האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

A CANTOR SET OF GAMES WITH NO SHIFT-HOMOGENEOUS EQUILIBRIUM SELECTION

By

YEHUDA (JOHN) LEVY

Discussion Paper # 607 April 2012

מרכז לחקר הרציונליות

CENTER FOR THE STUDY OF RATIONALITY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

A Cantor Set of Games with No Shift-Homogeneous Equilibrium Selection*

Yehuda (John) Levy†

April 28, 2012

Abstract

We construct a continuum of games on a countable set of players that does not possess a measurable equilibrium selection that satisfies a natural homogeneity property. The explicit nature of the construction yields counterexamples to the existence of equilibria in models with overlapping generations and in games with a continuum of players.

Keywords: Equilibrium Selection, Continuum of States

1 Introduction

Models that have a countable set of players can arise in many situations. For example, in repeated games (or, more generally, extensive-form games), it is often useful to think of each of the decisions a player may need to take as being taken by different agents (e.g., [7]); hence, even if there are only finitely many players, we can have infinitely many agents. Another is in games and markets of overlapping generations, in which each player has a finite lifetime, during which new players enter the market and older players leave (see [5],[4], and the references there). Another commonly occurring feature of both single-stage games and repeated games is choices made by Nature affecting the payoffs of the players (e.g., stochastic games [18], global games [2]).

We will consider a countable set of players. For convenience, we organize the set of players into teams: one team for each integer in $\mathbb{Z}$, where each team is a fixed finite set K_0. Each player will have actions 'left' and 'right'. Nature will assign a bit to each team, and the bits of all teams are common knowledge. The payoffs we define will possess a homogeneity property: the payoff of a team will not depend on its location on the integer lattice. Specifically, if we view the teams residing at points on the integer lattice (with team n having team $n-1$

*Research supported in part by Israel Science Foundation grants 1123/06 and 1596/10.

†Center for the Study of Rationality, and Department of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel.

to its left and team $n+1$ to its right), each player will get a payoff depending on the actions of his team, the team to his left, and the bit chosen for his team according to Nature. If teams in places $k, k-1$ and $l, l-1$ play the same action profiles, respectively, and teams k and l receive the same bit, then they receive the same payoff profiles.

The sequence of bits received by the set of teams can be viewed as an element of the Cantor set $X = \{-1, 1\}^{\mathbb{Z}}$; we will adopt the view that each sequence of bits determines a game. On the Cantor set, there is a natural shift-to-the-left operator S. We can also view the shift as acting on action profiles and on payoff profiles. Given the homogeneity properties of the payoffs we will define, as well as the homogeneous structure of the set of players, it would seem reasonable to expect that we can find a shift-homogeneous mapping that assigns to each element of X a Nash equilibrium profile of the associated game; that is, if the sequence of bits is shifted, then the equilibrium assignment shifts the action profile accordingly. The surprising result is that one cannot find an equilibrium selection that both satisfies this homogeneity and is Lebesgue-measurable.

The absence of a measurable shift-homogeneous equilibrium selection - henceforth, SHES - has several important consequences. Firstly, it implies that any (necessarily non-measurable) such equilibrium selection of the aforementioned example cannot be defined explicitly, for it is known that under the commonly adopted Zermelo-Fraenkel axioms one cannot explicitly construct a non-Lebesgue measurable function (see, e.g., Chapter 7 of [8] and the references there). Hence, despite the existence of a SHES (which is a consequence of the axiom of choice; see the discussion in Section 2), no SHES can be described.

Secondly, given any SHES of our example, one would not be able to answer some very natural questions of a probabilistic nature. For example, suppose we fix a player and we choose a game at random according to the Lebesgue-measure, and then ask our chosen player to choose an action (say, 'left' or 'right') via randomization as prescribed by the SHES. It is natural to question what is the probability that this process will yield the action 'left'; however, since the SHES is not measurable, this and similar questions are rendered mathematically meaningless. The Lebesgue-measure is not defined for all subsets of X, and as a result, for some questions like those above, the answer will not be defined.

Thirdly, we point out that measurable equilibrium selections that are not homogeneous can be easily and explicitly defined in our example (see Remark 8). However a nonhomogeneous equilibrium selection means that the players necessarily take into account information that affects no one's payoffs, precisely because the payoffs are homogeneous! It is remarkable that this dependence on irrelevant information can only be dispensed with at the cost of measurability.

Lastly, we again emphasize the pathological nature of nonmeasurable sets, and hence also of nonmeasurable functions; to quote Aumann (1964), [1], "[The] measurability assumption is of technical significance only and constitutes no economic restriction. Nonmeasurable sets are extremely 'pathological'; it is unlikely that they would occur in the context of an economic model." However, as I will discuss later in Sections 7 and 8, there are setups that arise in a number of economic applications in which our example shows that even if the data of the games are defined in a simple and explicit manner, the SHES may defy our intuition and escape the realm of measurable functions.

Simon (2003), [17], provies a good game-theoretical example of this phenomenon. In that work, Simon presented an example a Bayesian game where the set of states of the world is also the Cantor set, and the data pertaining to the payoffs and the knowledge of each player is also explicitly and simply described. Yet, despite the fact that the game possesses nonmeasurable equilibria, no Lebesgue-measurable equilibria exist. Simon's work has served as an inspiration for this one; see the further discussion in Section 8.

In Section 2, we present the preliminaries and state the main result. Sections 3, 4, 5, and 6 are dedicated to the construction and the proof. In Section 7, we discuss games with overlapping generations. Indeed, by a simple translation of our example, we will build a model in which participants last two generations (one in which they are 'young', and one in which they 'old' and do not actively make a choice) and receive a payoff that depends on the actions played during their lifetime and the bit of Nature during their 'youth', but which does not depend at on the time in which they lived; despite this stationarity, measurable stationary equilibrium need not necessarily exist. In Section 8, we use our construction to provide an example of a game with a continuum of players with no measurable equilibria. In section 9, we remark how our work parallels the example given by Rabin (1957), [14], pertaining to computability of strategies.

2 Preliminaries and The Main Theorem

Throughout this paper, $X = \{-1, 1\}^{\mathbb{Z}}$ will denote the Cantor set, endowed with the σ-algebra of Lebesgue-measurable sets (henceforth, 'measurable' will mean 'Lebesgue-measurable' unless otherwise specified) and μ will denote the Lebesgue measure on X, that is, the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure on X. For $x \in X$, $n \in \mathbb{Z}$, x_n denotes the n-th coordinate of x. $S : X \to X$ will denote the shift on X, given by $(Sx)_n = x_{n+1}$. For a finite set A, $\Delta(A)$ denotes the set of probability distributions on A, and if $p \in \Delta(A)$ and $a \in A$, $p[a]$ denotes the probability p associates with a.

Let $\mathcal{P} = \mathbb{Z} \times K_0$ be the set of players, where K_0 will be a fixed finite set whose size will be determined later. The action set for each player is $I = \{L, R\}$. $CG(\mathcal{P})$ denotes the set of cardinal games on $\mathcal{P}$, that is, the set of continuous functions from $I^{\mathcal{P}}$ to $\mathbb{R}^{\mathcal{P}}$. The shift map S on $\mathbb{Z}$, $S(k) = k - 1$, induces a map on $I^{\mathcal{P}}$ given by $(Sa)^{(n,k)} = a^{(n+1,k)}$ for $(n,k) \in \mathcal{P}$, which extends to a map on mixed profiles, and also induces a map on $CG(\mathcal{P})$, where $(S\rho)^{(n,k)}(a) = \rho^{(n+1,k)}(S^{-1}(a))$ for any $\rho \in CG(\mathcal{P})$.

In Section 4, we will define a (continuous) injection g from X to $CG(\mathcal{P})$ (we will denote g_x instead of $g(x)$), which commutes with the shift: $g \circ S = S \circ g$, or, more explicitly, for all $a \in I^{\mathcal{P}}$, $x \in X$, $(n,k) \in \mathcal{P}$, we have $g^{n,k}_{S(x)}(Sa) = g^{n+1,k}_x(a)$. The main result of this paper is:

Theorem 1. *For the function $g : X \to CG(\mathcal{P})$ defined in Section 4, there does not exist a mapping $\phi : X \to (\Delta(I))^{\mathcal{P}}$ such that:*

- *ϕ is measurable.*
- *ϕ commutes with the shift, i.e., $S \circ \phi = \phi \circ S$.*
- *For each $x \in X$, $\phi(x)$ is a Nash equilibrium of g_x.*

Remark 1. A mapping ϕ that satisfies the latter two conditions will be referred to as a shift-homogeneous equilibrium selection (SHES). Hence, the theorem states that g does not possess a measurable SHES.

Remark 2. Measurable equilibrium selections that do not satisfy the homogeneity condition do exist: the correspondence of Nash equilibria from X to $(\Delta(I))^{\mathcal{P}}$ is Borel with compact values (where $(\Delta(I))^{\mathcal{P}}$ has the Tychonoff topology, which makes it into a compact metric space), and hence classical measurable selection theorems ([12]; see also [9],[10]) yield a measurable equilibrium selection. In fact, from the construction of our example it will be clear that one can describe very explicitly many such nonhomogeneous selections; see Remark 8.

Remark 3. It is helpful, before we delve into the definition of the payoffs and the proofs themselves, to pinpoint an important underlying idea; although we will not directly make use of this idea, it is useful to keep it in the back of one's mind. Define an equivalence relation on X by $x \sim y$ if $S^n(x) = y$ for some $n \in \mathbb{Z}$; extend this to an equivalence relation of $\mathcal{P}$ by $(x,k) \sim (y,n)$ if $x \sim y$ (i.e., the second coordinate plays no role in the relation). Since each equivalence class is countable, one might attempt to choose an equilibrium for the entire game by choosing an equilibrium for each class. While this does yield an equilibrium, it will not be measurable. The situation is reminiscent of the following well-known result:[1]

Proposition 4. *There does not exist a measurable set $B \subseteq X$ that intersects each equivalence class in precisely one point.*

In fact, once our payoff function is defined, it will be immediately observable that if Proposition 4 were not true, then by choosing $\phi : B \to (\Delta(I))^{\mathcal{P}}$, which

[1]Proof: Suppose B were such a set as in Proposition 4; define $B_n = S^n(B)$ for $n \in \mathbb{Z}$. The B_n are pairwise disjoint sets whose union is X and which all have the same μ-measure, contradicting the fact that $\mu(X) = 1$.

is a measurable equilibrium selection for games $\{g_x \mid x \in B\}$, and then defining $\phi(y) = S^n(\phi(x))$ for $y = S^n(x)$ we would get a measurable SHES. Yet, such a measurable B does not exist.

We divide the method into four sections. In Section 3 we present an auxiliary construction in normal-form games. In Section 4 we define each g_x and characterize the equilibria; there will be three types, and hence, there will be three types of SHES's. Finally, we will show that none of these SHES's can be measurable - first for one type of SHES in Section 5, and then for the other two types in Section 6. In Sections 7 and 8 we discuss our construction in alternative settings.

3 Piecewise Linear Functions Realized via Normal Form Games

Let $0 < c < d < 1$, and let $f : [0,1] \to [c,d]$ be a continuous, piecewise linear function. Let N_f be the number of intervals that we need to divide $[0,1]$ such that f will be linear in each interval; i.e., suppose we have $0 = x_0 < x_1 < \ldots < x_{N_f} = 1$ with f being linear in each $[x_{j-1}, x_j]$.

Define a set of players K_0 of size $N_f + 1$, where players are denoted by $A, B, \alpha^1, \ldots, \alpha^{N_f - 1}$. (We denote the players differently because the roles of A, B will be different from those of (α^j).) We will build cardinal games G^L, G^R on the profile of K_0 players with action set $I = \{L, R\}$ each, which satisfy the following property: imagine the game $G(p)$, $p \in [0,1]$, in which Nature chooses an action $m \in \{L, R\}$, L with probability p or R with probability $1-p$, and simultaneously (i.e., without knowing Nature's choice) each player in K_0 chooses an action from I, resulting in a profile $a \in I^{K_0}$. The payoff to the players is then the payoff in the game G^m when action profile a is played. We will build G^L, G^R with the property that for any $p \in [0,1]$ and any equilibrium of $G(p)$, the probability that Player A plays L is $f(p)$.

Define, for all $q \in \mathbb{R}$, the game $H(q)$ by:

$H(q) =$

	L	R
L	$1, -1$	$1-4q, 3-4q$
R	$4q-3, 4q-1$	$1, -1$

For $0 < q < 1$, the unique equilibrium profile is $(q, 1-q) \times (q, 1-q)$, and hence the unique equilibrium of $H(f(p))$ is $(f(p), 1-f(p)) \times (f(p), 1-f(p))$. We will now define the payoffs in the games G^L, G^R. For each of the players α^j, $0 < j < N_f$, the payoff in game G^m, $m = L, R$, is given by

$(G^m)^{\alpha^j} =$

	$m = L$	$m = R$
L	$1 - x_j$	0
R	0	x_j

$H \circ f$ is linear in each interval $[x_{j-1}, x_j]$. For players A, B, we first implicitly define auxiliary 2×2 games H_j^L, H_j^R for $0 \le j < N_f$, such that, for $p \in [x_j, x_{j+1}]$,

$$H(f(p)) = p \cdot H_j^L + (1-p) \cdot H_j^R$$

This clearly defines the $(H_j^L), (H_j^R)$ uniquely. Now, let $\overline{a} = (a^1, \ldots, a^{N_f - 1})$ be a profile of pure actions of players $\alpha^1, \ldots, \alpha^{N_f-1}$; we will define the payoff matrix $(G^m)^{A,B}(\overline{a})$ to Players A, B. Denote

$$k(\overline{a}) = \max\{0 < j < N_f \mid a^j = L\}$$

where we take $\max(\emptyset) = 0$. Then, for $m \in \{L, R\}$, set

$$(G^m)^{A,B}(\overline{a}) = \begin{cases} H^L_{k(\overline{a})} & \text{if } m = L \\ H^R_{k(\overline{a})} & \text{if } m = R \end{cases}$$

We make several observations concerning the game $G(p)$:

- The payoff to each player α^j, $0 < j < N_f$, is determined by Nature's choice and his own action only. Therefore, we may refer to an action as being optimal for α^j given some $p \in [0,1]$; R is the only optimal action if $p < x_j$, L is the only optimal action if $p > x_j$, and both actions (as well as any mixtures) are optimal if $p = x_j$.

- If $p \in [x_k, x_{k+1}]$ and if each α^j plays an optimal action, then the expected payoff matrix to players A, B is

  $$pH_k^L + (1-p)H_k^R = H(f(p))$$

  Note that this is well defined, because

  $$H(f(x_k)) = x_k H_k^L + (1-x_k)H_k^R = x_k H_{k-1}^L + (1-x_k)H_{k-1}^R$$

  Thus, if some α^k has more than one optimal action, his choice will not effect the expected payoff matrix to A, B.

- Therefore, in any equilibrium of $G(p)$, A, B play an equilibrium profile in $H(f(p))$, and hence Player A (and also B) plays L with probability $f(p)$.

4 Defining The Game

We begin by defining two continuous piecewise linear functions, ξ, ζ:

$$\zeta(p) = \begin{cases} \frac{3}{4} & \text{if } p \le \frac{1}{4} \\ 1-p & \text{if } \frac{1}{4} \le p \le \frac{3}{4} \\ \frac{1}{4} & \text{if } p \ge \frac{3}{4} \end{cases}$$

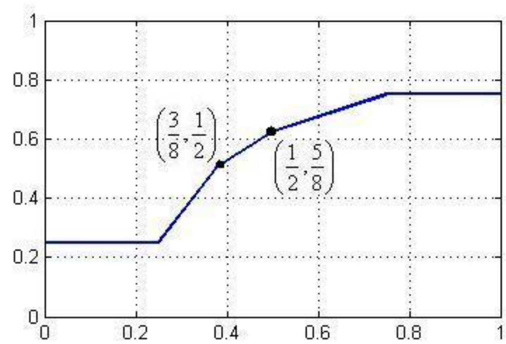

Figure 1: The Function ξ.

$$\xi(p) = \begin{cases} \frac{1}{4} & \text{if } p \leq \frac{1}{4} \\ 2p - \frac{1}{4} & \text{if } \frac{1}{4} \leq p \leq \frac{3}{8} \\ p + \frac{1}{8} & \text{if } \frac{3}{8} \leq p \leq \frac{1}{2} \\ \frac{1}{2}p + \frac{3}{8} & \text{if } \frac{1}{2} \leq p \leq \frac{3}{4} \\ \frac{3}{4} & \text{if } p \geq \frac{3}{4} \end{cases}$$

Note that both these functions, restricted to the interval $[\frac{1}{4}, \frac{3}{4}]$, are bijections from this interval to itself, with ξ strictly increasing and ζ strictly decreasing. Define a doubly infinite sequence $(c_n)_{n\in\mathbb{Z}}$ by starting with $c_0 = \frac{1}{2}$ and then continuing recursively by

$$c_{n-1} = c_n - \frac{1}{2^{(-n)+3}} \text{ for } n \leq 0$$

$$c_{n+1} = c_n + \frac{1}{2^{n+3}} \text{ for } n \geq 0$$

$(c_n)_{n\in\mathbb{Z}}$ is contained in the open interval $(\frac{1}{4}, \frac{3}{4})$, and is strictly increasing. We can write explicitly for $n \neq 0$:

$$c_n = \frac{1}{2} + sign(n) \cdot \frac{1}{4} \cdot (1 - 2^{-|n|})$$

Lemma 5. *$\zeta(c_n) = c_{-n}$ and $\xi(c_n) = c_{n+1}$ for all $n \in \mathbb{Z}$.*

Proof. The claim pertaining to ζ is immediate. If $n < 0$,

$$\xi(c_n) = 2c_n - \frac{1}{4} = 2(\frac{1}{2} - \frac{1}{4} \cdot (1 - 2^n)) - \frac{1}{4} = \frac{1}{2} - \frac{1}{4}(1 - 2^{n+1}) = c_{n+1}$$

since $n + 1 \leq 0$. On the other hand, if $n \geq 0$,

$$\xi(c_n) = \frac{1}{2}c_n + \frac{3}{8} = \frac{1}{2}(\frac{1}{2} + \frac{1}{4} \cdot (1 - 2^{-n})) + \frac{3}{8} = \frac{1}{2} + \frac{1}{4}(1 - 2^{-(n+1)}) = c_{n+1}$$

$\square$

Now, we will go about defining the games for each element of the Cantor set $X = \{-1,1\}^{\mathbb{Z}}$. The set of players is $\mathcal{P} = \mathbb{Z} \times K_0$, where

$$K_0 = \{A, B, \alpha^1, \alpha^2, \alpha^3, \alpha^4\}$$

Each player has an action set $I = \{L, R\}$. Now, let G^L_ζ, G^R_ζ, G^L_ξ, G^R_ξ be the normal-form games on K_0 constructed in the previous section for ζ and ξ, respectively, where $\alpha^1, \dots, \alpha^4$ are the auxiliary players. (The construction for ξ requires four auxiliary players, the construction for ζ requires two, and so two auxiliary players will be inconsequential in G^L_ζ, G^R_ζ.)

For fixed $x \in X$, we define the payoff g_x. For $a \in I^{\mathcal{P}}$ and $n \in \mathbb{Z}$, let $a|_n$ denote the restriction of a to the players $\{n\} \times K_0$. We define the payoff to a player $(n,k) \in \mathcal{P}$ when $a \in I^{\mathcal{P}}$ is played

$$g_x^{(n,k)}(a) = \begin{cases} (G^L_\xi)^k(a|_n) & \text{if } x_n = 1 \text{ and } a^{(n-1,A)} = L \\ (G^R_\xi)^k(a|_n) & \text{if } x_n = 1 \text{ and } a^{(n-1,A)} = R \\ (G^L_\zeta)^k(a|_n) & \text{if } x_n = -1 \text{ and } a^{(n-1,A)} = L \\ (G^R_\zeta)^k(a|_n) & \text{if } x_n = -1 \text{ and } a^{(n-1,A)} = R \end{cases}$$

For fixed x and $n \in \mathbb{Z}$, if Player $(n-1, A)$ plays L with probability p, then assuming players (n, α^i), $i = 1,2,3,4$, play a best-response, the expected payoff matrix facing $(n,A), (n,B)$ is either $G_\xi(p)$ or $G_\zeta(p)$ (recall the notation from Section 3), depending on whether $x_n = 1$ or $x_n = -1$. If ϕ is an SHES, then for all $n \in \mathbb{Z}$ and $k \in K_0$, $\phi^{(n,k)}(S^{-1}x) = \phi^{(n-1,k)}(x)$. Combining these two observations, we immediately deduce:

Proposition 6. *Let $\phi : X \to (\Delta(I))^{\mathcal{P}}$ be an SHES of g, and define $p : X \to [0,1]$ by $p(x) = \phi^{(0,A)}(x)[L]$. Then $\frac{1}{4} \le p \le \frac{3}{4}$, and*

$$p(x) = \begin{cases} \xi(p(S^{-1}(x))) & \text{if } x_0 = 1 \\ \zeta(p(S^{-1}(x))) & \text{if } x_0 = -1 \end{cases} \tag{4.1}$$

The rest of the proof is devoted to deriving a contradiction to the existence of a measurable function p that satisfies the conclusions of Proposition 6. This proposition, together with Lemma 5 and the fact that both ξ and ζ are strictly monotonic in $(\frac{1}{4}, \frac{3}{4})$, gives the following characterization:

Corollary 7. *Under the conditions of Proposition 6, for each $x \in X$, precisely one of the following holds:*

- *For all $n \in \mathbb{Z}$, $p(S^n(x)) \in \{\frac{1}{4}, \frac{3}{4}\}$, in which case $p(S^n(x)) = p(S^{n-1}(x))$ iff $x_n = 1$. (x will be called a quasi-pure point of ϕ.)*
- *For all $n \in \mathbb{Z}$, $p(S^n(x)) \in \{c_j \mid j \in \mathbb{Z}\}$. ($x$ will be called a regular mixed point of ϕ.)*
- *For all $n \in \mathbb{Z}$, $p(S^n(x)) \in (\frac{1}{4}, \frac{3}{4}) \backslash \{c_j \mid j \in \mathbb{Z}\}$. ($x$ will be called an irregular mixed point of ϕ.)*

A SHES for which μ-a.e. points of X are of the same type will be called a *quasi-pure / regular mixed / irregular mixed* SHES, in accordance with the type. Now, suppose $\phi : \mathcal{P} \to \Delta(I)$ is a measurable SHES. Then the set of quasi-pure / regular mixed / irregular mixed points is a shift-invariant measurable set in X. Since the shift S is μ-ergodic[2] on X, this shows us that, up to a set of μ-measure 0, only one of these types of points exist. Therefore, we may assume that a given measurable SHES is either quasi-pure, regular mixed, or irregular mixed (and we can assume that it has been altered on a set of measure 0 to use only one type of point everywhere on X). In the next sections, we will show that none of these can be.

Remark 8. As noted in Remark 2, we can construct explicitly many measurable equilibrium selections that are not homogeneous. Indeed, let $\eta : X \to [\frac{1}{4}, \frac{3}{4}]$ be any measurable map, let player $(0, A)$ (and $(0, B)$) play L with probability $\eta(x)$ in the game g_x, and define the actions of (n, A) (and (n, B)) by the recursive relation (4.1); the actions of the auxiliary players then follow.[3]

5 Quasi-Pure Points

Clearly, to show that that no measurable quasi-pure SHES exist, it suffices to prove the following theorem:

Theorem 2. *There does not exist a measurable function*[4] $f : X \to X$ *satisfying the following two conditions:*

- $f \circ S = S \circ f$.
- *For all* $x \in X$, $k \in \mathbb{Z}$, $(f(x))_k = (f(x))_{k-1} \iff x_k = 1$.

Indeed, if we had, as in Proposition 6, a measurable p associated with a quasi-pure SHES, we could define $f : X \to X$ by $(f(x))_n = p(S^n(x))$ after identifying $\{\frac{1}{4}, \frac{3}{4}\}$ with $\{-1, 1\}$; such f would be measurable and satisfy the conditions of Theorem 2, and this give rise to a contradiction. To prove Theorem 2, we first need the following:

Lemma 9. *Let* $f : X \to X$ *be measurable and satisfy the second given condition of* f *in Theorem 2. Then for all* μ*-measurable* $B \subseteq X$, $\mu(f(B)) \leq \mu(B)$.

Proof. First, we prove the claim for B which is determined by finitely many coordinates (i.e., which is clopen in X). Assume J is a finite subset of $\mathbb{Z}$ and $T \subseteq \{-1, 1\}^J$ is such that

$$B = \{z \in X \mid z_J \in T\}$$

[2]That is, for all μ-measurable $A \subseteq X$, $\mu(S^{-1}(A)\Delta A) = 0$ implies that $\mu(A) = 1$ or $\mu(A) = 0$, where Δ denotes the symmetric difference of sets.

[3]Up to degenerate cases of indifference, for which we can pre-select an action.

[4]That is, $f^{-1}(B)$ is Lebesgue-measurable for any Borel-measurable subset B of X.

where z_J denotes the projection of z to the coordinates in J. Then $\mu(B) = \frac{|T|}{2^{|J|}}$. Furthermore,

$$f(B) = U := \cup_{t\in T} \cap_{k\in J} \{u \in X \mid u_k = u_{k-1} \Longleftrightarrow t_k = 1\}$$

and $\mu(U) = \frac{|T|}{2^{|J|}}$. This proves the theorem for clopen B.

Since every open set in X is a countable union of clopen sets (X is separable and has a basis consisting of clopen sets), this proves the theorem for open B. Now, let B be μ-measurable, let $\varepsilon > 0$, and let V be open such that $B \subseteq V$ and $\mu(V \backslash B) < \varepsilon$. Then

$$\mu(f(B)) \leq \mu(f(V)) \leq \mu(V) \leq \mu(B) + \varepsilon$$

and taking $\varepsilon \to 0$ finishes the proof. $\square$

Proof. (Of Theorem 2) Suppose that there were such an f. It is easy to see that such an f must be injective (in fact, the second condition alone implies this). First, we maintain that we can find an S-invariant Borel subspace $Y \subseteq X$ with $\mu(Y) = 1$ such that $f|_Y$ (the restriction of f to Y) is Borel. Indeed, it is well known that we can find some such Y' that is not necessarily S-invariant, and then taking $Y = \cup_{k\in\mathbb{Z}} S^k(Y')$ can easily be seen to fit the bill. Since $f|_Y$ is injective and Borel, $f(Y)$ is Borel. Furthermore, $f(Y)$ is S-invariant, as $S(f(Y)) = f(S(Y)) = f(Y)$.

Define the involution $g : X \to X$ given by $(g(x))_k = 1 - x_k$. g clearly preserves μ. In addition, $g(f(X)) \cap f(X) = \emptyset$; one sees (again, only the second condition of Theorem 2 is needed) that z, $g(z)$ would have the same inverse image under f if they had both been in $f(X)$.

We also observe that $g(f(X)) \cup f(X) = X$. Indeed, let $y \in X$, and define $x \in X$ by $x_n = y_n \cdot y_{n-1}$. Then either $f(x) = y$ or $f(x) = g(y)$. We maintain that

$$\mu(g(f(Y)) \cup f(Y)) = 1$$

Using the injectivity of f and g, we see that

$$g(f(Y)) \cup f(Y) = (g(f(X)) \backslash g(f(X \backslash Y)) \cup (f(X) \backslash f(X \backslash Y))$$

It is enough to show that

$$\mu(g(f(X \backslash Y))) = 0 \text{ and } \mu(f(X \backslash Y)) = 0$$

Indeed, since g preserves μ, it is enough to show the latter inequality. This follows from Lemma 9, since $\mu(X \backslash Y) = 0$.

Since $\mu(g(f(Y)) \cup f(Y)) = 1$, $g(f(Y)) \cap f(Y) = \emptyset$, and g preserves μ, $f(Y)$ is an S-invariant set of measure $\frac{1}{2}$, contradicting the ergodicity of μ under S. $\square$

6 Mixed Points

The following theorem is similar to Lemma 4 of [17], and our proof uses similar ideas:

Theorem 3. *There does not exist a measurable function $f : X \to \mathbb{R}$ that satisfies*

$$f(x) = \begin{cases} f(S^{-1}x) + 1 & \text{if } x_0 = 1 \\ -f(S^{-1}x) & \text{if } x_0 = -1 \end{cases} \tag{6.1}$$

Before proving the theorem, we shall show why this implies that there does not exist a measurable mixed equilibrium selection.

Take the case where function p from Proposition 6 is associated with a regular mixed SHES. For each $x \in X$, there exists a unique $n \in \mathbb{N}$, such that $p(x) = c_n$; denote this n as $f(x)$. Then, by Lemma 5 and Proposition 6, $f : X \to \mathbb{R}$ satisfies (6.1), and hence it is not μ-measurable, and hence neither is p.

Take now the case where p is associated with an irregular mixed SHES. Define sets A_k for every odd $k \in \mathbb{Z}$ by $A_{2n+1} = (c_n, c_{n+1})$. By Lemma 5 and Proposition 6, along with the strong monotonicity of ξ and ζ, $p(x) \in A_k$ implies that $p(S^{-1}x) \in A_{k-2}$ if $x_0 = 1$ and $p(S^{-1}(x)) \in A_{-k}$ if $x_0 = -1$, as

$$A_{(2n+1)-2} = A_{2(n-1)+1} = (c_{n-1}, c_n)$$

and

$$A_{-(2n+1)} = A_{-2(n+1)+1} = (c_{-(n+1)}, c_{-n})$$

If $f : X \to \mathbb{R}$ is the unique function such that $p(x) \in A_{2 \cdot f(x)}$, then $f : X \to \mathbb{R}$ satisfies (6.1), and hence it is not μ-measurable, and again one sees that this implies that neither is p.

Proof. (Of Theorem 3) Suppose that there were such a function. Since f is μ-measurable, there is $M > 0$ such that $\mu(f^{-1}([-M, M])) > 0.9$. Denote $B = f^{-1}([-M, M])$.

Now, for every $n \in \omega = \{0, 1, 2, \ldots\}$, we will define $h_n : X \to \mathbb{Z}$ and $k_n : X \to \{-1, 1\}$ in the following recursive manner: $h_0(x) \equiv 0$ and $k_0(x) \equiv 1$. Hence,

$$(h_{n+1}(x), k_{n+1}(x)) = \begin{cases} (h_n(x) + k_n(x), k_n(x)) & \text{if } x_{n+1} = 1 \\ (h_n(x), -k_n(x)) & \text{if } x_{n+1} = -1 \end{cases}$$

It is immediate that if $x \in X$ and $n \in \mathbb{N}$ are such that $k_n(x) = 1$, then $f(S^n(x)) = f(x) + h_n(x)$. Clearly, it holds that for any $n \geq 1$,

$$\mu(\{x \in X \mid k_n(x) = 1\}) = \frac{1}{2}$$

and therefore, for each $n \in \mathbb{N}$,

$$\mu(E_n) \geq \frac{1}{2}, \text{ where } E_n = \{x \in X \mid f(S^n(x)) = f(x) + h_n(x)\}$$

It is clear that

$$\lim_{n\to\infty} \mu(Q_n) = 1, \text{ where } Q_n = \{x \in X \mid h_n(x) \notin [-2M, 2M]\}$$

Hence, we may choose $n \in \mathbb{N}$ such that

$$\mu(Q_n) > 0.9$$

Since $\mu(B) > 0.9$ (recall $B = f^{-1}([-M, M])$) and $\mu(E_n) \geq \frac{1}{2}$, it holds that,

$$\mu(B \cap Q_n \cap E_n) \geq \frac{1}{2} - 0.1 - 0.1 = 0.3$$

However, for $x \in B \cap Q_n \cap E_n$, $f(x) + h_n(x) = f(S^n(x))$, $f(x) \in [-M, M]$ and $|h_n(x)| \geq 2M$; hence, for such x, $f(S^n(x)) \notin [-M, M]$. Therefore,

$$\mu(x \in X \mid f(S^n(x)) \notin [-M, M]) \geq 0.3$$

which is equivalent to $\mu(S^{-n}(B^c)) \geq 0.3$. Since μ is S-invariant, this implies that $\mu(B^c) \geq 0.3$, contradicting the fact that $\mu(B) > 0.9$. $\square$

7 Overlapping Generations Games

Models of overlapping generations have been considered in several contexts. The oldest works in this direction are related to markets evolving over time: players live and participate in the market for several generations, with lifetimes of players overlapping with the lifetimes of others; see [4] and the references there. Gossner (1995), [5], and others have concentrated on repeated games: players of different types play repeatedly, each player participating for several stages, accumulating the stage-by-stage payoffs, until his death, and then he is replaced by another player of the same type. In addition, there have been several works which concentrate on the case where participants live for two-generations: One generation in which they are 'young', and another in which they are 'old'; see Galor (1992), [3], and the references there.

In this section, we introduce a simple model (easily generalized) that also uses the 'young'/'old' overlapping generations approach, and we will show that even though this model possesses time-stationarity, measurable stationary equilibria do not exist. By a Non-Cooperative Overlapping Generations Game with Deterministic Natural Process, we mean the following:

- A finite set of types K_0. The set of players is then $\mathcal{P} = \mathbb{Z} \times K_0$.
- Finite sets of actions, I_k, one for each $k \in K_0$. Denote $\overline{I} = \prod_{k \in K_0} I_k$.

- X_0 is the finite sets of the states of Nature. Denote $X = \prod_{n \in \mathbb{Z}} X_0$.
- μ is a distribution on X.
- A payoff function $g : \mathbb{Z} \times X_0 \times \overline{I}^2 \to \mathbb{R}^{K_0}$, where X denotes, as usual, the Cantor set.

For convenience,[5] denote $T = S^{-1}$. The game is said to be stationary if $\mu \circ T = \mu$, and there is $u : X_0 \times \overline{I}^2 \to \mathbb{R}^{K_0}$ such that for all $n \in \mathbb{Z}$, $\xi \in X_0$, $a, b \in \overline{I}$ $g(n, \xi, a, b) = u(\xi, a, b)$.

The interpretation is as follows: In each generation, the player set is K_0. There is a sequence of states that is determined by a known process chosen by the distribution μ. The players of generation n - who are 'young' in stage n - observe the choice and choose an action profile a. In stage $n+1$ - when the players of generation n are 'old' and do not actively participate in decisions - the players of generation $n+1$ play and choose an action profile b. The payoff profile to the players of generation n is then given by $g(n, x_n, a, b)$. The stationary condition means that not only is the distribution of the process that Nature uses time-invariant, but the payoff of each generation does not depend on what time the generation came along, only on what is done during that generation's lifetime and the state of Nature on that day.

A Nash equilibrium of such a game is a mapping[6] $\phi : X \to (\Delta(I))^{\mathcal{P}}$, which tells all players what to play given the process Nature chooses, such that for each fixed $x \in X$, $\phi(x)$ is a Nash equilibrium in the resulting game with players $\mathcal{P}$ in which player $q \in \mathcal{P}$ has payoff function $g(q, x, \cdot)$. The equilibrium can be viewed as the instructions given ex-ante to the players before Nature chooses the deterministic process. A stationary equilibrium is an equilibrium ϕ such that $(\phi(x))^{(n,k)} = (\phi(T^j x))^{(n+j,k)}$, for all $j \in \mathbb{Z}$; or, equivalently, is such that there is $\psi : X \to (\Delta(I))^{K_0}$ satisfying $(\phi(x))^{(n,k)} = \psi(T^{-n}x, k)$. Note that ϕ is measurable iff ψ is. In other words, the strategy profile of a generation should only depend on the sequence of bits that has been chosen up until now and the sequence of bits that will be chosen in the future.

Clearly, then, our example shows that stationary non-cooperative overlapping generations games with nature need not have a (Lebesgue-) measurable stationary equilibrium. As discussed in Section 1, this is a disturbing phenomenon; it implies that for an equilibrium strategy to be "well-behaved" in a very basic sense, it must take into account data that is completely irrelevant to the preferences of the players. This is despite the fact that - as noted in Remark 8 - many non-stationary equilibria can be given via very explicit descriptions.

[5] This is because it is more intuitive that the right-shift corresponds to a shift forward in time.

[6] It is clearly too restrictive to require that the strategy of generation depend only on the bit received by that generation.

8 Games with a Continuum of Players

Games with a continuum of players are extremely useful in modeling and approximating economics models (see [11] and the references there). Peleg (1969), [13], demonstrated that given an arbitrary player set[7] and finitely[8] many actions for each player, a Nash equilibrium exists if the payoff functions are continuous on the space of action profiles, which is endowed with the product topology. One can consider the case in which the set of players is endowed with a measurable structure, and the payoffs behave well with respect to the this structure;[9] we will show that in this case, it may not be possible to obtain a measurable equilibrium.

We should point out that the example we have here is of a very different nature than the type of games with a continuum of players introduced in Schmeidler, (1973) [16], and discussed in many works thereafter (see [11] for references). In the games Schmeidler considered, the player space is also a measure space, and measurable equilibria exist; however, the payoff of each player is dependent on his own action and on the statistics of the other players' actions, i.e., the measures of the sets of players who played each action. Other papers in this direction consider payoffs which are dependent on the responses of almost all players (with respect to a fixed measure); see Section 8 of [11]. In our game, however, each player's payoff depends on the specific actions of finitely many players - but the set of those whose actions affect a specific player varies from player to player.

For our example, $\mathcal{P} = X \times K_0$ will be the set of players, where K_0 is the same as in Section 4, and $I = \{L, R\}$ the set of actions that each player can choose from. For $a \in I^{\mathcal{P}}$, let $a|_x$ denote the restriction of a to the players $\{x\} \times K_0$. We now define the payoff function g:

$$g^{(x,k)}(a) = \begin{cases} (G_\xi^L)^k(a|_x) & \text{if } x_0 = 1 \text{ and } a^{(S^{-1}x,A)} = L \\ (G_\xi^R)^k(a|_x) & \text{if } x_0 = 1 \text{ and } a^{(S^{-1}x,A)} = R \\ (G_\zeta^L)^k(a|_x) & \text{if } x_0 = -1 \text{ and } a^{(S^{-1}x,A)} = L \\ (G_\zeta^R)^k(a|_x) & \text{if } x_0 = -1 \text{ and } a^{(S^{-1}x,A)} = R \end{cases}$$

Theorem 4. *There does not exist a measurable Nash equilibrium of g.*

Proof. Suppose that $\phi : \mathcal{P} \to \Delta(I)$ is a Nash equilibrium. Denote $p(x) = \phi^{(x,A)}[L]$. Such p would also satisfy the recursion relation of Proposition 6 as in Section 4, by the same reasoning given there, which gives rise to a contradiction in the same way. □

[7] No measurable structure is assumed.

[8] This was extended by Salonen (2010), [15], using a somewhat different argument, to the case of compact action spaces.

[9] E.g., we can require that the correspondence that assigns to each player the set of players whose actions affect him must have a Borel graph.

Remark 10. The main example of Simon (2003), [17], of a three-player Bayesian game with a continuum of states of the world - also the Cantor set X - possessing no measurable Bayesian equilibrium, can also give an example of a game with a continuum of players with no measurable equilibria. In Simon's game, each player has an action space $I = \{L, R\}$. The common prior μ is the Lebesgue-measure. Player 1 cannot differentiate between state x and $\sigma(x)$, where $(\sigma(x))_n = x_{-n}$, and Players 2 and 3 cannot differentiate between states x and $\tau(x)$, where $(\tau(x))_n = x_{1-n}$. Let X_1, X_2, X_3 be the resulting quotient spaces, i.e., $X_1 = X/(x \sim \sigma(x))$, and $X_2 = X_3 = X/(x \sim \tau(x))$, and let $[x]_i$ denote the equivalence class of x in X_i. (X_1, X_2, X_3 are the types for Players $1, 2, 3$.) Since the common prior is the Lebesgue measure, it follows that when a player is uncertain about which of two states is the true state, he attributes equal probability to each state. Let $p : X \times I^3 \to \mathbb{R}^3$ denote the payoff function defined there. Since σ,τ are continuous involutions,[10] the quotient σ-algebras induced on X_1, X_2 and X_3 are standard Borel spaces. Note that, by definition, the maps $x \to [x]_j$ are measurable.

Following an approach due to Harsanyi, (1967) [6], we can model the game as a normal-form game, with a set of players $\mathcal{P} = X_1 \cup X_2 \cup X_3$. The new payoff function, $g : \mathcal{P} \times I^{\mathcal{P}} \to \mathbb{R}$ is given by

$$g(t, a) = \sum_{x \in t} \frac{1}{|t|} p^j(a^{[x]_1}, a^{[x]_2}, a^{[x]_3}),\ t \in X_j$$

where $|t|$ is the size of t (either $|t| = 1$ or $|t| = 2$); the coefficient $\frac{1}{|t|}$ results from invariance of μ under σ, τ. It is now clear that if there were a measurable equilibrium φ of g, $\varphi : \mathcal{P} \to \Delta(I)$, then the maps $\psi^j : X \to \Delta(I)$, $j = 1, 2, 3$ given by $\psi^j(x) = \varphi([x]_j)$ would be a measurable Bayesian equilibrium of Simon's game, a contradiction. Note that Simon's game cannot give us an example of a setup as discussed in Section 2, as the shift is not well defined on equivalence classes under σ or τ.

9 Parallels with a Computability Example

We point out the similarity of our setup to an example given by Rabin, (1957) [14]. A game with the following simple dynamics is considered there: Player 1 chooses an integer, Player 2 chooses an integer in reply, and Player 1 chooses another integer. Player 1 (resp. 2) wins if the resulting triple is in (resp. not in) a specific set W of integer triples. Rabin shows that there exist winning sets W that are computable and for which Player 2 can force a win, but only by using non-computable strategies. Indeed, the situation parallels ours in some respects: Rabin's example can be described so explicitly that one could write an algorithm that decides which player wins the game given a triple of integers, Player 2 can

[10]Hence, the quotient maps, $x \to [x]_j$, viewed as correspondences from X to X, have Borel graphs and take on compact values.

always force a win, but no algorithm could be written which tells Player 2 how to win. Similarly, although we describe our game's payoffs explicitly, it is not possible to give an explicit description of any Nash equilibria.

10 Acknowledgements

I would like to thank Abraham Neyman, Robert Simon, Sergiu Hart, and Ilan Nehama for helpful feedback.

I would like to dedicate this paper to the memory of my wife, Ella Lois Pechony-Levy, who passed away from cystic fibrosis in the course of this research. Breathe easy, my love.

References

[1] Aumann, R. (1964), Markets with a Continuum of Traders, *Econometrica*, 32, 39–50.

[2] Carlsson, H. and van Damme, E. (1993) Global Games and Equilibrium Selection, *Econometrica*, 61, 989–1018.

[3] Galor, O. (1992), A Two-Sector Overlapping-Generations Model: A Global Characterization of the Dynamical System, *Econometrica*, 60, 1351–1386.

[4] Geanakoplos, J. and Polemarchakis, H.M. (1991), Overlapping Generations. In: Arrow, K. and Intriligator, D. (eds.), *Handbook of Mathematical Economics*, Elsevier. Ch. 35, pp. 1900–1960.

[5] Gossner, O. (1996), Overlapping Generations Games with Mixed Strategies, *Math. Oper. Res.*, 21, 477–486.

[6] Harsanyi, J. (1967), Games of Incomplete Information Played by Bayesian Players, Part I: The Basic Model, *Management Sci.*, 14, 159–182.

[7] Harsanyi, J. C., and Selten, R. (1988), *A General Theory of Equilibrium Selection in Games*, MIT Press.

[8] Herrlich, H. (2006), *Axiom of Choice*. Lecture Notes in Mathematics, Springer.

[9] Himmelberg, C. J. (1975), Measurable relations, *Fund. Math.*, 87, 53-72.

[10] Kechris, A. S. (1995), *Classical Descriptive Set Theory*, Graduate Texts in Mathematics 156. Springer.

[11] Khan, M. A., and Sun, Y. (2002), Non-Cooperative Games with Many Players. In: Aumann, R.J. and Hart, S. (eds.), *Handbook of Game Theory*, Elsevier Science B.V. Ch. 46, pp. 1761–1808.

[12] Kuratowski, K. and Ryll-Nardzewski, C. (1965), A General Theorem on Selectors, *Bull. Pol. Acad. Sci. Math (Ser. Math.)*, 13, 379-403.

[13] Peleg, B. (1969), Equilibrium Points for Games with Infinitely Many Players. *J. Lond. Math. Soc.*, 44, 292–294.

[14] Rabin, M. O. (1957), Effective Computability of Winning Strategies. In: *Contributions to the Theory of Games*, Vol. 3, Princeton University Press, Princeton, Annals of Mathematics Studies No. 39.

[15] Salonen, H. (2010), On the Existence of Nash Equilibria in Large Games, *Internat. J. Game Theory*, 39, 351–357.

[16] Schmeidler, D. (1973), Equilibrium Points of Nonatomic Games, *J. Stat. Phys.*, 4, 295-300.

[17] Simon, R. (2003), Games of Incomplete Information, Ergodic Theory, and the Measurability of Equilibria. *Israel J. Math.*, 138, 1, 73–92.

[18] Shapley, L. (1953), Stochastic Games, *Proc. Nat. Acad. Sci. USA* 39, 1095–1100.