesset along a spectrum from left to right, with the leftmost party appearing in the first position of the list and the rightmost party in the last position.

It was difficult if not impossible to imagine a party to the left of Labour-Alignment joining a coalition led by the Likud if that coalition did not include Labour-Alignment, and conversely it was difficult if not impossible to imagine a party to the right of Likud joining a coalition led by Labour-Alignment if that coalition did not include Likud, thus justifying the main assumption of coalitions being restricted to spectrum connectedness. The main exception to this rule chiefly involved ultra-Orthodox religious parties, such as Agudah and Shas, which were open to inclusion in either right-wing

or left-wing coalitions. We therefore place them in the centre of the spectrum between Labour-Alignment and Likud.

There are 120 seats in the Israeli Knesset. The coalitional game that is being played immediately after each general election is a weighted majority game, with the weights determined by the number of seats held by each party and the quota for forming a governing coalition being 61 members.

4.2. The Israeli General Election of 1981. Ten political parties won seats in the Knesset elections conducted on 30 June 1981.³

The election results are presented in Figure 1, with the parties listed in order according to their positioning on the spectrum starting from the extreme left to the extreme right.

	Party	Seats	Seats (%)	Spectrum (%)	Shapley (%)
1	Hadash	4	3.33	0	6.9841
2	Ratz	1	0.833	0	1.5873
3	Shinui	2	1.66	0	3.0952
4	Labour	47	39.166	31.25	26.4286
5	Agudah	4	3.33	7.8125	6.9841
6	Telem	2	1.66	1.5625	3.0952
7	Likud	48	40.0	34.375	29.9603
8	NRP ⁴	6	5.0	15.625	11.7857
9	Tami	3	2.5	7.8125	5.0397
10	Tehiya	3	2.5	1.5625	5.0397

FIGURE 1. The Israeli Knesset election results of 1981. The quota for forming a governing coalition is 61 seats.

As the table in Figure 1 shows, Likud and Labour-Alignment together won 95 seats, representing approximately 80 percent of the 120 Knesset

³ Cf. a detailed analysis of the 1981 Israeli elections in Rapoport and Golan (1985). In that paper the analysis is conducted using six indices, the Shapley-Shubik index, the Banzhaf index, the Deegan-Packel index, the generalised Shapley-Shubik index, the generalised Banzhaf index and the generalised Deegan-Packel index. The generalised indices, which are based on an analysis of a multi-dimensional “ideological space” that considers how the players in a coalitional game will vote with respect to various issues (which are combinations of parameters of the ideological space) appear to give weights that are closer to observed political power than the non-generalised indices. The generalised Shapley-Shubik index in that paper, in particular, indicates a preponderance of political weight on the right side of the political spectrum following the 1981 Knesset elections, as does the spectrum value index of this paper.

⁴ National Religious Party

seats. Individually they were of almost exactly equal size, with Labour-Alignment holding 47 seats (about 39 percent of seats) to Likud's 48 (40 percent of seats). Their spectrum values are also quite close: Labour-Alignment's spectrum value is 0.3125 compared to Likud's 0.34375. Looking at those numbers alone one might consider the election results a near dead heat.

An analysis of the spectrum value reveals a different result. The key to this analysis involves looking not only at the absolute spectrum value but also regarding in detail the connected coalitions leading to that number. In this case, the bulk of the weight of the spectrum value is on the right side of the spectrum, and this comes about because of the relatively large number of ways that right-wing or mostly right-wing connected coalitions can be formed with right-wing parties playing pivotal roles given these election results. In contrast, parties such as Ratz and Shinui get 0 under the spectrum value because they can never be pivotal relative to a connected coalition: any connected coalition that adds Ratz or Shinui to it will already have contain more members than the quota. Note that the Shapley value does not capture this at all. For obvious reasons, it grants equal value to Agudah and Hadash because they are symmetric, both having 4 seats. The spectrum value here strongly distinguishes between the two, giving Aguda nearly 8 percent but giving Hadash zero, since there is no connected coalition for which Hadash is pivotal.

The governing coalition formed on 5 August 1981 was a right-wing government comprised of the connected coalition Agudah, Telem, Likud, NRP and Tami.⁵

4.3. The Israeli General Election of 1984. Fifteen political parties won seats in the Knesset elections conducted on 23 July 1984. However, one of those parties, Yahad, with three seats, later merged with Labour-Alignment while Ometz (1 seat) and Tami (1 seat) merged with Likud. We will therefore regard the election results as including only 12 parties, with the seats held by the small merging parties counted among the total seats held by the larger parties. Based on that the election results were as presented in Figure 2, with the parties listed in order according to their positioning on the spectrum starting from the extreme left to the extreme right.

As in 1981, in 1984 the two largest parties split about 80 percent of the Knesset seats between them. The situation, however, was very different when one considers the spectrum values of the parties. Two extreme parties, Hadash on the left and Kach on the right, garnered non-zero spectrum

⁵ Tehiya later joined the governing coalition on 26 August 1981.

⁶ Progressive List for Peace

	Party	Seats	Seats (%)	Spectrum (%)	Shapley (%)
1	PLP ⁶	2	1.66	0	2.8319
2	Hadash	4	3.3	7.421875	6.0137
3	Ratz	3	2.5	0	4.347
4	Shinui	3	2.5	0	4.347
5	Labour	47	39.166	46.875	33.759
6	Shas	4	3.33	5.078	6.0137
7	Agudah	2	1.66	0	2.8319
8	Likud	43	35.833	39.0625	21.829
9	NRP	4	3.33	0	6.0137
10	Morasha	2	1.66	0	2.8319
11	Tehiya	5	4.166	0	7.8283
12	Kach	1	0.833	1.5625	1.3528

FIGURE 2. The Israeli Knesset election results of 1984. The quota for forming a governing coalition is 61 seats.

values because the only way to compose a purely left-wing coalition (respectively, a purely right-wing coalition) would involve including Hadash (respectively, Kach) in a pivotal position. Neither of these options was politically feasible given public opinion.

Note the extreme change in Hadash's payoff between 1981 and 1984 despite it garnering the same number of seats in the parliament. This change is due to the increase in the number of seats for the Labour which turned Hadash into a pivotal player within a potential connected left-wing coalition.

Disregarding Hadash and Kach leaves positive values concentrated in the center of the spectrum. Furthermore, a careful analysis reveals that the large spectrum value of Labour-Alignment arises solely from considering potential coalitions that contain both Labour-Alignment and Likud, and similarly the large spectrum value of Likud arises solely from considering potential coalitions that contain both Labour-Alignment and Likud. Again, the Shapley value, which does not take the relationships between the parties in terms of their ideological distances from each other, fails to capture any of these subtleties.

The governing coalition formed on 13 September 1984 was an unprecedented 'rotation government' that gave both Labour-Alignment and Likud equal power; the prime minister during the first half of the Knesset's four-year term was the leader of Labour-Alignment and the prime minister during the second half of the Knesset's four-year term was the leader of Likud.

The coalition itself was a connected coalition that included Shinui, Labour-Alignment, Shas, Agudah, Likud, NRP and Morasha.

5. FUTURE RESEARCH

The model presented here is entirely one-dimensional: it reduces all the differences between the players of a coalitional game to a single position along a strict linear ordering. The analysis of most political situations is much more complex and multi-dimensional. For example, a political party may be to the right of a rival party with respect to foreign affairs issues and to the left with respect to social policy.

We intend to study the multi-dimensional analogue to the spectrum value introduced here in follow-up research. In principle the model should be given to being extended to a general graph model in which each player is a vertex a connected coalition is formed by a connected subset of the graph.

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