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A GAME WITH NO BAYESIAN APPROXIMATE EQUILIBRIA

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ABSTRACT. Simon (2003) presented an example of a 3-player Bayesian games with no Bayesian equilibria but it has been an open question whether or not there are games with no approximate Bayesian equilibria. We present an example of a Bayesian game with two players, two actions and a continuum of states that possesses no approximate Bayesian equilibria, thus resolving the question. As a side benefit we also have for the first time an an example of a 2-player Bayesian game with no Bayesian equilibria and an example of a strategic-form game with no approximate Nash equilibria. The construction makes use of techniques developed in an example by Y. Levy of a discounted stochastic game with no stationary equilibria.

1. INTRODUCTION

One of the seminal contributions of Harsányi (1967) was the analysis of Bayesian games for studying games of incomplete information, which included showing that every finite Bayesian game (finite number of players, finite actions, finite states of the world) has a Bayes-Nash, or Bayesian, equilibrium. The fact that modellers could safely assume the existence of at least one equilibrium was undoubtedly an element in the widespread acceptance of Bayesian games in modelling a wide range of economic situations. Indeed, at this point it is impossible to imagine modern game theory and economic modelling without the theory of Bayesian games.

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The question of the existence or non-existence[1] of Bayesian equilibria in games with uncountably many states, however, remained open for many years, until Simon (2003) gave a negative answer by presenting an example of a three-player Bayesian game with no Bayesian equilibrium.

That important result left in its wake (at least) two open questions: (1) are there examples of games that have no Bayesian ε-equilibria?; (2) are there examples of two-player games that have no Bayesian equilibria? In particular, a negative answer to the first question would imply that modellers could always assume that Bayesian equilibria can be approximated as closely as desired in games with uncountably many states, thus significantly weakening Simon (2003)'s result,

We show here, however, that the answer to both questions is yes by constructing a two-player Bayesian[2] game with no Bayesian ε-equilibria. As a side-benefit, the example also shows that there there exist strategic-form games with a continuum of players and no Nash ε-equilibria[3] and that there exist two-player Bayesian games with no Harsányi equilibria (meaning *ex ante* Nash equilibria over the common prior of a Bayesian game), which had also been open questions.

We make extensive use of techniques developed in Levy (2012) in his paper on stochastic games without stationary Nash equilibria. The fact that these techniques have now been shown to be useful for generating counterexamples in separate subject fields (stochastic games and Bayesian games) may indicate that they have potential application in many other fields of interest.

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[1] By the existence of an equilibrium we mean the existence of a measurable equilibrium. There are several reasons for restricting attention to measurable strategies (and hence measurable equilibria); to consider just two reasons, if a strategy is not measurable it cannot be constructed explicitly, and the payoffs of non-measurable strategies haven't got well-defined expected values. Measurability has in fact been included as a basic requirement in the definition of an equilibrium over uncountable spaces since the earliest literature on the subject (see Schmeidler (1973) for one such example). We therefore throughout this paper use the term 'existence of an equilibrium' as synonymous with 'existence of a measurable equilibrium' without further qualification.

[2] The game constructed here is not only a Bayesian game, it is an *ergodic game* as defined in Simon (2003).

[3] This result does not contradict the result in Schmeidler (1973), which assumes that no deviation from equilibrium undertaken by a finite number (or even a measure zero set) of players can affect payoffs; we do not assume that here. Sion and Wolfe (1957) presents an example of a finite-player game with no equilibrium, but the example there assumes each player has a continuum of actions while we assume that each player has a finite action space.

The significance of counter-examples to the existence of equilibria and approximate equilibria such as the example here (and those in Simon (2003) and Levy (2012)) is that they serve as a sharp warning signal to modellers: although you routinely assume the existence of equilibria when you work with finite games, you cannot automatically do so in games with an uncountable number of states. A large percentage of economic models rely on the use of uncountably many states to represent quantities such as prices (as in models of auctions or bargaining, such as that of Chatterjee and Samuelson (1983) for example), profits and outputs in market models (for example Radner (1980)), time, accumulated wealth or stocks, population percentages, share percentages, and so forth.

In addition, an extensively-used approach to dealing with a Bayesian game with a finite but large number of states is to analyse instead a similar game with a continuum of states. Myerson (1997), for example, informs readers of Chapter 2 of his textbook on game theory, when referring to Bayesian games, that "it is often easier to analyze examples with infinite type sets than those with large finite type sets." Given this, it is important for modellers working with Bayesian games with uncountably many states to keep in mind that they cannot blindly rely on the well-known results in finite games guaranteeing the existence of equilibria and approximate equilibria.

2. Preliminaries and Notation

2.1. Information Structures and Knowledge.

A *space of states* is a pair $(\Omega, \mathcal{B})$ composed of a *set of states* Ω and a σ-field $\mathcal{B}$ of measurable subsets (*events*) of Ω.

We will work throughout this paper with a two-element set of players I. The two players will be denoted by Player Red and Player $Green$ (with capitalised initial letters). An *information structure* over the state space $(\Omega, \mathcal{B})$ is then given by a pair of partitions of Ω labelled Π_R and Π_G, respectively, of Player Red and Player $Green$. For $i \in I$ and for each state $\omega \in \Omega$ we denote by $\Pi_i(\omega)$ the element in Π_i that contains ω. Furthermore, denote by Γ_i the sub-σ-algebra of $\mathcal{B}$ generated by Π_i.

2.2. Types and Priors.

A *type function* t_i of player i for $(\Omega, \mathcal{B}, (\Pi_i)_{i \in I})$ is a function $t_i : \Omega \to \Delta(\Omega)$ from states to probability measures over $(\Omega, \mathcal{B})$ such that the mapping $t_i(\cdot)$ satisfies:

(1) $t_i(\omega)(E)$ is measurable for any fixed event E,
(2) $t_i(\omega)(\Pi_i(\omega)) = 1$,

(3) $t_i(\omega) = t_i(\omega')$ for all $\omega' \in \Pi_i(\omega)$.

For each ω, $t_i(\omega)$ is called player i's type at ω. Therefore, a quintuple $(I, \Omega, \mathcal{B}, (\Pi_i)_{i\in I}, (t_i)_{i\in I})$, where each t_i is a type function, is a *type space*.

A probability measure μ_i over $(\Omega, \mathcal{B})$ is a *prior* for a type function t_i if for each event A

$$\mu_i(A) = \int_\Omega t_i(\omega)(A)\, d\mu_i(\omega). \tag{2.1}$$

A probability measure μ that is a prior for each of the players' type function in a type space is a *common prior*.

2.3. **Agent Games.**

Recall the definition of the agent game $\mathbf{K}$ associated with a Bayesian game $\mathbf{B}$: $\mathbf{K}$ is a strategic-form game whose set of players, which is a measurable space, has a (measurable) bijection η with the the set of all types of all the players in $\mathbf{B}$. The action set of each player θ in $\mathbf{K}$ is equal to the action set of the player j in $\mathbf{B}$ associated with $\eta(\theta)$, and the payoff to player θ for an action profile is the corresponding expected payoff of j at $\eta(\theta)$. Every strategy $\hat{\psi}$ of $\mathbf{B}$ is naturally associated in this way with a strategy ψ in $\mathbf{K}$.

The analysis of the equilibria of a Bayesian game $\mathbf{B}$ can be accomplished by analysing the associated strategic-form game $\mathbf{K}$ in the sense that, for any $\varepsilon \geq 0$, the strategy ψ in $\mathbf{K}$ is a (measurable) Nash ε-equilibria if and only if $\hat{\psi}$ is a (measurable) Bayesian ε-equilibria in $\mathbf{B}$.

3. Constructing the Bayesian Game

We define in this section an ergodic game $\mathbf{B}$ that we will eventually show possesses no Bayesian ε-equilibrium.

3.1. **The State Space.**

Let X be the set of infinite sequences of 1 and -1, i.e. $X := \{-1, 1\}^{\mathbb{Z}_{\geq 0}}$. A generic element $x \in X$ is a sequence $x_0, x_1, x_2, \ldots$, where we denote the i-th coordinate of x by x_i.

Next, define the following two sets:

$$red = \{r\} \times X$$

and

$$green = \{g\} \times X,$$

that is, a generic element of red is $(r, x_0, x_1, x_2, \ldots)$ and a generic element of green is $(g, x_0, x_1, x_2, \ldots)$.

Our state space is $\Omega := red \cup green$. The measure μ we will work with over Ω gives each of red and $green$ the Lebesgue measure over X and independently gives equal probability to red and to $green$.

Notation 1.

- Denote by ι the operator on Ω defined by
$$\iota(r, x_0, x_1, x_2, \ldots) = (r, -1 \cdot x_0, x_1, x_2, \ldots)$$
$$\iota(g, x_0, x_1, x_2, \ldots) = (g, -1 \cdot x_0, x_1, x_2, \ldots).$$
Note that ι is colour-preserving.
- Denote by S the operator on Ω defined by
$$S(r, x_0, x_1, x_2, \ldots) = (g, x_1, x_2, x_3 \ldots)$$
$$S(g, x_0, x_1, x_2, \ldots) = (r, x_1, x_2, x_3 \ldots).$$
S is the product of the measure preserving involution $r \leftrightarrow g$ and the Bernoulli shift, which is well-known to be measure preserving,[4] hence S is measure preserving. Denote
$$S_+^{-1}(r, x_0, x_1, x_2, \ldots) = \{(g, 1, x_0, x_1, x_2, \ldots)\}$$
and
$$S_-^{-1}(r, x_0, x_1, x_2, \ldots) = \{(g, -1, x_0, x_1, x_2, \ldots)\}$$
with the same operator defined for $S_+^{-1}(g, x_0, \ldots)$ and $S_-^{-1}(g, x_0, \ldots)$ with the colours reversed . Then $S^{-1}(\omega) = S_+^{-1}(\omega) \cup S_-^{-1}(\omega)$, hence it maps each point to *two* points,

3.2. The Knowledge Partitions and Type Functions.

We next define the partitions of Player Red and Player $Green$.

A generic element of Π_R is of the form
$$\{(r, x_1, x_2, x_3, \ldots),\ (g, 1, x_1, x_2, \ldots),\ (g, -1, x_1, x_2, \ldots)\}$$

A generic element of Π_G is of the form
$$\{(g, x_1, x_2, x_3, \ldots),\ (r, 1, x_1, x_2, \ldots),\ (r, -1, x_1, x_2, \ldots)\}$$

Another way of expressing this, using the above-defined operators, is: If ω is a red state then
$$\Pi_G(\omega) = \{\omega, \iota(\omega), S(\omega)\}$$
$$\Pi_R(\omega) = \{\omega\} \cup S^{-1}(\omega)$$

[4] See, for example, Halmos (1956).

If ω is a $green$ state then the partition elements are exactly the same with colours reversed

$$\Pi_R(\omega) = \{\omega, \iota(\omega), S(\omega)\}$$
$$\Pi_G(\omega) = \{\omega\} \cup S^{-1}(\omega)$$

To complete the description of the types, all that's left is to give the probabilities, which for all partition elements are:

$$\begin{array}{c} \frac{1}{4}, \quad \frac{1}{4}, \quad \frac{1}{2} \\ \{\omega, \iota(\omega), S(\omega)\} \end{array}$$

In words, Player $Green$ ascribes 1/2 to the $green$ state in his partition element and 1/4 to each of the red states. Player Red ascribes 1/2 to the red state in her partition element and 1/4 to each of the $green$ states.

Lemma 3.1. *The functions t_R and t_G satisfy the conditions for being type functions with μ as their common prior.*

The proof of Lemma 3.1 is in the appendix.

3.3. **The Game Forms.**

The action set we will work with is identical at every state for both players: $A_R^\omega = A_G^\omega = \{U, D\}$, for all $\omega \in \Omega$.

The payoff functions are more complicated:

- At red (respectively, $green$) states, Player $Green$ (respectively, Player Red) gets payoff 0 *no matter what* actions are played by him or the other player; strategically, Player $Green$ (respectively, Player Red) can ignore the red (respectively, $green$) states.
- At a $green$ state $(g, x_0, x_1, \ldots)$ (respectively, red state $(r, x_0, x_1, \ldots)$), Player $Green$'s (respectively Player Red's) payoff is as follows, where Player $Green$ (respectively, Player Red) is row and Player Red (respectively, Player $Green$) is column:

If $x_0 = 1$:

	U	D
U	1	0
D	0.3	0.3

If $x_0 = -1$:

	U	D
U	0.7	0.7
D	1	0

TABLE 1. The payoff matrices.

This completes the construction of the Bayesian game $\mathbf{B}$.

3.4. **Reduction to an Agent Game.**

In the agents game, each type becomes an agent. A type of Player Red has the form

$$\{(r, x_0, x_1, x_2 \ldots),\ (g, 1, x_0, x_1, \ldots),\ (g, -1, x_0, x_1, \ldots)\}$$

We can therefore *uniquely* identify this type of Player Red by the state $(r, x_0, x_1, x_2 \ldots)$, and in this way we identify $\{\Pi_R(\omega)\}_{\omega \in \Omega}$ with $r \times X$. Hence we can consider Player Red's agents to be the set of red states and by the same reasoning, we can consider Player $Green$'s agents to be the set of $green$ states.

Formally, we have a bijection η between Ω and the collection of types $\{\Pi_R(\omega)\}_{\omega \in \Omega} \cup \{\Pi_G(\omega)\}_{\omega \in \Omega}$ as follows:

$$\eta(\omega) = \begin{cases} \Pi_R(\omega) & \text{if } \omega \in red \\ \Pi_G(\omega) & \text{if } \omega \in green. \end{cases} \tag{3.1}$$

To form the agent game $\mathbf{K}$ corresponding to $\mathbf{B}$, we use this bijection. Abusing notation, we will use ω to identify both the state and the agent associated with that state. We will use the same measure μ over $(\Omega, \mathcal{B})$ in both $\mathbf{B}$ and $\mathbf{K}$. The agents in $\mathbf{K}$ all share the same action set, $A_\omega = \{U, D\}$, hence each strategy profile ψ of $\mathbf{K}$ is given by $\psi : \Omega \to \Delta(\{U, D\})$.

The definition of the mixed strategy profile $\hat{\psi}$ in $\mathbf{B}$ associated with a mixed strategy profile ψ in $\mathbf{K}$ explicitly gives

$$\psi(\omega) = \begin{cases} \hat{\psi}_1(\omega) & \text{if } \omega \text{ is odd} \\ \hat{\psi}_2(\omega) & \text{if } \omega \text{ is even.} \end{cases}$$

The pure action payoff function u_ω of each agent ω depends only on the action $a(\omega)$ chosen by agent ω and the action $a(S(\omega))$ of agent $S(\omega)$; in effect, each agent ω perceives himself or herself as playing a game against $S(\omega)$.

To see this, consider for example Player Red's perspective when the true state of the world is a $green$ state ω. The relevant partition element for Player Red is $\Pi_R(\omega) = \{\omega, \iota(\omega), S(\omega)\}$. By construction, $S(\omega)$ is a red state and is the only red state in $S(\omega)$. Since Player Red's action choice can lead to a non-zero payoff for him only at $S(\omega)$, for calculating his expected payoff at $\Pi_R(\omega)$ he needs only to take into account the state of nature associated with $S(\omega)$ and accordingly to plan a best reply to Player $Green$'s actions at $S(\omega)$. Similar considerations then imply that at $S(\omega)$ Player $Green$ need only plan a best reply to Player Red's action at $S^2(\omega)$, and so forth.

The payoff function in the agents game is then given by the matrices of Table 2 at each ω, where the row player is ω and the column player is $S(\omega)$.

If $x_0 = 1$:

	U	D
U	1	0
D	0.3	0.3

If $x_0 = -1$:

	U	D
U	0.7	0.7
D	1	0

TABLE 2. The payoff matrices in the agents game, where the row player is ω and the column player is $S(\omega)$

3.5. **Observations on Equilibria in $\mathbf{K}$.**

Since there is an equivalence between the Bayesian ε-equilibria of a Bayesian game and the Nash ε-equilibria of the associated agents game, and in addition every strategy profile of the Bayesian game is measurable if and only if its associated profile of mixed strategies in the agents game is also measurable, for showing that there is no measurable Bayesian ε-equilibrium in $\mathbf{B}$ it suffices to show that there is no measurable Nash ε-equilibrium in $\mathbf{K}$.

From here to the end of the paper, fix the following values

$$0 < \varepsilon < \frac{1}{50};\ \delta = 10\varepsilon < \frac{1}{5}.$$

Assume, by way of contradiction, that there exists a measurable Nash ε-equilibrium in $\mathbf{K}$ denoted by ψ.

Definition 1. An agent $\omega \in \Omega$ is *U-quasi-pure* (respectively, *D-quasi-pure*) if under ψ he plays U with probability greater than $1 - \delta$ (respectively, less than δ).

Denote

$$\Xi_U = \{\omega \mid \omega \text{ is } U\text{-quasi-pure}\},$$

$$\Xi_D = \{\omega \mid \omega \text{ is } D\text{-quasi-pure}\}$$

and

$$\Xi_M = \Omega \setminus (\Xi_U \cup \Xi_D).$$

Lemma 3.2. *If $S(\omega)$ is quasi-pure under ψ then so is ω (i.e., $S(\omega) \in (\Xi_U \cup \Xi_D)$ implies $\omega \in (\Xi_U \cup \Xi_D)$). If the former is a-quasi-pure (for $a \in \{U, D\}$) then the latter is as well if and only if $x_0 = 1$ (i.e., if $S(\omega) \in \Xi_U$ then*

$\omega \in \Xi_U$ only if $x_0 = 1$ and similarly if $S(\omega) \in \Xi_D$, then $\omega \in \Xi_D$ only if $x_0 = 1$).

Proof. The proof is identical to the proof of Lemma 3.3.4 in Levy (2012). The intuition behind the proof is as follows: a careful analysis of the matrices in Table 2 shows that under an ε-equilibrium if $S(\omega)$ is quasi-pure then ω will want to 'quasi-match' iff $x_0 = 1$ and to 'quasi-mismatch' iff $x_0 = -1$. ∎

Lemma 3.3. *For all $\omega \in \Omega$, at least one of the two states in $S^{-1}(\omega)$ is in Ξ_U or Ξ_D (even if ω itself is not quasi-pure).*

Proof. The proof is identical to the proof of Lemma 3.3.5 in Levy (2012). The intuition behind the proof is as follows: if $S(\omega)$ plays U with probability greater than 2/5 then if $x_0 = 1$ agent ω will 'quasi-match' and play U with probability greater than $1 - \delta$. If agent $S(\omega)$ plays U with probability less than 3/5 (i.e. plays D with probability greater than 2/5) then if $x_0 = -1$ agent ω will 'quasi-mismatch' and play U with probability greater than $1 - \delta$. But agent $S(\omega)$ must play U with probability greater than 2/5 or less than 3/5 (or both). ∎

3.6. **Nonexistence of ε-Equilibria.**

The Main Theorem. *The game $\mathbf{B}$ has no Bayesian ε-equilibria.*

Proof. Although the following statement does not appear as an explicit theorem in Levy (2012), it is in effect what is proved by putting together the results Lemma 4.0.8, Lemma 4.0.9 and Proposition 4.0.10 of that paper:

Let F be a finite set and τ be a permutation of F. Denote $\Theta := X \times F$ and let S denote the measure-preserving operator defined by the cross product of the Bernoulli shift operator and τ over Θ. Then there cannot exist a decomposition of Θ into three disjoint measurable subsets Ξ_D, Ξ_U and Ξ_M satisfying

(1) $\Theta = \Xi_D \cup \Xi_U \cup \Xi_M$ up to a null set;
(2) Lemma 3.2;
(3) Lemma 3.3

But this is exactly the situation we have developed for $\mathbf{K}$. This is a contradiction. If $\mathbf{K}$ has no Nash ε-equilibrium then $\mathbf{B}$ has no Bayesian ε-equilibrium. ∎

3.7. **Robustness to Perturbations.**

For $\varepsilon > 0$, an ε-perturbation of a Bayesian game $\mathbf{B}$ is a Bayesian game $\mathbf{B}'$ over the same type space and action sets, with a set of payoff functions v_i^ω satisfying $\|v_i^\omega - u_i^\omega\|_\infty < \varepsilon$ for all $i \in I$.

Fixing a game $\mathbf{B}'$ that is an ε-perturbation of the ergodic game $\mathbf{B}$ defined above, it is straightforward to show that the inequalities that are essential for the proofs of Lemmas 3.2 and 3.3 continue to hold with respect to the agent game of $\mathbf{B}'$ (we need to restrict to ε because if we perturb a payoff from 0 to $-\varepsilon$ agent ω can still concentrate only on the actions of $S(\omega)$ and ignore the other states in calculating an ε-best reply). We therefore immediately have the following corollary.

Corollary 3.1. *For sufficiently small ε, an ε-perturbation of the game $\mathbf{B}$ has no Bayesian ε-equilibria.*

A similar result holds for sufficiently small perturbations of the posterior probabilities defining the types t_R and t_G.

4. End Remarks

An Harsányi ε-equilibrium of a Bayesian game with a common prior μ is a profile of mixed strategies $\Psi = (\Psi_i)_{i \in I}$ such that for each player i and any unilateral deviation strategy $\widehat{\Psi}_i$,

$$\int_\Omega u_i^\omega(\Psi(\omega))\, d\mu(\omega) \geq \int_\Omega u_i^\omega(\widehat{\Psi}_i(\omega), \Psi_{-i}(\omega))\, d\mu(\omega) - \varepsilon.$$

Simon (2003) shows that the existence of a measurable 0-Harsányi equilibrium implies the existence of a measurable 0-Bayesian equilibrium. However, this result is known not to hold for $\varepsilon > 0$. The example in this paper of a game without a measurable Bayesian ε-equilibrium does not, therefore, imply that there is no Harsányi ε-equilibrium. We leave the open question of whether or not there are examples of games that have no measurable Harsányi ε-equilibria for future research.

Furthermore, in the example here there is incomplete information on both sides: if we define the colour and value of x_0 to comprise the state of nature at a state of the world ω then neither player knows the true state of nature. Because of this ignorance of the state of nature and the way the payoff matrices are defined in Table 1, neither player ever knows the true payoffs.

This contrasts with the example in Simon (2003), where there is incomplete information on one and a half sides (that is, one player always knows the true payoff relevant data but not always what the other players might know) and by construction players know their own payoffs at each state. It is presently unknown whether an example can be constructed of a game in which players know their payoffs at each state but the game has no Bayesian ε-equilibrium.

5. Appendix

Proof of Lemma 3.1. By construction, $t_i(\omega)(\Pi_i(\omega)) = 1$ for all ω and $t_i(\omega) = t_i(\omega')$ for $\omega' \in \Pi_i(\omega)$. Two more items need to be checked: that for each event A, $t_i(\omega)(A)$ is measurable and that $\mu(A) = \int_\Omega t_i(\omega)(A)\, d\mu(\omega)$. We will prove these for $i = R$, with the proof for $i = G$ conducted similarly.

For the rest of this proof, denote by $1_A(\omega)$ the indicator function that returns 1 if $\omega \in A$ and 0 if $\omega \notin A$. Fix an event A. Then:

$$t_R(\omega)(A) = \begin{cases} \frac{1}{4}1_A(\omega) + \frac{1}{4}1_A(i(\omega)) + \frac{1}{2}1_A(S(\omega)) & \text{if } \omega \in green \\ \frac{1}{4}1_A(S_+^{-1}(\omega)) + \frac{1}{4}1_A(S_-^{-1}(\omega)) + \frac{1}{2}1_A(\omega)) & \text{if } \omega \in red \end{cases}$$

from which we conclude that $t_R(\omega)(A)$ is measurable.

Next, we divide up A as follows:

$$\begin{aligned} A_1 &:= \{\omega \in A \mid \omega \in green \text{ and } \iota(\omega), S(\omega) \notin A\}, \\ A_2 &:= \{\omega \in A \mid \omega \in green \text{ and } \iota(\omega) \in A, S(\omega) \notin A\}, \\ A_3 &:= \{\omega \in A \mid \omega \in green \text{ and } \iota(\omega), S(\omega) \in A\}, \\ A_4 &:= \{\omega \in A \mid \omega \in red \text{ and } S^{-1}(\omega) \subset A^c\}, \\ A_5 &:= \{\omega \in A \mid \omega \in red \text{ and } S^{-1}(\omega) \cap A \neq \emptyset, S^{-1}(\omega) \not\subset A\} \\ &\quad \cup \{\omega \in A \mid \omega \in green \text{ and } S(\omega) \in A, \iota(\omega) \notin A\}, \\ A_6 &:= \{\omega \in A \mid \omega \in red \text{ and } S^{-1}(\omega) \subset A\}. \end{aligned}$$

The sets (A_j) are all disjoint from each other. The proof proceeds by showing that $\mu(A_j) = \int t_R(\omega)(A_j)\, d\mu(\omega)$ for each $1 \leq j \leq 6$, which is straight-forward but tedious. We show how it is accomplished in two of the cases, trusting that the technique for the rest of the cases will be clear enough.

Case A_1. Define $B := A_1 \cup \iota(A_1) \cup S(A_1 \cup \iota(A_1))$. By the measure-preserving properties of ι and S, one has $\mu(\iota(A_1)) = \mu(A_1)$ and

$$\mu(S(A_1 \cup \iota(A_1))) = \mu(A_1 \cup \iota(A_1)),$$

hence $\mu(B) = 4\mu(A_1)$. On the other hand,

$$\int_\Omega t_R(\omega)(A_1)\, d\mu(\omega) = \int_B t_R(\omega)(A_1)\, d\mu(\omega) = \int_B \frac{1}{4}\, d\mu(\omega) = \mu(B)/4,$$

leading to the conclusion that $\int_\Omega t_R(\omega)(A_1)\, d\mu(\omega) = \mu(A_1)$.

Case A_5. Define $C := A_5 \cup (\iota(A_5 \cap green))$. Using similar reasoning as in the previous case, relying on measure-preserving properties, one deduces that $\mu(C) = \frac{4}{3}A_5$. On the other hand,

$$\int_\Omega t_R(\omega)(A_5)\, d\mu(\omega) = \int_C t_R(\omega)(A_5)\, d\mu(\omega) = \int_C \frac{3}{4}\, d\mu(\omega) = \frac{3}{4}\mu(C),$$

hence $\int_\Omega t_R(\omega)(A_5)\, d\mu(\omega) = \mu(A_5)$. ∎

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