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VALUES OF NONDIFFERENTIABLE VECTOR MEASURE GAMES

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Values of Nondifferentiable Vector Measure Games*

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Abstract

We introduce ideas and methods from distribution theory into value theory. This novel approach enables us to construct new diagonal formulas for the Mertens value [4] and the Neyman value [5] on a large space of non-differentiable games. This in turn enables us to give an affirmative answer to the question, first posed by Neyman [5], whether the Mertens value and the Neyman value coincide "modulo Banach limits"? The solution is an intermediate result towards a characterization of values of norm 1 of vector measure games with bounded variation.

1 Introduction

One of the most basic solution concepts of cooperative game theory is the Shapley value [7]. It was first introduced in the setting of n-players games, where it can be viewed as the players' expected payoffs. It has a wide range of applications in various fields of economics and political science (e.g., [1, Chapters 32–34]). In many such applications it is necessary to consider games that involve a large number of "individually insignificant" players. Among the typical examples are voting among stockholders of a corporation and markets with perfect competition. In such cases it is fruitful to model the game as a cooperative game with an underlying standard nonatomic space of players, i.e., as a nonatomic game. Aumann and Shapley [2] extended the definition of the value to nonatomic games. The value was defined using the axioms of additivity, efficiency, symmetry, and positivity.

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Once a solution concept on a space of games is defined, it is natural to ask whether this solution exists and is unique. The core of a cooperative game, the nucleus, and the Shapley value are examples of solution concepts of cooperative games whose existence and uniqueness is guaranteed under certain conditions. Aumann and Shapley [2] proved that the value exists and is in fact unique on some spaces of "differentiable" nonatomic games.

However, the problem of existence of a value on spaces of "nondifferentiable" games remained open for a long time after many trials. An example is the space generated by all market games, and of special interest are the "n-gloves" games. Tauman [8] proved the uniqueness of the value on the space Q^n, i.e., the minimal symmetric space spanned by "n-gloves" games that are functions of finitely many nonatomic and mutually singular probability measures. Tauman [8] also proved that this value can be extended to the minimal symmetric space generated by Q^n and pNA, the space of games that are "smooth" functions of finitely many nonatomic measures. But it was still unknown whether there is a value on larger spaces, such as the space generated by market games. In fact, it was not even known whether there is a value on the space generated by the union of the Q^n-s. Mertens [4] solved this problem and introduced a value on a very large space of games containing, among others, the space generated by market games. Neyman [5] introduced yet another value on the space spanned by games that are functions of finitely many measures.[1] It is straightforward, due to the use of Banach limits in Neyman's construction, that the value is not unique on the space of games on which the Neyman value was constructed. Yet, Neyman [5] asked whether the value is unique "modulo Banach limits," i.e., whether there is a unique value of norm 1 on the space of games for which the Neyman value exists without the use of a Banach limit. This problem has prooven to be extremely difficult. Consequently, Neyman [5] introduced an intermediate problem, namely, do the Mertens value and the Neyman value coincide "modulo Banach limits."

It is straightforward that the values coincide on subspaces of the intersection of their domains on which the value is known to be unique.[2] It is somewhat less obvious that these values coincide on the space generated by vector measure market games, as was proved by Neyman [5, Proposition 4]. In fact, Neyman [5, Proposition 5] proved that these values coincide on the space generated by games that are concave functions of finitely many nonatomic probability measures. However, the proof becomes extremely difficult for games that exhibit even the slightest singularities. For example, proving that these values coincide on the space LPS of piecewise smooth vector measure games[3] will immediately expose much of the difficulty inherent in the more general problem.

The main obstacle arises from the different methods used to construct both values. Essentially,

[1] That are also NA-continuous at the empty and the grand coalitions.

[2] E.g., the space of "differentiable" games pNA, and the space $bv'NA$.

[3] Introduced by Neyman and Smorodinsky [6].

Mertens' construction averages the marginal contribution of a coalition to some "infinitesimal" random perturbation of the diagonal s.t. the random perturbations are made to be independent of the computation of the marginal contribution. In contrast, Neyman's construction makes the random perturbations and the computation of the marginal contribution heavily dependent on each other. Another substantial difference is that while Mertens' construction takes the average over "infinitesimal" random perturbations, Neyman's construction considers an average on rather "large" perturbations. Thus, Neyman's [5] questions have an appealing interpretation. In effect, he asked whether the way in which the marginal contributions are computed and aggregated might influence the value. Nevertheless, the differences between the Mertens value and the Neyman value have the effect of turning even a rather simple exercise, like proving that the values coincide on the space LPS of piecewise smooth vector measure games, into a tedious and involved task. In fact, these major differences are the reason that the answer to Neyman's question is assumed to be negative.

Our approach to the problem is to develop neat "diagonal formulas" for games that lie in the intersection of the domains of both values, for which the Neyman value exists without the use of Banach limits. This task, which may seem hopeless at a first glance, is accomplished by an application of methods and ideas from distribution theory. As a consequence we prove the surprising result that the Merten value and the Neyman value coincide "modulo Banach limits," which yields an affirmative answer to Neyman's second question. Our result may considered to be a first step towards a general characterization of values of norm 1 on spaces consisting of vector measure games with bounded variation.

2 Background and Basic Definitions

2.1 Basic Definitions.

Let $(I, \mathcal{C})$ be a standard measurable space. The members of I are called *players* and the members of $\mathcal{C}$ are called *coalitions*. A *game* is a real valued function $v : \mathcal{C} \to \mathbb{R}$ with $v(\emptyset) = 0$. A *game* is a real valued function $v : \mathcal{C} \to \mathbb{R}$ s.t. $v(\emptyset) = 0$. A game v is:

1) *finitely additive* if it is bounded and $v(S \cup T) = v(S) + v(T)$ whenever S, T are two disjoint coalitions. We denote the space of all finitely additive games by FA, and its subspace of all nonatomic and countably additive measures by NA;

2) *monotonic* if $v(T) \geq v(S)$ whenever $S \subseteq T$. It is of *bounded variation* if it is the difference between two monotonic games. We denote the space of all games of bounded variation by BV.

The *variation* of a game $v \in BV$ is the supremum of the variation of v along all increasing chains $S_0 \subseteq S_1 \subset ... \subseteq S_m$ in $\mathcal{C}$, or equivalently

$$\|v\|_{BV} = \inf \{u(I) + w(I) : u, w \;\; are \;\; monotonic \;\; games \;\; s.t. \;\; v = u - w\}.$$

$\|\cdot\|_{BV}$ is a norm on BV, under which BV is a Banach algebra (see [2]).

Denote by Θ the group of measurable automorphisms of $(I, \mathcal{C})$; i.e., bijections $\theta : I \to I$ s.t. both θ and θ^{-1} are measurable. Each $\theta \in \Theta$ induces a linear mapping θ^* of BV onto itself by $(\theta^* v)(S) = v(\theta S)$. A set of games $Q \subset BV$ is *symmetric* if $\theta^* Q = Q$ for each $\theta \in \mathcal{G}$. Given a set of games Q we will denote by Q^+ its subset containing all monotonic games, and by Q^1 the subset $\{v \in Q^+ \ : \ v(I) = 1\}$.

Let Q be a symmetric space. A map $\Psi : Q \to BV$ is called *positive* iff $\Psi(Q^+) \subset BV^+$. It is *symmetric* iff $\theta^* \Psi = \Psi \theta^*$ for each $\theta \in \Theta$, and efficient iff $\Psi(v)(I) = v(I)$ for each $v \in Q$.

Definition 2.1. Let Q be a symmetric linear subspace of BV. A *value* on Q is a symmetric, positive, and efficient linear map $\Psi : Q \longrightarrow FA$.

Denote by $B(I, \mathcal{C})$ the space of bounded measurable real-valued functions on $(I, \mathcal{C})$ and let $B_1^+(I, \mathcal{C}) = \{f \in B(I, \mathcal{C}) : 0 \leq f \leq 1\}$ be the space of *ideal coalitions*. A function $\bar{v}$ on $B_1^+(I, \mathcal{C})$ is a *constant sum* if $\bar{v}(f) + \bar{v}(1 - f) = \bar{v}(1)$ for every $f \in B_1^+(I, \mathcal{C})$. It is *monotonic* if for every $f, g \in B_1^+(I, \mathcal{C})$ with $f \geq g$, $\bar{v}(f) \geq \bar{v}(g)$. It is *finitely additive* if it is bounded and for every $f, g \in B_1^+(I, \mathcal{C})$ with $f + g \leq 1$, $\bar{v}(f + g) = \bar{v}(f) + \bar{v}(g)$. It is *of bounded variation* if it is the difference between two monotonic functions, and its *variation norm* $\|\bar{v}\|_{IBV}$ is the supremum of the variation of $\bar{v}$ along all increasing sequences $0 \leq f_1 \leq f_2 \leq ... \leq f_m \leq 1$ in $B_1^+(I, \mathcal{C})$. The group Θ acts on $B_1^+(I, \mathcal{C})$ by $(\theta^* f)(s) = f(\theta s)$.

2.2 Cauchy Distributions

The Cauchy distribution with parameter $\alpha > 0$ is the distribution on $\mathbb{R}$ with density $\frac{\alpha}{\pi(\alpha^2 + x^2)}$. If X and Y are independent Cauchy random variables with parameters α and β respectively and $a, b \in \mathbb{R}$ s.t. $a^2 + b^2 \neq 0$, then $aX + bY$ is a Cauchy random variable with parameter $|a|\alpha + |b|\beta$. The characteristic function of the Cauchy distribution with parameter α is $\psi(t) = \exp(-\alpha|t|)$.

Recall that given a vector measure $\mu \in (NA^1)^k$ we defined $\overline{\mu} = \frac{1}{k}\sum_{i=1}^{k} \mu_i$. The μ *semi-norm* of $y \in \mathbb{R}^k$ is given by

$$\|y\|_\mu = \int |\sum_{i=1}^{k} (d\mu_i / d\overline{\mu}) y_i | d\bar{\mu}. \tag{2.1}$$

Denote the range of μ by $\mathcal{R}(\mu)$, and denote the affine space generated by $\mathcal{R}(\mu)$ by $AF(\mu)$. By [5, Lemma 1] the function $\phi_\mu : \mathbb{R}^k \longrightarrow \mathbb{R}$ given by $\phi_\mu(y) = exp(-\|y\|_\mu)$ is the characteristic function of a probability distribution P_μ on $AF(\mu)$, P_μ is absolutely continuous w.r.t. the Lebesgue measure on $AF(\mu)$, and its density ξ_μ is a $C_0(AF(\mu))$ function.

In [4] it is proved that the collection $\left\{P_\mu \circ \mu^{-1} : \mu \in (NA^1)^k, k \geq 1\right\}$ defines a cylindric set measure on $B(I,\mathcal{C})$. We denote this measure by P.

2.3 The Mertens Value

Definition 2.2. An *extension operator* is a linear and symmetric map ψ from a linear symmetric space of games to real-valued functions on $B_1^+(I,\mathcal{C})$, s.t. $(\psi v)(0) = 0$, $(\psi v)(1) = v(I)$, $\|\psi v\|_{IBV} \leq \|v\|$, ψv is finitely additive whenever v is finitely additive, and a constant sum whenever v is a constant sum.

Mertens [4] proved the existence of an extension operator ϕ_M on a large symmetric space EXT. The Mertens extension $\phi_M(v)$ of the game v can be extended to a function $\widetilde{v}$ on the space $B(I,\mathcal{C})$ by $\widetilde{v}(\chi) = \phi_M(v)(\max\{0,\min\{1,\chi\}\})$. Notice that, indeed, for every $\chi \in B_1^+(I,\mathcal{C})$ we have $\widetilde{v}(\chi) = \phi_M(v)(\chi)$. In the same paper he defined a value on a large space of games Q_M in the following way:

1) Map every game $v \in BV$ to the constant sum game $\phi_C(v)(S) = \frac{1}{2}(v(S) - v(S^c) + v(I))$.

2) Let Q be the space of all games with $\phi_M \circ \phi_C(v) \in EXT$. For $v \in Q$ let $\overline{v} = \widetilde{\phi_C(v)}$.

3) Define $Q_D \subseteq Q$ as the space of all games $v \in Q$ for which the following integral and limit exist:

$$(\Psi_D v)(\chi) = \lim_{\tau \searrow 0} \int_0^1 \frac{\overline{v}(t+\tau\chi) - \overline{v}(t-\tau\chi)}{2\tau} dt. \tag{2.2}$$

4) For every $w \in \Psi_D(Q_D)$ and every $\xi, \chi \in B(I,\mathcal{C})$ let

$$[w]_\xi(\chi) = \lim_{\tau \searrow 0} \frac{w(\chi+\tau\xi) - w(\chi-\tau\xi)}{2\tau}. \tag{2.3}$$

Next, let Q_M be the closed symmetric space generated by all games $v \in Q_D$ s.t. either $\Psi_D(v) \in FA$ or $\Psi_D(v)$ is a function of finitely many nonatomic measures.

Theorem 2.3. *[4, Section 2], Let $v \in Q_M$. Then for every $\xi \in B(I,\mathcal{C})$ the derivative $[\Psi_D]_\xi(\chi)$ exists for P almost every χ and is P-integrable in χ. In particular the map $\Psi_M : Q_M \to FA$ given* *by*

$$\Psi_M(v)(S) = \int [\Psi_D(v)]_S(\chi) dP(\chi), \tag{2.4}$$

is a value of norm 1 on Q_M*.*

2.4 The Neyman Value

Let $Q(\mu)$ be the space of all bounded variation games of the form $f \circ \mu$, where $\mu \in (NA^1)^k$ for some $k \geq 1$, and f is continuous at 0_k and at $\mu(I) = \mathbf{1}_k$. For any $\mathbb{R}^k$-valued nonatomic measure μ define a map Ψ^δ_μ from $Q(\mu)$ to BV as follows. For $\delta > 0$ let $I_\delta(t) = I(3\delta \leq t < 1 - 3\delta)$. For every sufficiently small $\delta > 0$, $x \in AF(\mu)$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$, and $S \in \mathcal{C}$, let

$$F_{f,\mu}(\delta, x, S) = \int_0^1 I_\delta(t) \frac{f(t\mu(I) + \delta^2 x + \delta^3 \mu(S)) - f(t\mu(I) + \delta^2 x)}{\delta^3} dt. \tag{2.5}$$

Let P^δ_μ be the restriction of P_μ to the set $B^\delta_\mu = \{x \in \mathbb{R}^k \ : \ \delta x \in 2\mathcal{R}(\mu) - \mu(I)\}$. The function $x \mapsto F_{f,\mu}(\delta, x, S)$ is continuous and bounded (see [5, Lemma 5]), and, therefore,

$$\Psi^\delta_\mu(f \circ \mu, S) = \int_{AF(\mu)} F_{f,\mu}(\delta, x, S) dP^\delta_\mu \tag{2.6}$$

(where $AF(\mu)$ is the affine space spanned by $\mathcal{R}(\mu)$) is well defined.

The space $Q(\mu)$ is neither symmetric nor mappable by Ψ^δ_μ to FA; its mapping is neither efficient nor symmetric, nor does its restriction to $Q(\mu) \cap Q(\nu)$ necessarily coincide with Ψ^δ_ν. However, these violations of the value axioms diminish as $\delta \longrightarrow 0$, and $\Psi^\delta_\mu(f \circ \mu) - \Psi^\delta_\nu(g \circ \nu) \longrightarrow 0$ as $\delta \longrightarrow 0^+$ whenever $f \circ \mu = g \circ \nu$. This remains true even if the limit exists only as some Banach limit $\mathcal{L}$ (see [5, Section 3.2] for a detailed construction of $\mathcal{L}$). Given $\mathcal{L}$, Neyman [5] defined

$$\Psi_N(f \circ \mu)(S) = \mathcal{L}(\Psi^\delta_\mu(f \circ \mu, S)). \tag{2.7}$$

It turns out that Ψ is a value of norm 1 on Q_N (hence also on its closure; see [5, Proposition 1]).

Let $Q_N = \bigcup Q(\mu)$ and define $\Psi_N : Q \longrightarrow \mathbb{R}^{\mathcal{C}}$ by

$$\Psi_N(v)(S) = \mathcal{L}(\Psi^\delta_\mu(v, S)) \tag{2.8}$$

whenever $v \in Q(\mu)$. Then Ψ_N is a value of norm 1 on Q_N ([5, Proposition 1]).

The value on Q_N is obviously not unique, due to the use of Banach limits in Neyman's construction. However, Neyman [5] asked whether the value is unique "modulo Banach limits." Namely, let Q'_N be the linear space consisting of games $v \in Q_N$ for which the limit on the right-hand side of equation (2.7) exists in the usual sense. Neyman asked [5, Section 4.5] whether there is a unique value of norm 1 on Q'_N or even on a large subspace of it. This question was left unanswered. A natural question in this context is whether the Mertens and Neyman values coincide on $Q'_N \cap Q_M$. This question was also raised by Neyman [5, Section 5.2]. Yet, despite the fact that Neyman had obtained a positive answer for smaller subspaces [5, Propositions 4–5], the question remains open in general.

2.5 Statement of the Main Results

The present paper concentrates on Neyman's second question, i.e., whether the Mertens and Neyman values coincide on the space $Q_0 = Q_M \cap Q'_N$. Although there are many reasons to suspect that the Mertens and Neyman values do not coincide on Q_0, we prove the following surprising theorem:

Main Theorem. $\Psi_M = \Psi_N$ *on* Q_0.

Our solution utilizes ideas and methods from distribution theory. This is a novel approach to the study of the value, and especially to the study of the notion of the derivative of a game. We consider games to be "distributions" *á la* Schwartz, and think of the derivative of the games as a *directional derivative* in the sense of distributions. The proof is then obtained by devising "diagonal formulas" for the Mertens and Neyman values on Q_0.

Unrigorously, we prove that for every game $f \circ \mu \in Q_0$ with μ of full dimension $n \in \mathbb{N}$, and every coalition $S \in \mathcal{C}$, there is a family of infinitely differentiable functions on $\mathbb{R}^n$ with a compact support $\left(\xi^\delta_\mu\right)_{\delta>0}$ and a family of measures $(\zeta_\delta(f, \mu(S)))_{\delta>0}$ that are interpreted as follows: every function ξ^δ_μ is approximately (depending on δ) the density of the averaging measure P_μ appearing in the Neyman value, and every measure $\zeta_\delta(f, \mu(S))$ is the average "distributional" directional derivative of the game $f \circ \mu$ in the direction of the coalition S in some small neighborhood (depending on δ) of the diagonal. As a result we get the following formulas for the values:

Proposition I. *The Neyman value admits the following representation on games* $f \circ \mu \in Q_0$ *with* μ *of full dimension:*

$$\Psi_N(f \circ \mu)(S) = \lim_{\delta \to 0^+} \int \xi^\delta_\mu(x) d\zeta_\delta(f, \mu(S))(x), \tag{2.9}$$

and

Proposition II. *The Mertens value admits the following representation on games $f \circ \mu \in Q_0$ with μ of full dimension:*

$$\Psi_M(f \circ \mu)(S) = \lim_{\delta \to 0^+} \int \xi^{\delta}_{\mu}(x) d\zeta_{\delta}(f, \mu(S))(x). \tag{2.10}$$

As every $v \in Q_0$ can be represented as $v = f \circ \mu$ with μ of full dimension, the Main Theorem is easily deduced.

3 Preliminaries

3.1 Some Distributional Calculus

Denote by $C_c^{\infty}(\mathbb{R}^n)$ the space of *test functions* (i.e., infinitely differentiable functions with compact support) on $\mathbb{R}^n$ and by $\lambda = \lambda^n$ the Lebesgue measure. For every test function $\phi \in C_c^{\infty}(\mathbb{R}^n)$ and $z \in \mathbb{R}^n$ denote the directional derivative of ϕ in the direction z by $\partial_z \phi$. Now let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be some function and assume that its directional derivative f'_z in direction $z \in \mathbb{R}^n$ exists λ-a.e. For every $\eta > 0$ and $x \in \mathbb{R}^n$ denote $F_\eta(x; z) = \frac{f(x+\eta z) - f(x)}{\eta}$. Denote by T_y the translation operator (i.e., $(T_y g)(x) = g(x + y)$).

Lemma 3.1. *Let $\phi \in C_c^{\infty}(\mathbb{R}^n)$. If f and f'_z are bounded λ-a.e. on supp(ϕ) and $F_\eta(\cdot; z)$ is uniformly bounded λ-a.e. on supp(ϕ) for any sufficiently small $\eta > 0$ (i.e., there are some $C = C(supp(\phi)) > 0$ and $\eta_0 > 0$ s.t. for each $0 < \eta < \eta_0$ we have $\lambda(\{x \in supp(\phi) : |F_\eta(x, z)| > C\}) = 0$. Then*

$$\int f'_z(x)\phi(x) d\lambda(x) = -\int f(x) \partial_z \phi(x) d\lambda(x).$$

Proof. Denote $K = supp(\phi)$ and let $\|\cdot\|_K$ be the $L^\infty(\lambda)$ norm on K. By our assumptions there exist some $C = C(K) > 0$ and $\eta_0 > 0$ s.t. for each $0 < \eta < \eta_0$ $\|F_\eta(\cdot; z)\|_K \leq C$. Thus

$$|F_\eta(x; z)\phi(x)| \leq C|\phi(x)| \tag{3.1}$$

for λ-a.e. $x \in K$ for every sufficiently small $\eta > 0$. As $f'_z(x) = \lim_{\eta \to 0^+} F_\eta(x; z)$ for λ-a.e. $x \in K$ and $C|\phi| \in L^1(K, \lambda)$ we deduce, by applying the dominated convergence theorem, that

$$\int f'_z(x)\phi(x) d\lambda(x) = \int \lim_{\eta \to 0^+} F_\eta(x; z)\phi(x) d\lambda(x) = \lim_{\eta \to 0^+} \int F_\eta(x; z)\phi(x) d\lambda(x). \tag{3.2}$$

By the additivity of the integral and a change of variables $x \mapsto x + \eta z$ we obtain that

$$\int F_\eta(x;z)\phi(x)d\lambda(x) = \int \frac{f(x+\eta z)-f(x)}{\eta}\phi(x)d\lambda(x) = \int f(x)\frac{\phi(x-\eta z)-\phi(x)}{\eta}d\lambda(x), \quad (3.3)$$

and hence

$$\lim_{\eta\to 0^+}\int F_\eta(x;z)\phi(x)d\lambda(x) = \lim_{\eta\to 0^+}\int f(x)\frac{\phi(x-\eta z)-\phi(x)}{\eta}d\lambda(x). \quad (3.4)$$

But as $\phi \in C_c^\infty(\mathbb{R}^n)$ we have $\frac{\phi(x-\eta z)-\phi(x)}{\eta} = -\int_0^1 \partial_z\phi(x-s\eta z)ds$, and hence $\left|\frac{\phi(x-\eta z)-\phi(x)}{\eta}\right| \le \|\partial_z\phi\|_\infty$ for every $x \in \mathbb{R}^n$. Together with our assumption on the a.e. boundedness of f on K, we deduce that for each sufficiently small $\eta > 0$,

$$\left|f(x)\cdot\frac{\phi(x-\eta z)-\phi(x)}{\eta}\right| \le \|f\|_K \|\partial_z\phi\|_\infty < \infty \quad (3.5)$$

for λ-a.e. $x \in K$. Notice that $\lim\limits_{\eta\to 0}\frac{\phi(x-\eta z)-\phi(x)}{\eta} = -\partial_z\phi(x)$ and that for any sufficiently small $\eta > 0$ the integration on the right-hand side of Equation (3.4) is supported on the compact set $K + B(0,\|z\|)$. Thus by applying the dominated convergence theorem to Equation (3.4) and combining that with Equation (3.2), we obtain

$$\int f_z'(x)\phi(x)d\lambda(x) = -\int f(x)\partial_z\phi(x)d\lambda(x). \quad (3.6)$$

□

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $z \in \mathbb{R}^n$. A function $g : \mathbb{R}^n \to \mathbb{R}$ is the *directional distributional derivative of f in the direction z* iff for each $\phi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int f(x)\partial_z\phi(x)d\lambda(x) = -\int g(x)\phi(x)d\lambda(x). \quad (3.7)$$

Thus Lemma 3.1 actually proves that if f is λ-a.e.-bounded on any compact set $K \subset \mathbb{R}^n$, if the directional derivative f_z' in the direction z exists and is bounded λ-a.e. on any compact set $K \subset \mathbb{R}^n$, and if $F_\eta(\cdot;z)$ is uniformly bounded for any sufficiently small $\eta > 0$ on any compact set $K \subset \mathbb{R}^n$, then f has a distributional directional derivative in the direction z, and this distributional directional derivative equals λ-a.e. to f_z'. We shall denote this distributional directional derivative by $\partial_z f$. In the following lemma we demonstrate a helpful property of $\partial_z f$:

Lemma 3.2. *For each $y \in \mathbb{R}^n$ the distributional directional derivative of $T_y f$ in the direction*

$z \in \mathbb{R}^n$ *is* $T_y \circ \partial_z f$.

Proof. We compute

$$\langle \partial_z (T_y \circ f), \phi \rangle = - \langle T_y f, \partial_z \phi \rangle = - \int (T_y f)(x) \partial_z \phi(x) d\lambda(x) = \tag{3.8}$$
$$- \int f(x+y) \partial_z \phi(x) d\lambda(x) = - \int f(x) \partial_z \phi(x-y) d\lambda(x) = \int f'_z(x) \phi(x-y) d\lambda(x) =$$
$$\int f'_z(x+y) \phi(x) d\lambda(x) = \langle T_y \circ \partial_z f, \phi \rangle ,$$

and the lemma follows. □

3.2 A Smooth Approximation with Compact Support of the Measures P_μ

Recall that P_μ is the measure on $AF(\mu)$ whose Fourier transform is $\phi_\mu = \exp(- \|y\|_\mu)$. We shall from now on suppose that μ is of full dimension[4]; hence $AF(\mu) = \mathbb{R}^n$. Denote by ξ_μ the density of P_μ w.r.t. the Lebesgue measure. It is well known that $\xi_\mu \in C_0(\mathbb{R}^n)$ (see [5]). We wish to approximate P_μ by measures Q^ϵ_μ with densities in $C^\infty_c(\mathbb{R}^n)$. Our first step is the following lemma:

Lemma 3.3. *For each* μ *we have* $\xi_\mu \in C^\infty_0(\mathbb{R}^n)$.

Proof. For each multi-index α, the function $g^\alpha_\mu(y) = 2\pi i \cdot y^\alpha \phi_\mu(y)$ is in $L^1(\mathbb{R}^n, \lambda)$. Thus by [3, Theorem 8.22] and the Fourier inversion theorem ([3, Theorem 8.26]) we have

$$\partial^\alpha \xi_\mu = \widehat{g^\alpha_\mu} \in C_0(\mathbb{R}^n), \tag{3.9}$$

and the lemma follows. □

Recall that $B^\epsilon_\mu = \frac{1}{\epsilon} (2\mathcal{R}(\mu) - \mu(I))$. Our next step is to prove that ξ_μ can be approximated in the $L^1(\mathbb{R}^n, \lambda)$ norm by nonnegative functions $\xi^\epsilon_\mu \in C^\infty_c(\mathbb{R}^n)$ that coincide with ξ_μ on some open sets $U^\epsilon_\mu \subset B^\epsilon_\mu$ whose diameter tends to infinity as $\epsilon \to 0^+$ and which satisfy $\xi^\epsilon_\mu \leq \xi_\mu$.

Lemma 3.4. *For any sufficiently small* $\epsilon > 0$ *there are open sets* $U^\epsilon_\mu, V^\epsilon_\mu \subset \mathbb{R}^n$ *with* $\overline{U^\epsilon_\mu} \subset V^\epsilon_\mu \subset B^\epsilon_\mu \subset \mathbb{R}^n$ *and a function* ξ^ϵ_μ *s.t. diam*(U^ϵ_μ), *diam*$(V^\epsilon_\mu) \underset{\epsilon \to^+}{\longrightarrow} \infty$, *supp*$(\xi^\epsilon_\mu) \subset V^\epsilon_\mu$, $\xi_\mu(x) = \xi^\epsilon_\mu(x)$ *for each* $x \in U^\epsilon_\mu$, $\xi^\epsilon_\mu \leq \xi_\mu$, *and* $\left\|\xi_\mu - \xi^\epsilon_\mu\right\|_1 \underset{\epsilon \to 0^+}{\longrightarrow} 0$.

[4]This is assumed in Propositions I and II.

Proof. For sufficiently small $\epsilon > 0$ choose open sets $U_\mu^\epsilon, V_\mu^\epsilon \subset \mathbb{R}^n$ with $\overline{U_\mu^\epsilon} \subset V_\mu^\epsilon \subset B_\mu^\epsilon$ s.t. $d\left(U_\mu^\epsilon, (B_\mu^\epsilon)^c\right), d\left(V_\mu^\epsilon, (B_\mu^\epsilon)^c\right) \in (\sqrt{n}\epsilon, 2\sqrt{n}\epsilon)$. First notice that indeed $diam(U_\mu^\epsilon)$, $diam(V_\mu^\epsilon) \underset{\epsilon\to 0^+}{\longrightarrow} \infty$. Also, by the smooth Urison Lemma ([3, Lemma 8.18]) there is a function $h_\mu^\epsilon \in C_c^\infty(\mathbb{R}^n)$ s.t. $supp(h_\mu^\epsilon) \subset V_\mu^\epsilon$, $0 \le h_\mu^\epsilon \le 1$, and $h_\mu^\epsilon|_{U_\mu^\epsilon} = 1$. Define $\xi_\mu^\epsilon = \xi_\mu h_\mu^\epsilon$. Then $\xi_\mu(x) = \xi_\mu^\epsilon(x)$ for each $x \in U_\mu^\epsilon$, $\xi_\mu^\epsilon \le \xi_\mu$, and

$$\left\|\xi_\mu - \xi_\mu^\epsilon\right\|_1 = \int\limits_{(U_\mu^\epsilon)^c} |(\xi_\mu - \xi_\mu^\epsilon)(x)| d\lambda(x) \le 2 \int\limits_{(U_\mu^\epsilon)^c} |\xi_\mu(x)| d\lambda(x) \underset{\epsilon\to 0^+}{\longrightarrow} 0. \tag{3.10}$$

□

Denote by Q_μ^ϵ the measure on $\mathbb{R}^n$, whose Radon–Nikodym derivative w.r.t. the Lebesgue measure is ξ_μ^ϵ.

Corollary 3.5. $\left\|P_\mu - Q_\mu^\epsilon\right\|_1 \underset{\epsilon\to 0^+}{\longrightarrow} 0.$

3.3 The Measures $\zeta_\delta(f, \mu(S))$

Here we introduce a family of measures that will be used in the proof of our Main Theorem. We begin with a useful lemma that is a analogue to [5, Lemma 4].

Lemma 3.6. *Suppose that μ is of full dimension. Let $\epsilon > 0$, $\omega \in \mathcal{B}(I, C)$, and $0 < \eta \le 1$. Then there exist $\delta_0 = \delta_0(\epsilon, \|\omega\|)$ s.t. for every $0 < \delta \le \delta_0$, and $\gamma \ge \delta\|\omega\|$,*

$$\sup_{x \in B_\mu^\epsilon} \left| \int I_\gamma(t) \frac{f(t\mu(I) + \delta x + \delta\eta\mu(\omega)) - f(t\mu(I) + \delta x)}{\delta\eta} dt \right| \le \|f \circ \mu\| \left(\|\omega_+\| + \|\omega_-\|\right). \tag{3.11}$$

Remark 3.7. It is sufficient to prove the lemma only for $\omega \ge 0$. Let $\omega = \omega_+ - \omega_-$. If $\gamma \ge \delta\|\omega\|$, then $\gamma \ge \delta\|\omega_\pm\|$, and thus by the triangle inequality,

$$\begin{aligned} \left| \int I_\gamma(t) \frac{f(t\mu(I) + \delta x + \delta\eta\mu(\omega)) - f(t\mu(I) + \delta x)}{\delta\eta} dt \right| \le \\ \left| \int I_\gamma(t) \frac{f(t\mu(I) + \delta x + \delta\eta\mu(\omega_+)) - f(t\mu(I) + \delta x)}{\delta\eta} dt \right| + \\ \left| \int I_\gamma(t) \frac{f(t\mu(I) + \delta(x + \eta\mu(\omega)) + \delta\eta\mu(\omega_-)) - f(t\mu(I) + \delta(x + \eta\mu(\omega)))}{\delta\eta} dt \right|. \end{aligned} \tag{3.12}$$

As $B_\mu^\epsilon + \eta\mu(\omega) \subset B_\mu^{\epsilon/(1+\eta\|\mu(\omega)\|)}$, the claim follows.

Furthermore, if we take $\delta' = \frac{\delta}{\epsilon}$, $\omega' = \epsilon\omega$, $x' = \epsilon x$, and $\gamma' = \epsilon\gamma$, it will be sufficient to prove that if $\omega' \geq 0$, then there is some $\delta_0' = \delta_0'(\|\omega'\|) > 0$ s.t, for every $0 < \delta' \leq \delta_0'$,

$$\sup_{x' \in B_\mu} \left| \int I_{\gamma'}(t) \frac{f(t\mu(I) + \delta' x' + \eta\delta'\mu(\omega')) - f(t\mu(I) + \delta' x')}{\delta'\eta} dt \right| \leq \|f \circ \mu\| \, \|\omega'\| \, . \tag{3.13}$$

We return now to the proof of Lemma 3.6.

Proof. Following Remark 3.7, choose $\delta_0(\|\omega\|) > 0$ s.t. for every $3\delta_0 \|\omega\| < t < 1 - 3\delta_0 \|\omega\|$ we have

$$t\mu(I) + \delta_0(1 + \|\mu(\omega)\|_\mu) B_\mu \subset \mathcal{R}(\mu). \tag{3.14}$$

Thus for every $0 < \delta \leq \delta_0$, $\gamma \geq \delta \|\omega\|$, $3\gamma < t < 1 - 3\gamma$, and $0 < \eta \leq 1$, we have

$$t\mu(I) + \delta(1 + \eta \|\mu(\omega)\|_\mu) B_\mu \subset \mathcal{R}(\mu). \tag{3.15}$$

Let $0 < \delta \leq \delta_0$, $\gamma \geq \delta \|\omega\|$, and let M be the smallest integer s.t. $3\gamma + M\delta \|\eta\omega\| > 1 - 3\gamma$. For every $1 \leq m \leq M$ define $I_\delta^m : [0,1] \mapsto \mathbb{R}$ as

$$I^m(t) = 3\gamma' + \mathbb{I}\left([(m-1)\delta\eta \|\omega\|, m\delta\eta \|\omega\|)\right). \tag{3.16}$$

Let $\xi \in B_+^1(I, C)$ be s.t. $2\mu(\xi) - \mu(I) = x$. For every $s \in [0, \delta\eta \|\omega\|)$ and every integer $0 \leq m \leq 2M$ consider the increasing chain of ideal coalitions g_m^s, where $g_0^s = 3\gamma + s + \delta(2\xi - 1)$, $g_{2m+1}^s = g_{2m}^s + \delta\eta\omega$, and $g_{2m}^s = g_0^s + m\delta\eta \|\omega\|$. Then

$$\begin{aligned} &\left| \int I_\gamma(t) \frac{f(t\mu(I) + \delta x + \delta\mu(\eta\omega)) - f(t\mu(I) + \delta x)}{\delta\eta} dt \right| \leq \\ &\frac{1}{\delta\eta} \int_0^1 \left(\sum_{m=1}^M I^m(t) \left| f(t\mu(I) + \delta x + \delta\mu(\eta\omega)) - f(t\mu(I) + \delta x) \right| \right) dt = \\ &\frac{1}{\delta\eta} \int_0^{\delta\eta\|\omega\|} \left(\sum_{m=1}^M \left| f(\mu(g_{2m-1}^t)) - f(\mu(g_{2(m-1)}^t)) \right| \right) dt \leq \\ &\frac{1}{\delta\eta} \int_0^{\delta\eta\|\omega\|} \|f \circ \mu\| \, dt = \|f \circ \mu\| \, \|\omega\| \, . \end{aligned} \tag{3.17}$$

$\delta_0(\|\omega\|)$ is independent of the particular choice of x; therefore, by taking the supremum over B_μ we are done. ∎

Proposition 3.8. *For each game $f \circ \mu \in Q_0$ and every sufficiently small $\delta > 0$, define the function*

$$G_\delta(x) = \begin{cases} \frac{1}{\delta^2} \int_0^1 I_\delta(t) f(t\mu(I) + \delta^2 x) dt, & x \in B_\mu^\delta, \\ 0, & \textit{otherwise}. \end{cases} \tag{3.18}$$

Then, for every sufficiently small $\delta > 0$ and every $S \in \mathcal{C}$, there is a measure $\zeta_\delta(f, \mu(S))$ on $\mathbb{R}^n (= AF(\mu))$ that is uniquely determined by f and $\mu(S)$ s.t. $\int G_\delta(x) \partial_{\mu(S)} \phi(x) d\lambda(x) = -\int \phi(x) d\zeta_\delta(f, \mu(S))$ for every $\phi \in C_c^\infty(\mathbb{R}^n)$.

Proof. Consider the following linear functional on $C_c^\infty(\mathbb{R}^n)$:

$$\Lambda_\delta(f, \mu(S))(\phi) = \int G_\delta(x) \partial_{\mu(S)} \phi(x) d\lambda(x). \tag{3.19}$$

This functional is well defined. As $\frac{\phi(x+\epsilon\mu(S)) - \phi(x)}{\epsilon}$ is bounded for every $x \in \mathbb{R}^n$ by $\left\| \partial_{\mu(S)} \phi \right\|_\infty$ for any sufficiently small $\epsilon > 0$, an application of the dominated convergence theorem yields

$$\Lambda_\delta(f, \mu(S))(\phi) = \lim_{\epsilon \to 0^+} \int G_\delta(x) \frac{\phi(x + \epsilon\mu(S)) - \phi(x)}{\epsilon} \, d\lambda(x). \tag{3.20}$$

Thus, by a change of variable $x \mapsto x + \epsilon\mu(S)$, we obtain

$$\Lambda_\delta(f, \mu(S))(\phi) = -\lim_{\epsilon \to 0^+} \int \frac{G_\delta(x) - G_\delta(x - \epsilon\mu(S))}{\epsilon} \phi(x) \, d\lambda(x). \tag{3.21}$$

Consider $0 < \epsilon \leq 1$. By setting $\omega = -\chi_S$ and $\gamma = \delta$ in Lemma 3.6 and by choosing there $\delta_0(\delta, 1) = \delta$, we obtain that for any small enough $\delta > 0$ we have for every $x \in B_\mu^\delta \setminus \partial B_\mu^\delta$

$$\limsup_{\epsilon \to 0^+} \left| \frac{G_\delta(x) - G_\delta(x - \epsilon\mu(S))}{\epsilon} \right| \leq \| f \circ \mu \| . \tag{3.22}$$

For any $x \in (B_\mu^\delta)^c$ we have $\lim_{\epsilon \to 0^+} \frac{G_\delta(x) - G_\delta(x - \epsilon\mu(S))}{\epsilon} = 0$. Thus as $\lambda(\partial B_\mu^\delta) = 0$ we obtain

$$\left\| \limsup_{\epsilon \to 0^+} \left| \frac{G_\delta(\cdot) - G_\delta(\cdot - \epsilon\mu(S))}{\epsilon} \right| \right\|_\infty \leq \| f \circ \mu \| . \tag{3.23}$$

Hence by applying first Fatou's lemma and then Hölder's inequality to Equation (3.21), we have

$$|\Lambda_\delta(f,\mu(S))(\phi)| \leq \int \limsup_{\epsilon\to 0^+} \left| \frac{G_\delta(x) - G_\delta(x-\epsilon\mu(S))}{\epsilon} \right| |\phi(x)| d\lambda(x) \leq \tag{3.24}$$
$$\left\| \limsup_{\epsilon\to 0^+} \left| \frac{G_\delta(\cdot) - G_\delta(\cdot-\epsilon\mu(S))}{\epsilon} \right| \right\|_\infty \|\phi\|_1 \leq \|f\circ\mu\| \, \|\phi\|_1 .$$

Thus $\Lambda_\delta(f,\mu(S))$ is a bounded linear functional on $C_c^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n,\lambda)$ of norm at most $\|f\circ\mu\|$. As $C_c^\infty(\mathbb{R}^n)$ is a dense subspace of $L^1(\mathbb{R}^n,\lambda)$ (see [3, Proposition 8.17]), it follows that $\Lambda_\delta(f,\mu(S))$ can be uniquely extended to a bounded linear functional $\widetilde{\Lambda}_\delta(f,\mu(S))$ on $L^1(\mathbb{R}^n,\lambda)$ whose norm is at most $\|f\circ\mu\|$. Hence there is a function $H_\delta(f,\mu(S)) \in L^\infty(\mathbb{R}^n,\lambda)$, uniquely determined up to a set of Lebesgue measure 0, that represents $\widetilde{\Lambda}_\delta(f,\mu(S))$, and we may write $\widetilde{\Lambda}_\delta(f,\mu(S))(g) = \int g(x)H_\delta(f,\mu(S))(x)d\lambda(x)$ for each $g \in L^1(\mathbb{R}^n,\lambda)$. Define $d\zeta_\delta(f,\mu(S)) = H_\delta(f,\mu(S))d\lambda$ and we are done. □

Recall that in Section 3.2, the density ξ_μ of P_μ was approximated by some $\xi_\mu^\epsilon \in C_c^\infty(\mathbb{R}^n)$ (see Lemma 3.4) and the measure Q_μ^ϵ was defined as the measure whose Radon–Nikodym density w.r.t the Lebesgue measure is ξ_μ^ϵ. It is well known that $\xi_\mu(x) = \xi_\mu(-x)$. The following helpful lemma gives a similar approximate result for the densities ξ_μ^ϵ and also proves a symmetry result for the distributional derivative of G_δ:

Lemma 3.9. *The following hold for any sufficiently small $\delta > 0$ for every $S \in \mathcal{C}$:*

(i) $|\int \xi_\mu^\epsilon(-x)d\zeta_\delta(f,\mu(S))(x) - \int \xi_\mu^\epsilon(x)d\zeta_\delta(f,\mu(S))(x)| = o(1)$, *as* $\epsilon \to 0^+$.

(ii) $\forall \phi \in C_c^\infty(\mathbb{R}^n)$, $\int G_\delta(-x)\partial_{\mu(S)}\phi(x)d\lambda(x) = \int \phi(-x)d\zeta_\delta(f,\mu(S))(x)$.

Proof. The measure $\zeta_\delta(f,\mu(S))$ is well defined for every sufficiently small $\delta > 0$ for every $S \in \mathcal{C}$.

(i) Recall that $\xi_\mu = \frac{dP_\mu}{d\lambda^n}$ and $\xi_\mu^\epsilon = \xi_\mu h_\mu^\epsilon$ for some $h_\mu^\epsilon \in C_c^\infty(\mathbb{R}^n)$ with values in $[0,1]$ (see Section 3.2). As $dP_\mu(x) = dP_\mu(-x)$, we deduce that $\xi_\mu(-x) = \xi_\mu(x)$, and hence $\xi_\mu^\epsilon(-x) = \xi_\mu(x)h_\mu^\epsilon(-x)$. Denote $g_\mu^\epsilon(x) = \xi_\mu^\epsilon(-x) - \xi_\mu^\epsilon(x) = \xi_\mu(x)(h_\mu^\epsilon(-x) - h_\mu^\epsilon(x))$. Hence, for every sufficiently small $\delta > 0$, we have for every $S \in \mathcal{C}$

$$\left| \int \xi_\mu^\epsilon(-x)d\zeta_\delta(f,\mu(S))(x) - \int \xi_\mu^\epsilon(x)d\zeta_\delta(f,\mu(S))(x) \right| = \tag{3.25}$$
$$\left| \int g_\mu^\epsilon(x)d\zeta_\delta(f,\mu(S))(x) \right| \leq \|f\circ\mu\| \, \|g_\mu^\epsilon\|_1 .$$

We compute

$$\left\|g_\mu^\epsilon\right\|_1 = \int |\xi_\mu(x)(h_\mu^\epsilon(-x) - h_\mu^\epsilon(x)|d\lambda(x) = \tag{3.26}$$

$$\int\limits_{U_\mu^\epsilon} |\xi_\mu(x)(h_\mu^\epsilon(-x) - h_\mu^\epsilon(x)|d\lambda(x) + \int\limits_{(U_\mu^\epsilon)^c} |\xi_\mu(x)(h_\mu^\epsilon(-x) - h_\mu^\epsilon(x)|d\lambda(x) \leq$$

$$2 \int\limits_{(U_\mu^\epsilon)^c} |\xi_\mu| d\lambda(x) = o(1)$$

as $\epsilon \to 0^+$.

(ii) By a change of variable $x \mapsto -x$ we obtain

$$\int G_\delta(-x)\partial_{\mu(S)}\phi(x)d\lambda(x) = \int G_\delta(x)(\partial_{\mu(S)}\phi)(-x)d\lambda(x). \tag{3.27}$$

Now if we set $\psi(x) = \phi(-x)$, then $\partial_{\mu(S)}\psi(x) = -(\partial_{\mu(S)}\phi)(-x)$, and hence by combining Lemma 3.1 with the definition of $\zeta_\delta(f, \mu(S))$ we get that

$$\int G_\delta(x)\partial_{\mu(S)}\phi(-x)d\lambda(x) = \int \phi(-x)d\zeta_\delta(f, \mu(S))(x), \tag{3.28}$$

which proves the lemma.

□

4 Proof of the Main Theorem

We are now ready to prove the Main Theorem. This is done in the following subsections. We will actually do more than prove the theorem; we shall prove "diagonal formula" representations for the Neyman and Mertens values, namely, prove Propositions I and II.

Recall that P_μ is the measure on $\mathbb{R}^n$ with Fourier transform $\exp(-\left\|y\right\|_\mu)$ and density ξ_μ, the measure P_μ^δ is its restriction to B_μ^δ, and the measure Q_μ^δ is an approximate measure to P_μ with density[5] $\xi_\mu^\delta \in C_c^\infty(\mathbb{R}^n)$.

[5] μ is assumed to be of full measure.

4.1 Proof of Proposition I

The main difficulty in the computation and application of the Neyman value is that it lacks a "neat" representation formula. Neyman [5, Lemma 10] proved the following representation formula for the Neyman value for games[6] in $Q_N \cap AC$:

Lemma 4.1. *If $f \circ \mu \in Q_N \cap AC$, then for every $y \in \mathcal{R}(\mu)$ the directional derivative f'_y exists a.e. in the relative interior of $\mathcal{R}(\mu)$, and for every sufficiently small $\delta > 0$ and every coalition $S \in \mathcal{C}$,*

$$\psi^\delta_\mu(f \circ \mu, S) = \int \int I_\delta(t) f'_{\mu(S)}(t\mu(I) + \delta^2 x) dt dP^\delta_\mu(x) \tag{4.1}$$

is well–defined and

$$\left|\psi^\delta_\mu(f \circ \mu, S) - \Psi^\delta_\mu(f \circ \mu, S)\right| \underset{\delta \to 0+}{\longrightarrow} 0. \tag{4.2}$$

Proposition I is a generalization of Lemma 4.1. First, notice that $Q_N \cap AC \subset Q_0$. Now, according to Lemma 4.1, if $f \circ \mu \in AC \cap Q_N$, then

$$\Psi_N(f \circ \mu)(S) = \int \int_0^1 I_\delta(t) f'_{\mu(S)}(t\mu(I) + \delta^2 x) dt dP^\delta_\mu(x) + o(1), \tag{4.3}$$

as $\delta \to 0^+$.

Notice that $d\zeta_\delta(f, \mu(S))(x) = \left(\int_0^1 I_\delta(t) f'_{\mu(S)}(t\mu(I) + \delta^2 x) dt\right) d\lambda(x)$, and that $\left\|P^\delta_\mu - Q^\delta_\mu\right\|_1 = o(1)$, as $\delta \to 0^+$. Thus we obtain

$$\Psi_N(f \circ \mu)(S) = \int \xi^\delta_\mu(x) d\zeta_\delta(f, \mu(S))(x) + o(1), \tag{4.4}$$

as $\delta \to 0^+$.

We shall now return to the proof of Proposition I.

Proof of Proposition I: Recall that T_y is the translation operator by a vector y (see Section 3.1). Recall that

$$\Psi^\delta_\mu(f \circ \mu, S) = \int \int_0^1 I_\delta(t) \frac{f(t\mu(I) + \delta^2(x) + \delta^3 \mu(S)) - f(\mu(I) + \delta^2 x)}{\delta^3} dt dP^\delta_\mu(x). \tag{4.5}$$

[6] AC is the space of games that are absolutely continuous w.r.t. some NA^1 measure.

By the definition of ξ_μ we obtain

$$\Psi_\mu^\delta(f\circ\mu,S)=\int\limits_{B_\mu^\delta}\left(\int\limits_0^1 I_\delta(t)\frac{f(t\mu(I)+\delta^2x+\delta^3\mu(S))-f(t\mu(I)+\delta^2x)}{\delta^3}dt\right)\xi_\mu(x)d\lambda(x)= \tag{4.6}$$

$$\int\limits_{B_\mu^\delta}\left(\int\limits_0^1 I_\delta(t)\frac{f(t\mu(I)+\delta^2x+\delta^3\mu(S))-f(t\mu(I)+\delta^2x)}{\delta^3}dt\right)\xi_\mu^\delta(x)d\lambda(x)+$$

$$\int\limits_{B_\mu^\delta}\left(\int\limits_0^1 I_\delta(t)\frac{f(t\mu(I)+\delta^2x+\delta^3\mu(S))-f(t\mu(I)+\delta^2x)}{\delta^3}dt\right)(\xi_\mu(x)-\xi_\mu^\delta(x))d\lambda(x)=$$

$$\int\left(\int\limits_0^1 I_\delta(t)\frac{f(t\mu(I)+\delta^2x+\delta^3\mu(S))-f(t\mu(I)+\delta^2x)}{\delta^3}dt\right)\xi_\mu^\delta(x)d\lambda(x)+ \tag{4.7}$$

$$\int\limits_{B_\mu^\delta}\left(\int\limits_0^1 I_\delta(t)\frac{f(t\mu(I)+\delta^2x+\delta^3\mu(S))-f(t\mu(I)+\delta^2x)}{\delta^3}dt\right)(\xi_\mu(x)-\xi_\mu^\delta(x))d\lambda(x). \tag{4.8}$$

In line (4.7) above no confusion should result from the omission of B_μ^δ as the integrand is supported on an open subset of B_μ^δ.

Consider now the summand in line (4.8). Recall that for a compact set $K\subset\mathbb{R}^n$ we denoted by $\|\cdot\|_K$ the $L^\infty(\lambda)$ norm on K (see the proof of Lemma 3.1). By applying first Hölder's inequality and then Lemma 3.6, and keeping in mind that $\xi_\mu(x)=\xi_\mu^\delta(x)$ for every $x\in U_\mu^\delta$ and that $0\le\xi_\mu^\delta\le\xi_\mu$, we obtain

$$\left|\int\limits_{B_\mu^\delta}\left(\int\limits_0^1 I_\delta(t)\frac{f(t\mu(I)+\delta^2x+\delta^3\mu(S))-f(t\mu(I)+\delta^2x)}{\delta^3}dt\right)(\xi_\mu(x)-\xi_\mu^\delta(x))d\lambda(x)\right|\le \tag{4.9}$$

$$\|F_{f,\mu}(\delta,\cdot,S)\|_{B_\mu^\delta}\int\limits_{B_\mu^\delta}|\xi_\mu(x)-\xi_\mu^\delta|d\lambda(x)\le\|f\circ\mu\|\int\limits_{B_\mu^\delta}|\xi_\mu(x)-\xi_\mu^\delta(x)|d\lambda(x)=$$

$$\|f\circ\mu\|\int\limits_{B_\mu^\delta\setminus U_\mu^\delta}|\xi_\mu(x)-\xi_\mu^\delta(x)|d\lambda(x)\le||f\circ\mu||\cdot\int\limits_{(U_\mu^\delta)^c}|\xi_\mu(x)-\xi_\mu^\delta(x)|d\lambda(x)\le$$

$$2\|f\circ\mu\|\int\limits_{(U_\mu^\delta)^c}|\xi_\mu(x)|d\lambda(x)=o(1)$$

as $\delta \to 0^+$. Thus

$$\Psi_\mu^\delta(f \circ \mu, S) = \tag{4.10}$$
$$\int \left(\int_0^1 I_\delta(t) \frac{f(t\mu(I) + \delta^2 x + \delta^3 \mu(S)) - f(t\mu(I) + \delta^2 x)}{\delta^3} dt \right) \xi_\mu^\delta(x) d\lambda(x) + o(1)$$

as $\delta \to 0^+$.

Recall that there is an open set $V_\mu^\delta \subset B_\mu^\delta$ s.t. $supp(\xi_\mu^\delta) \subset V_\mu^\delta$ and $diam(V_\mu^\delta) \underset{\delta\to 0^+}{\longrightarrow} \infty$ (see Lemma 3.4). Notice that[7] for each $x \in V_\mu^\delta$ we have $x + \delta\mu(S) \subset B_\mu^\delta$. Thus for each $x \in supp(\xi_\mu^\delta)$ we obtain by the definition of the function G_δ (see Proposition 3.8)

$$\int_0^1 I_\delta(t) \frac{f(t\mu(I) + \delta^2 x + \delta^3 \mu(S)) - f(t\mu(I) + \delta^2 x)}{\delta^3} dt = \frac{G_\delta(x + \delta\mu(S)) - G_\delta(x)}{\delta}; \tag{4.11}$$

hence

$$\Psi_\mu^\delta(f \circ \mu, S) = \int \left(\frac{G_\delta(x + \delta\mu(S)) - G_\delta(x)}{\delta} \right) \xi_\mu^\delta(x) d\lambda(x) + o(1), \tag{4.12}$$

as $\delta \to 0^+$. By a change of variable $x \mapsto x + \delta\mu(S)$ we obtain

$$\begin{aligned} \Psi_\mu^\delta(f \circ \mu, S) &= \int G_\delta(x) \frac{\xi_\mu^\delta(x - \delta\mu(S)) - \xi_\mu^\delta(x)}{\delta} d\lambda(x) + o(1) = \\ &- \int G_\delta(x) \left(\int_0^1 \partial_{\mu(S)} \xi_\mu^\delta(x - s\delta\mu(S)) ds \right) d\lambda(x) + o(1) = \\ &- \int \int_0^1 G_\delta(x) \partial_{\mu(S)} \xi_\mu^\delta(x - s\delta\mu(S)) ds d\lambda(x) + o(1) \end{aligned} \tag{4.13}$$

[7]By the choice of V_μ^δ we have $d\left(V_\mu^\delta, (B_\mu^\delta)^c\right) > \sqrt{n}\delta$

as $\delta \to 0^+$. By applying first Fubini's theorem and then Proposition 3.8 we obtain

$$\Psi_\mu^\delta(f \circ \mu, S) = \int_0^1 \left(-\int G_\delta(x) \partial_{\mu(S)} \xi_\mu^\delta(x - s\delta\mu(S)) d\lambda(x) \right) ds + o(1) = \tag{4.14}$$

$$\int_0^1 \int \xi_\mu^\delta(x - s\delta\mu(S)) d\zeta_\delta(f, \mu(S))(x) ds + o(1)$$

as $\delta \to 0^+$. As $\sup\limits_{s\in[0,1]} \left\|\xi_\mu^\delta - T_{-s\delta\mu(S)} \circ \xi_\mu^\delta\right\|_1 \underset{\delta\to 0}{\longrightarrow} 0$ and as, by Proposition 3.8, the measure $\zeta_\delta(f, \mu(S))$ induces a bounded linear functional[8] on $L^1(\mathbb{R}^n, \lambda)$ we have

$$|\int_0^1 \int \left(T_{-s\delta\mu(S)} \xi_\mu^\delta(x) - \xi_\mu^\delta(x)\right) d\zeta_\delta(f, \mu(S))(x) ds| \leq \tag{4.15}$$

$$\int_0^1 \left| \int \left(T_{-s\delta\mu(S)} \xi_\mu^\delta(x) - \xi_\mu^\delta(x)\right) d\zeta_\delta(f, \mu(S))(x) \right| ds \leq$$

$$\|f \circ \mu\| \sup_{s\in[0,1]} \left\|\xi_\mu^\delta - T_{-s\delta\mu(S)} \circ \xi_\mu^\delta\right\|_1 \underset{\delta\to 0}{\longrightarrow} 0.$$

By combining Equations (4.14)–(4.15) we obtain

$$\Psi_\mu^\delta(f \circ \mu, S) = \int \xi_\mu^\delta(x) d\zeta_\delta(f, \mu(S))(x) + o(1) \tag{4.16}$$

as $\delta \to 0^+$, which proves Proposition I.

4.2 Proof of Proposition II

Recall that the Mertens value of a game with an extension v is given by (see Section 2.3):

$$\Psi_M(v)(S) = \int \left[\lim_{\tau \to 0^+} \int_0^1 \frac{\bar{v}(t + \tau\chi) - \bar{v}(t - \tau\chi)}{2\tau} dt \right]_S dP(\chi). \tag{4.17}$$

The following proposition will give us a helpful representation for the Mertens value of a vector measure game $f \circ \mu \in Q_0$. It was already proved by Mertens [4] that if $v = f \circ \mu$, then for every

[8] Of norm $\leq \|f \circ \mu\|$.

$S \in \mathcal{C}$,

$$\Psi_M(v)(S) = \int \left[\lim_{\tau \to 0^+} \int_0^1 \frac{f(\mu(t)+\tau x) - f(\mu(t)-\tau x)}{2\tau} dt \right]'_{\mu(S)} dP_\mu(x). \tag{4.18}$$

Our proposition offers a helpful variation of this result. The methods of the proof are quite standard and the idea stems from the proof in [4].

Proposition 4.2. *For every $f \circ \mu \in Q_0$,*

$$\Psi_M(v)(S) = \int \left[\lim_{\tau \to 0^+} \int_0^1 I_{\sqrt{\tau}}(t) \frac{f(\mu(t)+\tau x) - f(\mu(t)-\tau x)}{2\tau} dt \right]'_{\mu(S)} dP_\mu(x). \tag{4.19}$$

Proof. For each $\chi \neq 0$ in $B(I, \mathcal{C})$, every $0 < \tau < 9 \|\chi\|^{-2}$, and every $t \in (3\sqrt{\tau}, 1-3\sqrt{\tau})$ we have $0 < t \pm \tau\chi < 1$; thus $\max\{0, \min\{1, t \pm \tau\chi\}\} = t \pm \tau\chi$. As for every $\chi \in B_1^+(I, \mathcal{C})$ we have $\overline{v}(\chi) = \frac{1}{2}(f(\mu(\chi)) - f(\mu(1-\chi)) + f(\mu(1)))$ we obtain

$$\Psi_D(v)(\chi) = \lim_{\tau \to 0^+} \Big(\int_0^1 I_{\sqrt{\tau}}(t) \frac{f(\mu(t)+\tau\mu(\chi)) - f(\mu(t)-\tau\mu(\chi))}{4\tau} dt + \tag{4.20}$$

$$\int_0^1 I_{\sqrt{\tau}}(t) \frac{f(\mu(1-t)-\tau\mu(\chi)) - f(\mu(1-t)+\tau\mu(\chi))}{4\tau} dt + \tag{4.21}$$

$$\int_0^{3\sqrt{\tau}} \frac{\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)}{2\tau} dt + \int_{1-3\sqrt{\tau}}^1 \frac{\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)}{2\tau} dt \Big).$$

By a change of variable $t \mapsto (1-t)$ in line (4.21) we obtain

$$\Psi_D(v)(\chi) = \lim_{\tau \to 0^+} \Big(\int_0^1 I_{\sqrt{\tau}}(t) \frac{f(\mu(t)+\tau\mu(\chi)) - f(\mu(t)-\tau\mu(\chi))}{2\tau} dt + \tag{4.22}$$

$$\int_0^{3\sqrt{\tau}} \frac{\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)}{2\tau} dt + \int_{1-3\sqrt{\tau}}^1 \frac{\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)}{2\tau} dt \Big). \tag{4.23}$$

Notice that it is sufficient to prove that the sum in line (4.23) diminishes to 0 as $\tau \to 0^+$ for each χ; if this is true then the substitution $x = \mu(\chi)$ proves the lemma. Denote this sum by

$G_\tau(\chi)$. Then by a change of variable $t \mapsto 1-t$ in the second summand we obtain

$$G_\tau(\chi) = \int\limits_0^{3\sqrt{\tau}} \frac{\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)}{2\tau} dt + \int_0^{3\sqrt{\tau}} \frac{\bar{v}(1-(t-\tau\chi)) - \bar{v}((1-(t+\tau\chi))}{2\tau} dt = \tag{4.24}$$
$$\int\limits_0^{3\sqrt{\tau}} \frac{\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)}{\tau} dt.$$

Let $\chi = \chi_+ - \chi_-$ where $\chi_\pm \geq 0$. Then

$$G_\tau(\chi) = \frac{1}{\tau} \int\limits_0^{3\sqrt{\tau}} [(\bar{v}(t+\tau\chi) - \bar{v}(t+\tau\chi_+)) + (\bar{v}(t+\tau\chi_+) - \bar{v}(t)) + \tag{4.25}$$
$$(\bar{v}(t) - \bar{v}(t-\tau\chi_-)) + (\bar{v}(t-\tau\chi_-) - \bar{v}(t-\tau\chi)]\, dt.$$

Consider the first summand in Equation (4.25). Let $m(\tau)$ be the minimal integer s.t. $m\tau \|\chi\| \geq 3\sqrt{\tau}$. Then

$$\left| \frac{1}{\tau} \int\limits_0^{3\sqrt{\tau}} (\bar{v}(t+\tau\chi) - \bar{v}(t+\tau\chi_+))\, dt \right| \leq \frac{1}{\tau} \sum_{i=0}^{m(\tau)-1} \int\limits_{i\tau||\chi||}^{(i+1)\tau||\chi||} |\bar{v}(t+\tau\chi) - \bar{v}(t+\tau\chi_+)|\, dt = \tag{4.26}$$
$$\frac{1}{\tau} \int\limits_0^{\tau||\chi||} \left(\sum_{i=0}^{m(\tau)-1} |\bar{v}(t+i\tau||\chi||+\tau\chi) - \bar{v}(t+i\tau||\chi||+\tau\chi_+)| \right) dt.$$

Denote by $V(\tau,\chi)$ the supremum of the variation of $\bar{v}$ along all finite chains $\Omega: \ g_0 \leq g_1 \leq ... \leq g_{2m-2} \leq g_{2m-1} \leq 3\sqrt{\tau}$ s.t.:

(a) $g_{2i} = t_i + \gamma_i\chi$, and

(b) $g_{2i+1} = g_{2i} + \gamma_i\chi_-$, where for every $0 \leq i \leq m-1$

  (i) $0 < \gamma_i \leq \gamma_{i+1} \leq \tau$, and

  (ii) $t_{i+1} - t_i \geq \gamma_{i+1} \|\chi\|$.

Thus,

$$\left| \frac{1}{\tau} \int\limits_0^{3\sqrt{\tau}} (\bar{v}(t+\tau\chi) - \bar{v}(t+\tau\chi_+))\, dt \right| \leq V(\tau,\chi). \tag{4.27}$$

We shall prove that $\lim_{\tau\to 0^+} V(\tau,\chi)=0$. Suppose, by contradiction, that we find some $c>0$ and a decreasing sequence of positive integers $\tau_n \searrow 0^+$ s.t $V(\tau_n,\chi)\geq c$ for each $n\in\mathbb{N}$. Choose $n_1=1$ and given $\tau_{n_1}=\tau_1$ choose an increasing chain $\Omega_1:\ g_0^1\leq g_1^1\leq ...\leq g_{2m_1-2}^1\leq g_{2m_1-1}^1\leq 3\sqrt{\tau_{n_1}}$ of the form $g_{2i}^1=t_i^1+\gamma_i^1\chi$, $g_{2i+1}^1=g_{2i}^1+\gamma_i^1\chi_-$, where t_i^1 and γ_i^1 satisfy the conditions (i) and (ii) above for each $1\leq i\leq m_1$, and s.t. the variation V_1 of $\bar{v}$ along this chain is at least $\frac{c}{2}$. Notice that $g_3^1>t_1>0$; thus we may choose $n_2>n_1$ s.t. $3\sqrt{\tau_{n_2}}\leq t_1<g_3^1$. Continue in this manner to choose an increasing sequence of integers $\{n_k\}_{k=1}^\infty$ s.t. for each $k\geq 1$ there is an increasing chain $\Omega_k:\ g_0^k\leq g_1^k\leq ....\leq g_{2m_k-1}^k\leq 3\sqrt{\tau_{n_k}}$ of the form $g_{2i}^k=t_i^k+\gamma_i^k\chi$, $g_{2i+1}^k=g_{2i}^k+\gamma_i^k\chi_-$, where t_i^k and γ_i^k satisfy conditions (i) and (ii) above for each $1\leq i\leq m_k$, the variation V_k of $\bar{v}$ along this chain is at least $\frac{c}{2}$, and $g_3^k\geq 3\sqrt{\tau_{n_{k+1}}}$. Consider now the variation V_k' of $\bar{v}$ along the increasing chain $\Omega_k':\ 3\sqrt{\tau_{n_{k+1}}}\leq g_3^k\leq g_4^k\leq ...\leq g_{2m_k-1}^k\leq 3\sqrt{\tau_{n_k}}$. Then $\lim_{k\to\infty} V_k'=0$, as otherwise we may use the sequence of increasing chains $\{\Omega_k\}_{k=1}^\infty$ to construct[9] a sequence of increasingly long chains along which the variation of $\bar{v}$ is unbounded, which yields a contradiction. Thus

$$\liminf_{k\to\infty}\sum_{i=1}^{3}|\bar{v}(g_i^k)-\bar{v}(g_{i-1}^k)|=\liminf_{k\to\infty}(V_k-V_k')\geq\frac{c}{2}>0, \tag{4.28}$$

which contradicts the continuity of $\bar{v}$ at 0. Applying the same reasoning to the rest of the summands that constitute $G_\tau(\chi)$ yields

$$\lim_{\tau\to 0^+} G_\tau(\chi)=0, \tag{4.29}$$

which proves the proposition. □

We are now ready to begin the proof of Proposition II. We start with the following lemma, a version of a lemma from [4] which we specialize to suit our own needs. The proof of the specialized version is a straightforward corollary of the proof given in [4] together with lemma 4.2. Nevertheless, we give the proof here for the benefit of the reader.

Lemma 4.3. *For every $f\circ\mu\in Q_0$ and λ-a.e. $x\in\mathbb{R}^n$,*

$$\left|\left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)}\right|\leq\|f\circ\mu\|\,. \tag{4.30}$$

Proof. By [4, Theorem 2] $\left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)}$ exists for λ-a.e. $x\in\mathbb{R}^n$ (as

[9]By simply concatenating them.

$P_\mu \ll \lambda$). Let $x \in \mathbb{R}^n$ s.t. the directional derivative above exists and $0 < \eta \leq 1$. Then by the triangle inequality we obtain for any sufficiently small $\tau > 0$

$$\frac{1}{2\tau\eta}\left|\int_0^1 I_{\sqrt{\tau}}(t)(f(\mu(t)+\tau x+\eta\tau\mu(S))-f(\mu(t)-\tau x-\eta\tau\mu(S))-f(\mu(t)+\tau x)+f(\mu(t)-\tau x))dt\right| \leq \tag{4.31}$$

$$\frac{1}{2\tau\eta}\left|\int_0^1 I_{\sqrt{\tau}}(t)\left(f(\mu(t)+\tau x+\eta\tau\mu(S))-f(\mu(t)+\tau x)\right)dt\right| +$$
$$\frac{1}{2\tau\eta}\left|\int_0^1 I_{\sqrt{\tau}}(t)\left(f(\mu(t)-\tau x-\eta\tau\mu(S))-f(\mu(t)-\tau x)\right)dt\right|.$$

By setting $\epsilon = \min\{\frac{1}{1+2||x||_\mu}, \frac{1}{1+2||x\pm\eta\mu(S)||_\mu}\}$ in lemma 3.6 we obtain

$$\frac{1}{2\tau\eta}\left|\int_0^1 I_{\sqrt{\tau}}(t)(f(\mu(t)+\tau x+\eta\tau\mu(S))-f(\mu(t)-\tau x-\eta\tau\mu(S))-f(\mu(t)+\tau x)+f(\mu(t)-\tau x))dt\right| \tag{4.32}$$

$$\leq \|f\circ\mu\|$$

for any sufficiently small $\tau > 0$, which may be chosen independently of η. Hence, taking first $\tau \to 0^+$ and then $\eta \to 0^+$ proves the lemma. □

Recall that the density ξ_μ of P_μ was approximated by functions $\xi_\mu^\epsilon \in C_c^\infty(\mathbb{R}^n)$ and that Q_μ^ϵ was the measure whose density is ξ_μ^ϵ (see Section 3.2). Let $\epsilon > 0$. The following two lemmata are crucial to the proof of our proposition. Here we replace P_μ in proposition 4.2 by its approximated measure Q_μ^ϵ and prove that the error term diminishes to 0 as $\epsilon \to 0^+$:

Lemma 4.4. *For every game $f\circ\mu \in Q_0$ and every coalition $S \in \mathcal{C}$*

$$\Psi_M(f\circ\mu)(S) = \int \left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)} dQ_\mu^\epsilon(x) + o(1) \tag{4.33}$$

as $\epsilon \to 0^+$

Proof. Recall that by proposition 4.2 we have

$$\Psi_M(f\circ\mu)(S) = \int \left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)} dP_\mu(x). \tag{4.34}$$

By lemma 4.3 we have $\left|\left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)}\right| \le \|f\circ\mu\|$ for λ-a.e. $x\in\mathbb{R}^n$, and by Corollary 3.5 $\left\|P_\mu - Q_\mu^\epsilon\right\|_1 = o(1)$ as $\epsilon\to 0^+$. Hence, as $P_\mu, Q_\mu^\epsilon \ll \lambda$, we obtain

$$\Psi_M(f\circ\mu)(S) = \int \left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)} dP_\mu(x) = \tag{4.35}$$

$$\int \left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)} dQ_\mu^\epsilon(x)+$$

$$\int \left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)} (dP_\mu(x)-dQ_\mu^\epsilon(x)) =$$

$$\int \left[\lim_{\tau\to 0^+}\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right]'_{\mu(S)} dQ_\mu^\epsilon(x) + o(1)$$

as $\epsilon\to 0^+$. ☐

Lemma 4.5. *For every $\epsilon > 0$ there is some $V(f\circ\mu,\epsilon) > 0$ s.t.*

$$\sup_{x\in B_\mu^\epsilon}\left|\lim_{\tau\to 0^+}\frac{1}{2\tau}\int_0^1 I_{\sqrt{\tau}}(t)(f(\mu(t)+\tau x)-f(\mu(t)-\tau x))dt\right| \le V(f\circ\mu,\epsilon). \tag{4.36}$$

Proof. Let $x\in B_\mu^\epsilon$ and let $\omega\in B(I,C)$ be s.t. $\mu(\omega)=2x$ and $\|\omega\|\le\frac{2}{\epsilon}$. Then

$$\left|\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau x)-f(\mu(t)-\tau x)}{2\tau}dt\right| = \tag{4.37}$$

$$\frac{1}{2}\left|\int_0^1 I_{\sqrt{\tau}}(t)\frac{f(\mu(t)+\tau(-x)+\tau\mu(\omega))-f(\mu(t)+\tau(-x))}{\tau}dt\right| \overset{1}{\le}$$

$$\frac{1}{2}\|f\circ\mu\|\left(\|\omega_+\|+\|\omega_-\|\right) \le \|f\circ\mu\|\,\|\omega\| \le \frac{2\,\|f\circ\mu\|}{\epsilon} \doteq V(f\circ\mu,\epsilon),$$

where inequality (1) above follows for any sufficiently small τ (whose choice depends on $f \circ \mu$, $\|w\|$, and ϵ) by applying lemma 3.6 (with $\eta = 1$). By taking $\tau \to 0^+$ and then the supremum we are done. $\square$

Proof of proposition II: Let $\epsilon > 0$. By lemma 4.4 we have

$$\Psi_M(f \circ \mu)(S) = \int \left[\lim_{\tau \to 0^+} \int_0^1 I_{\sqrt{\tau}}(t) \frac{f(\mu(t) + \tau x) - f(\mu(t) - \tau x)}{2\tau} dt \right]'_{\mu(S)} dQ^\epsilon_\mu(x) + o(1) = \tag{4.38}$$

$$\int \left[\lim_{\tau \to 0^+} \int_0^1 I_{\sqrt{\tau}}(t) \frac{f(\mu(t) + \tau x) - f(\mu(t) - \tau x)}{2\tau} dt \right]'_{\mu(S)} \xi^\epsilon_\mu(x) d\lambda(x) + o(1)$$

as $\epsilon \to 0^+$. For convenience, let $F(\tau, x) = \int_0^1 I_{\sqrt{\tau}}(t) \frac{f(t\mu(I) + \tau x) - f(t\mu(I) - \tau x)}{2\tau} dt$ and $F(x) = \lim_{\tau \to 0^+} F(\tau, x)$. By lemma 4.3 we have

$$\left| F'_{\mu(S)}(x) \right| \leq \|f \circ \mu\| \tag{4.39}$$

for λ-a.e. $x \in B^\epsilon_\mu$. By lemma 4.5 we have

$$|F(x)| \leq V(f \circ \mu, \epsilon) \tag{4.40}$$

for λ-a.e. $x \in B^\epsilon_\mu$. By equation (4.32) we have

$$\sup_{x \in B^\epsilon_\mu} \left| \frac{F(x + \eta\mu(S)) - F(x)}{\eta} \right| \leq 2 \|f \circ \mu\| \tag{4.41}$$

for every sufficiently small $\eta > 0$. Thus F and $F'_{\mu(S)}$ are bounded λ-a.e. on $supp(\xi^\epsilon_\mu) \subset B^\epsilon_\mu$ and $\frac{F(x+\eta\mu(S)) - F(x)}{\eta}$ is uniformly bounded λ-a.e. for any sufficiently small $\eta > 0$ on $supp(\xi^\epsilon_\mu) \subset B^\epsilon_\mu$. Hence by the distributional calculus lemma 3.1 we obtain

$$\Psi_M(f \circ \mu)(S) = \int F'_{\mu(S)}(x) \xi^\epsilon_\mu(x) d\lambda(x) + o(1) = \tag{4.42}$$

$$- \int F(x) \partial_{\mu(S)} \xi^\epsilon_\mu(x) d\lambda(x) + o(1)$$

as $\epsilon \to 0^+$.

Using first the $\|\cdot\|_\infty$ boundedness of $\partial_{\mu(S)}\xi_\mu^\epsilon$ and then applying lemma 4.5 we obtain

$$\left|F(\tau,x)\partial_{\mu(S)}\xi_\mu^\epsilon(x)\right| \leq |F(\tau,x)| \cdot ||\partial_{\mu(S)}\xi_\mu^\epsilon||_\infty \leq 2V(f\circ\mu,\epsilon)\cdot \left\|\partial_{\mu(S)}\xi_\mu^\epsilon\right\|_\infty \tag{4.43}$$

for any small enough τ, say $\tau < \epsilon^2$, for every x. Thus, by applying the dominated convergence theorem we obtain

$$\Psi_M(f\circ\mu)(S) = -\int F(x)\partial_{\mu(S)}\xi_\mu^\epsilon(x)d\lambda(x) + o(1) = \tag{4.44}$$
$$-\lim_{\tau\to 0^+}\int F(\tau,x)\partial_{\mu(S)}\xi_\mu^\epsilon(x)d\lambda(x) + o(1)$$

as $\epsilon \to 0^+$. As the integration is supported on B_μ^ϵ and we assume that $\tau < \epsilon^2$, then by the definition of $G_{\sqrt{\tau}}$ (see proposition 3.8) we have

$$F(\tau,x) = G_{\sqrt{\tau}}(x) - G_{\sqrt{\tau}}(-x). \tag{4.45}$$

Thus

$$\Psi_M(f\circ\mu)(S) = \tag{4.46}$$
$$\lim_{\tau\to 0^+}\frac{1}{2}\left[\int G_{\sqrt{\tau}}(-x)\cdot\partial_{\mu(S)}\xi_\mu^\epsilon(x)d\lambda(x) - \int G_{\sqrt{\tau}}(x)\cdot\partial_{\mu(S)}\xi_\mu^\epsilon(x)d\lambda(x)\right] + o(1)$$

as $\epsilon \to 0^+$.

Set $\delta = \sqrt{\tau}$. By applying part (ii) of lemma 3.9 to the first summand in Equation (4.46) and proposition 3.8 to the second summand in Equation (4.46) we obtain

$$\Psi_M(f\circ\mu)(S) = \lim_{\delta\to 0^+}\frac{1}{2}\left[\int \xi_\mu^\epsilon(x)d\zeta_\delta(f,\mu(S))(x) + \int \xi_\mu^\epsilon(-x)d\zeta_\delta(f,\mu(S))(x)\right] + o(1) \tag{4.47}$$

as $\epsilon \to 0^+$. Thus by part (i) of lemma 3.9,

$$\Psi_M(f\circ\mu)(S) = \lim_{\delta\to 0^+}\int \xi_\mu^\epsilon(x)d\zeta_\delta(f,\mu(S))(x) + o(1) \tag{4.48}$$

as $\epsilon \to 0^+$.

But

$$\lim_{\delta\to 0^+}\left|\int(\xi_\mu^\delta(x)-\xi_\mu^\epsilon(x))d\zeta_\delta(f,\mu(S))(x)\right| \le \lim_{\delta\to 0^+}\|f\circ\mu\|\left\|\xi_\mu^\delta-\xi_\mu^\epsilon\right\|_1 = \tag{4.49}$$
$$\|f\circ\mu\|\left\|\xi_\mu-\xi_\mu^\epsilon\right\|_1 = o(1)$$

as $\epsilon \to 0^+$. Therefore

$$\Psi_M(f\circ\mu)(S) = \lim_{\delta\to 0^+}\int \xi_\mu^\delta(x)d\zeta_\delta(f,\mu(S))(x) + o(1) \tag{4.50}$$

as $\epsilon \to 0^+$, which proves proposition II, and our Main Theorem follows.

References

[1] R.J. Aumann and S. Hart. *Handbook of Game Theory, with Economic Applications, Vol. III,*. North-Holland, 2002.

[2] R.J. Aumann and L.S. Shapley. *Values of Non-Atomic Games*. Princeton University Press, 1974.

[3] G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Weily-Interscience, 1999.

[4] J.F. Mertens. The Shapley value in the non-differentiable case. *International Journal of Game Theory*, 17:1–65, 1988.

[5] A. Neyman. Values of non-atomic vector measure games. *Israel J. Math*, 124:1–27, 1988.

[6] A. Neyman and R. Smorodinsky. Asymptotic values of vector measure games. *Mathematics of Operations Research*, 29, 2004.

[7] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, pages 307–371. Princeton University Press, 1953.

[8] Y. Tauman. Value on a class of non-differentiable market games. *International Journal of Game Theory*, 10:155–162, 1981.