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RISKINESS FOR SETS OF GAMBLES

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Riskiness for sets of gambles*

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Abstract

Aumann–Serrano (2008) and Foster–Hart (2009) suggest two new riskiness measures, each of which enables one to elicit a complete and objective ranking of gambles according to their riskiness. Hart (2011) shows that both measures can be obtained by looking at a large set of utility functions and applying "uniform rejection criteria" to rank the gambles in accordance with this set of utilities. We use the same "uniform rejection criteria" to extend these two riskiness measures to the realm of uncertainty and develop complete and objective rankings of sets of gambles, which arise naturally in models of decision making under uncertainty.

1 Introduction

Economic models enable one to rank risky prospects through the maximization of expected utility of a representative agent. However, in order to do so, one needs to know the probabilities of events and the utility function of a representative agent, both of which, if they exist, are difficult to measure in real life. Decision theorists and finance theorists developed alternative methods to rank prospects in the absence of either probabilities or utility functions. One strand of the literature deals with

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modeling decisions in situations where the utility of every final outcome is given, but a unique probability distribution over the events leading to these outcomes is either non existent or unknown. These are models of decision under uncertainty (to be distinguished from decisions under risk). In models of decision under uncertainty, lotteries are replaced with *acts*, where *acts* are defined as mappings from states of nature to objective lotteries or to deterministic outcomes. The models differ in the way they represent those *acts*, but they commonly share the assumption of a well-identified utility function. The simplified assumption that each decision maker uses his own unique and subjective probability distribution (as in Savage 1954) became highly unsatisfactory after Ellsberg introduced his famous paradox (Ellsberg 1961). Two significant models that incorporate an explanation to the Ellsberg Paradox are Schmeidler's capacities model (1989) and Gilboa and Schmeidler's multi-prior model (1989). The former uses non-additive probabilities and an order-dependent utility calculation, and the latter uses sets of probability distributions that the decision maker believes to contain the true probability distribution. Both models, when used to describe uncertainty-averse decision makers (henceforth DMs), imply a high degree of uncertainty aversion, focusing on the worst-case scenario. Other models, such as Bewley (2002), solved the paradox by maintaining that DMs have only partial preferences, so that they can be guaranteed to choose *act* a over *act* b only if $a \succ b$ for every possible prior.

A second strand of the literature models decisions under risk (i.e., assumes known probabilities) while maintaining only very general assumptions about the utilities of decision makers (e.g., risk aversion or decreasing absolute risk aversion). Since a utility function is a subjective characteristic of the decision maker, while probabilities may be objective, these models offer objective rankings of risky alternatives using the known characteristics of the given probability distribution. For example, the Sharpe Value is the expected value (not utility) of a random variable with a given probability distribution, divided by its standard deviation (to incorporate the risk aversion assumption). Riskiness measures and indices are widely used in finance to compare investment alternatives (regardless of the decision maker and his unique utility function), and lately have been explored also in economics. Two new riskiness measures were recently developed in a setup with known probabilities and a large class of utility functions: Aumann and Serrano's economic index of riskiness (2008) and Foster and Hart's operational measure of riskiness (2009).[1] These two new measures are superior to previous measures and

[1] Even more recently, Schreiber (2012) developed a similar economic index of *relative riskiness*. This index applies to shares (stocks), while the other two models of

indices (including the Sharpe Value) in two aspects: (1) They do not violate stochastic dominance; and (2) they are related to the concept of utility in a strong way: each of them can be induced by a *uniform rejection criterion*, as presented in Hart (2011). Each of the two *uniform rejection criteria* ranks a gamble g above a gamble h if a *uniform rejection* of g (uniform over wealth levels for one criterion and over utilities for the second) leads to a *uniform rejection* of h.[2]

The setup in Hart (2011) is that of risk; i.e., a gamble has a unique probability distribution that describes it. In Section 2 we apply these *uniform rejection criteria* to develop rankings in a setup of decision under uncertainty, and generate complete rankings of *acts*. Our chosen setup of decision under uncertainty is that of Gilboa et al. (2010), which is a recent incorporation of both Gilboa and Schmeidler's multi-prior model (1989) and Bewley's unanimity model (2002) into one unified model. In this setup, every *act* is associated with a set of gambles. Each of the two *uniform rejection criteria* is then applied to create a complete ranking of those sets of gambles. We show that each of the two resulting rankings can be directly generated by calculating for every set of gambles the maximal value of the corresponding set of riskiness measures.

In Section 3 we first apply the same technique of Section 2 to uncertainty-averse DMs in Schmeidler's capacity model (1989), and we get qualitatively similar results. We then relax the extent of uncertainty aversion following Gajdos et al. (2008), and get a ranking based on the maximal riskiness over a contracted set of gambles around the Steiner Point of the set. Next we substitute the extreme pessimism of the DMs in Gilboa et al. (2010) with equally extreme optimism, and finally we substitute both with minimax regret attitude. Those substitutions lead to new complete rankings, differing from those generated by the basic model. Section 4 binds most of the previous results together and generalizes them, by generating complete rankings of sets of gambles for any mixture of optimism and pessimism. In particular, DMs are assumed to put a weight of $\lambda \in [0,1]$ on the best outcome in the set and a weight of $1-\lambda$ on the worst outcome in the set (known in the literature as λ-Hurwicz[3] optimism). We show that for every $\lambda \in [0,1]$, applying each of the *uniform rejection criteria* induces a complete ranking of sets of gambles. The final step is performed in Section 5, where we expand the definition of *uniform rejection* to include uniformity over λ, the extent of optimism, and apply the new extended *uniform rejection criteria* to

riskiness apply to gambles.

[2] See Section 1.1.1 for further details.

[3] In the literature this criterion is often noted by α-Hurwicz, but we reserve the symbol α to denote the Arrow–Pratt coefficient of CARA utilities.

generate the most generally motivated complete rankings. Sections 6 and 7 discuss the results and conclude.

The rankings presented in this paper are probably the first complete and objective rankings of alternatives that use neither utility functions nor unique probability distributions. Proofs are relegated to the appendix.

1.1 Definitions

Following Aumann and Serrano (2008) and Foster and Hart (2009), a *gamble* g is a real-valued random variable with finitely many values having some negative values - losses are possible - and positive expectation, i.e., $P[g<0]>0$ and $E[g]>0$ (Foster and Hart 2009, page 4).[4]

Let $\mathcal{G}$ denote the collection of all such gambles. We will say that a *set of gambles* $G\subset\mathcal{G}$ has *finite support* if there exists a finite set $Z\subset R$ s.t. $\text{supp}\{g\}\subset Z$ for all g in G. Let $\mathbb{G}$ denote the collection of all such *sets of gambles.*[5]

Let u be a vN-M utility function. We say that u *rejects* the gamble g at wealth w, if and only if $u(w)\geq u(w+g)$.

Let L be the set of all real-valued random variable with finitely many values. The elements of L are called *lotteries*, and we have $\mathcal{G}\subset L$. Compound lotteries are lotteries themselves, i.e., $\Delta L=L$. In this paper we will interpret the outcomes of *lotteries* as final wealth levels, and the outcomes of *gambles* as net values (i.e., gains when positive and losses when negative).

Let S denote the set of states of nature, and let $\mathcal{P}$ denote all the probability distributions over the set S. *Acts* are mappings from states of nature to lotteries, $A:S\to L$. $F(a,p)\in L$ denotes the compound lottery that results from applying probability distribution $p\in\mathcal{P}$ to the set of states of nature S on which the *act* $a\in A$ is defined.

1.1.1 Riskiness

For every gamble $g\in\mathcal{G}$, we have:

$R^{AS}(g)$, the *Aumann–Serrano index of riskiness* of g, is given implicitly by

$$E\left[\exp\left(-\frac{g}{R^{AS}(g)}\right)\right]=1$$

[4]Note that the outcomes of *gambles* are *net* outcomes - gains and losses - in contrast to *lotteries*, where the outcomes are final outcomes.

[5]Throughout the paper we work with *sets of gambles* in $\mathbb{G}$, in order to avoid the cumbersome use of supremum and infimum. Naturally all propositions and theorems can be generalized to *any* (not necessarily finite) subset of $\mathcal{G}$ with the proper adjustments.

$R^{FH}(g)$, the *Foster–Hart measure of riskiness* of g, is given implicitly by

$$E\left[\log\left(1+\frac{g}{R^{FH}(g)}\right)\right]=0$$

Furthermore, Hart (2011) shows that each of these two riskiness measures can be induced by a *uniform rejection criterion* as follows.

A gamble g is *wealth-uniformly rejected* by u if g is rejected by u at *all* wealth levels w.

Let $g, h \in \mathcal{G}$. We say that g wealth-uniformly dominates h, which we write $g \geq_{WU} h$, if the following holds:

For every utility $u \in U^*$,[6]

If g is rejected by u at all $w > 0$

Then h is rejected by u at all $w > 0$.

Theorem 1(i) in Hart (2011) states that for any two gambles g and h in $\mathcal{G}$, $g \geq_{WU} h$, if and only if $R^{AS}(h) \geq R^{AS}(g)$.

Similarly, a gamble g is *utility-uniformly rejected* at wealth w if g is rejected by *all* utility functions $u \in U^*$ at w.

Let $g, h \in \mathcal{G}$. We say that g utility-uniformly dominates h, which we write $g \geq_{UU} h$, if the following holds:

For every wealth level $w > 0$,

If g is rejected by all $u \in U^*$ at w

Then h is rejected by all $u \in U^*$ at w.

Theorem 1(ii) in Hart (2011) states that for any two gambles g and h in $\mathcal{G}$, $g \geq_{UU} h$, if and only if $R^{FH}(h) \geq R^{FH}(g)$.

2 Basic model

2.1 From *acts* to sets of gambles

In the framework of decision under uncertainty, the objects to be ranked are *acts*. The first axiomatization of decision under uncertainty is that of Savage (1954), but the first setup to include *lotteries* and *acts* together is that of Anscombe and Aumann (1963). In both setups, it is possible to elicit a subjective probability for every state of nature, as long as the preference relation satisfies certain reasonable axioms. However, Ellsberg (1961) showed that the very assumption of having a unique subjective probability corresponding to every state of nature is questionable. The decision maker in Ellsberg (1961) violates the Savage axioms and demonstrates an aversion towards "uncertainty" in addition to the widespread aversion towards risk. In the aftermath of the Ellsberg Paradox, new models tried to incorporate this "uncertainty aversion".

[6] For the definition of U^* see Section 2.2.

One such fundamental model is the *multi-prior utility* (*mpu* in short) model of Gilboa and Schmeidler (1989). The *mpu* model solves the Ellsberg Paradox in the following way: the decision maker behaves as if he has a *set* of probability distributions over the states of nature, and when faced with a certain *act* in the context of a decision problem, he attaches to it the minimal utility over the probability distributions in his set.[7] As a result, the probability attached to a state of nature is no longer independent of the payoff corresponding to it, as is the case in Savage's framework (1954).

More formally, let $\succsim$ be a complete preference relation over standard lotteries. If $\succsim$ satisfies:

(-) Few standard axioms (transitivity, completeness, continuity, monotonicity);

(-) Uncertainty aversion ($a \simeq b \Leftrightarrow \lambda a + (1-\lambda) b \succsim a$ $\forall a, b \in A, \lambda \in (0,1)$);

(-) Certainty independence ($a \succ b \Rightarrow \lambda a + (1-\lambda) c \succ \lambda b + (1-\lambda) c$ $\forall a, b \in A$, constant *act* c, and $\lambda \in (0,1)$);

then there exists an affine function $u : A \rightarrow \mathcal{R}$ and a non-empty, closed and convex set $P \subseteq \mathcal{P}$, such that:

$$a \succsim b \Leftrightarrow \min_{p \in P} \int E\left[u\left(a(s)\right)\right] dp(s) \geq \min_{p \in P} \int E\left[u\left(b(s)\right)\right] dp(s).$$

Using the notations defined previously we can write:

$$a \succsim b \Leftrightarrow \min_{p \in P} E\left[u\left(F\left(a, p\right)\right)\right] \geq \min_{p \in P} E\left[u\left(F\left(b, p\right)\right)\right].$$

Denote now by G and H the sets of lotteries corresponding to *acts* a, b respectively under the set of probability distributions P, i.e., $G \equiv \{F(a,p) : p \in P\}$ and $H \equiv \{F(b,p) : p \in P\}$. Then $G, H \subseteq L$, and

$$a \succsim b \Leftrightarrow \min_{g \in G} E[u(g)] \geq \min_{h \in H} E[u(h)].$$

The last correspondence states that every decision maker compares any two *acts* by comparing the minimal utility over the sets of lotteries corresponding to those *acts*. While Gilboa and Schmeidler are agnostic about the way in which the set P of probability distributions is determined for every *act*, we explicitly treat P as independent of u. When the different utility functions are considered as representing one decision maker (see discussion in Section 6), this interpretation is natural. Alternatively, if different utility functions are thought of as representing

[7] Given an act, every probability distribution over the states of nature is associated with a random variable, i.e., a lottery, the utility of which is uniquely determined given a utility function.

different decision makers, we can follow Gilboa et al. (2010), and require objective and subjective rationality with respect to a common set P.[8] This ensures that all DMs judge every *act* according to the minimal utility over the same set P. As a result, every *act* is linked to a unique set of lotteries, common to all DMs, and we end up with a criterion that ranks well-defined sets of lotteries according to the worst lottery in every set. As already pointed out in Gilboa and Schmeidler (1989), this result stands in line with the minimax loss criterion of Wald (1950): "A minimax solution seems, in general, to be a reasonable solution of the decision problem when an *a priori* distribution in Q does not exist or is unknown to the experimenter".

At this point, the maximin decision rule is defined on sets of lotteries, not *gambles*, the basic objects in Hart (2011). Therefore, in order to be able to apply the concept of riskiness to the sets G and H, we would first require that all lotteries be defined in net values (as opposed to final outcomes). We further need to restrict the possible combinations of priors and *acts* to include only *gambles*, i.e., have positive expectation and a strictly positive probability to lose. Thus not every pair of *acts* in the framework of Gilboa et al. (2010) can be compared.[9] However, this loss of generality can be compensated by broadening the set of objects for comparison. In particular, the framework of Gilboa et al. (2010) implies that each of the sets G and H is created by attaching a closed and convex set of priors to an *act*.[10] But, by allowing G and H to be *any* set in $\mathbb{G}$ (not necessarily connected to one specific *act*, see Section 1.1), we can get a stronger comparison tool.

[8] These are the properties of "consistency" and "caution" in the vocabulary of Gilboa et al. (2010). The model in Gilboa et al. (2010) can be thought of as a modified version of the original one in Gilboa and Schmeidler (1989). From this point on, we refer to Gilboa et al. (2010) (instead of Gilboa & Schmeidler 1989) as the underlying framework for our basic model. We do so not only in order to support both interpretations for the source of different utility functions, but also in order to get a representation of the unanimity rule (see Section 3).

[9] One can remove this restriction by setting zero riskiness for every gamble with no probability to lose, and infinite riskiness for every gamble with a negative expectation, but then it will not be possible to rank gambles belonging to any of this classes. Moreover, propositions 3 and 4 rank sets of *gambles* according to the riskiest *gamble* in the set, so even if only one of the *gambles* in the set has a negative expectation, the riskiness of the set will go to infinity. Similarly, propositions 9 and 10 rank sets of *gambles* according to the least risky *gamble* in the set, so even if only one of the *gambles* in the set has a zero probability to lose, the riskiness of the set will be set to zero.

[10] This is a set of priors over the states of nature, hence can be thought of as independent of the specific *act*, but we do not pose such a restriction.

The resultant decision rule would then be[11]: $\forall u$ and $\forall G, H \in \mathbb{G}$,

$$G \succsim H \Leftrightarrow \min_{g \in G} E[u(g)] \geq \min_{h \in H} E[u(h)].$$

Consequentially, u *rejects* the set of gambles G at wealth w if $u(w) \geq \min_{g \in G} u(w+g)$. Otherwise, u *accepts* the set of gambles G at wealth w if $u(w) < u(w+g)$, $\forall g \in G$.

2.2 Utilities and riskiness

At this stage, we have decision makers who evaluate sets of gambles according to the utility they get from taking their subjectively worst gamble in the set. If we restrict ourselves to the set U^* of utilities to which the *uniform rejection criteria* of Hart (2011) apply,[12] then we have a tool for comparing not only single gambles but also sets of gambles, as long as the decision makers agree on the identity of the worst gamble in every set. Unfortunately, this is hardly the case. Two arbitrary decision makers with utilities in U^* may disagree not only on the ranking of two well specified gambles, each representing a different set of gambles, but also on the ranking of gambles *inside* each of these sets. As a result, they will usually disagree on the identity of the worst gamble in each set, hence on the objects to be compared in order to rank those sets. The direct consequence of this complication, is that the existence of a criterion which leads to a complete objective ranking of *gambles* does not necessarily guarantee that applying this criterion to *sets of gambles* in the setup proposed here will eventually lead to a complete objective ranking of those sets.

Figure 1 exemplifies this complication. We want to compare two sets of gambles - set G and set H, where both are created by attaching a corresponding set of priors of the form (p_1, p_2, p_3) to the gamble that pays $-N$ with probability p_1, 0 with probability p_2, and M with probability $p_3 = 1 - p_1 - p_2$ (where M and N are strictly positive amounts). u_1 judges set G according to his subjective worst case g_1, and set H according to his subjective worst case h_1, resulting in a preference for set G. Similarly, u_2 judges set G according to *his* subjective worst case g_2, and set H

[11] In section 4 we provide a different framework in which *sets of gambles* are *directly* compared (i.e., not as substitutes of *acts*), and this decision rule becomes a special case.

[12] U^* is the set of "regular utilities" in Hart (2011). It includes all strictly rising and strictly risk-averse utility functions that satisfy (a) the two "Arrow conditions" (Arrow 1965, Lecture 2 and Arrow 1971, page 96) - decreasing absolute risk aversion (DARA) and increasing relative risk aversion (IRRA), and (b) no gamble should always (i.e., at any wealth level) be accepted by $u \in U^*$.

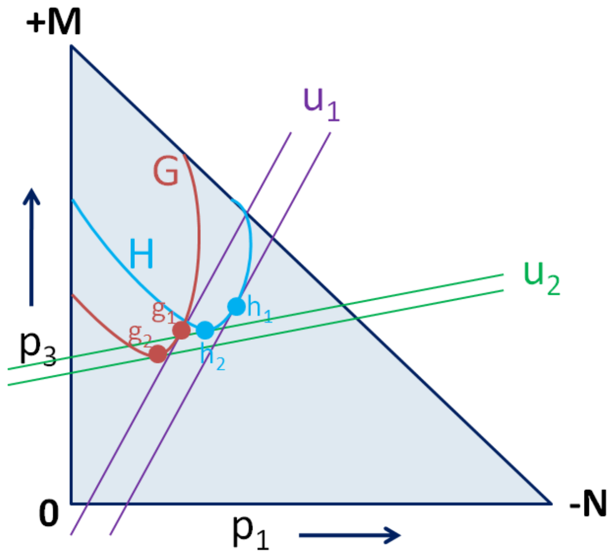

Figure 1: A Machina-Marshak triangle demonstrating the comparison problem. Every point inside the triangle corresponds to a unique gamble, with probability p_1 to get $-N$, probability p_3 to get $+M$, and probability $1 - p_1 - p_3$ to stay with the initial endowment. G is the set of gambles that are restricted from below by the orange curve, and H is the set of gambles that are restricted from below by the blue curve. The figure demonstrates that $u_1(g_1) > u_1(h_1)$ while $u_2(g_2) < u_2(h_2)$, and so u_1 prefers set G over set H while u_2 prefers set H over set G.

according to *his* subjective worst case h_2, resulting in a preference for set H.

Eventually, despite this problem, it is indeed possible to achieve two parallel complete rankings of sets based on the two parallel *uniform rejection criteria* of Hart (2011). Each of these two rankings is based on the gamble which maximizes the corresponding riskiness measure over the set. It is important though to realize, that this is not a trivial consequence of applying these criteria, since individual decision makers compare the sets according to their subjective worst gamble, and not according to this focal point of maximal riskiness, unlike the setup of Hart (2011), where the riskiness is computed for the same gambles that are used by the DMs for making a choice. Proposition 3 and proposition 4 state this result, following a few necessary definitions.

Definition 1 *Let $G, H \in \mathbb{G}$. We say that G wealth-uniformly dominates H, which we write $G \geq_{WU} H$, if the following holds:*

For every utility $u \in U^*$,

If G is rejected by u at all $w > 0$
Then H is rejected by u at all $w > 0$.

Definition 2 *Let $G, H \in \mathbb{G}$. We say that G utility-uniformly dominates H, which we write $G \geq_{UU} H$, if the following holds:*

For every wealth level $w > 0$,
If G is rejected by all $u \in U^*$ at w
Then H is rejected by all $u \in U^*$ at w.

Proposition 3 *For any two sets G and H in $\mathbb{G}$, $G \geq_{WU} H$, if and only if $\max_{h \in H} R^{AS}(h) \geq \max_{g \in G} R^{AS}(g)$.*[13]

Proposition 4 *For any two sets G and H in $\mathbb{G}$, $G \geq_{UU} H$, if and only if $\max_{h \in H} R^{FH}(h) \geq \max_{g \in G} R^{FH}(g)$.*[14]

Notice that both $R^{AS}(G) \equiv \max_{g \in G} R^{AS}(g)$ and $R^{FH}(G) \equiv \max_{g \in G} R^{FH}(g)$ are homogeneous of degree 1, i.e., $R^{AS}(\mu G) = \mu R^{AS}(G)$ and $R^{FH}(\mu G) = \mu R^{FH}(G)$ for every $\mu > 0$, where μG is the set containing the same gambles as G with all outcomes multiplied by μ. This follows from the homogeneity of degree 1 of $\max(\cdot)$, $R^{AS}(g)$ and $R^{FH}(g)$.

Concentrating on the riskiest gamble in the set may seem too extreme in first glance, in line with the *prima facie* extreme uncertainty aversion of the decision makers in the model of Gilboa et al. (2010). However, this extremity preserves an important aspect of the riskiness measure[15] $R^{FH}(\cdot)$.

2.3 Foster–Hart riskiness as critical wealth

In Foster and Hart (2009), the authors interpret $R^{FH}(g)$ as the minimal reserve needed to guarantee no bankruptcy when taking the gamble g sequentially. In their setup, one faces a series of consecutive *gambles* that are realized one after the other. Similarly, in our setup one faces a series of *sets of gambles*, where the realization of every set in the series is a realization of one gamble that is drawn from that set.

[13] $R^{AS}(\cdot)$ is continuous hence gets its minimum and maximum values over compact sets.

[14] $R^{FH}(\cdot)$ is not continuous hence does not necessarily get minimum and maximum values over compact sets. However, we prove in the appendix that if the *set of gambles* G is compact and has a fixed and finite support, then $R^{FH}(\cdot)$ indeed gets its minimum and maximum values over G. This guarantees that $R^{FH}(\cdot)$ gets its minimum and maximum values over every $G \in \mathbb{G}$.

[15] We discuss the properties of $R^{AS}(G)$ as a riskiness measure for *sets of gambles* after we generalize this measure in section 4.

Therefore, if one's objective is to guarantee no bankruptcy, one should look at the riskiest possible scenario and avoid bankruptcy in this extreme scenario. This can be guaranteed only by judging a set G according to its riskiest component, i.e., by $\max_{g \in G} R^{FH}(g)$.

More formally, a process $(G_t)_{t=1,2,\dots}$ is generated by a sequence of elements of $\mathbb{G}$. When a $G \in \mathbb{G}$ is accepted by the DM, a gamble $g \in G$ is chosen (with no restriction on the choice process) and one of its outcomes is realized. A *simple strategy* s_Q *yields no-bankruptcy* for the process $(G_t)_{t=1,2,\dots}$ and the initial wealth w_1 if $P\left[\lim_{t\to\infty} w_t = 0\right] = 0$, and *guarantees no-bankruptcy* if it yields no-bankruptcy for every process $(G_t)_{t=1,2,\dots}$ and every initial wealth W_1 (see Foster and Hart (2009) for further details).

Proposition 5 *Following Foster and Hart (2009), let a simple strategy* s_Q *denote rejection of the set* G *if* $w < Q(G)$*, and acceptance of the set* G *if* $w \geq Q(G)$*, where the critical-wealth function* $Q(G) \in [0, \infty]$ *is homogeneous of degree* 1*. Then for every set* $G \in \mathbb{G}$ *there exists a unique real number* $R^{FH}(G) \equiv \max_{g \in G} R^{FH}(g)$*, such that a simple strategy* s_Q *with a critical-wealth function* Q *guarantees no-bankruptcy if and only if* $Q(G) \geq R^{FH}(G)$ *for every set* $G \in \mathbb{G}$*.*

Consequentially, using Gilboa et al (2010) to represent decision makers leads to a Foster–Hart riskiness measure of sets which not only gives a complete ranking of those sets, but also maintains an important aspect of the Foster–Hart riskiness measure of gambles, i.e., being the minimal reserve needed to guarantee no bankruptcy.

3 Extensions

3.1 Relaxing the "caution" axiom

3.1.1 Objective rationality

The extreme degree of uncertainty aversion in Gilboa et al. (2010) stems from the "caution" axiom, which connects the set of priors to which *subjective rationality* applies (i.e., the relevant set for choosing the worst case) with the set of priors to which *objective rationality* applies (i.e., the set that can be used to persuade others that their preferences are irrational).[16] In fact, one can do entirely without *subjective rationality* and still have a decision model. Indeed, if we settle for *objective rationality* alone, we get that decision makers use *unanimity* rules à la Bewley (2002).

[16] See Gilboa et al. (2010) for further details.

In other words, when comparing two *acts* a and b, their decision rule is: $a \succsim b \Leftrightarrow \int E\left[u\left(a(s)\right)\right] dp(s) \geq \int E\left[u\left(b(s)\right)\right] dp(s),\ \forall p \in P$ (obviously this is a very partial ranking). However, in contrast to Bewley (2002), where every DM may have a different set P, in this setup P is common to everyone.[17] Surprisingly, this modification neither prevents the existence, nor changes the order, of the two complete rankings of the sets of gambles introduced earlier. This is true even though now decision makers have only partial preferences.

The reason is quite simple - in this setup, one accepts an *act a* if and only if one prefers this *act* over one's initial state under every possible prior in P (this is the "inertia"[18] feature of Bewley 2002), i.e., one accepts a if and only if $u(w) < u(w+g)$, $\forall g \in G \equiv \{F\left(a,p\right) : p \in P\}$, just as in the setup based on Gilboa et al. (2010).

3.1.2 Weighting with the Steiner Point

A second attempt to relax the extreme degree of uncertainty aversion may take the form of weighting the worst case with a "mean" case. Gajdos et al (2008) may then become a good candidate to the role of the underlying setup of decision under uncertainty. In Gajdos et al. (2008), the DM evaluates *act a* using a weighted average of the worst case and the Steiner Point of the set of priors (its "center of gravity"), i.e., $u(a,P) = \epsilon \min\limits_{p \in P} \int E\left[u\left(a(s)\right)\right] dp(s) + (1-\epsilon) \int E\left[u\left(a(s)\right)\right] dSt(P)$, where $St(P)$ is the Steiner Point of set P.

Following some further axiomatization, Gajdos et al. (2008) introduce a contraction representation, under which this weighted average is equivalent to taking the point of minimal utility in $\varphi_\epsilon(P) \equiv (1-\epsilon)s(P) + \epsilon P$, which is the set of priors contracted around the Steiner Point. Obviously, if all decision makers use the same weighted average, then $\varphi_\epsilon(P)$ can replace the original set of priors P, resulting in G_ϵ and H_ϵ - reduced forms of the original sets of gambles G and H respectively. Since decision makers in this setup evaluate sets according to the worst case scenario, the resulting rankings will be based on the riskiest gambles in these reduced sets.

More formally, let $G_\epsilon \equiv \{F\left(a,p\right) : p \in \varphi_\epsilon(P)\}$ and $H_\epsilon \equiv \{F\left(b,p\right) : p \in \varphi_\epsilon(P)\}$.

[17] This is not explicitly written in Gilboa et al. (2010), but otherwise the concept of objective rationality is vacuous: consider a DM who prefers a to b although $u(b) > u(a)$ over P; if P is not common, he can always claim that his subjective relevant set of priors is bigger and contains additional priors where $u(a) > u(b)$, rendering the whole notion of objective rationality useless.

[18] The "inertia" criterion roughly states that if act a was initially chosen (or previously adopted), and neither $a \succsim b$ nor $b \succsim a$ holds, then act a will remain the choice of the DM.

Then $G_\epsilon \geq_{WU} H_\epsilon$, if and only if $\max_{h\in H_\epsilon} R^{AS}(h) \geq \max_{g\in G_\epsilon} R^{AS}(g)$, and $G_\epsilon \geq_{UU} H_\epsilon$, if and only if $\max_{h\in H_\epsilon} R^{FH}(h) \geq \max_{g\in G_\epsilon} R^{FH}(g)$. The proofs are identical to the proofs of propositions 3 and 4 respectively.

3.2 Capacities instead of sets of priors

Another related model of decision under uncertainty is the *rank-dependent utility* (*rdu* in short) model (called sometime the "capacities" model) of Schmeidler (1989). The objects for comparison here are *acts*, just as in the setups of Gilboa and Schmeidler (1989) and Gilboa et al. (2010). However, whereas in those setups an *act* corresponds to a set of additive probability measures over the states of nature in S, in Schmeidler (1989) an *act* corresponds to a non-additive probability measure over $\sum$, the algebra of subsets on S. This non-additive probability measure is called "capacity", and is denoted by $v(\cdot)$.

The set of axioms from Gilboa and Schmeidler (1989) is partly maintained, with two major changes. The first is the replacement of the Certainty-Independence Axiom with a Co-monotonic-Independence Axiom. Two *acts* a,b, are said to be co-monotonic if for no s,$t \in S$, $a(s) \succ a(t)$ and $b(t) \succ b(s)$. The Co-monotonic-Independence Axiom demands that $a \succ b \Rightarrow \lambda a + (1-\lambda)c \succ \lambda b + (1-\lambda)c$, for every pairwise co-monotonic *acts* $a, b, c \in L$, and $\forall \lambda \in (0,1)$.

The second major change is the removal of the uncertainty-aversion axiom. Unlike the Gilboa–Schmeidler model, Schmeidler's *rdu* model is not restricted to uncertainty averse decision makers. Nevertheless, it does include a specific characterization of uncertainty aversion through a property of the capacity, which will be introduced shortly.

Schmeidler's main theorem (1989) states that if $\succeq$ is a preference relation which satisfies these axioms, then there exist a unique non-additive probability measure (capacity) on $\sum$ and an affine function $u : L \rightarrow R$, such that for any two *acts* a and b,

$$a \succsim b \Leftrightarrow \int_S u(a(\cdot))\,dv \geq \int_S u(b(\cdot))\,dv.$$

The integral $\int_S u(a(\cdot))\,dv$, which may be referred to as "the utility of *act* a", is the Choquet Integral. It is calculated as a weighted sum of the utilities of the (finitely many) outcomes of the *act* a:

$$u(a) \equiv \int_S u(a(\cdot))\,dv = \sum_{k=1}^{N} \pi_k u(\alpha_k),$$

where $\alpha_1 \succeq \alpha_2 \succeq ... \succeq \alpha_N$ is a (subjective) ranking of the outcomes of a (according to u) under the N states of nature in S, and $\pi_k =$

$v(s_1 \cup s_2 \cup ... \cup s_k) - v(s_1 \cup s_2 \cup ... \cup s_{k-1})$ for $k \neq 1$ (the indices correspond to that of α_k, and $\pi_1 = v(s_1)$).

It can be shown that switching the order between any two states of nature i,j for which $\alpha_i \backsimeq \alpha_j$, does not change the resultant utility of a. Notice that the vector π is an additive probability measure (because $v(S) = 1$), and π_k represents the marginal contribution of state k to the states of nature leading to "better" outcomes. Hence, every *act* is in fact associated, through the concept of capacities, with a set of $N!$ order-dependent priors.

We will focus now on uncertainty-averse preference relations (or decision makers). This case is known to be a special case of the *mpu* model, where the core of $v(\cdot)$ is the convex set on which the minimum utility is computed (see Gilboa and Schmeidler 1989). However, we analyze it separately because we avoid using the core entirely, and adjust the decision rule to the realm of $\mathcal{G}$ directly.

In the *rdu* model, uncertainty aversion is characterized by a convex capacity $v(\cdot)$: $\forall E, F \in \sum, v(E) + v(F) \leq v(E \cup F) + v(E \cap F)$, i.e., $v(E \cup R) - v(R)$ is increasing in R, where $E, R \in \sum$ are disjoint. This property of the capacity function may thus be understood as an increasing marginal contribution of E, for every $E \in \sum$. One result of this property is that for every $E \in \sum$, $v(E)$ is a lower bound on the "probability" allocated to E (think of π_k as the probability allocated to s_k). Combining this property of increasing marginal contribution with the formula for calculating the utility of an *act*, we get that a risk-averse decision maker puts the minimal weight π_1 on his most preferred state of nature (out of the legitimate weights), and the maximal weight π_N on his least preferred state of nature.

This allows for a new representation of the calculation of $\int_S u(a(\cdot))\, dv$ for uncertainty averse decision makers:

Let X be the set of $N!$ permutations of the N states of nature in S, $(s_j)_{j=1}^N$. Then x_{ij} denotes the j^{th} state of nature in permutation $x_i \in X$. Now let P be the set of $N!$ additive probability measures corresponding to the set of $N!$ permutations of the N states of nature in S. $p_i \in P$ is the probability measure corresponding to the order x_i, whose elements $p_{ij} = v(x_{i1} \cup x_{i2} \cup ... \cup x_{ij}) - v\left(x_{i1} \cup x_{i2} \cup ... \cup x_{i(j-1)}\right)$ represent the probabilities attached to the states x_{ij}. Obviously, the additive probability measure $\pi = (\pi_1, \pi_2, ..., \pi_N)$ belongs to P. Moreover, Lemma 6 states that calculating the expected utility of *act* a using π as a prior, gives the minimum expected utility over all the priors in P:

Lemma 6 $u(a) = \int_S u(a(\cdot))\, dv = \sum\limits_{k=1}^{N} \pi_k u(\alpha_k) = \min\limits_{i \in [1,N!]} \left\{ \sum\limits_{j=1}^{N} p_{ij} u(a(x_{ij})) \right\}$, *where* $a(x_{ij}) \in \{\alpha_k\}_{k=1}^{N}$.

Proof. Schmeidler (1989) shows that $u(a)$ is the minimum utility over the (non-empty) core generated by $v(\cdot)$. Since the core is convex, the minimum utility must lie on one of the vertices. Hence $u(a)$ is the minimum utility over the vertices of the core, which are the $N!$ additive probability measures corresponding to the set of $N!$ permutations of the N states of nature in S. ■

Lemma 6 allows one to represent *acts* in the *rdu* model as sets of lotteries, just as in the Gilboa–Schmeidler's *mpu* model, but now these sets are not convex as required in the *mpu* model. Instead, each of these sets contains $N!$ distinctive lotteries.

Let g_i be a compound lottery that corresponds to the probability measure p_i, by assigning probability p_{ij} to the lottery $a(x_{ij})$ for $j = 1, ..., N$. Then $u(g_i) = \sum_{j=1}^{N} p_{ij} u(a(x_{ij}))$. Now let $G := \{g_i\}_{i=1}^{N!}$. Then $u(a) = \min_{g_i \in G} u(g_i)$. Similarly, $H := \{h_i\}_{i=1}^{N!}$, where lottery h_i assigns probability p_{ij} to the lottery $b(x_{ij})$ for $j = 1, ..., N$, so that $u(h_i) = \sum_{j=1}^{N} p_{ij} u(b(x_{ij}))$. Consequently,

$$u(a) \geq u(b) \Leftrightarrow \min_{g_i \in G} u(g_i) \geq \min_{h_i \in H} u(h_i).$$

The sets G and H depend on the non-additive measure $v(\cdot)$ through the set of probability measures P. However, just as in the framework based on Gilboa–Schmeidler's *mpu* model, in order to objectively compare *acts* a and b, we need the sets G and H to be common to all decision makers. This can be guaranteed by letting $v(\cdot)$ be common to everyone or be part of the decision problem itself (like a given lower bound on the probabilities of the events), just as P was common to all decision makers in the basic model of Section 2. A common (and objective) $v(\cdot)$ links every *act* to a unique set of lotteries, and specifically links the *act* a to the set G and *act* b to the set H.

In order to apply the *uniform rejection criteria* of Hart (2011) to get objective and complete rankings of *acts* in the *rdu* model, we will concentrate now on the set of utilities U^*, and will require that all lotteries in the sets G and H be *gambles*, i.e., have a positive expectation and a non-zero probability of losing.

We know by now that $\forall u \in U^*, u(a) = \min_{g_i \in G} \{u(g_i)\}$. Hence u rejects the set of gambles G at wealth w, if $u(w) \geq \min_{g_i \in G} E[u(w + g_i)]$ and u accepts the set of gambles G at wealth w if $u(w) < E[u(w + g_i)]$, $\forall g_i \in$

G. Therefore we can apply propositions 3 and 4 to get the following result:

Proposition 7 *For any two acts a, b in Schmeidler (1989) with corresponding sets $G, H \in \mathbb{G}$, $G \geq_{WU} H$ if and only if* $\max_{h \in H} R^{AS}(h) \geq \max_{g \in G} R^{AS}(g)$.

Proposition 8 *For any two acts a, b in Schmeidler (1989) with corresponding sets $G, H \in \mathbb{G}$, $G \geq_{UU} H$ if and only if* $\max_{h \in H} R^{FH}(h) \geq \max_{g \in G} R^{FH}(g)$.

Proof. Propositions 7 and 8 follow directly from propositions 3 and 4 respectively. ■

We interpret the relations $G \geq_{WU} H$ and $G \geq_{UU} H$ as a dominance relation between the corresponding *acts* in Schmeidler (1989), i.e., *act a* dominates (ranked above) *act b* if $G \geq_{WU} H$, when ranking is based on *wealth-uniform rejection*, and *act a* dominates *act b* if $G \geq_{UU} H$, when ranking is based on *utility-uniform rejection*.

3.3 Optimists

At this point, it is natural to examine if the *uniform rejection criteria*, when applied to sets of gambles, can lead to a complete ranking also when decision makers use a maximax decision rule instead of the maximin. We can refer to such decision makers as "optimists". An optimist u rejects the set of gambles G at wealth w, if and only if $u(w) \geq \max_{g \in G} \{u(w+g)\}$. Indeed, the following propositions state that applying the same *uniform rejection criteria* to the maximax decision rule leads to two complete rankings of sets, this time based on the *least* risky gamble in each set:

Proposition 9 *For any two sets G and H in $\mathbb{G}$, $G \geq_{WU} H$, if and only if* $\min_{h \in H} R^{AS}(h) \geq \min_{g \in G} R^{AS}(g)$.

Proposition 10 *For any two sets G and H in $\mathbb{G}$, $G \geq_{UU} H$, if and only if* $\min_{h \in H} R^{FH}(h) \geq \min_{g \in G} R^{FH}(g)$.

So just as in the basic model, each of these two new rankings is based on an extremum of the riskiness measure over the whole set, although individual decision makers compare the sets according to their *subjective* most/least preferred gamble, and not according to this focal point of extremum riskiness.

3.4 Minimax Regret

Another popular way to model decision under uncertainty is the minimax regret rule. There are various ways to model minimax regret (cf. Stoye 2011a, 2011b, and Berger 1985). In the context of *sets of gambles*, the most natural way seems to be the following:

Let $\succcurlyeq_{u,w}$denote the (complete and transitive) preference relation of u at wealth w. Then $G \succcurlyeq_{u,w} H$ if and only if

$$\max_{h\in H, g\in G} \{E[u(w+h] - E[u(w+g]\} \leq \max_{g\in G, h\in H} \{E[u(w+g] - E[u(w+h]\}$$

That is, u prefers set G to set H at w, if and only if the maximal disadvantage (in utility terms) of a gamble in G with respect to a gamble in H, indicating the maximal regret u may feel by preferring G to H, is smaller than the maximal disadvantage of a gamble in H with respect to a gamble in G.

From this definition follows the condition for rejection: u rejects G at w iff:

$$\max_{g\in G} \{E[u(w+g] - u(w)\} \leq \max_{g\in G} \{u(w) - E[u(w+g]\}.$$

Hence, for every pair (u, w), we can attach a binary tag to G - "accepted" or "rejected", based on the comparison of the maximal possible regret u can feel due to accepting G at w, and the maximal possible regret u can feel due to rejecting G at w.

We can then compare sets G and H by looking for pairs (u, w) where one set is rejected while the other is accepted. If whenever G is rejected by u at all $w > 0$ we get that H is rejected by u at all $w > 0$ too (but not vice-verse), and this holds for every $u \in U^*$, then G *wealth-uniformly dominates* H. Similarly, if whenever G is rejected by every $u \in U^*$ at w, we get that H is rejected by every $u \in U^*$ at w too (but not vice-verse), and this holds for every $w > 0$, then G *utility-uniformly dominates* H.

Just as with maximin and maximax decision rules, there is no preliminary reason to believe that either one of these two methods leads to a complete ranking, but once again they both do. First notice that we can state the condition for rejection as follows: u rejects G at w iff:

$$\frac{1}{2}\max_{g\in G} \{E[u(w+g]\} + \frac{1}{2}\min_{g\in G} \{E[u(w+g]\} \leq u(w).$$

This means that practically, a DM with utility u judges G according to the average between his subjective worst case scenario (as the pessimist does) and his best case scenario (as the optimist does). Consequentially, it seems that a natural way to rank sets in the minimax regret framework would be to set $R^{AS}(G) \equiv \frac{1}{2}\max_{g\in G} R^{AS}(g) + \frac{1}{2}\min_{g\in G} R^{AS}(g)$ and $R^{FH}(G) \equiv \frac{1}{2}\max_{g\in G} R^{FH}(g) + \frac{1}{2}\min_{g\in G} R^{FH}(g)$.

However, this resultant ranking, despite being complete, does not follow from applying the *uniform rejection criteria*, hence is not based on the tendency of utility functions in U^* to reject sets of gambles. Nevertheless, the idea of judging a set of gambles according to the average between the most preferred and least preferred gambles, does suggest that this minimax regret rule is a special case of the well known λ-Hurwicz criterion[19] (with $\alpha = 0.5$). In Section 4 we present the solution to this more general case, while using the setup in Olszewski (2007) as the foundation for the behavior of decision makers. We show that for every value of λ-Hurwicz there is a complete ranking based on *wealth-uniform rejection*, and a complete ranking based on *utility-uniform rejection*.

4 λ-maximizers

Let G be a *set of gambles* in $\mathbb{G}$. We define $V \equiv V_{u,\lambda}$ and[20]:

$$V(w+G) \equiv \lambda \max_{g\in G} E[u(w+g)] + (1-\lambda)\min_{g\in G} E\left[u(w+g)\right].$$

Then u who is a "λ-maximizer" (represented by the functional V) rejects G at w if and only if $V(w+G) \leq u(w)$. Olszewski (2007) axiomatizes a similar preference relation over a family of closed *sets of lotteries*[21] with $\lambda \in (0,1)$.

The motivation for modeling "λ-maximizers" goes beyond the necessity to generalize the previous special cases of "optimists" and "pessimists", or to find a general solution from which to derive the minimax regret-based ranking. As elaborated in Section 6, a further motivation can be to supply a normative criterion for ranking sets to help a DM (or a small group of DMs) who is unaware of his specific utility function, beyond its belonging to U^*, but who can evaluate his attitude toward uncertainty, represented in this case by the value of λ.

Consequentially, we are interested in finding $R_\lambda^{AS}(G)$ and $R_\lambda^{FH}(G)$, which are (respectively) the Aumann–Serrano riskiness index of G for λ-maximizers and the Foster–Hart riskiness measure of G for λ-maximizers.

[19] λ-Hurwicz is usually used in the context of acts, where the average is between the prior leading to the best outcome and the prior leading to the worst outcome. This is of course equivalent to translating the act and its corresponding set of priors to a set of gambles, and averaging between the gambles yielding the maximal utility and minimal utility respectively.

[20] The dependency of $V(w+G)$ on $u(\cdot)$ guarantees that $V(w+G)$ is unique up to affine transformations, i.e., $V(w+G) > V(w+H) \iff mV(w+G)+n > mV(w+H)+n$.

[21] Of which $\mathbb{G}$ is a subset.

Thus, each of the rankings of the sets G and H for λ-maximizers will be uniquely determined by $R_{\lambda}^{AS}(G)$ or by $R_{\lambda}^{FH}(G)$. We will refer to λ as the *uncertainty-aversion parameter* of the decision maker.

4.1 Aumann–Serrano riskiness index of sets for λ-maximizers

Theorem B in Aumann and Serrano (2008) states that $R^{AS}(g)$ is the reciprocal of the (unique) α such that a $CARA$ person with risk-aversion parameter α is indifferent between accepting and rejecting the gamble g. Theorem 1 in Hart (2011) further states that for any two gambles g and h, $g \geq_{WU} h$ if and only if $R^{AS}(g) \leq R^{AS}(h)$. Theorem 13, following the next definitions, extends this connection between the $CARA$ risk-aversion parameter α on the one hand, and the *wealth-uniform domination* on the other hand, from *gambles* to *sets of gambles*.

Definition 11 *Let $G, H \in \mathbb{G}$ and let $\lambda \epsilon [0,1]$. We say that the set of gambles G λ-wealth-uniformly dominates the set of gambles H, denoted $G \geq_{WU}^{\lambda} H$, if the following holds:*

For every $u \in U^*$,
If G is rejected by $V_{u,\lambda}$ at all $w > 0$,
Then H is rejected by $V_{u,\lambda}$ at all $w > 0$.

Definition 12 *For every set of gambles $G \in \mathbb{G}$, let $R_{\lambda}^{AS}(G)$ be the reciprocal of the value of*[22] *α such that a $CARA$ person with risk-aversion parameter α and uncertainty-aversion parameter λ is indifferent between accepting and rejecting the set G.*

Theorem 13 *For any two sets of gambles G and H in $\mathbb{G}$, $G \geq_{WU}^{\lambda} H$ if and only if $R_{\lambda}^{AS}(G) \leq R_{\lambda}^{AS}(H)$.*

4.1.1 The duality of the Aumann–Serrano riskiness index

Let L_g denote the maximal loss in the gamble g, and let $L_G \equiv \max_{g \in G} \{L_g\}$. Recall that for any $u \in U^*$, $u(0) \to -\infty$, hence at every wealth $w \leq L_G$ there exists at least one gamble in G that yields expected utility of $-\infty$ to every $u \in U^*$. As a result, for any $\lambda < 1$, every $V_{u,\lambda}$ with $u \in U^*$ rejects the set G at $w \leq L_G$ (due to putting a non-zero weight on the least-preferred gamble in the set).

[22] This value of α is unique. See corollary 25 following the proof to Lemma 24 (in the appendix).

Following Aumann and Serrano (2008), call $V_{u_1,\lambda}$ at least as risk-averse as $V_{u_2,\lambda}$ (written $V_{u_1,\lambda} \trianglerighteq V_{u_2,\lambda}$), if whenever $V_{u_2,\lambda}$ rejects a *set of gambles* $G \in \mathbb{G}$ at some $w' > L_G$, $V_{u_1,\lambda}$ rejects this set G at all $w > 0$.[23]

The Aumann–Serrano index of riskiness for *gambles* is characterized by a property of *duality*: $\forall g, h \in \mathcal{G}$ and every pair of utilities $\{u_1, u_2\}$, if[24] $u_1 \triangleright u_2$ (i.e., $u_1 \trianglerighteq u_2$ and $u_2 \ntrianglerighteq u_1$), u_2 rejects h at some w', and $R^{AS}(g) > R^{AS}(h)$, then u_1 rejects g at all $w > 0$. We next show that for every $\lambda \epsilon [0,1]$, the Aumann–Serrano index of riskiness for *sets of gambles,* $R^{AS}_{\lambda}(G)$, is characterized by a property of *duality* either. The only modification is that u_1 and u_2 are replaced by $V_{u_1,\lambda}$ and $V_{u_2,\lambda}$ respectively, where λ is the same for both.

Proposition 14 *Set $\lambda \epsilon [0,1]$. Then $\forall G, H \in \mathbb{G}$ and every pair $\{V_{u_1,\lambda}, V_{u_2,\lambda}\}$ with $u_1, u_2 \in U^*$, if $V_{u_1,\lambda} \triangleright V_{u_2,\lambda}$, $V_{u_2,\lambda}$ rejects H at some $w' > L_G$, and $R^{AS}_{\lambda}(G) > R^{AS}_{\lambda}(H)$, then $V_{u_1,\lambda}$ rejects G at all $w > 0$.*

4.2 Foster–Hart riskiness measure of sets for λ-maximizers

In Section VI(B) of Foster and Hart (2009), the authors show that $R^{FH}(g)$ is the (unique) wealth level w such that the logarithmic utility decision maker is indifferent between accepting and rejecting the gamble g at w. Theorem 3 in Hart (2011) further states that for any two gambles g and h, $g \geq_{UU} h$ if and only if $R^{FH}(g) \leq R^{FH}(h)$. Theorem 17, following the next definitions, extends this connection between the wealth level of indifference for the log utility on the one hand, and the *utility-uniform domination* on the other hand, from *gambles* to *sets of gambles.*

Definition 15 *Let $G, H \in \mathbb{G}$ and let $\lambda \epsilon [0,1]$. We say that the set of gambles G λ-utility-uniformly dominates the set of gambles H, denoted $G \geq^{\lambda}_{UU} H$, if the following holds:*

For every wealth level $w > 0$,
If for every $u \in U^*$ G is rejected by $V_{u,\lambda}$ at w,
Then for every $u \in U^*$ H is rejected by $V_{u,\lambda}$ at w.

[23] Aumann and Serrano (2008) call i at least as risk-averse as j, if for all levels w_i and w_j of wealth, j accepts at w_j any gamble that i accepts at w_i. In our corresponding definition, we replaced *gambles* with *sets of gambles*, and we gave the logically-equivalent definition based on *rejection* instead of *acceptance.* We have done it in order to be able to apply the definition also to utilities which are defined only for positive values. Moreover, the discussion in the previous section clarifies why (at least for $\lambda < 1$) only rejections at $w > L_G$ are of interest.

[24] Notice that when applying the relation $V_{u_1,\lambda} \trianglerighteq V_{u_2,\lambda}$ to single gambles, λ has no effect, and this relation boils down to $u_1 \trianglerighteq u_2$, as defined as a relation over gambles in Aumann and Serrano (2008).

Definition 16 *For every set of gambles $G \in \mathbb{G}$, let $R_{\lambda}^{FH}(G)$ be the wealth level*[25] *w such that a logarithmic utility decision maker with uncertainty-aversion parameter λ is indifferent between accepting and rejecting the set G at w.*

Theorem 17 *For any two set of gambles G and H in $\mathbb{G}$, $G \geq_{UU}^{\lambda} H$ if and only if $R_{\lambda}^{FH}(G) \leq R_{\lambda}^{FH}(H)$.*

5 Uniformity over Uncertainty Aversion

The essence of a *uniform rejection criterion* is to create a ranking which is independent of specific utility functions or specific wealth levels. In this light, having a ranking which depends on a specific *uncertainty-aversion parameter* λ, as presented in Section 4, may seem inadequate. A solution to this inadequacy would be either to assume that all DMs follow a set of axioms which is translated to a unique uncertainty-aversion parameter λ, such as was done in the basic model using Gilboa et al. (2010) to induce a "pessimist" behavior (i.e., $\lambda = 0$); or to create a ranking which is uniform over λ too. To perform the latter, we introduce here an extended uniformity concept, which applies the *uniform rejection* method to the level of *uncertainty-aversion* too. Let (u, λ) denote a DM with utility function u and *uncertainty-aversion* parameter λ.

Definition 18 *Let $G, H \in \mathbb{G}$. We say that G wealth-uniformly and λ-uniformly dominates H, denoted $G \geq_{WU\lambda U} H$, whenever:*

> if G is rejected by (u, λ) for every $\lambda \in [0, 1]$ and at all $w > 0$,
> then H is rejected by (u, λ) for every $\lambda \in [0, 1]$ and at all $w > 0$,
> for every utility $u \in U^*$.

Proposition 19 *For any two set of gambles G and H in $\mathbb{G}$, $G \geq_{WU\lambda U} H$ if and only if $\min_{h \in H} R^{AS}(h) \geq \min_{g \in G} R^{AS}(g)$.*

Definition 20 *Let $G, H \in \mathbb{G}$. We say that G utility-uniformly and λ-uniformly dominates H, denoted $G \geq_{UU\lambda U} H$, whenever:*

> if G is rejected at w by every (u, λ) with $u \in U^*$ and $\lambda \in [0, 1]$,
> then H is rejected at w by every (u, λ) with $u \in U^*$ and $\lambda \in [0, 1]$,
> for every $w > 0$.

Proposition 21 *For any two set of gambles G and H in $\mathbb{G}$, $G \geq_{UU\lambda U} H$ if and only if $\min_{h \in H} R^{FH}(h) \geq \min_{g \in G} R^{FH}(g)$.*

[25] This wealth level is unique. See Lemma 28.

The proof for both propositions is immediate - if the rejection of set G is independent of the degree of *uncertainty aversion*, then even the most optimistic ones (those with parameter $\lambda = 1$) reject G. The opposite direction also holds: if, for some $u \in U^*$, G is rejected by (u, λ) with $\lambda = 1$, then G is rejected by (u, λ) for every $\lambda \in [0, 1]$ too. Eventually, we can substitute "every $\lambda \in [0, 1]$" in each of these propositions with $\lambda = 1$, hence the ranking is determined solely by the "optimists". Finally, propositions 9 and 10 tell us that when ranking sets of gambles according to the preferences of "optimists", the minimal riskiness of the gambles in the set is the key for ranking.

6 Discussion

The motivation given up till now to the proposed rankings was taken from Hart (2011), i.e., create objective measures and objective rankings by using uniformity over a large class of utility functions. However, a quite different interpretation and motivation may fit as well. In the decision theory literature, it is often assumed that completeness of preferences is too strict a requirement (cf. Bewley 2002). Relaxing this requirement may take the form of assuming DMs do not have a well-specified utility function. We follow this route by letting DMs admit only to having $u \in U^*$. In particular, this means that with regard to risk, DMs know just that they satisfy the two "Arrow conditions"[26] ($DARA$ and $IRRA$), and that they reject every *gamble* at some w. With regard to ambiguity, each DM knows only his general approach: optimist, pessimist, λ-optimist, minimax regret, status quo à la Bewley (2002), etc.. We can then interpret the proposed rankings as normative recipes for generating complete rankings of *sets of gambles* for such DMs.

Combining these two approaches - an attempt to apply uniformity over a large class of utility functions on the one hand, and constructing a tool for ranking alternatives[27] for DMs with incomplete preferences on the other hand - we can treat the rankings as tools to support group decision making, as the following example tries to illustrate.

Imagine a board of a certain firm, faced with a decision problem under uncertainty, going to consult with an external advisor. In this decision problem, the probabilities of outcomes are (naturally) undetermined, but the board members are willing to give assessments in the form of ranges for the probabilities (e.g., an estimate that a competi-

[26] Arrow (1965, Lecture 2; 1971, page 96).

[27] The word "alternative" is used in this section to describe a potential action route in a decision problem. Generally, any such alternative may lead to a bet with unknown probabilities, which in turn is translated (using all the possible priors) to a set of gambles.

tor will be forced to leave the area with probability between 10% to 35%). The advisor then collects all the data, and computes the relevant *sets of gambles* (where each alternative in the original problem is translated to such a set). Moreover, the board managers can hardly come up with a precise and agreed-upon utility function, but they can admit to being subject to some general all-encompassing properties such as the two "Arrow conditions" (in their weak sense, which includes all $CRRA$ functions).

Had the gamble been well defined in terms of probabilities, the advisor could have used Foster–Hart's riskiness measure for recommendation: it would have given an objective ranking of gambles in the absence of a utility function, by looking for a *uniform rejection* over U^*, the given class of utilities.[28] However, the probabilities are not known, so there is a need to choose between *sets of gambles*. Hence, the board members should be requested to supply also a general attitude toward ambiguity: if they adopt a pessimist (optimist) attitude, the advisor should focus on the gambles with maximum (minimum) riskiness in every set; if they are λ-optimists, the advisor should use the $R_{\lambda}^{FH}(G)$ measure; if they can specify only a range of λ's that they want to take into account, the advisor should apply uniformity over this range of λ's;[29] if they want to minimize the maximal regret, the advisor should use the $R_{0.5}^{FH}(G)$ measure; and if they wish to use the center (Steiner Point) of every set of gambles as an anchor, the advisor should use the maximum riskiness in the ϵ-contracted set corresponding to every alternative.

This paper shows that for every pair of alternatives that can be translated to *sets of gambles*, either alternative A *utility-uniformly dominates* alternative B, or alternative B *utility-uniformly dominates* alternative A (and the order is transitive). The same holds for *wealth-uniform domination*. Finally, it is worth noting that the advisor does not need to go over the utilities and the wealth levels and compute rejection rates, but only to compute the Foster–Hart (or Aumann–Serrano) measure of riskiness for every *set of gambles*, and choose the alternative whose corresponding set has the lowest riskiness measure.

7 Conclusion

Since a utility function is a subjective characteristic of the decision maker, while probabilities may be objective, riskiness measures offer

[28] We concentrate here on the Foster–Hart riskiness measure, because it is more natural in this example to treat w as given, and to look for uniformity over the class of utilities, i.e., *utility-uniform domination*, which induces a ranking based on R^{FH}.

[29] It is carried out in the same way as the uniformity over all $\lambda \in [0,1]$ in Section 5. We omit this trivial extension to keep the analysis parsimonious.

objective rankings of risky alternatives using the known characteristics of the given probability distribution. Riskiness measures are widely used in finance to compare investment alternatives (regardless of the decision maker and his unique utility), and recently been explored in economics too.

Two new riskiness measures were recently developed in a setup with known probabilities and a large class of utility functions: Aumann and Serrano's economic index of riskiness (2008), and Foster and Hart's operational measure of riskiness (2009). These two new measures are superior to previous measures and indices (including the famous Sharpe Value) in two respects: (1) They do not violate *stochastic dominance*; and (2) they are related to the concept of utility in a strong way: they can be induced by *uniform rejection criteria*. Hart (2011) looks at U^*, a large set of utility functions, and ranks *gambles* according to two *uniform rejection criteria*: *wealth-uniform rejection* and *utility-uniform rejection*. These rankings turn out to be complete because eventually every gamble is characterized by a unique number, its riskiness measure. In this paper we extend the usage of these two *uniform rejection criteria* to the realm of uncertainty.

In the uncertainty literature, DMs usually differ in both their attitude toward risk (captured by u) and their attitude toward ambiguity. The attitude toward ambiguity is often captured by two distinct properties: the belief about the extent of ambiguity, e.g., the relevant set of priors to consider, and the degree of ambiguity aversion, e.g., a cautious attitude that concentrates on the worst case.

Most of the rankings presented in this paper assume that DMs differ only by their attitude toward risk. In the basic model, based on Gilboa et al. (2010), both the extent of ambiguity and the degree of ambiguity aversion are determined uniquely in the decision model. In the extension based on Gajdos et al. (2008), only the degree of ambiguity aversion is determined by the model, but a complete ranking can be generated for DMs with the same parameter ε defining their belief about the extent of ambiguity. Conversely, in the extension based on Olszewski (2007), it is the extent of ambiguity that is determined by the model (through the specified set of gambles), and a complete ranking can be generated for DMs with the same parameter λ designating their degree of ambiguity aversion (known in the literature as λ-Hurwicz). Finally, we present two rankings based on uniformity over this parameter λ. As opposed to the assumptions underlying the previous rankings, this time not only does the attitude toward risk vary, but so does the degree of ambiguity aversion, as captured by λ.

All the rankings presented in this paper are complete. Since they are based on *uniform rejection* by decision makers, they can be treated as objective criteria to rank the "attractiveness" of *acts*, sets of gambles, or, generally, alternatives that are not uniquely determined in terms of probabilities.

8 Appendix

In our proofs we will use K_g to denote the maximal gain in the *gamble* g, and will use L_g to denote the maximal loss in this gamble.

For each of the two riskiness measures, the Aumann–Serrano riskiness index and the Foster–Hart riskiness measure, we will first provide the proofs for λ-maximizers. Then the proofs for the maximin and maximax decision rules will follow as special cases. We assume throughout that $G, H \in \mathbb{G}$.

8.1 Aumann–Serrano riskiness index of sets for λ-maximizers

Fix $\lambda \in [0,1]$ and let $V \equiv V_{u,\lambda}$.

We will focus for a while only on utilities that are characterized by constant absolute risk aversion ($CARA$). The set containing all these utilities will be denoted by U^{CA}.

Every $u \in U^{CA}$ can be fully characterized by its risk aversion coefficient $\alpha > 0$, and will be denoted by u_α. For $u = u_\alpha$ with uncertainty-aversion coefficient λ we use V^α.

Lemma 22 *$\forall g_1, g_2 \in G, u_\alpha$ either prefers g_1 to g_2 at all $w > 0$, or prefers g_2 to g_1 at all $w > 0$.*

Proof. u_α prefers g_1 to g_2 at w if and only if $E\left[-e^{-\alpha(w+g_1)}\right] \geq E\left[-e^{-\alpha(w+g_2)}\right] \Leftrightarrow E\left[-e^{-\alpha g_1}\right] \geq E\left[-e^{-\alpha g_2}\right]$, and this inequality is independent of w. ■

Following Lemma 22, we will perform the whole analysis of U^{CA} behavior for some arbitrary w, and denote $M^\alpha(G) \in \arg\max_{g \in G} E\left[u_\alpha(w+g)\right]$ and $m^\alpha(G) \in \arg\min_{g \in G} E\left[u_\alpha(w+g)\right]$.

Lemma 23 *$\forall \alpha$, V^α either accepts G at all $w > 0$, or rejects G at all $w > 0$.*

Proof. V^α accepts G at w if and only if $\lambda E\left[-e^{-\alpha(w+M^\alpha(G))}\right] + (1-\lambda) E\left[-e^{-\alpha(w+m^\alpha(G))}\right] > -e^{-\alpha w}$. Multiplying by $e^{\alpha w}$ yields a condition that is independent of w. If $\lambda E\left[-e^{-\alpha M^\alpha(G)}\right] + (1-\lambda) E\left[-e^{-\alpha m^\alpha(G)}\right] \leq 1$, V^α (weakly) rejects G at all $w > 0$. ■

Lemma 24 *Let $u_{\alpha_1}, u_{\alpha_2} \in U^{CA}$ with $\alpha_1 < \alpha_2$ such that V^{α_1} (strictly) accepts G at w and V^{α_2} (strictly) rejects G at w. Then $\exists \alpha^*$ s.t. $\alpha_1 < \alpha^* < \alpha_2$ and:*

(i) $V^{\alpha^*}(w+G) = u_{\alpha^*}(w)$.
(ii) $\forall \alpha < \alpha^*$, V^{α} accepts G at w, and $\forall \alpha > \alpha^*$, V^{α} rejects G at w.

We remind here that w in Lemma 24 is arbitrary and kept only for the coherence of the formulation. In fact the Lemma says that every V^{α} with $\alpha < \alpha^*$ accepts G at all $w > 0$, and that every V^{α} with $\alpha > \alpha^*$ rejects G at all $w > 0$.

Proof. (i) Let $\Delta u(\alpha) \equiv V^{\alpha}(w+G) - u_{\alpha}(w) = \lambda \max\limits_{g \in G} E[u_{\alpha}(w+g)] + (1-\lambda)\min\limits_{g\in G} E[u_{\alpha}(w+g)] - u_{\alpha}(w)$. $\Delta u(\alpha)$ is continuous in α because the functions $E[\cdot]$, $\max()$ and $\min()$ are all continuous, and because u_{α} is continuous $\forall \alpha$. Moreover $\Delta u(\alpha_1) > 0$ and $\Delta u(\alpha_2) < 0$. The continuity of $\Delta u(\alpha)$ guarantees that $\exists \alpha^*$ s.t. $\alpha_1 < \alpha^* < \alpha_2$ for which $\Delta u(\alpha^*) = 0$, i.e., $V_{\alpha^*}(w+G) = u_{\alpha^*}(w)$.

(ii) W.l.o.g., let $u_{\alpha}(x) = \frac{1-e^{-\alpha(x-w)}}{\alpha}$. If $\dot{\alpha} < \ddot{\alpha}$ then[30] $u_{\dot{\alpha}}(w+x) > u_{\ddot{\alpha}}(w+x)$ for all $x \neq 0$. Hence $E[u_{\dot{\alpha}}(w+g)] > E[u_{\ddot{\alpha}}(w+g)]$ for all $g \in G$ and $\dot{\alpha} < \ddot{\alpha}$. Since G is compact, we thus have $\max\limits_{g\in G} E[u_{\dot{\alpha}}(w+g)] > \max\limits_{g\in G} E[u_{\ddot{\alpha}}(w+g)]$ and $\min\limits_{g\in G} E[u_{\dot{\alpha}}(w+g)] > \min\limits_{g\in G} E[u_{\ddot{\alpha}}(w+g)]$. Finally, noticing that $u_{\alpha}(w) = 0$ for all $\alpha > 0$, we get that $\lambda \max\limits_{g\in G} E[u_{\dot{\alpha}}(w+g)] + (1-\lambda)\min\limits_{g\in G} E[u_{\dot{\alpha}}(w+g)] - u_{\dot{\alpha}}(w) > \lambda \max\limits_{g\in G} E[u_{\ddot{\alpha}}(w+g)] + (1-\lambda)\min\limits_{g\in G} E[u_{\ddot{\alpha}}(w+g)] - u_{\ddot{\alpha}}(w)$, i.e., $\Delta u(\dot{\alpha}) > \Delta u(\ddot{\alpha})$. So $\Delta u(\alpha^*) = 0$ implies that $\forall \alpha < \alpha^*$ we have $\Delta u(\alpha) > 0$, i.e., V^{α} accepts G at w, and $\forall \alpha > \alpha^*$ we have $\Delta u(\alpha) < 0$, i.e., V^{α} rejects G at w. ■

Corollary 25 *There is a unique[31] $u \in U^{CA}$ s.t. u is indifferent to G (at all w). This u is characterized by the unique $\alpha > 0$ which solves the equation $\lambda E[e^{-\alpha M^{\alpha}(G)}] + (1-\lambda) E\left[e^{-\alpha m^{\alpha}(G)}\right] = 1$.*

We now return to discuss the set of utilities U^*. We denote by $\rho_u(w)$ the local (absolute) risk aversion coefficient of $u \in U^*$ at w. Let $g^{u,w}_{\max}$ be a maximizer of the expected utility of u at wealth level w over the set of gambles G, i.e., $g^{u,w}_{\max} \in \arg\max\limits_{g\in G} E[u(w+g)]$. Similarly let $g^{u,w}_{\min} \in \arg\min\limits_{g\in G} E[u(w+g)]$.

Lemma 26 compares the acceptance and rejection criteria of a general u to those of u_{α^*}, and sets the ground for the main theorem. For every

[30] See Proof of Theorem A in Aumann and Serrano (2008).
[31] Up to affine transformations.

set G we denote by g_{α^*} the compound gamble $\lambda * M^{\alpha^*}(G) + (1-\lambda) * m^{\alpha^*}(G)$ (i.e., gamble $M^{\alpha^*}(G)$ with probability λ and gamble $m^{\alpha^*}(G)$ with probability $1-\lambda$), where α^* is the unique risk aversion coefficient of the $CARA$ utility that satisfies $V^{\alpha^*}(w+G) = u(w)$ (as was shown in Lemma 24(i)).

Lemma 26 *(i) If $\rho_u(w) \geq \alpha^*$ $\forall w > 0$ then u rejects G $\forall w > 0$.*

(ii) If $\rho_u(w) < \alpha^*$ $\forall w \in [\hat{w} - L_{g_{\alpha^*}}, \hat{w} + K_{g_{\alpha^*}}]$ then u accepts G at $\hat{w}$.

Proof. (i) Assume that $\rho_u(w) \geq \alpha^*$ $\forall w > 0$ for some u. Then obviously every gamble g which is rejected by u_{α^*} is rejected by u either. Assume now by negation that $\exists w$ s.t. u accepts G at w. Then u in fact accepts $\lambda * g_{\max}^{u,w} + (1-\lambda) * g_{\min}^{u,w}$ at w, and therefore accepts $\lambda * g_{\max}^{u,w} + (1-\lambda) * m^{\alpha^*}(G)$ at w too.[32] On the other hand, we know that u_{α^*} is indifferent to G at w, thus (weakly) rejects $\lambda * g_{\max}^{u,w} + (1-\lambda) * m^{\alpha^*}(G)$ at w (remember that if u_{α^*} is indifferent to G at some w', u_{α^*} is also indifferent to G at any other w). So the compound gamble $\lambda * g_{\max}^{u,w} + (1-\lambda) * m^{\alpha^*}(G)$ is accepted at w by u but rejected by u_{α^*}, in contradiction to the assumption.

(ii) We repeat almost the same proof: now we denote $\widehat{\alpha} \equiv \max \rho_u(w)$ at the range $[\hat{w} - L_{g_{\alpha^*}}, \hat{w} + K_{g_{\alpha^*}}]$ for some u. Let $u_{\widehat{\alpha}}$ be the $CARA$ utility with risk aversion coefficient $\widehat{\alpha}$. Since $\widehat{\alpha} < \alpha^*$ we get that $u_{\widehat{\alpha}}$ accepts $g_{\alpha^*} = \lambda * M^{\alpha^*}(G) + (1-\lambda) * m^{\alpha^*}(G)$ at $\hat{w}$, and from $\rho_u(w) \leq \widehat{\alpha}$ $\forall w \in [\hat{w} - L_{g_{\alpha^*}}, \hat{w} + K_{g_{\alpha^*}}]$ we know that also u accepts $\lambda * M^{\alpha^*}(G) + (1-\lambda) * m^{\alpha^*}(G)$ at $\hat{w}$. Therefore, if u were to reject G at $\hat{w}$, u would have to reject $\lambda * M^{\alpha^*}(G) + (1-\lambda) * g_{\min}^{u,\hat{w}}$ at $\hat{w}$ either. On the other hand, since u_{α^*} is indifferent to G at $\hat{w}$, and since $u_{\widehat{\alpha}}$ accepts every single gamble that u_{α^*} accepts, we know that $u_{\widehat{\alpha}}$ accepts $\lambda * M^{\alpha^*}(G) + (1-\lambda) * m^{\alpha^*}(G)$ at $\hat{w}$, hence accepts $\lambda * M^{\alpha^*}(G) + (1-\lambda) * g_{\min}^{u,\hat{w}}$ at $\hat{w}$. So the compound gamble $\lambda * M^{\alpha^*}(G) + (1-\lambda) * g_{\min}^{u,\hat{w}}$ is accepted at $\hat{w}$ by u but rejected at $\hat{w}$ by $u_{\widehat{\alpha}}$, in contradiction to the assumption. ■

We are now ready for the main proposition on Aumann–Serrano riskiness of sets. Proposition 27 identifies every set of gambles G with a unique compound gamble g_{α^*} on the grounds of wealth-uniform rejection. As a result, the complete ranking of gambles induces a complete ranking of sets of gambles.

Proposition 27 *For every $u \in U^*$, u rejects g_{α^*} $\forall w > 0$ if and only if u rejects G $\forall w > 0$.*

Proof. Direction (1): u rejects g_{α^*} $\forall w > 0 \Rightarrow u$ rejects G $\forall w > 0$.

Indeed if u rejects g_{α^*} at some w', then from Proposition 4(iii) in Hart (2011) we know that $\rho_u(w' - L_{g_{\alpha^*}}) \geq \alpha^*$. Since u rejects g_{α^*} $\forall w > 0$, we

[32] Follows from vN-M's Independence Axiom.

get that $\rho_u(w) \geq \alpha^* \ \forall w > 0$, and from Lemma 26(i) we conclude that u rejects $G \ \forall w > 0$.

Direction (2): u rejects $G \ \forall w > 0 \Rightarrow u$ rejects $g_{\alpha^*} \ \forall w > 0$.

Assume by negation that u accepts g_{α^*} at some w'. Then from Proposition 4(ii) in Hart (2011) we get that $\rho_u(w' + K_{g_{\alpha^*}}) < \alpha^*$. From the decreasing absolute risk aversion of u we then get that $\forall w > w' + K_{g_{\alpha^*}}$, $\rho_u(w) < \alpha^*$. In particular let $w'' \equiv w' + K_{g_{\alpha^*}} + L_{g_{\alpha^*}}$. Then $\rho_u(w) < \alpha^*$ $\forall w \in [w'' - L_{g_{\alpha^*}}, w'' + K_{g_{\alpha^*}}]$, hence from Lemma 26(ii) we know that u accepts G at w'', in contradiction to the assumption. ■

As defined in the paper, $R^{AS}_{\lambda}(G)$ is the reciprocal of the (unique) α such that a $CARA$ person with risk-aversion parameter α and uncertainty-aversion parameter λ is indifferent between accepting and rejecting the set G. Since we know that this $CARA$ person has parameter $\alpha^*(G)$, and since $R^{AS}\left(g_{\alpha^*(G)}\right)$ is the reciprocal of $\alpha^*(G)$ (from Theorem B in Aumann and Serrano (2009), while recalling that a $CARA$ person with parameter $\alpha^*(G)$ is indifferent between accepting and rejecting $g_{\alpha^*(G)}$), we get that $R^{AS}_{\lambda}(G) = R^{AS}\left(g_{\alpha^*(G)}\right)$.

Proof of Theorem 13 From proposition 27 we know that $G \geq^{\lambda}_{WU} H$ if and only if $g_{\alpha^*(G)} \geq_{WU} h_{\alpha^*(H)}$. Then from Hart (2011) we get $g_{\alpha^*(G)} \geq_{WU} h_{\alpha^*(H)} \Leftrightarrow R^{AS}_{\lambda}(g_{\alpha^*}) \leq R^{AS}_{\lambda}(h_{\alpha^*}) \Leftrightarrow R^{AS}_{\lambda}(G) \leq R^{AS}_{\lambda}(H)$.■

8.1.1 The duality of the Aumann–Serrano riskiness index

Proof of Proposition 14 Let $V_{u_1,\lambda} \rhd V_{u_2,\lambda}$ where $u_1, u_2 \in U^*$, and let $V_{u_2,\lambda}$ reject a *set of gambles* $H \in \mathbb{G}$ at some $w' > L_G$. Then from $V_{u_1,\lambda} \rhd V_{u_2,\lambda}$ it follows that $V_{u_1,\lambda}$ rejects H at all $w > 0$. Since $R^{AS}_{\lambda}(G) > R^{AS}_{\lambda}(H)$, we get that $V_{u_1,\lambda}$ rejects G at all $w > 0$.■

8.1.2 Aumann–Serrano riskiness index of sets for maximin and maximax decision rules

We are now ready to prove Propositions 3 and 9 as special cases of Theorem 13. Specifically, when DMs judge every set G by their (subjective) worse gamble in the set (in terms of expected utility), $\lambda = 0$, and we will prove that $R^{AS}_0(G) = \max_{g \in G} R^{AS}(g)$. similarly, when DMs judge every set G by their (subjective) best gamble in the set, $\lambda = 1$, and we will prove that $R^{AS}_1(G) = \min_{g \in G} R^{AS}(g)$.

Proof of Proposition 3 Let $\bar{g} \in \arg\max_{g \in G} R^{AS}(g)$ and $\bar{\alpha} \equiv \frac{1}{R^{AS}(\bar{g})}$. Then by Theorem B in Aumann and Serrano (2009), $u_{\bar{\alpha}}$ is indifferent to

$\bar{g}$ (at all $w > 0$). Therefore, every $CARA$ utility u_α with $\alpha > \bar{\alpha}$ rejects $\bar{g}$ at all $w > 0$ hence rejects G at all $w > 0$ (judging G by the worst case). Similarly, every u_α with $\alpha < \bar{\alpha}$ accepts $\bar{g}$ at all $w > 0$ hence accepts G at all $w > 0$ (otherwise u_α rejects $m^\alpha(G)$ while accepting $\bar{g}$, in contradiction to $R^{AS}(m^\alpha(G)) \leq R^{AS}(\bar{g})$). Then from Lemma 24, $u_{\bar{\alpha}}$ is indifferent to G (at all $w > 0$). Denote by $R_0^{AS}(G)$ the reciprocal of the (unique) α such that a $CARA$ person with risk-aversion parameter α and uncertainty-aversion parameter $\lambda = 0$ is indifferent between accepting and rejecting the set G. Then $R_0^{AS}(G) = \frac{1}{\bar{\alpha}} = R^{AS}(\bar{g}) = \sup_{g \in G} R^{AS}(g)$. Applying Theorem 13 we get $G \geq_{WU} H$, if and only if $\max_{h \in H} R^{AS}(h) \geq \max_{g \in G} R^{AS}(g)$.■

Proof of Proposition 9 Let $\underline{g} \in \arg\min_{g \in G} R^{AS}(g)$ and $\underline{\alpha} \equiv \frac{1}{R^{AS}(\underline{g})}$. Then by Theorem B in Aumann and Serrano (2009), $u_{\underline{\alpha}}$ is indifferent to $\underline{g}$ (at all $w > 0$). Therefore, every $CARA$ utility u_α with $\alpha < \underline{\alpha}$ accepts $\underline{g}$ at all $w > 0$ hence accepts G at all $w > 0$ (judging G by the best case). Similarly, every u_α with $\alpha > \underline{\alpha}$ rejects $\underline{g}$ at all $w > 0$ hence rejects G at all $w > 0$ (because rejection of $\underline{g}$ implies rejection of every $g \in G$, hence of G itself). Then from Lemma 24, $u_{\underline{\alpha}}$ is indifferent to G (at all $w > 0$). Denote by $R_1^{AS}(G)$ the reciprocal of the (unique) α such that a $CARA$ person with risk-aversion parameter α and uncertainty-aversion parameter $\lambda = 1$ is indifferent between accepting and rejecting the set G. Then $R_1^{AS}(G) = \frac{1}{\underline{\alpha}} = R^{AS}(\underline{g}) = \min_{g \in G} R^{AS}(g)$. Applying Theorem 13 we get $G \geq_{WU} H$, if and only if $\min_{g \in G} R^{AS}(h) \geq \min_{g \in G} R^{AS}(g)$.■

8.2 Foster–Hart riskiness measure of sets for λ-maximizers

Fix $\lambda \in [0, 1]$ and let $V \equiv V_{u,\lambda}$.

For the log utility (i.e., $u_{\lg}(w) \equiv \log(w)$) we will denote $g_{\max}^{u,w}, g_{\min}^{u,w}$ simply by $M_w(G), m_w(G)$ respectively. For $u = u_{\lg}$ with uncertainty-aversion coefficient λ we use $V_{\lg}$.

Lemma 28 *There is a unique w' s.t. $V_{\lg}(w' + G) = \log(w')$ (i.e., $u_{\lg}(w)$ is indifferent towards G at w'). Moreover, $u_{\lg}(w)$ accepts G for every $w > w'$ and rejects G for every $w < w'$.*

Proof. Denote $\Delta u \equiv V_{\lg}(w + G) - u_{\lg}(w) = \lambda E[\log(w + M_w(G))] + (1 - \lambda) E[\log(w + m_w(G))] - \log(w)$. Δu is continuous in w because the functions $E[\cdot], \log(), \max()$ and $\min()$ are all continuous. By definition, $\Delta u > 0$ if and only if $u_{\lg}(w)$ accepts G at w. Let $R(G), r(G)$ denote the gambles in G with maximal and minimal values of R^{FH} respectively. Then $\forall w < r(G)$, $u_{\lg}(w)$ rejects every $g \in G$ hence rejects G (i.e., $\Delta u < 0$), and $\forall w > R(G)$, $u_{\lg}(w)$ accepts every $g \in G$ hence accepts[33] G (i.e., $\Delta u > 0$). From the continuity of Δu we conclude that $\exists w' > 0$ s.t. $\Delta u = 0$ at w', i.e., $V_{\lg}(w' + G) = u_{\lg}(w')$.

Assume now by negation that $\exists w'' < w'$ s.t. $u_{\lg}(w)$ (weakly) accepts G at w''. $u_{\lg}(w)$ is indifferent towards $\hat{g} \equiv \lambda * M_{w'}(G) + (1-\lambda) * m_{w'}(G)$ at w', hence (weakly) rejects $g_1 \equiv \lambda * M_{w''}(G) + (1-\lambda) * m_{w'}(G)$ at w' (g_1 is identical to $\hat{g}$ except for replacing the most preferred gamble at w' with potentially a different one). By assumption, $u_{\lg}(w)$ (weakly) accepts G at w'', i.e., (weakly) accepts $\lambda * M_{w''}(G) + (1-\lambda) * m_{w''}(G)$ at w'', hence (weakly) accepts g_1 at w'' (this time replacing the *least* preferred gamble with potentially a different one). So we get that $u_{\lg}(w)$ (weakly) accepts g_1 at $w'' < w'$ while (weakly) rejecting g_1 at w', in contradiction with the strictly decreasing absolute risk aversion property of the log utility.

Similarly, assume now by negation that $\exists w'' > w'$ s.t. $u_{\lg}(w)$ (weakly) rejects G at w''. $u_{\lg}(w)$ is indifferent towards $\hat{g}$ at w', hence (weakly) accepts $g_2 \equiv \lambda * M_{w'}(G) + (1-\lambda) * m_{w''}(G)$ at w'. By assumption, $u_{\lg}(w)$ (weakly) rejects G at w'', i.e., (weakly) rejects $\lambda * M_{w''}(G) + (1-\lambda) * m_{w''}(G)$ at w'', hence (weakly) rejects g_2 at w''. So we get that $u_{\lg}(w)$ (weakly) rejects g_2 at $w'' > w'$ while (weakly) accepting g_2 at w', in contradiction with the strictly decreasing absolute risk aversion property of the log utility. ■

Denote $l(G) \equiv \lambda * M_{w'}(G) + (1-\lambda) * m_{w'}(G)$ (where w' is the unique w s.t. $u_{\lg}(w)$ is indifferent towards G at w). Then we have:

Corollary 29 *$u_{\lg}(w)$ accepts G at some w if and only if $u_{\lg}(w)$ accepts $l(G)$ at that w.*

Proof. The strictly decreasing absolute risk aversion property of the log utility implies that $\exists w^*$ such that $u_{\lg}(w)$ is indifferent towards $l(G)$ at w^*, accepts $l(G)$ $\forall w > w^*$, and rejects $l(G)$ $\forall w < w^*$.[34] Since $u_{\lg}(w)$ is indifferent towards G and therefore towards $l(G)$ at w', we get that $w^* = w'$, hence $u_{\lg}(w)$ accepts both G and $l(G)$ $\forall w > w'$, and rejects both G and $l(G)$ $\forall w < w^*$. ■

We are now ready for the main proposition on Foster–Hart riskiness of sets, encompassing the more general set of utilities U^*. Proposition 30 identifies every set of gambles G with the unique compound gamble $l(G)$

[33] $u_{\lg}(w)$ is a special case: for every gamble g, $u_{\lg}(w)$ accepts g whenever $w > R^{FH}(g)$, and rejects g whenever $w < R^{FH}(g)$. At $w = R^{FH}(g)$ $u_{\lg}(w)$ is indifferent to g (see Foster and Hart 2009, section VI(B)).

[34] Foster and Hart (2009) show that $w^* = R^{FH}(l(G))$.

on the grounds of utility-uniform rejection. As a result, the complete ranking of gambles induces a complete ranking of sets of gambles.

Proposition 30 $\forall w > 0$, *every* $u \in U^*$ *rejects* $l(G)$ *at* w *if and only if every* $u \in U^*$ *rejects* G *at* w*.*

Proof. Direction (1): $\forall w > 0$, every $u \in U^*$ rejects G at $w \Rightarrow$ every $u \in U^*$ rejects $l(G)$ at w.

Let $w > 0$ be an arbitrary wealth level. Every $u \in U^*$ rejects G at w, so $u_{\lg}(w) \in U^*$ rejects G at w either, and by corollary 29 $u_{\lg}(w)$ rejects $l(G)$ at w. Therefore $R^{FH}(l(G)) \geq w$ (Foster and Hart 2009, Section VI(B)), and by Lemma 10 in Hart (2011) we get that every $u \in U^*$ rejects $l(G)$ at w.

Direction (2): $\forall w > 0$, every $u \in U^*$ rejects $l(G)$ at $w \Rightarrow$ every $u \in U^*$ rejects G at w.

If every $u \in U^*$ rejects $l(G)$ at w, then $u_{\lg}(w)$ rejects $l(G)$ at w either, and by corollary 29 $u_{\lg}(w)$ rejects G at w. Assume now by negation that $\exists u' \in U^*$ s.t. u' accepts G at w. That is, u' accepts $\lambda * g^{u',w}_{\max} + (1-\lambda) * g^{u',w}_{\min}$ at w. Consequentially, u' accepts $g' \equiv \lambda * g^{u',w}_{\max} + (1-\lambda) * m^{w}(G)$ at w too. $u_{\lg}(w)$ rejects G at w, hence rejects $\lambda * M_w(G) + (1-\lambda) * m_w(G)$ at w. Consequentially, $u_{\lg}(w)$ rejects g' at w too. But this means that $R^{FH}(g') \geq w$, and by Lemma 10 in Hart (2011) every $u \in U^*$ should reject g' at w, in contradiction to the assumption that u' accepts g' at w. ■

As defined in the paper, we denote by $R^{FH}_{\lambda}(G)$ the (unique) wealth level w at which a logarithmic utility decision maker with uncertainty-aversion parameter λ is indifferent between accepting and rejecting the set G. But this wealth level is exactly w', and since the indifference of $u_{\lg}(w)$ toward G at w' implies the indifference of $u_{\lg}(w)$ toward $l(G)$ at w', w' equals $R^{FH}(l(G))$, and we get that $R^{FH}_{\lambda}(G) = R^{FH}(l(G))$.

Proof of Theorem 17 From proposition 30 we know that $G \geq^{\lambda}_{UU} H$ if and only if $l(G) \geq_{UU} l(H)$. Then from Hart (2011) we get $l(G) \geq_{UU} l(H) \Leftrightarrow R^{FH}_{\lambda}(l(G)) \leq R^{FH}_{\lambda}(l(H)) \Leftrightarrow R^{FH}_{\lambda}(G) \leq R^{FH}_{\lambda}(H)$.■

8.2.1 Foster–Hart riskiness measure of sets for maximin and maximax decision rules

We are now ready to prove Propositions 4 and 10 as special cases of Theorem 17. Specifically, when DMs judge every set G by their (subjective) worse gamble in the set (in terms of expected utility), $\lambda = 0$, and we will prove that $R^{FH}_{0}(G) = \max_{g \in G} R^{FH}(g)$. similarly, when DMs judge every set G by their (subjective) best gamble in the set, $\lambda = 1$, and we will prove that $R_1^{FH}(G) = \min_{g\in G} R^{FH}(g)$.

Proof of Proposition 4. $R_0^{FH}(G)$ equals the wealth level w' at which a log utility $u_{\lg}(w)$ with uncertainty-aversion parameter $\lambda = 0$ is indifferent to G. Let $\bar{g} \in \arg\max_{g\in G} R^{FH}(g)$. $\forall w > R^{FH}(\bar{g})$ we have $w > R^{FH}(g)$ $\forall g \in G$, hence by Foster and Hart (2009) $u_{\lg}(w)$ accepts every gamble in G at w, i.e., $u_{\lg}(w)$ accepts G at w. Conversely, $\forall w < R^{FH}(\bar{g})$ we know that $u_{\lg}(w)$ rejects $\bar{g}$ at w, and therefore also rejects G (judged by the worst case) at w. From the continuity of $u_{\lg}(w)$ we then get that $R_0^{FH}(G) = R^{FH}(\bar{g}) = \max_{g\in G} R^{FH}(g)$. Applying Theorem 17 we finally get that $G \geq_{WU} H$ if and only if $\max_{h\in H} R^{FH}(h) \geq \max_{g\in G} R^{FH}(g)$.■

Proof of Proposition 10 $R_1^{FH}(G)$ equals the wealth level w'' at which a log utility $u_{\lg}(w)$ with uncertainty-aversion parameter $\lambda = 1$ is indifferent to G. Let $\underline{g} \in \arg\min_{g\in G} R^{FH}(g)$. $\forall w < R^{FH}(\underline{g})$ we have $w < R^{FH}(g)$ $\forall g \in G$, hence $u_{\lg}(w)$ rejects every gamble in G at w, i.e., $u_{\lg}(w)$ rejects G at w. Conversely, $\forall w > R^{FH}(\bar{g})$ we know that $u_{\lg}(w)$ accepts $\underline{g}$ at w, and therefore also accepts G (judged by the best case) at w. From the continuity of $u_{\lg}(w)$ we then get that $R_1^{FH}(G) = R^{FH}(\underline{g}) = \min_{g\in G} R^{FH}(g)$. Applying Theorem 17 we finally get that $G \geq_{WU} H$ if and only if $\min_{h\in H} R^{FH}(h) \geq \min_{g\in G} R^{FH}(g)$.■

We complete by proving that $R^{FH}(g)$ attains the maximum and the minimum in every compact *set of gambles* $G \subset \mathcal{G}$ having a *finite support.*

Foster–Hart riskiness as critical wealth

Proof of Proposition 5 Accepting a set G with wealth $w \geq R^{FH}(g)$ for every $g \in G$ guarantees no-bankruptcy (Theorem 1 in Foster and Hart 2009), regardless of the way g is chosen from G. On the other hand, if G is accepted when $w < \max_{g\in G} R^{FH}(g)$, then $\bar{g} \in \arg\max_{g\in G} R^{FH}(g)$ may be chosen, and then no-bankruptcy cannot be guaranteed.■

8.2.2 Extrema of $R^{FH}(g)$ in sets of gambles

At last we prove that if the *set of gambles* G is compact, and has a fixed and finite support, then $R^{FH}(\cdot)$ gets its minimum and maximum values over every $G \in \mathbb{G}$, although $R^{FH}(\cdot)$ is not a continuous function.

Lemma 31 *If $G \subset \mathcal{G}$ is compact and has finite support, then both* $\sup R^{FH}(g)$ *and* $\inf R^{FH}(g)$ *are attained in* G*.*

Proof. (i) Let $\underline{\rho} \equiv \inf_{g\in G} R^{FH}(g)$, and let $(g_n)_{n=1,2,\dots}$ be a sequence in G with $R^{FH}(g_n) \to \underline{\rho}$. Since G is compact, and its support is finite, we can take w.l.o.g. a subsequence g_n' in g_n such that $g_n' \to g_0 \in G$ and $L_{g_n'} \equiv L_0$ for all n. Proposition 10 in Foster and Hart (2009) implies that $\underline{\rho} = \lim R^{FH}(g_n') = \max\left\{R^{FH}(g_0), L_0\right\} \geq R^{FH}(g_0) \geq \underline{\rho}$. Therefore all inequalities are equalities, and so $\underline{\rho} = R^{FH}(g_0)$.

(ii) Let $\bar{\rho} \equiv \sup_{g\in G} R^{FH}(g)$, and let $(g_n)_{n=1,2,\dots}$ be a sequence in G with $R^{FH}(g_n) \to \bar{\rho}$. W.l.o.g. let g_n' a subsequence in g_n such that $R^{FH}(g_n')$ is increasing, $g_n' \to g_0 \in G$ and $L_{g_n'} \equiv L_0$ for all n. Since $L_0 = L_{g_n'} < R^{FH}(g_n')$, and $R^{FH}(g_n')$ increases to $\bar{\rho}$, we have $\bar{\rho} > L_0$. By applying Proposition 10 in Foster and Hart (2009) we get that $\bar{\rho} = \lim R^{FH}(g_n') = \max\left\{R^{FH}(g_0), L_0\right\} > L_0$, so $\bar{\rho} = R^{FH}(g_0)$. ■

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