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A DISCOUNTED STOCHASTIC GAME WITH NO STATIONARY EQUILIBRIA: THE CASE OF ABSOLUTELY CONTINUOUS TRANSITIONS

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A Discounted Stochastic Game with No Stationary Equilibria: The Case of Absolutely Continuous Transitions*

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Abstract

We present a discounted stochastic game with a continuum of states, finitely many players and actions, such that although all transitions are absolutely continuous w.r.t. a fixed measure, it possesses no stationary equilibria. This absolute continuity condition has been assumed in many equilibrium existence results, and the game presented here complements a recent example of ours of a game with no stationary equilibria but which possess deterministic transitions. We also show that if one allows for compact action spaces, even games with state-independent transitions need not possess stationary equilibria.

Keywords: Stochastic Game, Discounting, Stationary Equilibrium

1 Introduction

1.1 Background

Recent work, [16], has shown that discounted stochastic games with uncountable state spaces need not, in general, possess stationary equilibria. The purpose of this paper is to show that stationary equilibria need not exist even in a restricted, but much studied, class of stochastic games; that is, in the class of games in which all transitions are absolutely continuous with respect to a fixed measure.

Stochastic games were introduced by Shapley (1953), [31]. In a stochastic game, players play in discrete stages, with stochastic transitions between states chosen by Nature via distributions determined by the state and action. In the

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β-discounted game, each player receives the β-discounted sum of the stream of his stage-by-stage payoffs. A particular class of strategies, the *stationary strategies*, in which a player makes his decision based only on the current state, has been particularly studied in games with discounted payoffs.

There are two main reasons for this focus. First of all, stationary strategies are the natural class of strategies for the discounted payoff evaluation, as sub-games that are defined by different histories but begin at the same state are strategically equivalent: players will have the same preferences over plays in one sub-game as in the other. The view that strategies should depend only on payoff-relevant data in the discounted game motivated the development of the concept of *Markov perfect equilibria*, [19]. In [12] this view is called the *subgame-consistency principle*, which is described succinctly in [13] as "the behaviour principle according to which a player's behaviour in strategically equivalent subgames should be the same, regardless of the different paths by which these subgames might be reached." Another reason stationary strategies are the subject of much study is because of their simplicity and easy implementation; to quote [11], "an equilibrium which does not display minimal regularity through time - maybe stationarity - is unlikely to generate the coordination between agents that it assumes."

Results for existence of equilibrium in stationary strategies have appeared in increasing generality: for zero-sum games with finite state spaces, [31]; for zero-sum games with general state spaces, [18]; for non-zero-sum games with finite state space, [10, 32, 28, 29]; for non-zero-sum games with countable state space, [25]. The question of existence of stationary equilibria for games with general state spaces remained open until [16] presented a counter-example.[1]

The example presented in [16], however, does not fit into the framework that many of the previous works on stochastic games with a general state space had assumed: Specifically, many works assumed that all transition measures are absolutely continuous with respect to a fixed measure. We have termed this assumption the *Absolute Continuity Condition* (henceforth, *ACC*); see Definition 2.1 below. Under this ACC assumption, the existence of stationary ε-equilibria, [21], and of stationary extensive-form correlated equilibria,[2] [24], had been shown. Both of these results, when contrasted with the properties of the example given in [16] (see discussion in Section 1.2) show that the ACC model can behave very differently than the general model.[3] Conditions similar to (and usually stronger than) ACC have also been used to derive existence results in several works with applications in capital accumulation, dynamic competitions, games with noisy transitions, and other settings; see, e.g., [2, 5, 6, 7, 14, 22, 23].

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[1] [30] presented an argument for the existence of stationary equilibria in the non-zero-sum game with general state space, but the proof is flawed; this was already pointed out in [9].

[2] I.e., stationary equilibria when the state space is enlarged to include an element of $[0, 1]$, which is chosen i.i.d. uniformly each stage.

[3] Indeed, the example given in [16] does not possess ε-equilibria nor extensive-form stationary correlated equilibria.

1.2 Our Example

The purpose of this paper is to present an example of a stochastic games which satisfies ACC and yet does not possess a stationary equilibrium, regardless of the discount factor used. This example has finite action spaces, and payoffs and transitions that are continuous in the state (the latter referring to norm-continuity).

We also show that a modification of example gives us a game with compact action spaces and state-independent transitions[4] that does not possess stationary equilibria. (In this framework, the payoffs are jointly continuous in state and actions, and the transitions are jointly continuous in actions.) The existence of stationary equilibria in games with finite action spaces and state-independent transitions had been established in [27], and it had been unknown if this extends to the case of compact action spaces.

The example of the present paper takes advantage of the structure of the manifold of Nash equilibria for normal-form games. It relies on the existence of a two-person game whose set of equilibrium is homeomorphic to a circle (thus connected but not simply connected) and each equilibrium of it is stable in the appropriate sense; see, [15, pp. 1034].

We note that the example of the present paper possesses, as a result of ACC, both stationary ε-equilibria for any $\varepsilon > 0$ ([21]) and stationary extensive-form correlated equilibria ([24]). In this sense, it differs from the example presented in [16], which possesses neither.

Also unlike the example presented in [16], the example of the present paper can be shown to posses Markovian[5] equilibria, and it is not robust to perturbations of the payoffs or transitions. It is not known, however, if this is a result of the ACC condition, and hence it is still of interest to search for stronger counterexamples and/or existence results. (We mention that [1] and [4] provide incorrect proofs for the existence of Markovian equilibria under ACC.)

We also remark that in a forthcoming work, using the same normal-form game we construct in Section 4, we construct a non-zero-sum continuous-time Markov game (see [34]) with fixed duration which does not possess Markovian equilibria.

1.3 Layout of this Paper

In Section 2 we present the model of discounted stochastic games. Section 3 informally gives the idea of our construction. Section 4 constructs the fundamental normal form game, which is the backbone of our example, except for several technical claims which are proved in Appendix A and Appendix B. In Section 5, the example of a stochastic game without a stationary equilibrium is presented. In Section 6, we modify it to a game with state-independent transi-

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[4] That is, for all action profiles a and any two states ω, ω', $q(\omega, a) = q(\omega', a)$, where q is the transition kernel.

[5] Markovian strategies are those in which players choose their actions as a function of both the current state.

tions and compact action spaces. (We remark that Section 5 can be read after having only read the description and the properties of the normal-form game provided in 4.2; that is, Section 5 does not depend directly on Section 4.1.)

2 Stochastic Game Model

A discounted stochastic game with a general state space and finitely[6] many actions is composed of:

- A standard Borel[7] space Ω of states.
- A finite set $\mathcal{P}$ of players.
- A finite set of actions I^p for each $p \in \mathcal{P}$. Denote $\overline{I} = \prod_{p\in\mathcal{P}} I^p$
- A discount factor $\beta \in (0, 1)$.
- A bounded payoff function $r : \Omega \times \overline{I} \to \mathbb{R}^{\mathcal{P}}$, which is Borel-measurable.
- A transition function $q : \Omega \times \overline{I} \to \Delta(\Omega)$, which is Borel-measurable (where $\Delta(\Omega)$, the space of regular Borel probability measures on Ω, possesses the Borel structure induced from the topology of narrow convergence).

The game is played in discrete time. If $z \in \Omega$ is a state at some stage of the game and the players select an $a \in \overline{I}$, then $q(z, a)$ is the probability distribution of the next state of the game. A *behavioral strategy* for a player is a Borel-measurable mapping that associates with each given history a probability distribution on the set of actions available to him. A *stationary strategy* for Player p is a behavioral strategy that depends only on the current state; equivalently, it is a Borel-measurable mapping that associates with each state $s \in \Omega$ a probability distribution on the set I^p.

Let $H^\infty = (S \times \overline{I})^{\mathbb{N}}$ be the space of all infinite histories of the game, endowed with the product σ-algebra. For any profile of strategies $\sigma = (\sigma^p)_{p\in\mathcal{P}}$ of the players and every initial state $z_1 = z \in \Omega$, a probability measure P_z^σ and a stochastic process $(z_n, a_n)_{n\in\mathbb{N}}$ are defined on H^∞ in a canonical way, where the random variables z_n, a_n describe the state and the action profile chosen by the players, respectively, in the n-th stage of the game (see, e.g., Chapter 7 in [3]). The expected payoff vector under σ, in the game starting from state z, is

$$\gamma_\sigma(z) = E_z^\sigma\big(\sum_{n=1}^{\infty} \beta^{n-1} r(z_n, a_n)\big). \tag{2.1}$$

Let Σ^p denote the set of behavioral strategies for Player $p \in \mathcal{P}$, and $\Sigma = \prod_{p\in\mathcal{P}} \Sigma^p$. A profile $\sigma \in \Sigma$ will be called a Nash equilibrium if

$$\gamma_\sigma^p(z) \geq \gamma_{(\tau,\sigma^{-p})}^p(z),\ \forall p \in \mathcal{P}, \forall z \in \Omega, \forall \tau \in \Sigma^p. \tag{2.2}$$

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[6] This is a particular case of the general model, which allows for compact actions spaces that are state-dependent; see, e.g., [20].

[7] That is, a space that is homeomorphic to a Borel subset of a complete, separable metrizable space.

Denote, for stationary $\sigma \in \Sigma$, for a state $z \in \Omega$, and for a mixed action profile $a \in \prod_{p \in \mathcal{P}} \Delta(I^P)$,

$$X_\sigma(z,a) := r(z,a) + \beta \int_\Omega \gamma_\sigma(t) dq(z,a)(t). \tag{2.3}$$

By way of iterations, one can show that

$$\gamma_\sigma(z) = X_\sigma(z, \sigma(z)). \tag{2.4}$$

For stationary $\sigma \in \Sigma$, it is easily shown that (2.2) is equivalent to

$$X_\sigma^p(z, \sigma(z)) \geq X_\sigma^p(z, (b, \sigma^{-p}(z))),\ \forall p \in \mathcal{P}, \forall z \in \Omega, \forall b \in I^p, \tag{2.5}$$

i.e., that for all z, $\sigma(z)$ is an equilibrium in the game with payoff $X_\sigma(z, \cdot)$.

In this paper, we study games with the following property:

Definition 2.1. *A stochastic game is said to satisfy the Absolute Continuity Condition (ACC) if there is $\nu \in \Delta(\Omega)$ such that for all $z \in \Omega$, $a \in \overline{I}$, $q(z,a)$ is absolutely continuous w.r.t. ν.*

Remark 2.2. One might think to relax the definition of Nash equilibrium in stationary strategies in games satisfying ACC by requiring that (2.2) only hold for ν-a.e. $z \in \Omega$. However, [26] shows that existence of this weaker equilibrium concept would imply existence of the stronger concept, via a simple modification of the "a.e.-equilibrium" on a ν-null set.

3 The Idea of The Construction

The game we will construct will have state space $[0,1]$, where 1 is an absorbing state with payoff 0. The payoffs decrease linearly as one moves towards 1, and the transitions from state t are of two types (or some mixture thereof): uniformly in $[t,1)$, or quitting to 1. As such, the game progresses towards the right.

The transitions will be controlled by a pair of players, who we denote C, D. These players have no influence over their stage payoff, and each of them influences whether the game is to "continue," i.e., if the transition should be uniform in $[t,1)$, or is to "quit," i.e., go all the way to the absorbing state. Clearly, then, in state $t < 1$, each of these players chooses which way he wishes to influence depending on whether his future average expected payoff in the states to his right is positive or negative.

We seek to build a group of players around C, D with which to implement a mechanism with two main properties in each state $t < 1$. First, the action that each of the players C, D plays in response to a future expected positive (resp. negative) payoff in $[t,1)$ induces the other players, in any stationary equilibrium, to award that player a negative (resp. positive) stage payoff. From this mechanism (and the particular structure of the game) it will follow that, in any stationary equilibrium, each of the players C, D must always receive a payoff of 0. However, this contradicts the other main property of the mechanism: the stage payoff to at least one of the players C, D must be non-zero in any stage of play of any stationary equilibrium.

To achieve a mechanism with both these properties, we take advantage of anomalies in the manifold of Nash equilibria. In particular, we take advantage of the fact that the set of equilibria of a normal-form game can have connected components that are not simply connected, and such that enough equilibria of this component are stable in an appropriate sense. Although we will list what are the essential properties we need, for simplicity the construction will center around a particular example presented in [15] in relation to stability properties of equilibria, in which the set of equilibria is homeomorphic to a circle and all equilibria satisfy an appropriate stability property.

4 Construction of the Normal-Form Game

4.1 Preliminaries

We begin with some notations and conventions:

- For a bounded real-valued function f, $||f||_\infty = \sup |f|$, where the supremum is taken over the entire domain of f.
- Distances in any Euclidian spaces (including spaces of games and spaces of mixed action profiles) are always w.r.t. to the $|| \cdot ||_\infty$ norm.
- If p is a mixed action over an action space A and $a \in A$, then $p[a]$ denotes the probability that p chooses a.
- If g is some payoff vector to some set of players $\mathcal{P}$, and $T \subseteq \mathcal{P}$, then g^T denotes the restriction of the vector to the players in T.
- If a is an action profile of the players in $\mathcal{P}$, and $T \subseteq \mathcal{P}$, then a^T denotes the vector of strategies of players in T.
- If Λ is a normal form game on some set of players, and α is a strategy profile of those players, then $\Lambda(\alpha)$ denotes the resulting payoff vector. If $T \subseteq \mathcal{P}$, then $\Lambda^T(\alpha)$ (resp. $\Lambda^{-T}(\alpha)$) denotes the payoff to the players in T (resp. in T^c).
- For such Λ, α, and $T \subseteq \mathcal{P}$, $\Lambda^T(\cdot, \alpha^{-T})$ denotes the expected normal-form game facing the players in T when the other players are restricted to playing α^{-T}.
- For a normal-form game Λ, $NE(\Lambda)$ is the set of Nash equilibria of Λ.
- We let S denote the boundary of the square,

$$S = \{(p, q) \mid -1 \leq p, q \leq 1, \ (|p| = 1) \vee (|q| = 1)\}. \tag{4.1}$$

We denote the four closed edges of S by $S_{\mathcal{N}}, S_{\mathcal{E}}, S_{\mathcal{S}}, S_{\mathcal{W}}$ for the north, east, south, and west edges, respectively. Note that $S_{\mathcal{N}} = -S_{\mathcal{S}}$, $S_{\mathcal{E}} = -S_{\mathcal{W}}$.

The multi-player normal game is built around a 'base' normal-form game G_0 with the following properties:

(1) The set of equilibria $NE_0 = NE(G_0)$ contains a unique hyperstable set H_0. By a hyperstable set, defined in [15], we mean a set that is minimal w.r.t. the following property: for any $\varepsilon > 0$, there is $\delta > 0$ such that the equilibria of any game G' that is in a δ-neighborhood of a game G that is equivalent[8] to G_0 are in an ε-neighborhood of H_0.

(2) H_0 is connected but not simply connected.

(3) Furthermore, there exists:

- A continuous semi-algebraic injection $\psi : S \to H_0$, which is not nulhomotopic in H_0.
- A semi-algebraic retract $\rho : NE_0 \to \psi(S)$.
- For all $\varepsilon > 0$, a semi-algebraic mapping Γ_ε from S to the ε-neighborhood of G_0, such that for each edge of S and any equilibria of any game in $\Gamma_\varepsilon(E)$ is in an ε-neighborhood of $\rho^{-1}(\psi(-E))$.

Remark 4.1. For later purposes, we remark that the upper-semicontinuity of the Nash equilibrium correspondence implies that for each $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon)$ such that if $||H - G_0||_\infty < \eta$, then $NE(H)$ is contained in the ε-neighborhood of $NE(G_0)$.

Properties (1)-(3) would be enough for us to build a normal-form game with the desired properties. Yet, relying on these properties alone would require the some very cumbersome and technical machinery. For the sake of simplicity, we take advantage of some further properties of the game given in Appendix B of [15]:

The Game G_0

$A\backslash B$	L	M	R
L	$1, 1$	$0, -1$	$-1, 1$
M	$-1, 0$	$0, 0$	$-1, 0$
R	$1, -1$	$0, -1$	$-2, -2$

Table 4.2.a

Equilibria of G_0

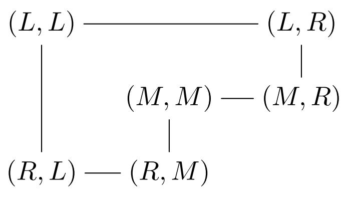

Figure 4.2.b

(4.2)

The additional properties of G_0 are the following:

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[8] Two games are equivalent if they have the same reduced form, where the reduced form is achieved by eliminating actions that are payoff-equivalent to a convex combination of other actions.

(4) There are two players, A, B, with action spaces $I = \{T, M, B\}$, $J = \{L, C, R\}$, respectively.

(5) The set of Nash equilibria is hyperstable and homeomorphic to S.

(6) For each $\varepsilon > 0$, the maps[9] ψ and Γ_ε can be taken to be piecewise linear,[10] 8ε-Lipshitz, and satisfying the following property:

$$||\Gamma_\varepsilon - G_0||_\infty < 2\varepsilon, \tag{4.3}$$

such that for any edge E of S, and for any equilibrium (x, y) of any game in $\Gamma_\varepsilon(E)$ it holds that

$$||E_{x\otimes y}[\vartheta] - (-E)||_\infty < 4\varepsilon, \tag{4.4}$$

where ϑ is defined by

$\vartheta :=$

$A\backslash B$	L	M	R
L	$1, 1$	$0, 0$	$1, -1$
M	$0, 0$	$1, -1$	$1, -1$
R	$-1, 1$	$-1, -1$	$0, 0$

ϑ can be understood graphically:

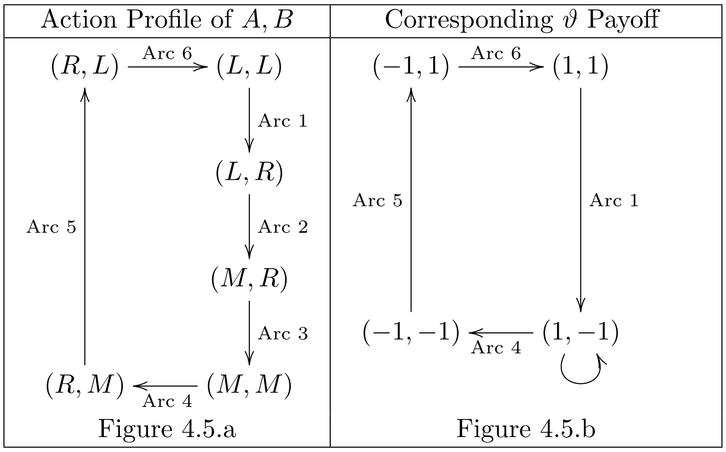

Figure 4.5.a Figure 4.5.b

(4.5)

The latter part of (6) can be stated informally: For any equilibria of a game assigned to a point on E via Γ_ε, the expected payoff under ϑ is not too far from the edge opposite E. Indeed, in Appendix A we verify that property[11] (6) holds for the game given in (4.2). We remark that property (4) only serves to make concrete the notation in property (6) and below, and is completely irrelevant otherwise.

In Appendix B, we prove the following proposition:

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[9] In this case, ψ is a homeomorphism, and the retract ρ is the identity.

[10] In the sense that each edge of the square is viewed as an interval.

[11] Where ψ is described by Figure 4.5.a., and ρ is the identity.

Proposition 4.2. *Let I, J be finite sets*[12]*, and let $Q : S \to \mathbb{R}^{2|I\times J|}$ be a continuous and piecewise linear*[13] *map to bimatrix games on these action sets. Then for some integer M, there exist 4 normal-form games on the set of players $A, B, \theta^1, \ldots, \theta^M$, denoted $\mathfrak{K}^k$ for $k \in \{1, -1\}^2$, such that:*

- *(I) A, B have action spaces I, J respectively; each θ^j has an action space $\{L, R\}$. The players $\{\theta^1, \ldots, \theta^M\}$ will be called auxiliary players.*
- *(II) The payoffs of $\theta^1, \ldots, \theta^M$ are not affected*[14] *by the actions of A, B in any of the games; let $\mathfrak{K}^k_\Theta$ denote the well-defined restriction of $\mathfrak{K}^k$ to the Players $\Theta^1, \ldots, \Theta^M$.*
- *(III) For $(p,q) \in [-1,1]^2$, let $\mathfrak{K}(p,q)$ (resp. $\mathfrak{K}_\theta(p,q)$) denote the convex combination of the $\{\mathfrak{K}^k\}_k$ (resp. $\{\mathfrak{K}^k_\Theta\}_k$), with weights given by*[15] *$(\frac{1+p}{2}, \frac{1-p}{2}) \otimes (\frac{1+q}{2}, \frac{1-q}{2})$. If $(p,q) \in S$, and a_θ is an equilibrium in the game $\mathfrak{K}_\Theta(p,q)$, then the expected payoff matrix facing A, B, given by $\mathfrak{K}^{A,B}(p,q)(\cdot, a_\theta)$, is $Q(p,q)$.*
- *(IV) If L denotes a Lipshitz constant of Q and if $||Q(p,q) - Q_0||_\infty \leq \kappa$ for some Q_0, some κ, and all $(p,q) \in S$, then*

$$||\mathfrak{K}^{A,B}(p,q)(\cdot, a_\theta) - Q_0||_\infty \leq L \cdot \kappa,\ \forall (p,q) \in [-1,1]^2, \forall a_\theta \in NE(\mathfrak{K}_\Theta(p,q)) \tag{4.6}$$

4.2 The Normal-Form Game

We now turn to our normal-form game. Fix $\varepsilon \leq \min[\frac{1}{16}, \eta(\frac{1}{4})]$, where $\eta(\cdot)$ is defined in Remark 4.1. The payoff is dependent on a parameter $\omega = (\omega^C, \omega^D) \in \mathbb{R}^2$:

- The Players are $A, B, \theta^1, \ldots, \theta^M$, where M as in Proposition 4.2 for the function Γ_ε, as well as Players C, D.
- As in Proposition 4.2, Players A, B have action sets $I = \{T, M, B\}$, $J = \{L, C, R\}$, and each player θ^j has action sets $\{L, R\}$; furthermore, Players C, D have action sets $\{1, -1\}$.
- The payoff r_ω, will be the sum of two payoffs, $r_\omega := r_1 + r_{2,\omega}$, defined separately as follows:
- The first payoff function r_1 satisfies $r_1^{C,D}(a) = G^{C,D}(a^{A,B}) := \vartheta[a^{A,B}]$, where ϑ is defined in property (6) of Section 4.1, and the payoff to the other players is the same as in the game of Proposition 4.2 when the profile

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[12] The proposition also extends with almost no change in the proof to the case that Q is a map to games with any finite set of players.

[13] I.e., piecewise linear on each edge of S.

[14] Unlike the (β_j) of Proposition 6.3, their payoffs can be affected by each other's actions.

[15] $(\phi, 1-\phi)$ denotes the probability distribution choosing 1 with probability ϕ, and choosing -1 with probability $1-\phi$.

$a^{-\{C,D\}}$ is played and the choice $a^{C,D} \in \{+1,-1\}^2$ is made by Nature; namely,

$$r_1^{C,D}(a) = G^{C,D}(a^{A,B}) := \vartheta[a^{A,B}], \quad r_1^{-\{C,D\}}(a) = \mathfrak{K}^{a^{C,D}}(a^{-\{C,D\}})$$

- The second payoff function $r_{2,\omega}$ is dependent on ω. It gives a payoff of 0 to all players other than C, D: That is, $r_{2,\omega}^{-\{C,D\}} \equiv 0$. To players C, D, $r_{2,\omega}$ is dependent only on $a^{C,D}$ and is given by:

$$r_{2,\omega}^{C,D}(a) = \begin{array}{|c|c|c|} \hline C\backslash D & 1 & -1 \\ \hline 1 & \omega^C, \omega^D & \frac{1}{2}\omega^C, \frac{1}{2}\omega^D \\ \hline -1 & \frac{1}{2}\omega^C, \frac{1}{2}\omega^D & 0 \\ \hline \end{array}$$

For each $(p,q) \in S$, let $a_{p,q}$ be an equilibrium profile in the game with payoff r_1 for the players $A, B, \theta^1, \ldots, \theta^M$ when Players C, D are restricted to playing $b_{p,q} := (\frac{1+p}{2}, \frac{1-p}{2}) \otimes (\frac{1+q}{2}, \frac{1-q}{2})$; that is $a_{p,q}$ is an equilibrium in $r_1^{-\{C,D\}}(\cdot, b_{p,q})$. By applying property (III) of Proposition 4.2 to the mapping Γ_ε which has the properties given in property (6), we get with the help of Figure 4.5 the following relationship between p, q and the payoff in r_1 to C, D under the profile $a_{p,q}$, $r_1^{C,D}(a_{p,q}, b_{p,q})$:

p, q

$$\begin{tikzcd} (-1,1) \arrow[r, "S_{\mathcal{N}}"] & (1,1) \arrow[d, "S_{\mathcal{E}}"] \\ (-1,-1) \arrow[u, "S_{\mathcal{W}}"] & (1,-1) \arrow[l, "S_{\mathcal{S}}"] \end{tikzcd}$$

Figure 4.7.a

$r_1^{\{C,D\}}(a_{p,q}, b_{p,q})$

$$\begin{tikzcd} (1,-1) \arrow[r, leftrightarrow, "S_{\mathcal{S}}"] \arrow[d, leftrightarrow, "S_{\mathcal{E}}"'] & (-1,-1) \arrow[d, leftrightarrow, "S_{\mathcal{W}}"] \\ (1,1) \arrow[r, leftrightarrow, "S_{\mathcal{N}}"] & (-1,1) \end{tikzcd} \tag{4.7}$$

Figure 4.7.b

Intuitively, as the point (p,q) goes around the square, the payoff $r_1^{C,D}(a_{p,q}, b_{p,q})$ (which is not uniquely determined) must also go 'around' the square 'close to it' - at a distance of at most 4ε from the edge on which (p,q) lies, because of (4.4).

Proposition 4.3. *Let $\omega \in \mathbb{R}^2$, and let a be an equilibrium profile in the game r_ω. Denote $p = 2a^C[1] - 1$, $q = 2a^D[1] - 1$. Then:*

(i) If $\omega^C > 0$, then $p = 1$; if $\omega^C < 0$, then $p = -1$. The same holds for q w.r.t. ω^D.

(ii) Hence, if $\omega^C > 0$, then $r_1^C(a) \leq -\frac{1}{2}$. If $\omega^C < 0$, then $r_1^C(a) \geq \frac{1}{2}$. Similarly, if $\omega^D > 0$, then $r_1^D(a) \leq -\frac{1}{2}$. If $\omega^D < 0$, then $r_1^D(a) \geq \frac{1}{2}$.

(iii) Let H be the expected matrix facing players A, B; that is, $H = r_\omega^{A,B}(\cdot, a^{-\{A,B\}})$. Then $||H - G_0||_\infty < \varepsilon$ (regardless of the values of ω^C, ω^D; this includes the case where one or both are 0), and $r_1^{C,D}(a) \neq 0$.

Proof. The first part follows from (4.4), and because C, D consider only the payoff from $r_{2,\omega}$ when making a decision. The second part follows from the first part and from (4.4), since we had chosen $\varepsilon \leq \frac{1}{16}$. For the last part, first note that since $\varepsilon \leq \frac{1}{16}$ and Γ_ε is 8ε-Lipshitz, Γ_ε is $\frac{1}{2}$-Lipshitz; hence, from property IV of Proposition 4.2 and (4.3), we see that $||H - G_0||_\infty < \varepsilon$. Finally, since $\varepsilon \leq \eta(\frac{1}{4})$, with η as in Remark 4.1, and $\max\vartheta - \min\vartheta = 2$, we see that $r_1^{C,D}(a) = \vartheta[a^{A,B}]$ is in the $\frac{1}{2}$-neighborhood of the square S. $\square$

5 The Example

The stochastic game has the following components:

- The players are $\mathcal{P} = \{A, B, C, D, \theta^1, \ldots, \theta^M\}$ as in Section 4.2, along with the actions sets given there.
- The state space Ω is $[0, 1]$, with the Borel σ-algebra.
- The payoff function $r(s, \cdot)$ in state s is given by $(1 - s)r_1(\cdot)$, where r_1 is defined in Section 4.2. Note that $r(1, \cdot) \equiv 0$.
- The transitions $q(t, a)$ are controlled by Players C, D and are given by:

  $$q(t, a) = (1 - \zeta(1 - t))\delta_1 + \zeta(1 - t) \cdot \tilde{q}(t, a)$$

  where $0 < \zeta \leq 1$ is fixed and satisfies

  $$\frac{\zeta \cdot ||r||_\infty}{1 - \zeta} < \frac{1}{2} \tag{5.1}$$

  and

  $\tilde{q}(t, a) =$

$C\backslash D$	L	R
L	$U(t,1)$	$\frac{1}{2}U(t,1) + \frac{1}{2}\delta_1$
R	$\frac{1}{2}U(t,1) + \frac{1}{2}\delta_1$	δ_1

  where $U(a, b)$ is the uniform distribution on $[a, b]$, and δ_c is the Dirac measure at c; we interpret $U(1, 1) = \delta_1$. Note that 1 is an absorbing state.
- $\beta \in (0, 1)$ is a discount factor.

By way of contradiction, fix a stationary equilibrium σ. Recall the notations γ_σ and X_σ from Section 2. We will denote $V^C = \gamma_\sigma^C$, $V^D = \gamma_\sigma^D$, and for $j = C, D$, $W^j(t) = \int_t^1 V^j(s)ds$. For $j = C, D$, (2.4) becomes

$$V^j(t) = X_\sigma^j(t, \sigma(t)) = r(t, \sigma(t)) + \beta\zeta\big(1 - q(\{1\} \mid t, \sigma(t))\big)W^j(t)dt. \tag{5.2}$$

It is immediate to verify that:

Lemma 5.1. *For* $0 \leq t \leq 1$,

$$X_\sigma(t,\cdot) = (1-t)\cdot r_{\omega(t)}(\cdot) + \xi_\sigma(t,\cdot)$$

where r_ω *is defined in Section 4.2,* $\xi_\sigma^C \equiv \xi_\sigma^D \equiv 0$*, and*

$$\xi_\sigma^{-\{C,D\}}(t,a) = \beta\zeta(1 - q(\{1\} \mid t, \sigma(t))) \int_t^1 \gamma_\sigma^{-\{C,D\}}(t)$$

and ω *is given by*

$$\omega(t) = (\omega^C(t), \omega^D(t)) := \zeta\beta \int_t^1 \gamma_\sigma^{\{C,D\}}(z')dq(z,\sigma(z)) = \zeta\beta\cdot(W^C(t), W^D(t)).$$

Lemma 5.2. *For* ω *as in Lemma 5.1, we have* $||\omega||_\infty < \frac{1}{2}$.

Proof. Since $q([0,1) \mid \cdot) \leq \zeta$, it follows for $j \in \{C,D\}$,

$$|V^j| \leq \sum_{j=1}^{\infty} ||r||_\infty \cdot \zeta^{j-1} = \frac{||r||_\infty}{1-\zeta}$$

and hence (5.1) implies that

$$|\zeta \cdot W^j| < \frac{\zeta \cdot ||r||_\infty}{1-\zeta} < \frac{1}{2}$$

□

It is immediate that:

Lemma 5.3. *Let* g_1, g_2 *be two payoff functions on the same player set, such that for any Player* p *and any pair of pure action profiles* a, b *that differ (at most) in Player* p*'s action,*

$$g_1^p(a) - g_1^p(b) = g_2^p(a) - g_2^p(b)$$

Then the set of Nash equilibria under g_1 *is the same as the set of Nash equilibria under* g_2*.*

Note that under $\xi_\sigma(t,\cdot)$, each player's payoff is independent of his own action. Combining this observation with Lemma 5.3 (where $g_1(\cdot) = X_\sigma(t,\cdot)$ and $g_2(\cdot) = (1-t)r_{\omega(t)}(\cdot)$), Lemma 5.1, and Proposition 4.3, we deduce that:

- If $W^C(t) > 0$ (resp. < 0), $r^C(t,\sigma(t)) \leq -\frac{1}{2}$ (resp. $\geq \frac{1}{2}$).
- If $W^D(t) > 0$ (resp. < 0), $r^D(t,\sigma(t)) \leq -\frac{1}{2}$ (resp. $\geq \frac{1}{2}$).
- Regardless of the values of $W^C(t), W^D(t)$,

$$r^C(t,\sigma(t)) \neq 0 \text{ or } r^D(t,\sigma(t)) \neq 0 \tag{5.3}$$

Therefore, we can further deduce

- Since Lemma 5.2 implies that:

$$||r^{C,D}(z,\cdot) - X_{\sigma}^{C,D}(z,\cdot)||_{\infty} = ||\omega||_{\infty} < \frac{1}{2}, \tag{5.4}$$

  it follows from (5.2) that if $W^C(t) > 0$ (resp. < 0), then $V^C(t) < 0$ (resp. > 0), and similarly for V^D w.r.t. W^D.

- We deduce that for at least one $j \in \{C, D\}$, $V^j(t) \neq 0$: If $W^C(t) = W^D(t) = 0$, we deduce this from (5.3) and (5.2), while otherwise it follows from the case above.

Furthermore, it is known that for *a.e.* t, $\frac{dW^j}{dt}(t) = -V_j(t)$ for $j = C, D$. Define $G = (W^C)^2 + (W^D)^2$. The assumptions show that for at least one $j \in \{C, D\}$, W^j is non-zero somewhere, and hence G is not uniformly 0. Furtheremore, it holds a.e. that

$$G' = 2 \cdot W^c \cdot \frac{dW^C}{dt} + 2 \cdot W^D \cdot \frac{W^D}{dt} \geq 0$$

G is absolutely continuous, because both W^C, W^D are absolutely continuous and hence also bounded. Therefore, since $G' \geq 0$ a.e. and G is positive at some point, we deduce that $G(1) > 0$, a contradiction since $G(1) = 0$.

6 Games with State-Independent Transitions

Let $\Gamma = (\Omega, \mathcal{P}, (I^p)_{p \in \mathcal{P}}, r, q, \beta)$ be any stochastic game in with finite (or compact metric) action spaces, and compact metric state space, such that r, q are both continuous jointly in the state and action profile (where the space $\Delta(\Omega)$ is endowed with the norm-topology), and such that Γ does not possess a stationary equilibrium. Indeed, the example presented in Section 5 is such a game. We will use a standard trick to construct a new stochastic game $\Gamma' = (\Omega, \tilde{\mathcal{P}}, (J^p)_{p \in \tilde{\mathcal{P}}}, \tilde{r}, \tilde{q}, \beta)$, with the same state space and the same discount factor as Γ, with compact action spaces and jointly continuous $\tilde{r}, \tilde{q}$, such that $\tilde{q}$ is state-independent and Γ' possesses no stationary equilibria. We shall simply add an auxiliary player, whose action space is the state space, and who is incentivized in any sub-game perfect equilibrium to truthfully play the current space.

The details are as follows. Fix some metric d on Ω and define:

- $\tilde{\mathcal{P}} = \mathcal{P} \cup \{Z\}$.
- $J^p = I^p$ for each $p \in \mathcal{P}$, and $J^Z = \Omega$. A profile of actions will be denoted $(\omega^Z, (a^p)_{p \in \mathcal{P}})$.
- $\tilde{r}^p(\omega, (\omega^Z, (a^p)_{p \in \mathcal{P}})) = r(\omega, (a^p)_{p \in \mathcal{P}})$ for $p \in \mathcal{P}$, $\tilde{r}^Z(\omega, (\omega^Z, (a^p)_{p \in \mathcal{P}})) = -d(\omega, \omega^Z)$.

- $\tilde{q}(\omega, (\omega^Z, (a^p)_{p\in\mathcal{P}})) = q(\omega^Z, (a^p)_{p\in\mathcal{P}})$. (This is clearly independent of ω.)

It is clear that in any stationary[16] equilibrium, Player Z always chooses action $\omega^Z = \omega$ when in state ω. It is then immediate that if Γ' had a stationary equilibrium σ', then by defining $\sigma(\omega)$ to be the projection of $\sigma'(\omega)$ to $\prod_{p\in\mathcal{P}} I^p$ for each $\omega \in \Omega$, we would derive a stationary equilibrium in Γ, a contradiction.

Appendix A: Construction from Kohlberg and Mertens' Game

Let G_0 be the game defined in Figure 4.2; let $E_1, \dots, E_6$ denote the 6 equilibria, beginning with (L, L) and proceeding clockwise, and let A_i denote the arc from E_i to $E_{i+1,mod\ 6}$. Also, for a two-player game G, the game G', defined by $G'^i(a, b)) = G^{3-i}(b, a)$, is the game where the players and action profiles are switched.

Fix $\varepsilon > 0$; we begin by defining mappings $G^1, \dots, G^6, G^Z : [0, 1] \to \mathbb{R}^{2\times I\times J}$, and from these we will define Γ_ε.

- $G_1(t) :=$

$A\backslash B$	L	M	R
L	$1+\varepsilon, 1+(1-t)\varepsilon$	$\varepsilon, -1$	$-1+\varepsilon, 1+t\cdot\varepsilon$
M	$-1, (1-t)\varepsilon$	$0, 0$	$-1, t\cdot\varepsilon$
R	$1, -1$	$0, -1$	$-2, -2$

All equilibria in $G_1(t)$ lie on the arc A_1.

- $G_2(t) :=$

$A\backslash B$	L	M	R
L	$1+(1-t)\varepsilon, 1$	$(1-t)\varepsilon, -1$	$-1+(1-t)\varepsilon, 1+\varepsilon$
M	$-1, 0$	$t\cdot\varepsilon, 0$	$-1+t\cdot\varepsilon, \varepsilon$
R	$1-t\cdot\varepsilon, 0$	$-t\cdot\varepsilon, -1$	$-2, -2$

All equilibria of $G_2(t)$ lie along A_2.

- $G_3(t) :=$

$A\backslash B$	L	M	R
L	$1, 1-2t\cdot\varepsilon$	$-t\cdot\varepsilon, -1$	$-1, 1-2(t-\frac{1}{2})\varepsilon$
M	$-1, -t\cdot\varepsilon$	$\varepsilon, t\cdot\varepsilon$	$-1+\varepsilon, -2(t-\frac{1}{2})\varepsilon$
R	$1-\varepsilon, -1$	$-\varepsilon, -1+t\cdot\varepsilon$	$-2, -2$

All equilibria of $G_3(t)$ lie along A_3.

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[16] Or in any sub-game perfect behavioral equilibrium, for that matter.

- $G_Z(t) =$

$A\backslash B$	L	M	R
L	$1-2t\varepsilon, 1-2(1-t)\varepsilon$	$-\varepsilon, -1$	$-1, 1-\varepsilon$
M	$-1, -\varepsilon$	ε, ε	$-1+\varepsilon, -\varepsilon$
R	$1-\varepsilon, -1$	$-\varepsilon, -1+\varepsilon$	$-2, -2$

For $t < \frac{1}{2}$ or $t > \frac{1}{2}$, the unique equilibrium of $G_4(t)$ is (M, M).

$G_Z(\frac{1}{2}) =$

$A\backslash B$	L	M	R
L	$1, 1$	$-\varepsilon, -1$	$-1, 1-\varepsilon$
M	$-1, -\varepsilon$	ε, ε	$-1+\varepsilon, -\varepsilon$
R	$1-\varepsilon, -1$	$-\varepsilon, -1+\varepsilon$	$-2, -2$

which has pure equilibria (L, L) and (M, M), and the mixed equilibrium,

$$(x^*, y^*) = \big((\frac{\varepsilon}{1+\varepsilon}, \frac{1}{1+\varepsilon}, 0), (\frac{\varepsilon}{1+\varepsilon}, \frac{1}{1+\varepsilon}, 0)\big) \tag{6.1}$$

which satisfies $||(x^*, y^*) - (M, M)||_\infty = \frac{\varepsilon}{1+\varepsilon} < \varepsilon$.

- Since $G_3(0) = G'_Z(1)$, retrace our steps in the transposed games; we get

$$G_4(t) := G'_3(1-t)$$

$$G_5(t) := G'_2(1-t)$$

$$G_6(t) := G'_1(1-t)$$

In each of these cases, all equilibria of G_j lie along A_j.

We then define

$$\Gamma_\varepsilon(p, q) = \begin{cases} G_4(\frac{1}{2}(1+p)) & \text{if } q = 1 \\ G_6(\frac{1}{2}(1-q)) & \text{if } p = 1 \\ G_6(\frac{1}{2}(1-p)) & \text{if } q = -1 \\ G_1(2(p+1)) & \text{if } q = -1, p \leq -\frac{1}{2} \\ G_2(2(p+\frac{1}{2})) & \text{if } q = -1, -\frac{1}{2} \leq p \leq 0 \\ G_3(2p) & \text{if } q = -1,\ 0 \leq p \leq \frac{1}{2} \\ G_Z(2(p-\frac{1}{2})) & \text{if } q = -1,\ \frac{1}{2} \leq p \leq 1 \end{cases} \tag{6.2}$$

To see this more clearly, denote $H_j = G_j(0)$ for $j = 1, \ldots, 6, Z$. Then the map Γ_ε is the piecewise linear map given by the following diagram:

$(p,q) \in S$	$\Gamma_\varepsilon(p,q) \in \mathbb{R}^{2\times\lvert I\rvert\times\lvert J\rvert}$
$\begin{tikzcd} (-1,1) \arrow[r, "S_{\mathcal{N}}"] & (1,1) \arrow[d, "S_{\mathcal{E}}"] \\ (-1,-1) \arrow[u, "S_{\mathcal{W}}"] & (1,-1) \arrow[l, "S_{\mathcal{S}}"] \end{tikzcd}$	$\begin{tikzcd} H_4 \arrow[r, "G_4"] & H_5 \arrow[dddd, "G_5"] \\ H_Z \arrow[u, "G_Z"] & \\ H_3 \arrow[u, "G_3"] & \\ H_2 \arrow[u, "G_2"] & \\ H_1 \arrow[u, "G_1"] & H_6 \arrow[l, "G_6"] \end{tikzcd}$
Figure 6.3.a	Figure 6.3.b

(6.3)

From these figures and the explicit forms $G_1, \ldots, G_6, G_Z$ and their equilibria properties listed above, it is immediate that Γ_ε satisfies property (6) in Section 4.1.

Appendix B: Piecewise Linear Games on the Square

In this section, we prove Proposition 4.2. We recall the following proposition, from [17]:

Proposition 6.1. *We use the notations and conventions (in particular, that all metrics are w.r.t. the supremum norm) introduced at the beginning of Section 4.1.*

Let $f : [0,1] \to (0,1)$ *be a continuous, piecewise linear function. Then there exist*[17] *an integer* $N \geq 0$ *and two normal-form games,* $\mathfrak{G}^L$ *and* $\mathfrak{G}^R$*, on the set of players* $A, B, \alpha^1, \ldots, \alpha^{N-1}$*, each with action space* $\{L, R\}$*, such that for any* $p \in [0,1]$*, denoting*

$$\mathfrak{G}(p) := p \cdot \mathfrak{G}^L + (1-p)\mathfrak{G}^R$$

it holds that in any equilibrium of $\mathfrak{G}(p)$*, Players* A, B *play a mixed action profile* $(f(p), 1-f(p)) \times (f(p), 1-f(p))$*.*

Remark 6.2. The construction above has other properties:

(i) The payoffs of each of the (α^j) - these players will be referred to as auxiliary players - are independent of the actions of any other player; hence, we can refer to the matrix $G(p)$, which is the expected matrix facing players A, B

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[17] $N+1$ is the number of segments into which $[0,1]$ has to be divided into in order for f to be linear in each segment.

when each of the α^j plays an optimal action; this turns out to be well-defined, as when any α^j are indifferent in $\mathfrak{G}(p)$ for some p, any choices yields the same expected payoffs for players A, B.

(ii) Suppose f, g are two continuous, piecewise linear functions, and $G(p), H(q)$ are the associated normal-form games facing players A, B when auxiliary players maximize for the parameters $p, q \in [0, 1]$. Then $f(p) = g(q)$ implies that $G(p) = H(q)$.

(iii) If L is a Lipschitz constant of f, and $\max_{[0,1]}[f] - \min_{[0,1]}[f] \leq \kappa$, then $\max[\mathfrak{G}^k] - \min[\mathfrak{G}^k] \leq L\kappa$ for $k \in \{A, B\}$; the maximum and minimum are taken over all action profiles.

Proposition 6.3. *Let S be the boundary of the square:*

$$S = \{(p, q) \mid -1 \leq p, q \leq 1,\ (|p| = 1) \vee (|q| = 1)\}$$

and let $g : S \to (0, 1)$ be a continuous and piecewise linear[18] *map. Then for some integer K, there exists four normal form games on the set of players $A, B, \gamma, \delta, \beta^1, \ldots, \beta^K$, denoted $\mathfrak{H}^k$ for $k \in \{-1, 1\}^2$, such that:*

- *A, B and also each of the (β^j) has the action set $\{L, R\}$, and for each j and $k \in \{L, R\}$, the payoff of β^j in $\mathfrak{H}^k$ is independent of any other player's action.*
- *γ, δ have action set $\{-1, 1\}$.*
- *If Nature chooses $k \in \{1, -1\}^2$ with distribution*[19] *$(\frac{1+p}{2}, \frac{1-p}{2}) \otimes (\frac{1+q}{2}, \frac{1-q}{2})$, $(p, q) \in S$, and $\beta^1, \ldots, \beta^k$ all play best responses, then the unique equilibrium of the expected game facing A, B is $(g(p), 1 - g(p)) \times (g(p), 1 - g(p))$.*

Proof. We denote the vertices of the square S by

$$\begin{tikzcd}
V_{-,+} = (-1, 1) \arrow[r, "S_{\mathcal{N}}"] & V_{+,+} = (1, 1) \arrow[d, "S_{\mathcal{E}}"'] \\
V_{-,-}(-1, -1) \arrow[u, "S_{\mathcal{W}}"] & V_{+,-} = (1, -1) \arrow[l, "S_{\mathcal{W}}"]
\end{tikzcd}$$

For $i \in \{-, +\}^2$, let $i+$ be such that V_{i+} follows V_i in the clockwise orientation. for $i \in \{-, +\}^2$, let (g_i) be function of one parameter, which is the restriction of g to the arc extending clockwise from V_i; that is, $g_i(0) = g(V_i)$ and $g_i(1) = g(V_{i+})$.

For $j \in \{-, +\}^2$, let N_j correspond to g_j as in Proposition 6.1. Then let $K = \sum_j (N_j - 1)$; and also treat K as the set $\{1, \ldots, K\}$ partitioned into subsets $N_{V_{\pm,\pm}}$ of sizes $N_{\pm,\pm} - 1$. For each $k \in \{-, +\}^2$, let $\mathfrak{G}_k^m$, $m = L, R$, be

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[18] That is, piecewise linear on each of the four edges of S

[19] $(\phi, 1 - \phi)$ denotes the probability distribution choosing 1 with probability ϕ, and choosing -1 with probability $1 - \phi$.

the two games that correspond to g_k on the set of players $A, B, \beta^1, \ldots, \beta^K$, as in Proposition 6.1 (the auxiliary players which were there denoted $(\alpha^j)_{j<N_{V_k}}$ are now $(\beta^j)_{j\in N_{V_k}}$ - i.e., $(\beta^j)_{j<K} = \cup_{k\in\{-,+\}^2}(\alpha^j)_{j<N_{V_k}}$, where the union is disjoint - and β^j is given a payoff of 0 in $\mathfrak{G}_k^m$ for each $j \notin N_{V_k}$.) For each $k \in \{-,+\}^2$, let $G_k(p)$ denote the corresponding matrix to A, B when auxiliary players play optimally in $\mathfrak{G}_k(p)$; as we have mentioned in property (i) of Remark 6.2, this bimatrix game is well defined. Finally, note that $G_k(1) = G_{k^+}(0)$ by property (ii) of Remark 6.2.

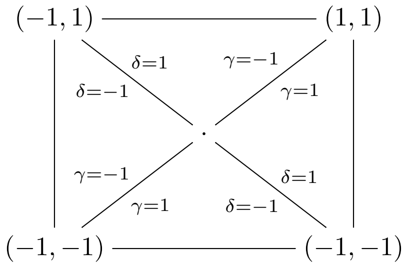
We can now define $(\mathfrak{H}_k)_k$ from the $(\mathfrak{G}_k)_k$ as follows. First, define the payoffs to γ, δ. For each of these players, the payoff is determined only by k and his own action. The payoffs to γ in the various games are given by the following table:

	$k = (1,1)$	$k = (1,-1)$	$k = (-1,-1)$	$k = (-1,1)$
γ plays 1	0	1	0	-1
γ plays -1	0	-1	0	1

and the payoffs to δ by

	$k = (1,1)$	$k = (1,-1)$	$k = (-1,-1)$	$k = (-1,1)$
δ plays $+1$	1	0	-1	0
δ plays -1	-1	0	1	0

The diagram below describes the best-replies of γ, δ when Nature chooses $k \in \{+1,-1\}^2$ via the distribution $(p, 1-p) \otimes (q, 1-q)$ (with γ,δ, and Nature making their choices simultaneously). In the diagram, this (mixed) choice of Nature is represented by the point with coordinates $(2p-1, 2q-1)$, and the best-reply profile of γ, δ depends on which of the four regions in the square Nature chooses.



(6.4)

On one diagonal, γ will be indifferent; on the other, δ will be.

Now, we define the payoffs to Players $A, B, \beta^1, \ldots, \beta^K$. Given the choice of Nature $k \in \{+1,-1\}^2$, the actions of γ and δ determine which game $A, B, \beta^1, \ldots, \beta^K$ face, as depicted in the following table ($*$ denotes an arbitrary action):

Game	Action of γ	Action of δ	Game Facing $A, B, \beta^1, \ldots, \beta^K$
$\mathfrak{H}_{1,1}$	-1	$*$	$(\mathfrak{G}_{-,+})^R$
$\mathfrak{H}^{1,1}$	1	$*$	$(\mathfrak{G}_{+,+})^L$
$\mathfrak{H}^{1,-1}$	$*$	-1	$(\mathfrak{G}_{+,+})^R$
$\mathfrak{H}^{1,-1}$	$*$	1	$(\mathfrak{G}_{+,-})^L$
$\mathfrak{H}^{-1,-1}$	-1	$*$	$(\mathfrak{G}_{+,-})^R$
$\mathfrak{H}^{-1,-1}$	1	$*$	$(\mathfrak{G}_{-,-})^L$
$\mathfrak{H}^{-1,1}$	$*$	-1	$(\mathfrak{G}_{-,-})^R$
$\mathfrak{H}^{-1,1}$	$*$	1	$(\mathfrak{G}_{-,+})^L$

Since we have already noticed that $G_k(1) = G_{k^+}(0)$ for all k, it is easy to verify that these are the required games.

□

Proof. (of Proposition 4.2) It's straightfoward to see that it suffices to prove the case[20] $0 < Q < 1$. For each $(p, i, j) \in \{A, B\} \times I \times J$, let $Q_{p,i,j} : S \to \mathbb{R}$ be the corresponding component of Q; and for each such piecewise linear function, let $(\mathfrak{H}^k_{p,i,j})_k$ be the four corresponding games from Proposition 6.3, on the list of players $P_{p,i,j} := A_{p,i,j}, B_{p,i,j}, \gamma_{p,i,j}, \delta_{p,i,j}, \beta^1_{p,i,j} \ldots, \beta^{N_{p,i,j}}_{p,i,j}$ for some $N_{p,i,j}$. When nature chooses $k \in \{-1, 1\}^2$, each set of players $P_{p,i,j}$ plays $\mathfrak{H}^k_{p,i,j}$, and the payoff to Player A (resp. B) when action profile (i, j) is played is 1 if $A_{p,i,j}$ plays L, and 0 if he plays R. We then take $\theta^1, \ldots, \theta^M$ to be some enumeration of $\cup_{p,i,j} P_{p,i,j}$. Property (IV) follows from property (iii) of Remark 6.2. □

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