An Evolutionary Bargaining Model

Murali Agastya*
Center for Rationality and
Interactive Decision Theory
Feldman Building
Hebrew Univ., Givat Ram
Jerusalem 91904
ISRAEL
Email: murali@math.huji.ac.il

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Abstract

Varying quantities of a single good can be produced using at least two and at most *n* factors of production. The problem of allocating the surplus among the factors is studied in a dynamic model with adaptive behavior. Representatives for the factors (called players) make wage demands based on precedent and ignorant of each other's utilities for this good. Necessary and sufficient conditions are provided under which the long-run equilibria coincide with the core allocations. A global convergence result is proved to show that players do learn to reach a core allocation in the long run. Moreover, allowing for the possibility of mistakes by the players, all the *stochastically stable outcomes* are characterized. The main result shows that in the limit, these stable allocations for a particular set of players converges to the allocation that maximizes the product of all the players' utilities over all core allocations.

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1 Introduction

The problem of allocating the surplus among various factors of production, or equivalently characteristic function bargaining has been been studied in a variety of non-cooperative and/or axiomatic models. While there can hardly be anything indeterminate in the predictions of axiomatic models, the story is quite different for the non-cooperative models with three or more players. Many natural adaptations of the Rubinstein (1982) result in a variety of folk theorems[1]. A problem of considerable interest then is as to which of the many equilibria is (are) most likely to be played.

Analysis of strategic interaction places great demands on the knowledge and rationality of the players. These demands are magnified when there are several possible equilibria to choose from. Although it is true for more general games, this problem is especially severe in bargaining situations where there are several pareto-optimal equilibria[2]. For, there is a paucity of eductive explanations to select between several pareto-optimal equilibria. Consequently, and only recently, the study of explicit dynamic learning processes through which agents learn the way to play a game has commanded greater attention. The idea is to see which equilibrium, if any, the players will learn to play under simple rules of behavior and minimal assumptions on their knowledge.

In this paper, I study the allocation problem in the context of a simple dynamic learning model. At each date a single good can be produced. The production possibilities in a period are described by a characteristic function f defined on the class of all subsets of a finite set of factors. Here, $f(S)$ is the quantity of a single output that can be produced using the resources of the subset S alone. At each date, one representative for each factor, henceforth referred to as a player, demands

[1] See Sutton(1985) for a survey. See also Perry and Reny (1991), Serrano and Vohra(1993), Chatterjee *et.al* (1993), Winter(1992).

[2] For example, in Perry and Reny (1991), every core allocation is a subgame-perfect equilibrium of the negotiation process described therein.

a certain amount of the surplus as a wage for his services. Demands are made simultaneously and players are committed to these demands for the period. There is no assumption of common knowledge of players' utilities. Players use statistical data from a random sample from recent history to demand a wage for the current period. Demands are made only in integral multiples of δ, a smallest money unit. Moreover, players are myopic and play a best response by maximizing the one period expected utility. The best response rules of the above dynamic process, induce a stationary Markov Process referred to as Evolutionary Bargaining Process (EBP). This dynamic process itself is due to Young(1993a) and is a variation on fictitious play.

There is a one to one correspondence between the set of absorbing states of the EBP, referred to as conventions, and the Strict Nash equilibria of a simultaneous move demand commitment game. In this game, players simultaneously demand a part of the surplus and are committed to their demands. This is of course the original Nash demand commitment game. The difference and difficulty arises from the fact that unlike in Nash's original demand game where a pie is to be split between several players, here players make demands over a general characteristic function. Consequently a given set of demands can perhaps be met in several coalitions. This uncertainty regarding coalition formation coupled with the fact that we are dealing with *individual demands* and *not proposals directed towards coalitions* makes the problem interesting and somewhat more complex. In section 2, the game is analyzed. It is shown that if the technology displays increasing returns to scale in the *number of factors* [3], then core allocations coincide with the (strict) Nash equilibria. Needless to say, certain assumptions regarding coalition formation are also required.

Section 3 studies the dynamics of the EBP. Theorem 1 presents sufficient conditions under which the only persistent states are absorbing. In other words,

[3]Notice that this does not necessarily require increasing returns in the extent of use of particular factor(s)

under the hypotheses of Theorem 1, players learn to reach a core allocation regardless of the starting point.

Section 3.3 then studies the dynamics of the EBP by allowing for the possibility of the players making *mistakes*. Players play a best response with a very high probability but with a small but positive probability, they commit errors. Consequently there are no longer any absorbing states. It is possible to move from one convention to another. However, some conventions are harder to displace than others. When the probability of making errors is small, those conventions that are hardest to displace that will be observed most often. These conventions are referred to as *Stochastically Stable Conventions*, hereafter SSC, and constitute the main notion of refinement of equilibria.

Theorem 3 characterizes the set of all SSCs. There may be several such allocations. However, (Theorem 4) when the size of the smallest money unit δ shrinks to zero, the allocations in the SSCs, for a particular subset of players, converge to the allocation that maximizes the product of all the players' utilities over core allocations, a seeming generalization of the Nash Bargaining solution. For the three player case however, one has uniqueness.

Young (1993b) is the closest relative of this paper. Young considers a very special technology and restricts attention to the case of two players. With three or more players, the possibility of cycles in the dynamic process makes analysis here much more complex. To obtain a tractable proof, I assume that players only make 'small mistakes'. But this in turn creates a different technical problem. The perturbed EBP is no longer strongly ergodic. Since the refinements in models that study naive behavior [4] are based on the convergence properties of the invariant distributions of the perturbed process, their techniques are not immediately applicable.

Binmore (1987) and Carlsson (1991) analyze perturbations of the original

[4] See for e.g., Foster and Young (1990), Kandori *et.al* (1993), Young (1993a and 1993b)

Nash demand game and study the properties of non-cooperative equilibria of the perturbed game as the perturbations become very small. They show that the Pareto optimal Nash equilibria of the perturbed game converge to the original Nash Bargaining solution. The model presented below (as does Young (1993b)) differs from the above models in that they presuppose the equilibrium of both the original as well as the unperturbed game. In this model on the other hand, players reach long-run equilibria as the likelihood of mistakes becomes very small, without any knowledge of the utilities of other players.

The paper is organized as follows. Section 2 presents the one shot demand game and characterizes the set of equilibria. Section 3 studies the EBP. Section 4 concludes with a few examples. Most formal proofs appear in Appendix A and Appendix B.

2 The Demand Game

Let $N = \{1, 2, \ldots n\}$ denote the set of factors of production. I will assume $n \geq 3$. The technological possibilities are described by a non-negative function f defined on the class of all coalitions of N. By way of interpretation, $f(S)$ is the total quantity of a single good that can be produced using the services of the factors in S alone. I will assume that the function is normalized so that $f(i) = 0$ for all $i \in N$.

Let Δ be the half open interval $(0, f(N)]$. One representative for each factor of production demands a wage from Δ. We will refer to these representatives as players. Let ω_i denote a typical wage demand of player i. Given a vector of wage demands W, let $W(S)$ denote the total wages demanded by the coalition S . The demands of S are said to be *compatible* at W, if $W(S) \leq f(S)$. The demands are *strictly compatible* if the inequality is strict. Let $\beta(W)$ ($\hat{\beta}(W)$) denote the set of all coalitions in whose demands are compatible (strictly compatible).

Players are assumed to make their demands simultaneously and are committed to their demands. A player' demand is not met unless his services are used. By the assumption of commitment, a player cannot have his demand met unless there is a coalition S to which he belongs and the demands of the coalition as a whole are compatible. However, a particular W may be compatible in several coalitions. Thus even under the reasonable hypothesis that only compatible coalition(s) will eventually form, there is still some uncertainty whether a player's demand ω_i will be met. To capture this uncertainty, let $p_i(\omega_i|W_{-i})$ denote the probability belief that player i's demand ω_i is met if others demand W_{-i}. This probability may be thought of as the result of an exogenous matching process. By allowing for p to depend on i and making assumptions directly on the p_i, we allow for possibility a certain amount of subjective uncertainty on the part of the players regarding this matching process. The following will be assumed of the matching process (or beliefs) at the very outset.

Assumption 1 *Fix W. Then for each $i \in N$,*

1. *If i is not the member of <u>any</u> coalition whose demands are compatible, then $p_i(\omega_i|W_{-i}) = 0$.*
2. *For all $\omega > \hat{\omega}$, $p_i(\hat{\omega}|W_{-i}) \geq p_i(\omega|W_{-i})$).*

Assumption 1 is natural enough and requires no further comment. Now, assuming that a player obtains zero if his demand is not met, the expected utility of player i if he demands ω_i when others have demanded W_{-i} is

$$U_i(\omega_i|W_{-i}) = p_i(\omega_i|W_{-i})v_i(\omega_i) + [1 - p_i(\omega_i|W_{-i})]v_i(0) \tag{1}$$

where the function v_i determines the utility derived from consuming ω_i. I will assume that v_i is a strictly increasing, continuously differentiable and concave function. Finally, assuming that players maximize expected utility and normalizing $v_i(0) = 0$, we have the following usual definition of a strict Nash equilibrium:

Definition 1 *A vector of wage demands W^* is a strict Nash equilibrium iff*

$$U_i(\omega_i^*|W_i^*) > p_i(\omega_i|W_{-i}^*)v(\omega_i).$$

if $\omega_i^* \neq \omega_i$, for all $i \in N$.

In this paper, Nash equilibrium will always be taken to mean the equilibrium defined above.

Definition 2 *A vector of wage demands W belongs to the core of the technology f,*

1. *$f(N) = W(N)$. (Feasibility)*
2. *For all $S \subseteq N$, $W(S) \geq f(S)$. (No blocking sub- coalitions.)*

Let $\mathcal{C}(f)$ denote the set of all core allocations.

At a first glance it seems somewhat intuitive that the core and the set of Nash equilibria are related. However, recall that in this model, players make demands and not proposals directed towards a particular coalition. When a player makes a proposal, not only does he seek a payoff for himself, but also specifies a payoff for a subset of other players as well. Hence, implicitly he also suggests which coalition is to form. In the present model, however, players independently and simultaneously make their demands. Hence it it possible that individual players may make demands that are not pareto-optimal. At this point it may not pay for any of them to deviate by responding with higher demands as it will, by Assumption 1, will reduce the likelihood with which it is met. Similarly, it may also be the case that the players may demand wages much higher than what the grand coalition can afford but expect to obtain these in smaller coalitions.

Example 1 Let $N = \{1, 2, 3\}$, $f(ij) = 1$ for all i, j and $f(123) = 2$. Let $v_i = v$ for all i. Assume that the matching process is such that for a given W, all coalitions in $\beta(W)$ are equally likely. It may be checked, that this matching process is consistent with Assumption 1. Now consider $W^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Since the demands of all the three two player coalitions are compatible as well as those of the grand coalition, the payoff of player i on demanding $\frac{1}{2}$ at W^* is

$$U_i(\frac{1}{2}|W^*_{-i}) = \frac{3}{4}v(\frac{1}{2}) \tag{2}$$

Now consider a deviation $\hat{\omega}_1$ by 1. Clearly, since the set of compatible coalitions does not change, a demand strictly less than 1/2 is strictly dominated by 1/2. Furthermore, any demand $\hat{\omega} > 1/2$, is strictly dominated by a demand of 1, unless $\hat{\omega} = 1$. Indeed, if he responds with 1, the new set of demands are compatible in the $\{2, 3\}$ coalition and N. Since each is equally likely to form, the payoff is

$$U_i(1|W^*_{-i}) = \frac{1}{2}v(1) \tag{3}$$

For an appropriately concave v, the demand 1 can be seen to be strictly dominated by 1/2. Since all players are symmetric, we have shown that on the one hand $(1/2, 1/2, 1/2)$ that is not in the core is a Nash equilibrium while on the other hand, $(1, 1/2, 1/2)$, an element of the core is

not a Nash equilibrium.

The above example demonstrates two things. First, there is no immediate relation between core allocations and Nash equilibria. Second, Assumption 1 is not sufficient to rule out rather unintutive outcomes. Note that in the above example, every feasible coalition is assumed to form with a strictly positive probability. Thus, the problem is not one of beliefs which assign probability zero to coalitions that can form. In the example, players believe that the two player coalitions form although no player is "hurt" by the formation of the grand coalition. Since the players are assumed to be committed to their demands, it does not seem unreasonable to assume that ties will be broken if favor of the larger coalition. Assumption 2 below is a weaker statement of this intuition.

Assumption 2 *Fix W. Suppose that there is a set S^* such that*

- *The demands of S^* are compatible,*
- *For every coalition T whose demands are compatible, the demands of $T \cup S^*$ are also compatible.*

Then, $p_i(\omega_i|W_{-i}) = 1$ *for all* $i \in S^*$.

The demands of S^* are compatible. However, the demands of a different coalition T may also be compatible and this might in fact require the services of a subset of players in S^*. Assumption 2 then requires that if it never "hurts" anyone to accommodate the demands of the remaining players in S^*, then all the players in S^* will be matched.

Proposition 1 *Under Assumption 1 and Assumption 2, every element of the core is a Nash equilibrium.*

Proof: Let W^* be an element of the core. Hence $W^*(N) = f(N)$. Take $S^* = N$ in Assumption 2. It then follows that $p_i(\omega_i^*|W_{-i}^*) = 1$ for all $i \in N$. A deviation by any player i by demanding $\omega_i > \omega_i^*$, implies the demands are not compatible in any coalition containing him. Consequently, $p_i(\omega_i|W_{-i}^*) = 0$. Hence, it is not profitable to ask for more than ω_i^*.

The only reason that a player may ask for less than ω_i^* if it can be met with a higher probability than ω_i^*. But since $p_i(\omega_i^*|W_{-i}^*) = 1$, ω_i^* is the unique best response. □

The following example shows that the converse of Proposition 1 does not hold without further restrictions.

Example 2 Let $f(12) = f(13) = 300$, $f(123) = 302$. Assume that $v_i(\omega) = \omega$. Under the hypotheses of Proposition 1, all core allocations are Nash equilibria. In any core allocation, neither player 2 nor 3 gets more than 2.

Now note that for all $\omega > 4$, all demands of the form $(300 - \omega, \omega, \omega)$ are compatible in exactly the $\{1, 2\}$ or $\{1, 3\}$ coalitions. Assume that each of these are equally likely to form. Again, this is consistent with Assumptions 1 and 2. It is straightforward to check that the above demands constitute a Nash equilibrium. Indeed, the highest payoff of either player 2 (or 3) can get from deviation is 2 which is strictly smaller than his expected payoff with the current demands.

Proposition 2 *Under Assumption 1 and 2, if the technology is convex, i.e.,*

$$f(S) + f(T) \leq f(S \cup T) + f(S \cap T) \qquad \text{for all } S, T \subseteq N.$$

then a Nash equilibrium demand vector W^* is in the core of the technology.

Proof. The proof involves showing that if the demands of a certain coalition are strictly compatible at W, then it is possible to find a coalition S^* such that whenever T is compatible, $S^* \cup T$ is strictly compatible. Thus, a player in this coalition can increase his demand by a positive amount and yet ensure that it is met with probability one, by virtue of Assumption 2. Hence, at a Nash equilibrium, no coalition's demands are strictly compatible. However, for each player, there must be at least one coalition in which his demands are compatible. This is sufficient to now conclude that the demands must be feasible in the grand coalition. Details appear in Appendix A.

3 Dynamic Adjustment Process: Adaptive Play

In this section I describe a dynamic process. I consider the case where the demand game of section 2 is played repeatedly at discrete dates. Players are assumed to be myopic, have a bounded memory and make demands based on precedent.

More precisely, time is denoted by, $t = 1, 2, \ldots$. At each date t, players simultaneously demand a wage. To steer clear of the complications that arise from infinite dimensional strategy spaces, the strategy spaces are discretized.

Towards this end, I first assume that each $f(S)$ is a rational number, say p_S/q_S. Let $M = \prod_{S\subseteq N} q_S$. For a given positive integer p, the number let $\delta = 1/(10^p M)$ is the precision[5] of the model. Players will be assumed to demand in integral multiples of δ. Let Δ_δ denote the subset of Δ consisting of integral multiples of δ. A central concern of the paper is the behavior of model as $\delta \to 0$ (or equivalently large values of $p \to \infty$).

Let $\mathcal{C}_\delta(f)$ denote the set of core allocations with each $\omega_i \in \Delta_\delta$. It is important to note that with the above method of discretization, $\mathcal{C}_\delta(f)$ can be defined exactly like $\mathcal{C}(f)$. That is, we do not encounter problems of demands in $\mathcal{C}_\delta(f)$ being strictly compatible in coalition because players are allowed to demand from Δ_δ only.

In the sequel, $\omega_i \in \Delta_\delta$ unless specified otherwise.

Let ω_{it} and W^t denote a typical demand of a player i and the vector of wage demands at date t respectively. The complete *history* up to and including period t is a sequence of demands $W^1, W^2, \ldots W^t$. Consider a typical player i who has to make a demand at date t. Players have no knowledge or a prior regarding the utility of the other players. To determine an optimal response, they have to rely on the historical records.

Fix $\alpha_1 \leq \alpha_2 \leq \ldots \alpha_n$ where each α_i is a rational number. An integer m is said to be *admissible* if $\alpha_i m$ is an integer for all i. Let $k_i = \alpha_i m$

Fix an admissible m. Player i samples at random a fraction of at most α_i of the last m records, $\mathbf{s}= (W^{t-m+1}, W^{t-m+2}, \ldots W^t)$. It is important for this model that every subsample at most of size $\alpha_i m$ is sampled with a positive probability. However one need make no assumptions on the relative probabilities with which different parts of the history are sampled. The variable α_i is a measure of player i's *information*.

Recall that the history up to date t consisted of only past demands. In particular, the history was silent as to which coalition has formed when similar demands were

[5] The dependence of δ on p is suppressed for conserving on notation.

in place. Consider a player who has picked the sample $\sigma_i = (W^1, W^2, \ldots W^{k_i})$. Player i believes that his demand ω is met with probability $F_i(\omega|\sigma_i)$ where

$$F_i(\omega|\sigma_i) = \frac{1}{k_i}[p_i(\omega|W_{-i}^1) + p_i(\omega|W_{-i}^2) + \ldots + p_i(\omega|W_{-i}^{k_i})].$$

The expected payoff of player i upon demanding ω following the sample σ_i is

$$F_i(\omega|\sigma_i)v_i(\omega) \tag{4}$$

Hence if the state at time t is $\mathbf{s}$, the wage demand at $t + 1$ must satisfy

$$\omega_{i(t+1)} = \arg\max F(\omega|\sigma_{it})v_i(\omega) \tag{5}$$

for some sample σ_{it} of size k_i from the state $\mathbf{s}$. If there are several values of ω that solve Eq. (5) above, then each of them is played with a strictly positive probability.

The above model is similar to fictitious play in the sense that players make their demands naively based on empirical distributions. Unlike in fictitious play, where a player samples the entire history, in the above process, a player samples only a fraction of the most recent history. This process has been termed *adaptive play* by Young (1993a). Since players are sophisticated enough to actually play a best response at each date, it seems that the above behavioral process, as is fictitious play, makes sense only if one assumes that players do not know each other's utility functions.

The response rules of the players (determined by Eq. 5), determine a stationary Markov chain. The state space Ω consists of all sequences $\mathbf{s}$ of length m. Each entry of $\mathbf{s}$, is a vector of wage demands by the agents.

Let $p_i^*(\omega_i|\mathbf{s})$ denote the probability with which player i demands ω_i in the state $\mathbf{s}$. For each i, p_i^* is a best response distribution, i.e., $p^*(\omega_i|\ \mathbf{s}) > 0$ iff ω_i solves equation 5 for some sample σ_i in $\mathbf{s}$.

For a state **s**$= (W^1, W^2, \ldots W^m)$, a state $\mathbf{s}^{'} = (W^2, W^3, \ldots, W^m, W)$ is said to be its successor. The probabilities are

$$P^0_{\mathbf{ss}^{'}} = \begin{cases} \prod_{i \in N} p_i^*(\omega_i|\mathbf{s}) & \text{if } \mathbf{s}^{'} \text{ is a successor of } \mathbf{s} \\ 0 & \text{otherwise.} \end{cases}$$

Let P^0 denote the matrix of the above transition probabilities. As in Young (1993b), the above Markov process, with the state space Ω, and transition probability matrix P^0 is said to be an *evolutionary bargaining process* (EBP) with memory m, precision δ, information parameters $\{\alpha_i\}$ and best reply distributions $\{p_i^*\}$.

3.1 Conventions

Definition 3 (Convention) *A state* **s** *is said to be a convention iff it is an absorbing state of the evolutionary bargaining process,* i.e., $P^0_{\mathbf{ss}} = 1$.

From a given state, there is a positive probability of reaching only an immediate successor. Thus, if **s** is a convention and is the state at time t, then **s** is the state at time $t + 1$ as well. For this to be true, it is clear that **s** must be a sequence of m identical demands. Now, let **w** denote the state in which each entry is the vector of demands W. Suppose that the process is in state **w** at time t. Then regardless of which sample player i picks, the probability with which his demand will be met is given by

$$F_i(\omega|\sigma) = p_i(\omega_i|W_{-i}) \tag{6}$$

An easy application of Eqs. 1, 4, 5 and yields Proposition 3 below.

Proposition 3 *A state* **w** *is a convention iff W is a Nash equilibrium of the demand game.*

Proposition 3 identifies the isomorphism between the conventions and Nash equilibria of the one shot demand game. Once the players reach a convention, they are locked into playing the game in a particular way. Thus, by establishing the global convergence from an arbitrary initial state to a convention, we describe a particular way in which players learn to play the Nash equilibrium of the demand game. This is the object of the next subsection.

3.2 Convergence of the EBP

In a Markov Process the set of all aperiodic states are either transient or persistent. If the set of all persistent states are absorbing, then the EBP converges from an arbitrary state to some absorbing state, with probability one. Theorem 1 below presents sufficient conditions under which the only persistent states are conventions.

Theorem 1 *Suppose that* $\alpha_i \leq \frac{2}{(n^2-n+4)}$ *for all i. Then the EBP converges with probability one to a convention.*

It is useful to point out that the bound in the Theorem is only a sufficient condition that is independent of f. For particular cases it is possible to give a much sharper bound. For example, for the standard Nash Bargaining game where $f(N) = 1$ and $f(S) = 0$ if $S \neq N$, $\frac{k_i}{m} \leq \frac{1}{3}$ is sufficient. The same bound is also sufficient for the three player case with an arbitrary f. This is proved below. However, for all subsequent results I assume that the hypothesis of Theorem 1 is met. The proof of the general case appears in Appendix B.

Proof. Assume, for ease of exposition, that $\alpha_i = \alpha \leq 1/3$ for all i. Let the process be in state **s** at date t. Let σ denote the last k elements of this state. Let W^1 denote a best-response vector to the sample σ. Since every sample of size k has a positive probability of being sampled, there is a positive probability of observing W^1 at date $t+1$. In fact, there is a positive probability of observing a run of W^1 between $t+1$ and $t+k$. If $\mathbf{w}^1$ is a convention, we are done. For, between $t+1$

and $t+2k$ all the players will sample the records containing W^1 alone and respond with ω_i^1. Hence there is a positive probability p of reaching a convention in $m+2k$ periods. So the probability of not reaching a convention in $r(m+2k)$ periods at most $(1-p)^r$, which goes to zero as $r \to \infty$. Suppose that W^1 is not in the core but is compatible in at least one of the three two player coalitions say $\{12\}$. Then, between $t+k+1$ and $t+2k$, there is a positive probability that players 1 and 2 will continue to sample σ, while player three will sample the records consisting of W^1 alone. His best response [6], by virtue of Assumption 2 is $f(123)-\omega_1-\omega_3$. Hence, between period $t+k+1$ and $t+2k$, there is a positive probability of seeing a run of $W^2 = (\omega_1, \omega_2, f(123)-\omega_1-\omega_2)$.

It is useful to note that until now we needed a history of length at most $3k$. From now on will make use of samples of size k that appear from dates $t+k+1$ onwards.

Now between $t+2k+1$ and $t+3k$, there is a positive probability all the three players will sample demand W^2. The best response of player 3 continues to be $f(123)-\omega_1-\omega_2$ while player 1 and player 2 must demand $f(12)-\omega_2$ and $f(12)-\omega_1$ respectively. Hence we will see a run of $W^3 = (f(12)-\omega_2, f(12)-\omega_1, f(123)-\omega_1-\omega_2)$ for k periods with a positive probability.

Between $t+3k+1$ and $t+4k$, there is a positive probability of players 2 and 3 continuing to sample demands of W^2 alone while player 1 samples W^3. The best responses of players 2 and 3 remain unchanged while that of player 1 is now ω_1. Hence, there is a positive probability of seeing a run of $W^4 = (\omega_1, f(12)-\omega_1, f(123)-\omega_1-\omega_2)$.

Finally, there is a positive probability of all three players sampling the most recent k records consisting of W^4 alone. The best response of players 1 and 2 continue to be ω_1 and $f(12)-\omega_1$ respectively while player three must now demand $f(123)-f(12)$. It may be verified that the demands $W^5 = (\omega_1, f(12)-\omega_1, f(123)-$

[6] It may be verified that since the technology is convex, $f(123)-\omega_i$ is not feasible in either of the smaller coalitions that contain player 3.

$f(12)$) is an element of the core. Now we repeat arguments similar to those found in the first paragraph of this proof to conclude that we the EBP converges with probability one to a convention if the information is less than or equal to 1/3.

The only other case to consider is when W^1 is feasible and is strictly compatible in the grand coalition but is not compatible in any of the smaller coalitions. In this case, it is clear that W^2 above is in the core.

3.3 Stochastically Stable Conventions

This section studies the same dynamic process as in the previous section but allows for the possibility of players making mistakes, much in the spirit of the models in Kandori, et. al (1993), Young (1993a) and Young (1993b). The aim of this section is to introduce the notion of stochastic stability and provide a characterization of such states. I start with a few definitions. Fix the sample sizes k_i and the memory m.

Definition 4 (Mistake) *Suppose that the EBP is in state* $\mathbf{s} = (W^{t-m+1}, W^{t-m+2}, \ldots, W^t)$ *at time t and* $\mathbf{s}' = (W^{t-m+2}, W^{t-m+3}, \ldots W^t, W)$. *The transition* $\mathbf{s}$ *to* $\mathbf{s}'$ *is said to involve exactly one mistake, if there is exactly one player, say i, for whom there is no sample in* $\mathbf{s}$ *of size* k_i *for which* ω_i *is a best-response, i.e.,* $p_i^*(\omega_i|\mathbf{s}) = 0$.

Clearly the number of mistakes involved in a transition from a state to its successor is between zero and n, depending on the number of players that have made errors.

Suppose that the probability with which player i makes a mistake is given by $\epsilon\lambda_i > 0$. Conditional on the fact that player i has made a mistake, let $q_i(\omega_i|\mathbf{s})$ be the probability with which he demands the wage ω_i in state $\mathbf{s}$. Clearly, q_i is different from p_i^*. The parameter ϵ is the absolute probability with which players make mistakes and λ_i/λ_j is the relative probability of players i and j making mistakes. The event that player i makes a mistake is assumed to be independent of the event j makes a mistake.

Now suppose that the process is in state **s** at time t. The probability that exactly the members in the coalition S make mistakes is $\epsilon^{s}(\prod_{i\in S}\lambda_i)(\prod_{i\notin S}(1-\epsilon\lambda_i))$. Conditional on this event, the transition probability of moving from a state **s** to a state $\mathbf{s}'$ is

$$Q^{S}_{ss'} = \begin{cases} \prod_{i\in S} q_i(\omega_i|\mathbf{s}) \prod_{i\notin S} p^*_i(\omega_i|\ \mathbf{s}) & \text{if } \mathbf{s}' \text{ is a successor of } \mathbf{s} \\ & \text{and the demands to the far right are } W \\ 0 & \text{otherwise.} \end{cases}$$

If none of the players make mistakes, then the transition probability of moving from state **s** to a state $\mathbf{s}'$ is given by the earlier transition probabilities $P^0_{ss'}$. This event has the probability $\prod_{i=1,2,3}(1-\epsilon\lambda_i)$.

Allowing for the possibility of mistakes, we now obtain a new Markov process with the same state space Ω as before but with the transition function:

$$P^{\epsilon}_{ss'} = \left(\prod_{i\in N}(1-\epsilon\lambda_i)\right) P^0_{ss'} + \sum_{S\subseteq N} \epsilon^{|S|}(\prod_{i\in S}\lambda_i)(\prod_{i\notin S}(1-\epsilon\lambda_i))Q^S_{ss'}.$$

Let P^{ϵ} denote the above matrix of transition probabilities. In most models similar to the one presented here, including Kandori, et. al. (1993) and Young (1993a) and Young (1993b), it is assumed that when players make mistakes, every feasible strategy is played with a strictly positive probability. Mistakes then, constantly perturb the process away from a convention. Now there are no absorbing states. However, since the transition probabilities of the perturbed process converge to those of the unperturbed process as ϵ converges to zero, for small values of ϵ, the perturbed process continues to be attracted to conventions, without actually settling down. Which of the conventions that the process stays at for the most part depends on the number of mistakes that are required to move it far enough to a state from which it would gravitate toward a different another convention. Hence, in the long run, if when the probability of mistakes is very small, the convention that is observed most of the time will be the one that requires the largest number of mistakes to displace.

The asymptotic (or long run) behavior of a Markov Process is captured completely by its invariant distributions. When one assumes that there is a positive probability of *every* strategy being played, the perturbed process is irreducible. It is easy to show that each of the states is aperiodic as well. Hence there is a unique invariant distribution μ^ϵ for the perturbed process, for each $\epsilon > 0$. For a state **s**, μ_s^ϵ is the limit of the the relative frequency with it is observed in the first t periods as $t \to \infty$. Since the invariant distributions (perhaps along a subsequence), converge to the invariant distribution of the unperturbed process, the conventions that are observed most often, when the probability of mistakes is small, are those in the support of the limit of the invariant distributions of the perturbed process (which is of course an invariant distribution of the unperturbed process). This motivates the following refinement of the set of conventions, first introduced by Foster and Young (1990).

Definition 5 (Stochastically Stable Convention) *A convention* ***s*** *is stochastically* stable *if* $\lim_{\epsilon\to 0} \mu_s^\epsilon$ *exists and is positive. A state is strongly stable, if* $\lim_{\epsilon\to 0} \mu_s^\epsilon = 1$.

Identification of the SSC(s) is considerably simplified by a certain equivalence theorem initially due to Friedlin and Wentzell (1984) and adapted for finite processes by Young (1993a). In order to introduce this, certain definitions are required.

Definition 6 (**w**-tree) *Fix a convention* ***w****. A* ***w****-tree is a directed graph with the set of conventions as its vertices such that from each convention* $\mathbf{w}' \neq \mathbf{w}$*, there is a unique path directed to* ***w*** *and there are no cycles.*

Definition 7 (Resistance) *Let* $\mathbf{s}'$ *be a successor of* ***s****. The resistance between these two states, denoted by* $\mathbf{r(s,s')}$*, is the minimum number of mistakes required in the one period transition* $\mathbf{s} \longrightarrow \mathbf{s}'$*. Similarly, for any two states* $\mathbf{s}^1$ *and* $\mathbf{s}^2$*,* $\mathbf{r(s^1,s^2)}$ *is the minimum number of mistakes required to reach* $\mathbf{s}^2$ *from* $\mathbf{s}^1$ *through a sequence of one period transitions.*

The resistance of a **w**-tree is naturally defined as the total resistance of each of its edges. Let $\mathcal{T}_w$ denote the set of all **w**-trees.

Definition 8 (Stochastic Potential) *The stochastic potential of a convention* $\mathbf{w}$ *is the least resistance among all* **w***-trees:*

$$\gamma(\boldsymbol{w}) = \min_{T \in \mathcal{T}_w} \sum_{(\boldsymbol{w}^1, \boldsymbol{w}^2) \in T} \boldsymbol{r}(\boldsymbol{w}^1, \boldsymbol{w}^2).$$

Theorem A (Young (1993a)) *The sequence of stationary distributions* μ^ϵ *converge to a stationary distribution* μ^0 *of* P^0 *as* $\epsilon \to 0$*. Moreover, a state* $\boldsymbol{s}$ *is stochastically stable iff* $\boldsymbol{s} = \boldsymbol{w}$ *is a convention and has the minimum stochastic potential amongst all conventions.*

In order to allow for a parsimonious construction of the tree of minimum stochastic potential, I make the following assumption.

Assumption 3 *Players only make mistakes that are a distance* δ *away from a best response. That is, if* $\boldsymbol{s}$ *is the state at time* t*, then for every* i*,* $q_i(\omega_{i,t+1} | \boldsymbol{s}) > 0$ *iff there is a* $\hat{\omega}$ *such the*

- $p_i^*(\hat{\omega}|\boldsymbol{s}) > 0$
- $|\hat{\omega} - \omega_i| \leq \delta$.

Young (1993b) constructs a tree of minimum stochastic potential without resorting to any assumptions on players' mistakes in the two player case. With three or more players, however, the allocation space has is at least two dimensional. Consequently, one has to contend with the possibility of cycles that do not arise in Young's compact, one dimensional space.

Two further remarks are in order. First, note that Assumption 3 is made on the players' strategies rather than the payoffs. However, when in a convention, say $\mathbf{w}^*$, the unique best-response is ω_i^*. With the above assumption, player i is assumed to demand for $\omega_i^* + \delta$ and $\omega_i^* - \delta$ as well as ω_i^* with a positive probability. In terms of payoffs, $\omega_i^* - \delta$ will constitute as "small mistake" as it will be met for sure while $\omega^* + \delta$ which will be rejected for sure is a "large mistake". So, in a sense, Assumption 3 allows only for extremes in terms of payoffs. Of course, nothing is implied in states that are not conventions as the payoff functions in this model are not continuous in strategies.

Second, recall that the definition of stochastic stability was based on the assumption that the perturbed transition probability matrix had a unique invariant distribution. With the above assumption, it is no longer clear that the P^ϵ is irreducible. Consequently, it is now not immediate that a unique invariant distribution exists and hence stochastic stability may be an ill-defined concept. Theorem 2 below, which uses a special bound on the extent of players' information establishes the validity of the solution concept.

Theorem 2 *Under Assumption 3, P^ϵ admits a unique invariant distribution for every $\epsilon > 0$ if $\alpha_i \leq 1/4$ for all i. Moreover, the support of this invariant distribution contains the set of all conventions.*

Proof. See Appendix B.

The following notation is useful:

$$g(W) = \sum_{i \in N} \alpha_i \log(v_i(\omega))$$

$$g_i(\omega) = \frac{\partial \alpha_i \log(v_i(\omega))}{\partial \omega}$$

$$r_i(\omega, \delta) = \alpha_i[1 - \frac{v_i(\omega - \delta)}{v_i(\omega)}]$$

$$r(W, \delta) = \min_{i \in N} \; r_i(\omega_i, \delta).$$

Theorem 3 *There exists a level of precision δ^* such that for all $0 < \delta \leq \delta^*$ a convention $\mathbf{w}^\delta$ is stochastically stable for every admissible m iff W^δ maximizes the function $r(., \delta)$ over all*

$W \in \mathcal{C}_\delta(f)$.

Proof. (Sketch) Starting from a convention **w**, The function $[mr(W, \delta)]$ is the minimum number of mistakes that are required before some player has a best response that is different from that in the convention (Corollary B.1). Hence $\mathbf{r(w,w^{'})}$ cannot fall below $[mr(W, \delta)]$. The proof then involves showing that one can construct a $\mathbf{w}^\delta$-tree that involves exactly $[mr(W, \delta)]$ mistakes at each edge, $\mathbf{w} \to \mathbf{w}^{'}$. The resistance of any other **w**-tree must then exceed that of this particular $\mathbf{w}^\delta$ tree by at least $[mr(W^\delta, \delta)] - [mr(W, \delta)]$. For large enough m, if W is not another maximum of $r(., \delta)$, then by Theorem A, **w** cannot be stochastically stable. Details appear in Appendix B.

Theorem 3 extends the results obtained by Young (1993b). The result here is weaker than in Young (1993) in two ways. First, an upper bound on δ is required. Second, there can be several stochastically stable conventions leading to an indeterminacy in the allocation for a subset of players even when δ converges to zero.

Example 3 Let $N = \{1, 2, 3, 4\}$, $f(12) = 100, f(N) = 101$ and $f(S) = 0$ otherwise. In every core allocation, $\omega_1 + \omega_2 \geq 100$. Hence, $r(W, \delta) = \min\{r_1(\omega_1, \delta), r_2(\omega_2, \delta)\}$ for a core allocation W and attains a maximum when $\omega_1 = \omega_2 = 50$. By Theorem 3, every convention **w** with $W = (50, 50, \omega, 1-\omega)$ with $\delta \leq \omega \leq 1-\delta$ is stochastically stable.

Theorem 4 *Let W^* maximize $g(.)$ over $\mathcal{C}(f)$. Let $K \subset N$ be the set of players such that $i \in K$ implies $g_i(\omega_i^*) \leq g_j(\omega_j^*)$ for all $j \in N$. For every every $\epsilon > 0$, there is a $\delta^* > 0$ such that for all $0 < \delta < \delta^*$, if $\mathbf{w}^\delta$ is stochastically stable, then*

$$| \omega_i^\delta - \omega_i^* | < \epsilon,$$

for all $i \in K$.

Proof. See Appendix B.

4 Some examples

4.1 Non-Convex Technology

Even when the technology is not convex, it is still relatively easy to compute the minimum number of mistakes that are required before some player has a best response different from the conventional one. These turn out to be functions that look like $r(., \delta)$. But now, this minimum number of mistakes is no longer sufficient to lead one out of the domain of attraction of the convention as the following example demonstrates.

Note that the following example may appear somewhat terse for the reader who has skipped Appendix B.

Example 4 In Example 2, set $\alpha_i = 1/4$ for all i. The minimum number of mistakes $\hat{r}_\delta(\mathbf{w})$ before which some player has a best response different from the conventional demand in a convention $\mathbf{w}$ is

$$\frac{4}{\delta}\hat{r}(W, \delta) = \begin{cases} \min_{i=1,2,3} \frac{1}{\omega_i} & \text{if } W \text{ is in the core} \\ \\ \min\{\frac{2}{\omega}, \frac{1}{300-\omega}\} & \text{for a convention such as } W = (300 - \omega, \omega, \omega) \end{cases}$$

where $\omega > 4$. This bound may be obtained by constructing arguments similar to those found in Appendix B. For elements in the core, $\hat{r}(., \delta)$ coincides with $r(., \delta)$ that appears in Theorem 3. Among the set of conventions that are not in the core, the fewest number of mistakes required before player 1 demands a δ less is $2/\omega_1$, whereas for players 2 and 3 the minimum number of mistakes continues to be $1/\omega_i$. This is because for player 1 to demand δ less as a best response in a convention such as $(300 - \omega, \omega, \omega)$, we require both player 2 and player 3 to make the mistake of asking for $\omega + \delta$. Hence, the correction of 2 for player 1.

The least player 1 obtains in the core is in the allocation $(298, 2, 2)$. In the convention involving these demands, player 1 requires the minimum number of mistakes to be the first player to have a best response different from 298. This corresponds to the case where both players 2 and 3 have demanded $298 - \delta$ for some L^* periods, determined by $\hat{r}_\delta$. To this, the best response of player 1 is to ask for $298 + \delta$. Suppose now, that from this point on, say T, no further mistakes occur. Assume, without loss of generality, that all mistakes actually occurred in the last $T - L^*$ dates.

It is easy to see, that from this point onwards, the demands of player 1 will not be anything other than 298 or $298 - \delta$ while those of players 2 and 3, will not be anything other than 2 or $2 + \delta$.

Now, consider a sample of m/4 in which every demand of player 1 is $298 - \delta$, while that of player 3 is 2 in $m/4 - 1$ entries. The one demand is $2 + \delta$. To this sample, a demand of 2 by player 2 will be a unique best response if

$$\begin{aligned} 2 &> \frac{(m/4 - 1)}{m/4}(2 + \delta) + \frac{1}{m/4}(2 + \delta)/2 \quad \text{or} \\ 2 &> (1 - 2/m)(2 + \delta) \end{aligned}$$

The last inequality holds for an appropriate δ. Thus, the only sample for which $2 + \delta$ is a best-response is if player 3 always has asked for 2 and player 1 always has asked for $2 - \delta$. Since $(298 - \delta, 2 + \delta, 2)$ is not a convention, the process eventually returns to $(298, 2, 2)$ in the absence of further mistakes.

The problem in this example appears due to the fact that the set of conventions is not connected. When δ is small, the minimum number of mistakes required to lead to a best response is not sufficient to lead away to a different convention. Indeed, the number of mistakes required to required to reach a convention involving demands of the form $(300 - \omega, \omega, \omega)$, starting from a demand in the core, turns out to be very large. This is because we require a series of mistakes on the part of players 2 and 3 each of

them demanding a δ higher at each instant. On the other hand, the number of mistakes required to reach an element of the core from the convention with demands $(296, 4, 4)$ is at least $298m/[4(298 + \delta)]$. For a fixed δ, this is a large number. Hence, one cannot obtain a bound independent of δ and m. Questions of existence and characterization of stochastically stable conventions for examples such as these are left open as possibilities for future research.

4.2 Competition and Evolution

The competitive allocations of an economy continue to be the benchmark for comparing those obtained under other mechanisms. A frequent criticism of the Walrasian mechanism is the absence of a dynamic story of adjustment [7] The function f can be thought of as being the game generated by an endowment economy in which agents have quasilinear preferences for money. The EBP can then be thought of a process of adjustment to equilibrium in such an economy. Hence it is of interest to compare the competitive outcomes with the SSC conventions.

Example 5 Let $N = \{1, 2, 3\}$, $\alpha_i = 1/4$, $v_i(x) = x$, $f(12) = f(13) = f(123) = 300$ and $f(S) = 0$, otherwise. The technology corresponds to the well-known representation of a game with two sellers and one buyer. When one does not allow for zero demands, the core of this technology is empty. But the set of absorbing states corresponds to m repetitions of the form $(300 - \omega, \omega, \omega)$, where $\delta \leq \omega \leq 300 - \delta$. Player 1 obtains $300 - \omega$ in one of the two smaller coalitions while 2 and 3 obtain ω with some probability. It can also be shown that, with the same bound as in Theorem 1, all other states are transient. Of course, it is being assumed that both the two player coalitions are equally likely.

As mentioned before, the problem in Example 4 appears because the set of absorbing states is not a connected set. However, when the set of absorbing states is a connected set, as it is here, the techniques in the proof of Theorem 3 can still be applied. A stochastically stable convention maximizes the function $\hat{r}_\delta$ over $\delta \leq \omega \leq 300 - \delta$ where .

$$\hat{r}_\delta(\mathbf{w}) = \frac{\delta}{4} \min \left\{ \frac{2}{300 - \omega}, \frac{1}{\omega} \right\}.$$

The competitive outcome on the other hand, is one in which player 1 yields the least amount to either player 2 or 3. In the present case this corresponds to $(300 - \delta, \delta, \delta)$.

[7] See Gale (1987a and 1987b) however.

When $\delta \to 0$, the stochastically stable outcomes converge to $(200, 100, 100)$ whereas the competitive outcome is the one in which player 1 gets the whole surplus, i.e., $(300, 0, 0)$. In expected terms, players 2 and 3 obtain 50 in the bargained outcome while the competitive outcome gives them 0.

4.3 Does the Utility function matter?

As in Young(1993b), let (v_i, α_i) denote the type of a player. An interesting result of Young(1993b) is emergence of a fifty-fifty split as the SSC regardless of the types of the players. In the model above, the result continues to hold in the sense that a particular division will be observed regardless of the utilities. However, if the technology is not convex, as it is in the earlier example, this is not the case.

Example 6 In Example 5, suppose that the utility that player i derives from consuming ω is given by $v_i(\omega) = 1 - e^{-\omega}$. Conduct the entire analysis as before to deduce that the minimum number of mistakes that are required before some player has a best response different from the conventional one is given by

$$\hat{r}_\delta(\mathbf{w}) = \frac{e^\delta - 1}{4} \min\left\{\frac{2}{e^{300-\omega} - 1}, \frac{1}{e^\omega - 1}\right\}.$$

The stochastically stable outcome is the convention that maximizes the above function. When $\delta \to 0$, the stochastically stable outcome converge to $(300 - \omega^*, \omega^*, \omega^*)$ where ω^* solves

$$2(e^{\omega^*} - 1) = (e^{300-\omega^*} - 1).$$

It may be checked that $\omega^* > 100$.

The competitive outcome on the other hand, converges to be $(300, 0, 0)$. Relative to the competitive outcome, player 1 continues to fare worse in the stochastically stable outcome. In fact, he does worse in the stochastically stable outcomes with constant absolute risk-aversion than he did with under similar outcomes when all the players had linear utilities in Example 5.

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A Appendix

For the sake of clarity, the proof of Proposition 2 will be preceded by a couple of lemmas. Recall that $\beta(W)$ was the set of all coalitions in which the demands were just compatible (at W) and $\hat{\beta}(W)$, the set of all coalitions in which the demands are strictly compatible.

$$\beta(W) = \{S \subseteq N : W(S) \leq f(S)\}$$

$$\hat{\beta}(W) = \{S \subseteq N : W(S) < f(S)\}.$$

Lemma A.1 *Fix W. If the demands of S and T are compatible, then*

$$W(S \cap T) - f(S \cap T) \leq f(S \cup T) - W(S \cup T).$$

Moreover, the above inequality is strict if the demands of either S or T are strictly compatible.

Proof. Immediate from covexity and the fact the for every W,

$$W(S \cup T) + W(S \cap T) \equiv W(S) + W(T).$$

Corollary A.1 *Suppose that the demands of S and T are compatible but there is no coalition whose demands are strictly compatible. Then the demands of $S \cap T$ and $S \cup T$ are also compatible.*

Corollary A.2 *If there are no coalitions whose demands are strictly compatible at W, then there can be at most one largest coalition whose demands are compatible.*

Proof. Immediate from the Corollary A.1.

Lemma A.2 *Fix W and let $W(S) < f(S)$. Then, there is an $S^* \subseteq S$ such that*

1. *The demands of S^* are strictly compatible.*
2. *For every compatible coalition T that does not contain S^*, $T \cup S^*$ is strictly compatible.*

Proof. Let $S_1 \equiv S$. If the Lemma holds with $S_1 = S^*$, we are done. Otherwise there is a compatible coalition T that does not contain S_1 such that $W(S \cup T) \geq f(S \cup T)$. Then $S_2 = S_1 \cap T$ is a strict, non-empty subset of S_1. Furthermore, a straightforward application of Lemma A.1 reveals that, $S_2 = W(S_2) < f(S_2)$. If the Lemma holds with $S_2 = S^*$, we are done. If not, we can repeat the above process finitely many times to obtain an $S_k = \{i\}$ such that $\omega_i < f(i) = 0$. This contradicts the fact that $\omega_i > 0$. $\Box$

Corollary A.3 *Under the hypotheses of Lemma A.2, for all $i \in S^*$,*

1. $\hat{\omega}_i > \omega_i$, *where* $\hat{\omega}_i = \arg\max p_i(\omega|W_{-i})v_i(\omega)$.
2. $\hat{\beta}(\hat{W})$ *is a strict subset of* $\hat{\beta}(W)$, *where* $\hat{W}_{-i} = W_{-i}$ *and* $\hat{\omega}_i$ *is as defined above.*

Proof. Let $i \in S^*$. By virtue of Part 2 of Lemma A.2, player i can increase his demand by a positive amount say to $\hat{\omega}$ and still ensure that if a coalition T continues to be compatible, then $T \cup S^*$ is also compatible. Hence, by Assumption 2, $p(\hat{\omega}|W_{-i}) = 1$. This proves Part 1. Part 2 is must hold since player i must increase his demand until one of the coalitions in which the demands were strictly compatible is just compatible. □

Corollary A.4 *Let W^* be a Nash equilibrium. Then, $\hat{\beta}(W^*) = \phi$.*

Proof. Immediate from the definition of a Nash equilibrium and Part 1 of Corollary A.3. □

Proof. (Proposition 2) Let W^* be a Nash equilibrium. Given Corollary A.4, we need only show that $W^*(N) = f(N)$. It is clear that for W^* to be a Nash equilibrium, for every player i, there is at least one $S \in \beta(W^*)$ that contains i. Now use Lemma A.2 to conclude that $W^*(N) = f(N)$. □

B Appendix B

In this Appendix, players' demands are in Δ_δ, unless stated otherwise. All the results from the Appendix A continue to hold with this restriction on players demands because of our assumption the each $f(S)$ is a rational number and the particular method of discretization employed.

Given W, W', we will write $W \xrightarrow{S} W'$ if

$$\omega_i' = \begin{cases} \arg\max p_i(\omega|W_{-i})v_i(\omega) & \text{if } i \in S \\ \omega_i & \text{if } i \notin S \end{cases}$$

Lemma B.1 *Suppose $\beta(W^0) \neq \phi$. If there is a set T such that whenever $W(S) < f(S)$, S contains a player not in T, then there is a sequence*

$$W^0 \xrightarrow{i_1} W^1 \xrightarrow{i_2} \dots \xrightarrow{i_{L-1}} W^{L-1} \xrightarrow{i_L} W^L$$

such that

1. *The demands of no coalition are strictly compatible at W^L but there is a unique largest coalition whose demands are compatible.*
2. *Furthermore, the sequence of players above can be chosen so that $i_l \notin T$ for all $l = 1, 2, \dots L$.*

Proof. Assume that $\hat{\beta}(W^0) \neq \phi$ (otherwise there is nothing to prove). We can find an S^* as in Lemma A.2. By hypothesis, S^* contains a player not in T, say i_1. W^1 is obtained by $W^0 \xrightarrow{i_1} W^1$. By Corollary A.3, $\hat{beta}(W^1)$ is a strict subset of $\hat{\beta}(W^0)$. Clearly $\beta(W^1) \neq \phi$ because there must be at least one coalition in which player i_1's demand is compatible. Now repeat the above procedure to obtain W^1 to obtain i_2 and W^2 and so on. In finitely many steps, $\hat{\beta}(W^L) = \phi$. Now use Corollary A.2 to conclude that there is a unique largest coalition at W^L in which the demands are compatible. □

Lemma B.2 *Suppose $\beta(W^0) \neq \phi$. Then, there is a sequence $\{W^l, S_l\}$, $l = 1, 2 \dots L_1, L_1 + 1 \dots L_2, L_2 + 1 \dots, L_{Q-1} + 1, \dots, L_Q$ such that;*

1. *For all $W^l \xrightarrow{S_{l+1}} W^{l+1}$, for all $0 \leq l \leq L_{Q-1}$.*
2. *$S_{L_k+1} = N$, for all $k = 1, 2, \dots Q-1$.*
3. *The demand vector W^{L_Q} is in the core.*

Proof. Set $T = \phi$ in Lemma B.1 and obtain the sequence up to W^{L_1}. Let T_1 be the unique largest coalition in which the demands are compatible. If $T_1 = N$, we are done. Otherwise, consider $W^{L_1} \xrightarrow{N} W^{L_1+1}$. Since the demands of no coalition are strictly compatible at W^{L_1}, no player will increase his demand. Moreover, by Assumption 2, $p_i(\omega_i^{L_1}|W_{-i}^{L_1}) = 1$ for all $i \in T_1$. Hence, $\omega_i^{L_1+1} = \omega_i^{L_1}$, for all $i \in T_1$. Furthermore, a player not in T_1 will reduce his demand so that at W^{L_1+1}, every player will have his demand met in at least one coalition. If now $\hat{\beta}(W^{L_1+1}) = \phi$, we are done as W^{L_1+1} is in the core.

Suppose $\hat{\beta}(W^{L_1+1}) \neq \phi$. Set $T_1 \equiv T$, $W^0 \equiv W^{L_1+1}$ in Lemma B.1 to obtain the sequence leading to W^{L_2}. Again we have a unique largest coalition T_2 whose demands are compatible but the demands of no coalition are strictly compatible. Since the demands of the players in T_1 have not changed, and at least one more player's demand is now being met, it follows that T_2 is a strict superset of T_1. If W^{L_2} is in the core, we are done. Otherwise we need repeat the above process only finitely many times to reach a situation where $T_Q = N$. $\Box$

Proof. (Theorem 1) Again, assume, for the sake of exposition, that $\alpha_i = \alpha_j$. Let **s** be the state at time t and let σ denote the sample consisting of the k most recent records. Let W^0 be the demand vector resulting from every player sampling σ. Assume that $\beta(W^0)$ is not empty. The other case will be dealt with shortly. Now, let $\{W^l, S_l\}$ be the sequence as in Lemma B.2.

Let **A** be the following event:

1. Between t and $t + k$, all players sample σ.

2. Between $t + lk + 1$ and $t + (l + 1)k$, all players except those in S_l continue to sample what they were sampling between $t + (l - 1)k + 1$ and $t + lk$. Players in S_l sample the k most recent demands, $1 \leq l \leq L_{Q-1}$.

It is easy to verify that if the **A** were to occur, it will lead to k repetitions of W^0 between $t + 1$ and $t + k$, followed successively by k repetitions of W^1 between $t + k + 1$ and $t + 2k$ and so on finally ending with k repetitions of W^{L_Q} between

$t+L_{Q-1}k+1$ and $t+L_Qk$. Indeed, given our assumptions that each player samples every k sample with a positive probability, the event **A** would occur with a positive probability if the initial sample σ were available in its entirety between until date $t+k(L_1+1)$ and since $S_{L_q}=N$, for the next $k(L_q+1)$ periods W^{L_q} were available. This would be possible if $m \geq \max_q\{(L_q+2)k\}$. Note that L_q's depend only on the number of coalitions in which the demands were strictly compatible. Hence, $L_q \leq \frac{n(n-1)}{2}$, which is the number of two player coalitions with n players.

Now, assuming that the bound on k/m is met, there is then a positive probability of observing k repetitions of W^{L_Q}, which is a core allocation, in finitely many stages. Furthermore, now there is a positive probability that all the players sample this k-run of W^{L_Q} to achieve another k-run of W^{L_Q} and so on to finally obtain the convention $\mathbf{w}^{L_Q}$ in some M periods altogether.

Thus, there is a positive probability p, bounded away from zero by our assumption of stationarity, and an integer M, such that a convention is established in M periods. Hence the probability of not reaching a convention in rM periods is $(1-p)^r$, which goes to zero as $r \to \infty$.

Now suppose $\hat{\beta}(W^0)=\phi$. If W^0 is in the core, we are done. If the demands of no coalition are compatible at W^0, then consider $W^0 \xrightarrow{N} \hat{W}^0$ and conduct the above analysis starting from somewhere in the middle of the sequence obtained in Corollary B.1. □

Lemma B.3 *Fix a convention* $\mathbf{w}^*$. *Starting from this convention, the minimum number of mistakes required for player* i_0 *to be the <u>first player</u> to have a best response different from* ω_i^* *for the <u>first time</u> is given by*

$$\min\left\{\frac{k_{i_0}v_i(\omega_{i_0}^*)}{v_{i_0}(\omega_{i_0}^*+(n-1)\delta)}, k_{i_0}[1-\frac{v_{i_0}(\omega_{i_0}^*-\delta)}{v_{i_0}(\omega_{i_0}^*)}]\right\}$$

Proof. Starting from a convention $\mathbf{w}^*$, let $\mathbf{s}$ be the first state in which player i_0 is the first player to have a best-response different from the conventional demand.

For concreteness, set $i_0 = 1$. Note that by the definition of **s** , every W_{-1} in **s** that is not W^*_{-1} involves at least one mistake. Let there be L such demands that differ from W^*_{-1}. In fact, without loss of generality assume that these

are be the L most recent demands. Let $\hat{\omega}$ denote the best response of player 1.

<u>*Case 1:*</u> ($\hat{\omega} < \omega_i^*$) Now consider a sample of size k_1 from **s** that includes the L demands. Thus $(k_1 - L)$ of the demands in this sample are W^*_{-1} while the last L demands are W^l_{-1}, $l = 1, 2, \ldots L$. By playing ω_1^* player 1's expected utility is

$$(1 - \frac{L}{k_1})v_1(\omega_1^*) + \sum_{L=k_1-L+1}^{k_1} p_1(\omega_1^*|W^l_{-1})v_1(\omega^*), \tag{7}$$

which is no larger than

$$(1 - \frac{L}{k_1})v_1(\hat{\omega}) + \sum_{l=k_1-L+1}^{k_1} p_1(\hat{\omega}|W^l_{-1})v_1(\hat{\omega}), \tag{8}$$

the payoff if he plays $\hat{\omega}$.

Define

$$S_1 = \bigcap_{S \in \beta(W^*), 1 \in S} S. \tag{9}$$

Since there are no strictly compatible coalitions at W^*, Corollary A.1 applies and, $W^*(S_1) = f(S_1)$. Without loss of generality, assume that $2 \in S_1$. Consider a new demand vector $\overline{W}$ that differs from W^* only in the demand of player 2. Player 2 demands $\omega_2^* + (\omega_1^* - \hat{\omega})$.

Now consider the new sample constructed from the original sample with the last L entries as $\overline{W}$. Note well that this new sample has exactly L mistakes, no more than those in the original sample. These are of course the L demands of player 2 that differ from w_2^*. I will now show that given $\hat{\omega}$ is a best-response to the original sample, $\hat{\omega}$ is a best-response to this new sample.

Note that by our choice of player 2, player 1's demand ω_1^* is not compatible in any coalition when 2 demands $\omega_2^* + (\omega_1^* - \hat{\omega})$. The payoff of player 1, then on playing ω_1^* is by

$$(1 - \frac{L}{k_1})v_1(\omega_1^*).$$

However on demanding $\hat{\omega}$, he obtains it for sure and hence achieves a payoff of $v_i(\hat{\omega})$. Since the expression in Eq. 8 is at least as large as that in Eq. 7, it is easy to see that

$$v_1(\hat{\omega}) \geq (1 - \frac{L}{k_1})v_1(\omega_1^*).$$

Hence, if $\hat{\omega}$ was a best response to the initial sample, it continues to be a best response to the new sample. Thus,

$$\begin{aligned} L \geq k_1[1 - \frac{v_1(\hat{\omega})}{v_1(\omega_1^*)}] & \\ \geq \quad mr_1(\omega_1^*, \delta). \end{aligned} \tag{10}$$

<u>*Case 2:*</u> ($\hat{\omega} > \omega_1^*$) As before consider a sample of size k_1 from the state $\mathbf{s}$ for which $\hat{\omega}$ is a best response. Now as in the earlier case the payoff on demanding ω_1^* is given by Eq. 7. However on demanding $\hat{\omega}$, the payoff is

$$\sum_{l=n+1}^{k_1} p_1(\hat{\omega}|W_{-1}^l)v_1(\hat{\omega}). \tag{11}$$

Since $\hat{\omega}$ is a best response, Eq. (11) is at least as large as Eq.(7).

As in the earlier case, construct a new sample from the initial sample by replacing the last L demands of every player except 2 with ω_j^*. Replace those of player 2 with $\overline{\omega}_2 = \omega_2^* - (\hat{\omega} - \omega_1^*)$. In this new sample, there are exactly L mistakes, no larger than those in the initial sample.

Now, it remains to show that $\hat{\omega}$ continues to be a best response for player 1 in this new sample. That is, we need to show that

$$\frac{L}{k_1}v_1(\hat{\omega}) \geq v_1(\omega_1^*).$$

The above follows from the following sequence of inequalities.

$$
\begin{aligned}
\frac{L}{k_1} v_1(\hat{\omega}) - v_1(\omega_1^*) &= \frac{L}{k_1}[v_1(\hat{\omega}) - v_1(\omega^*)] - (1 - \frac{L}{k_1}) v_1(\omega_1^*) \\
&\geq \sum_{l=k_1-L+1}^{k_1} p_1(\hat{\omega}|W_{-1}^l)[v_1(\hat{\omega}) - v_1(\omega_1^*)] - (1 - \frac{L}{k_1}) v_1(\omega_1^*) \\
&\geq \sum_{l=k_1-L+1}^{k_1} p_1(\hat{\omega}|W_{-1}^l) v_1(\hat{\omega}) - \\
&\quad \sum_{l=k_1-L+1}^{k_1} p_1(\omega_1^*|W_{-1}^l) v_1(\omega_1^*) - (1 - \frac{L}{k_1}) v_1(\omega_1^*) \qquad (12) \\
&\geq 0.
\end{aligned}
$$

Eq. (12) follows from Assumption 1 and the non-negativity follows from the fact that the expression in Eq. (11) is at least as large as that in Eq. (7). Hence, the number of mistakes is given by

$$L \geq k_1 v_1(\omega_1^*)/v_1(\hat{\omega}).$$

The minimum number occurs when $\hat{\omega}$ takes the maximum value. But since we allow only for small mistakes, it follows that the largest value of $\hat{\omega}$ cannot be more than $\omega_1^* + (n-1)\delta$. So, the number of mistakes cannot fall below

$$k_1 \frac{v_1(\omega^*)}{v_1(\omega^* + (n-1)\delta)}.$$

The proof is complete on taking the minimum of the two lower bounds obtained in the two different cases above. □

Lemma B.4 *Let* $0 < \delta < \alpha_1 f(N)/(\alpha_n n^2)$. *If* $\mathbf{w}$ *is a convention, then*

$$\alpha_i \frac{v_i(\omega_i)}{v_i(\omega_i + (n-1)\delta)} \geq r(W, \delta).$$

Proof. Using concavity of v_i and the fact that $\omega_i \geq \delta$, we have

$$\begin{aligned} \alpha_i \frac{v_i(\omega_i)}{v_i(\omega_i + (n-1)\delta)} &\geq \alpha_i \frac{\omega_i}{\omega_i + (n-1)\delta} \\ &\geq \frac{\alpha_1}{n} \end{aligned}$$

Again by concavity of v_i and the fact that there is at least one $\omega_i \geq f(N)/n$, we have

$$\begin{aligned} r(W,\delta) &\leq r_i(\omega_i,\delta) \\ &\leq \frac{\alpha_i \delta}{\omega_i} \\ &\leq \frac{\delta n \alpha_n}{f(N)} \end{aligned}$$

The rest of the proof is immediate upon comparing the above two inequalities. □

Corollary B.1 *Let* $0 < \delta \leq \alpha_1 f(N)/(\alpha_n n^2)$*. At a convention* **w** *and for any state* **s**,

$$\mathbf{r(w,s)} \geq [mr(W,\delta)]$$

Proof. By virtue of the previous two lemmas, $[mr_i(\omega_i,\delta)]$ is the minimum number of mistakes for player i to be the first player to have a best response different from ω_i in the convention **w**. Hence, **r(w,s)** cannot fall below $[mr(W,\delta)]$, the minimum number of mistakes required before *some* player has a different best response for the first time. □

Now, let e_{ij} denote the vector in R^n_+ where the ith component is $-\delta$, the jth component is δ and every other component is zero. Given a convention **w**, the convention **w**+e_{ij} (if it is a convention) involves a transfer of δ from player i to j.

Lemma B.5 *Let* $\delta \leq \alpha_1 f(N)/(\alpha_n n^2)$ *and* W, $W + e_{ij} \in \mathcal{C}_\delta(f)$. *Suppose that* $r(W,\delta) = r_{i_0}(\omega_{i_0},\delta)$. *Then*

1. **r(w,w**+e_{ij}**)** $= [mr(W,\delta)] = [mr_{i_0}(\omega_{i_0},\delta)]$, *if* $j \neq i_0$ *and*
2. **r(w,w**+e_{ij}**)** $\leq \min_{i \neq i_0}[r_i(\omega_i,\delta)]$, *if* $j = i_0$.

Proof. I will prove only Part 1 of the Lemma. Part 2 can be proved by constructing arguments almost identical to those of provided for Part 1 below.

For concreteness, take $i_0 = 1$ and pick $2 \in S_1$ where S_1 is obtained as in Eq. 9. Now consider the following three demand vectors:

1. W^1 differs from W^* only in player 1's demand: $\omega_1^1 = \omega_1^* - \delta$.
2. W^j differs from W^* only player j's demand: $\omega_j^j = \omega_j^* + \delta$.

By Corollary B.1, it suffices to exhibit a path between $\mathbf{w}$ and $\mathbf{w}+e_{ij}$ that involves exactly $[mr(W, \delta)$ mistakes.

Suppose the EBP is in state $\mathbf{w}$ at time t. Now, suppose that between time $t+1$ and $t^* = t + [mr(W, \delta)]$ player 2 plays $\omega_2^* + \delta$ instead of ω_2^*. Each of these demands is a mistake. Let σ denote the sample consisting of the $k = \max_{i \in N} k_i$ most recent demands. By our choice of player 1, σ contains a subsample for which the best response of every player other than 1 continues to be ω_i^*. For player 1, given our choice of player 2, the best response is $\omega_1^* - \delta$.

Now there is a positive probability between $t^* + 1$ and $t^* + k$ all the players will sample from σ leading to a run of k repetitions of W^1. It is useful to remind the reader, that the entire sample σ will be available until date $t^* + 3k$ by our assumption that $k \leq m/4$.

Following this run of W^1, there is a positive probability that between $t^* + k + 1$ and $t^* + 2k$ all the players except 1 and j will continue to sample σ while 1 and j sample from the k most recent records consisting of W^1. It is straightforward to check that this should lead to a run of k demands of W^j.

Between $t^* + 2k + 1$ and $t^* + 3k$, there is a positive probability that all the players except players $1, i$ and j will sample σ; players 1 and j continue to sample W^1 as before and player i samples the k most recent records consisting of W^j. It may be checked that this will lead to a run of $W + e_{ij}$.

Finally, from $t^* + 3k$ onwards, there is a positive probability that all the players will sample the k most records consisting of $W + e_{ij}$ thereby establishing the convention $\mathbf{w}+e_{ij}$ in due course. Since the above path involved a best response by the players at every stage following the first $t^* - t$ mistakes by player 2, the proof is complete. □

Proof. (Theorem 2)[8] Consider an aperiodic Markov Chain with state space Ω. A state can either be persistent or transient. Now, we know from that the persistent states of a finite Markov chain can be uniquely decomposed into disjoint closed sets, $C_1, C_2, \ldots C_K$ such that *from any state of a given set* C_k*, all states of this set and no other can be reached*. In particular, if we show that the decomposition must involve a unique closed set, say Ω^*,

it then follows that all the states $\Omega \setminus \Omega^*$ are transient and that the states corresponding to Ω^* form an irreducible sub- chain. This sub-chain admits a unique stationary distribution. Since a transient state cannot be in the support of a stationary distribution of the original Markov Chain, it then follows that there is a unique stationary distribution of the original chain; this is of course the invariant distribution corresponding to the the states in Ω^* extended with zeros corresponding to the states in $\Omega \setminus \Omega^*$.

That the perturbed EBP is aperiodic is obvious; from an arbitrary state, a convention can be reached following which a return to the original state if possible, can be achieved in either an even number or an odd number of periods. I now claim that there is a unique closed set of recurrent states, say Ω^*, for the perturbed EBP and this contains all conventions. Indeed, let $C_1, C_2, \ldots C_K$ the decomposition. Let $\mathbf{s}$ be a state in C_k. By Theorem 1, there is a positive probability of reaching a convention in finitely many steps. Hence, C_k must contain at least one convention,

[8]In this proof, I use, without providing formal definitions and proofs certain terminology and assertions that are rather standard in the theory of finite Markov Processes. See for e.g. Feller (1957)

say **w**. Moreover, by virtue of Lemma B.5, from a given convention **w**, there is a positive probability of reaching every convention of the form **w**+e_{ij}. Since the core is a convex set, it follows that any convention $\mathbf{w}'$ can be reached from an arbitrary convention **w**. Hence C_k must in fact contain all the conventions. Since C_k was arbitrarily chosen, it follows that the decomposition must involve a unique closed set of persistent states. The proof is now complete by the arguments of the preceding paragraph.

Lemma B.6 *Suppose that W^δ maximizes $r(.,\delta)$ over $\mathcal{C}_\delta(f)$. Let $W \in \mathcal{C}_\delta(f)$ be such that $\omega_i < \omega_i^\delta$. Then $r(W,\delta) < r_i(\omega_i,\delta)$.*

Proof. Suppose, by way of contradiction, that $r(W,\delta) = r_i(\omega_i,\delta)$. Since $r_i(.,\delta)$ is a strictly decreasing function, $r(W,\delta) > r_i(\omega_i^\delta,\delta) \geq r(W^\delta,\delta)$, thereby contradicting the fact that W^δ maximizes $r(.,\delta)$.

Proof. (Theorem 3) Let W^δ be a maximum of $r(.,\delta)$. Now construct a $\mathbf{w}_\delta$-tree by placing a directed edge from each **w** ($\neq \mathbf{w}_\delta$) to a convention **w**+e_{ij}, where $\omega_i > \omega_i^\delta$ and $\omega_j < \omega_j^\delta$. Intuitively, this procedure involves transferring δ at each stage from player *i* who is demanding "too much" to a player *j* who is demanding "too little". Since the core is a convex set, this "equalizing" principle can be followed to obtain a $\mathbf{w}^\delta$-tree. Let T^* denote this tree.

At an edge $\mathbf{w} \rightarrow \mathbf{w}+e_{ij}$, $\omega_j < \omega_j^\delta$. Hence by Lemma B.6, $r(W,\delta) < r_j(W,\delta)$. Hence, by Part 1 of Lemma B.5, the resistance of each edge in the tree T^* is

$$\mathbf{r}(\mathbf{w},\mathbf{w}+e_{ij}) = r(W,\delta).$$

So, the resistance of the tree T^* is

$$\sum_{W \neq W^*} [mr(W,\delta)]. \tag{13}$$

For any other convention, say $\hat{\mathbf{w}}$, the resistance of a $\hat{\mathbf{w}}$-tree, say T is

$$\sum_{(\mathbf{w}^1,\mathbf{w}^2)\in T} \mathbf{r}(\mathbf{w}^1,\mathbf{w}^2) \geq \sum_{W\neq\hat{W}} [mr(W,\delta)] \geq \sum_{W\neq W^\delta} [mr(W,\delta)].$$

The first inequality follows from Corollary B.1 and the second inequality is strict if $\hat{W}$ does not maximize $r(.,\delta)$.

Hence, $\mathbf{w}^\delta$ has the least stochastic potential with the resistance given Eq. (13). By Theorem A, it is then a Stochastically stable state.

If there are several maxima of $r(.\delta)$, each of them is a SSC.

Proof. (Theorem 4) Let $W^* \in C(f)$ be the maximum of g and let K be as in the statement of the Theorem. For a player i, let S_i denote the smallest coalition S that contains i and $W^*(S) = f(S)$. Such an S_i is well defined by Corollary A.1.

For concreteness, suppose that $1 \in K$. Since, $g_1(\omega_1^*) \leq g_j(\omega_j^*)$ for all $j \in N$, it is easy to show[9] that for every $\epsilon_1 > 0$, there exists an ϵ_2 in the open interval $(0, \epsilon_1)$ such that

$$g_1(\omega_1^* + \epsilon_1) < g_j(\omega_j^* + \epsilon_2). \tag{14}$$

Now, suppose that there is a player $j \neq 1$ such that $1 \in S_j$. (The case when no such j exists will be analyzed shortly.) At W^* then, we can reduce the demand of j by an arbitrarily small $\epsilon > 0$ and correspondingly increase the demand of 1 and yet remain in the core. Hence for all $0 < \epsilon_2 < \epsilon_1 < \epsilon/(n+1)$, there exists a $W \in \mathcal{C}(f)$ such that $\omega_1^* + \epsilon_1 < \omega_1 < \omega_1^* + \epsilon/(n+1)$ and $\omega_j < \omega_j^* + \epsilon_2$, for all $j \neq 1$. Note that with our method of discretization, $\mathcal{C}_\delta(f) \subset \mathcal{C}_{\delta'}(f)$ when $\delta < \delta'$ and $\cup_\delta \mathcal{C}_\delta(f) = \mathcal{C}(f)$. Hence, we can find a $\delta^* > 0$ and a $W \in \mathcal{C}_{\delta^*}(f)$ so that for all $\delta < \delta^*$, $\omega_1^* + \epsilon_1 \leq \omega_1 - \delta < \omega_1 \leq \omega_1^* + \epsilon/(n+1)$ and for $j(\neq 1) \in N$, $\omega_j \leq \omega_j^* + \epsilon_2$. In

[9] Using continuity and the fact that g_i's are strictly decreasing

fact, pick ϵ_1 and ϵ_2 such that Eq. 14 is satisfied. Then,

$$\begin{aligned} r_1(\omega_1, \delta) &\leq g_1(\omega_1^* + \epsilon_1) \\ &< g_j(\omega_j^* + \epsilon_2) \\ &\leq r_j(\omega_j, \delta), \end{aligned} \tag{15}$$

for all $j \in N$. Hence, $r(W, \delta) = r_1(\omega_1, \delta)$. Now apply Lemma B.6 to conclude that if W^δ maximizes $r(., \delta)$ over $\mathcal{C}_\delta(f)$, then

$$\omega_1^\delta \leq \omega_1^* + \frac{\epsilon}{n+1}. \tag{16}$$

Now if $1 \notin S_1$ for every $j \neq 1$, then $S_j \subseteq N \backslash 1$. Apply Corollary A.1 to conclude that $W^*(N \setminus 1) = f(N \setminus 1)$ and hence $\omega_1^* = f(N) - f(N \setminus 1)$, the largest payoff that player 1 can obtain in the core. Hence, Eq. 16 follows immediately.

To obtain a lower bound on ω_1^δ, first note that there in no loss in generality in assuming that Eq. 16 holds for all players in K for the same ϵ. Second, I claim that $W^*(K) = f(K)$. To see this, consider a player $j \in S_1$. If $g_1(\omega_1^*) < g_j(\omega_j^*)$, we can increase the demand of player j by a small amount while decreasing the demand of player 1 while remaining in the core. By[10] doing this, we increase the value of g and this contradicts the fact that g is maximized at W^*. Hence, $g_j(\omega_j^*) \leq g_1(\omega_1^*)$. But this implies $S_1 \subseteq K$. Hence $\cup_{i \in K} S_i = K$ and using Corollary A.1, $W^*(K) = f(K)$.

It is now easy to verify that if $\omega_1^\delta < \omega_1^* - \frac{n}{n+1}\epsilon$, then there must be a player $j \in K$ such that $\omega_i^\delta > \omega_j^* + \frac{\epsilon}{n+1}$, thereby violating Eq. 16.

Hence, for all $i \in K$, and for all $0 < \delta \leq \delta^*$,

$$| \omega_i^\delta - \omega_i^* | < \epsilon.$$

[10] A formal argument involves a standard application of Taylor's theorem