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AN ECONOMIC INDEX OF RELATIVE RISKINESS

By

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An Economic Index of Relative Riskiness

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Abstract

In their seminal works, Arrow (1965) and Pratt (1964) defined two aspects of risk aversion: absolute risk aversion and relative risk aversion. Based on their definitions, we define two aspects of risk: absolute risk and relative risk. We consider situations in which, by making an investment, an agent exchanges a certain amount of wealth w by a random distributed level of wealth $\widetilde{w}$. In such situations, we define *absolute risk* as the riskiness of a gamble that is distributed as $\widetilde{w}-w$, and *relative risk* as the riskiness of a security that is distributed as $\widetilde{w}/w$. We measure absolute risk by the Aumann and Serrano (2008) index of riskiness and relative risk by an equivalent index that we develop in this paper. The two concepts of risk do not necessarily agree on which one of two investments is riskier, and hence they capture two different aspects of risk.

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1 Introduction

In many situations of decision making under risk, individuals take a decision in relation to some risky assets. We distinguish between two types of risky assets: assets whose returns are absolute ("gambles") and assets whose returns are relative ("securities"). To clarify this distinction, note that accepting a gamble g at initial wealth w, causes the wealth to be distributed as $w + g$, and investing w in a security r causes the wealth to be distributed as wr. We call the riskiness of gambles *absolute riskiness* and that of securities *relative riskiness*.

Measuring risks has been widely discussed in the literature. In general, the goal of this research is to find rules that can help decision makers determine which one of two given investments is riskier. We identify two different approaches to measuring risks. The first approach focuses on the final distribution of wealth. According to this approach, given two risky assets, s_1 and s_2, we compare the distribution of wealth after investing in s_1 ($\widetilde{w}_1$) with the distribution of wealth after investing in s_2 ($\widetilde{w}_2$). The literature that developed the concept of stochastic dominance[1] took this approach. It showed that if $\widetilde{w}_1$ stochastically dominates $\widetilde{w}_2$, then any individual who is averse to risk will prefer $\widetilde{w}_1$ to $\widetilde{w}_2$. The second approach compares the risky assets themselves. For instance, the beta of the CAPM[2] enables investors to compare any two securities. The value of the beta depends only on the distributions of the available securities, not on the wealth of individuals.

In recent years, Aumann and Serrano (2008) extended this approach to gambles. They present an economic index of riskiness of gambles that is based on the Arrow–Pratt concept of absolute risk aversion. The riskiness of a gamble is defined as the reciprocal of the parameter of an individual who has a constant absolute risk aversion (CARA) utility function and is indifferent between taking the gamble or rejecting it. The concept of riskiness of Aumann–Serrano has an economic interpretation, it reflecting the idea that "risk is what risk averters hate".

In this paper we apply the idea of Aumann and Serrano (2008) to securities. We define the riskiness of a security as the reciprocal of (-1 plus) the parameter of an individual who has a constant relative risk aversion (CRRA)

[1]See e.g., Hadar and Russell (1969); Levy and Hanoch (1969); Rothschild and Stiglitz (1970).

[2]The capital asset pricing model (CAPM) was introduced by Sharpe (1964) and Lintner (1965).

utility function and is indifferent between investing w in the security or rejecting it. Since the Aumann–Serrano index is defined on gambles, we call their concept of riskiness *absolute riskiness*. In contrast, we call our concept of riskiness *relative riskiness*.

The existence of two different concepts of riskiness leads us to a third approach to measuring the risk that arises from making any investment. According to this approach, the risk of an investment depends on two parameters: the "status quo", i.e., the initial wealth level w, and the randomly distributed wealth $\widetilde{w}$ that is the result of making the investment. Obviously, the risky asset that causes the exchange of w by $\widetilde{w}$ could be either a gamble, distributed as $\widetilde{w} - w$, or a security, distributed as $\widetilde{w}/w$ (where the investor invests w in this security). We measure the absolute risk of the investment by the absolute riskiness of the gamble $\widetilde{w} - w$ and the relative riskiness of the investment by the relative riskiness of the security $\widetilde{w}/w$. The absolute risk (AR) of the investment and the relative risk (RR) of the investment refer to the Arrow–Pratt concepts of absolute risk aversion (ARA) and relative risk aversion (RRA), respectively. The two concepts of risk, AR and RR, do not always agree on which one of two investments is riskier and hence, as we show later in the paper, they capture two different aspects of risk.

Most of the paper is devoted to studying the properties of our new index of relative riskiness. One of its important properties is that it extends the well-known (second-degree) *stochastic-dominance order*. For any two risky assets s_1 and s_2, s_1 (second-degree) stochastically dominates s_2 if and only if all risk-averse decision makers prefer s_1 to s_2. This is correct for both, gambles and securities. Hence, if s_1 stochastically dominates s_2 then it makes sense to say that s_1 is less risky than s_2. However, as Hart (2011) notes, it seldom happens that all decision makers agree which one of the two assets is preferred to the other. Formally, this means that the stochastic dominance order between assets is a partial order. Hart (2011) presents two complete stochastic orders on gambles, which extend the stochastic dominance order: "wealth-uniform dominance" and "utility-uniform dominance". He shows that "wealth-uniform dominance" is equivalent to the order induced by the Aumman–Serrano index of riskiness and "utility-uniform dominance" is equivalent to the order induced by the Foster–Hart measure of riskiness (Foster and Hart (2009)). In this paper, we define similar stochastic orders that extend the stochastic dominance, but this time on securities. We show that the "wealth-uniform dominance" in the setup of securities is equivalent to the order induced by our index of relative riskiness. Moreover, in Appendix B we show that the "utility-uniform dominance" is equivalent to one of the variants of the Foster–Hart measure of riskiness that appears in Foster and Hart (2009).

The paper is organized as follows. Section 2 is devoted to the basic axiomatic definition of our index of relative riskiness and its numerical characterization. We chose to present the Aumann–Serrano index together with our index in order to emphasize the similarities as well as the differences between the indices. Section 3 sets forth some desirable properties of the index of relative riskiness. Section 4 defines the concepts of "acceptance dominance" in relation to securities. We present some conditions on the relationship between two securities so that, once they are satisfied, all decision makers who reject one security will also reject the other security. In addition, we extend the definition of Hart (2011) of "wealth uniform dominance" to the set of securities and show that it coincides with the order induced from our index of riskiness. Section 5 defines the concepts of the absolute risk and relative risk of an investment, based on the two indices of riskiness. In addition, we study in Section 5 the use of the two concepts of risk. We conclude in Section 6. The proofs are relegated to Appendix C. In addition, Appendix A suggests an alternative axiomatic characterization of our index. Appendix B deals with the Foster–Hart measure of riskiness in the setup of securities (instead of gambles).

2 Axiomatic Characterization

In this section, we present the characterization of our index of relative riskiness which is equivalent to the characterization of the Aumann–Serrano index of absolute riskiness. In order to emphasize the equivalency between absolute and relative riskiness, we formulate all definitions, axioms, and theorems in both absolute and relative terms. The letters (AS) in the end of a statement denotes that this statement appears already in Aumann and Serrano (2008).

2.1 The Indices

Throughout this paper, a utility function is a von Neumann–Morgenstern utility function for money; it is strictly monotonic, strictly concave, and twice continuously differentiable.

We consider two types of risky assets, namely, gambles and securities.

A *gamble* g is a random variable with real values —interpreted as dollar amounts— some of which are negative and have positive expectation. Say that an agent with utility function u accepts a gamble g at wealth w if $Eu(w + g) > u(w)$ (E stands for expectation), that is, if she prefers taking the gamble at w to refusing it. Other wise, she rejects it. A *security* $r = [x_1, p_1; x_2, p_2; ...\ x_n, p_n]$ is a random variable with finitely many real values, where $x_1, x_2, ..._n$ are the values with respective probabilities $p_1, p_2, ...\ p_n$. The values of x_i, $1 \leq i \leq n$, are interpreted as (gross) returns, some of which are less than one while their weighted geometric mean is greater than one, i.e., $\Pi x_i^{p_i} > 1$. We say that an investor with utility function u accepts (invests in) a security r at wealth w if she prefers investing all her wealth w in r to refusing this investment, i.e., $Eu(wr) > u(w)$.[3]

Gambles and securities are differ in terms of how they affect the wealth of agents who accept them. If an agent has an initial wealth w, accepting a gamble g causes the wealth to be distributed as $w + g$ and accepting a security r causes the wealth to be distributed as wr. Since a gamble has absolute returns and a security has relative returns, we call the riskiness of gambles *absolute riskiness* and the riskiness of securities *relative riskiness*. As we will show latter, these names are compatible with the well-known concepts of absolute and relative risk aversion of Pratt and Arrow.

Following Aumann and Serrano (2008) who define an (incomplete) order relation on the set of agents based on accepting or rejecting gambles, we define another (incomplete) order relation, based on accepting or rejecting securities. The orders are defined as follows:

Definition 2.1.

1. *An agent i is said to be uniformly no less absolute risk averse than agent j, denoted by $i \succeq_A j$, if whenever i accepts a gamble at some wealth, j accepts that gamble in any wealth. (AS)*

2. *An agent i is said to be uniformly no less relative risk averse than agent j, denoted by $i \succeq_R j$, if whenever i accepts a security at some wealth, j accepts that security in any wealth.*

[3]In Section 4 we consider the case where there exists a non-risky alternative in the economy. In this case, the condition for investing w in r is $Eu(wr) > u(wr_f)$ where r_f indicates the risk-free interest rate.

We call agent i uniformly more absolute- (relative-) risk averse than j, denoted by $i \succ_A j$ ($i \succ_R j$), if whenever i accepts a gamble (security) at some wealth, j accepts it in any wealth.

Define an index as a positive real-valued function on risky assets (to be thought of as measuring riskiness). Given an index Q, say that asset s_i is riskier than asset s_j if $Q(s_i) > Q(s_j)$. Aumann and Serrano (2008) consider two axioms for Q where Q is defined on gambles. Here, we extend each one of the axioms to relate also to securities.

The first axiom posits a kind of duality between riskiness and risk aversion, such that less risk-averse agents accept riskier assets. The definition of less risk-averse can be made in terms of absolute riskiness as in Aumann and Serrano (2008), or in relative terms as we do here. Note that gambles and securities are two different subclasses of random variables and as such it is not necessarily the case that an index defined on one set of assets (either gambles or securities) is well defined on the other set. As a matter of fact, the Aumann–Serrano index is not well defined on securities and our new index of relative riskiness is not well defined on securities.

Let Q_A and Q_R be two indices of riskiness of gambles and securities, respectively. Let g and h be two gambles and let r and k be two securities. The original axioms in relation to gambles with the equivalent axioms in relation to securities are the following.

Axiom. *DUALITY*

1. *If $i \succ_A j$, i accepts g at w, and $Q_A(g) > Q_A(h)$, then j accepts h at w. (AS)*

2. *If $i \succ_R j$, i accepts r at w, and $Q_R(r) > Q_R(k)$, then j accepts k at w.*

Axiom. *SCALING.*

1. *$Q_A(tg) = tQ_A(g)$ for all positive numbers t. (AS)*

2. *$Q_R(r^t) = tQ_R(r)$ for all positive numbers t.*

As Aumann and Serrano (2008) write, duality says that if the more risk-averse of two agents accepts the riskier of two assets, then a fortiori the less risk-averse agent accepts the less risky asset. The scaling axiom embodies the cardinal nature of riskiness. If s_1 is a risky asset, we define a new asset, s_2, the acceptance of which is equivalent to accepting s_1 twice. This means that if s_1 and s_2 are gambles then $s_2 = 2s_1$, and if s_1 and s_2 are securities, then $s_1 = s_2^2$. It makes sense to say that s_2 is "twice as risky as s_1", not just "more" risky.

We now define two indices of riskiness. The first is the Aumman–Serrano index of absolute riskiness and the second is our proposed index of relative riskiness. If g is a gamble, $R(g)$ denotes the absolute riskiness of g and if r is a security, $S(r)$ is the relative riskiness of r. $R(g)$ and $S(r)$ are defined implicitly as follows:

$$Ee^{-g/R(g)} = 1. \tag{1}$$

$$Er^{-1/S(r)} = 1. \tag{2}$$

(1) and (2) imply that for any security r,

$$S(r) = R(\log r). \tag{3}$$

Equivalently, for any gamble g, $R(g) = S(e^g)$. Note that R is not well defined on securities and S is not well defined on gambles.[4]

Theorem 2.2.

1. *For each gamble g, there is a unique positive number R that solves for* (1)*. R satisfies duality and scaling, and any index of gambles satisfying these two axioms is a positive multiple of R. (AS)*

2. *For each security r, there is a unique positive number S that solves for* (2)*. S satisfies duality and scaling, and any index of securities satisfying these two axioms is a positive multiple of S.*

Aumann and Serrano (2008) note that in relation to R, duality and scaling are both essential: omitting either one of them results in admitting indices that are not positive multiples of R. But duality is by far more central: together with certain weak conditions of continuity and monotonicity —but not scaling— it already implies that the index is ordinally equivalent to R. The same statement is correct in relation to S.

[4]Recall that the probability for negative values is positive in gambles but zero in securities.

2.2 Risk Aversion and Duality

To understand the concept of uniform comparative risk aversion that underlies this treatment, recall first that Arrow (1965) and Pratt (1964) define two coefficients of risk aversion, one for absolute risk aversion (ARA) $\rho_i(w) := \rho(w, u_i) := -u_i''(w)/u_i'(w)$, and one for relative risk aversion (RRA) $\varrho_i(w) := \varrho(w, u_i) := -w\rho(w, u_i)$.

Lemma 2.3.

1. *i is uniformly no less absolute-risk averse than j if and only if $\rho_i(w_i) \geq \rho_j(w_j)$ for all w_i and w_j. (AS)*
2. *i is uniformly no less relative-risk averse than j if and only if $\varrho_i(w_i) \geq \varrho_j(w_j)$ for all w_i and w_j.*

As Aumann and Serrano (2008) say, the Arrow–Pratt concept of absolute and relative risk aversion is a "local" concept in that it concerns i's attitude toward infinitesimally small risky assets at a specified wealth only; in contrast, the concepts of being uniformly no less absolute-risk averse and uniformly no less relative-risk averse are "global" in two senses: (1) they apply to risky assets of an arbitrary, finite size, which (2) may be taken at any wealth. However, these are only partial orders, whereas Arrow and Pratt define a numerical index (and hence a total order).

2.3 CARA and CRRA

An agent i is said to have constant absolute risk aversion (CARA) if her ARA is a constant α that does not depend on her wealth. In that case, i is called a CARA agent and her utility u a CARA utility, both with parameter α. There is an essentially unique CARA utility with parameter α, given by $u(w) = -e^{-\alpha w}$.

Similarly, an agent i is said to have constant relative risk aversion (CRRA) if the value of $\varrho_i(w)$ is constant for all w. CRRA expresses the idea that wealthier people are less risk averse. Here, wealth is assumed to be positive. There is an essentially unique[5] CRRA utility with parameter α, given by

$$u_\alpha(x) = \begin{cases} \frac{(x^{1-\alpha}-1)}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(x) & \text{if } \alpha = 1 \end{cases}$$

[5]Up to additive and positive multiplicative constants.

While defined in terms of local concept of risk aversion, CARA and CRRA may in fact be characterized in global terms, as follows.

Lemma 2.4.

1. *An agent i has CARA if and only if for any gamble g and any two wealth levels, i either accepts g at both levels or rejects g at both levels. (AS)*
2. *An agent i has CRRA if and only if for any security r and any two wealth levels, i either accepts r at both levels or rejects r at both levels.*

Lemma 2.5.

1. *If a CARA agent accepts a gamble, then any CARA agent with a smaller parameter of CARA also accepts the gamble. Equivalently, if a CARA agent rejects a gamble, then any CARA agent with a larger parameter also rejects the gamble. (AS)*
2. *If a CRRA agent accepts a security, then any CRRA agent with a smaller parameter of CRRA also accepts the security. Equivalently, if a CRRA agent rejects a security, then any CRRA agent with a larger parameter also rejects the security.*

From Lemma (2.5) it follows that for each gamble g (security r), there is precisely one "cutoff" value of the parameter, such that g (r) is accepted by CARA (CRRA) agents with a smaller parameter and rejected by CARA (CRRA) agents with a larger parameter. The larger the parameter, the more absolute- (relative-) risk averse the agent; so the duality axiom indicates that this cutoff might be a good inverse measure of absolute (relative) riskiness. And indeed we have the following theorem.

Theorem 2.6.

1. *The riskiness $R(g)$ of a gamble g is the reciprocal of the number α such that a CARA person with parameter α is indifferent between taking and not taking the gamble. (AS)*
2. *The riskiness $S(r)$ of a security r is the reciprocal of the number $\alpha - 1$ such that a CRRA person with parameter α is indifferent between investing and not investing in the security.*

Proof. Follows from (1) and (2) and the form of CARA and CRRA utilities.

Lemma 2.7.

1. *If $\rho_i(x) < 1/R(g)$ for all x between $w + \min g$ and $w + \max g$, then i accepts g at w; if $\rho_i(x) > 1/R(g)$ for all such x then i rejects g at w. (AS)*

2. *If $\varrho_i(x) < 1/S(r) + 1$ for all x between $w \cdot \min r$ and $w \cdot \max r$, then i accepts r at w; if $\varrho_i(x) > 1/S(r) + 1$ for all such x then i rejects r at w.*

3 Properties

In this section we present some properties of relative riskiness. For the properties of absolute riskiness, see Aumann and Serrano (2008).

3.1 The Non-Risky Alternative

The relative riskiness of r depends only on the distribution of r. A more general definition of riskiness might take into account the non-risky alternative available for investors. Let r_f be the risk-free (gross) return instrument available for investors. Investing an amount of w in r_f gives wr_f in the next period. In the presence of r_f, we consider a slightly different set of securities. A security now is a finite-valued random variable whose geometric expectation is greater than r_f and there is a positive probability of getting a return lower than r_f. Conceptually, if the geometric expectation of a security is lower than r_f, we would consider its riskiness as infinity. Similarly, if the return of r is surely higher than r_f, we consider the riskiness of r to be zero.

Given r_f, we define the relative riskiness of r as

$$S_f(r_f, r) = S(r/r_f). \tag{4}$$

It is easy to see that if $r_f = 1$ we return to the original setup and $S_f(r_f, r) = S(r)$. For any value of r_f, S_f inherits the properties of S. Moreover, S_f satisfies the axioms duality and scaling, and any index satisfying these two axioms is a positive multiple of S_f.

It is only reasonable to expect that a higher risk-free interest rate makes any security riskier. In the range of values of r_f, where $\min(r) < r_f <$

max(r) and where the geometric mean of r/r_f is greater than one, we get the following:

Lemma 3.1. *$S_f(r_f, r)$ increases in r_f.*

Proof. If $r_{f1} > r_{f2}$ then r/r_{f2} stochastically dominates r/r_{f1}, which implies that the riskiness of r/r_{f2} is lower.

We proceed now to study the properties of S.

3.2 Monotonicity with Respect to the Invested Fraction of Wealth

If $\alpha < 1$, investing only αw in a security r is less risky than investing w in r. In order to formalize this claim, we define $r(\alpha)$ to be the security $r(\alpha) = 1 + \alpha(r - 1)$. Obviously, investing w in $r(\alpha)$ is equivalent to investing only $w\alpha$ in r.

Lemma 3.2. *$S(r(\alpha)) < S(r)$ for $0 < \alpha < 1$.*

Another interesting result in relation to $r(\alpha)$ is the following:

Lemma 3.3.

$$\lim_{\alpha \to 0} \frac{\widehat{S}(r(\alpha))}{\alpha} = R(r-1), \tag{5}$$

where $\widehat{S}(r) \equiv S(r)/(1 + S(r))$. As Aumann and Serrano note, $R(r - 1)$ can be interpreted as the absolute riskiness of investing \$1 in r.

3.3 Log-Normal Securities

If the security r has a log-normal distribution[6] with parameters μ and σ, then $S(r) = \sigma^2/2\mu$, where σ^2 is the variance of $\log r$ and μ is the expectation of $\log r$. Indeed, the density of r's distribution is $e^{(\ln x - \mu)^2/2\sigma^2}/x\sigma\sqrt{2\pi}$, so

$$Er^{-1/(\sigma^2/2\mu)} = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty x^{-1} e^{-(\ln x - \mu)^2/2\sigma^2} x^{-1/(\sigma^2/2\mu)} dx$$

[6]By our earlier definition, a security has only finitely many values so its distribution cannot be log-normal. We therefore redefine a security as a random variable r for which S is well defined.

By substituting $y = \ln x$ we get

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(y-\mu)^2/2\sigma^2} e^{-y/(\sigma^2/2\mu)} dy$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(y+\mu)^2/2\sigma^2} dy = 1$$

3.4 Repeated Investments

Investing repeatedly in independent identically distributed (i.i.d) securities is just as risky as investing once in one of these securities.

Lemma 3.4. *If $r_1, r_2 ... r_n$ are independent identically distributed (i.i.d) securities with riskiness s, then Πr_i also has riskiness s.*

Lemma 3.5. *If r and k are independent, then the riskiness of rk lies between those of r and k.*

Even without independence, we still have subadditivity:

Lemma 3.6. *$S(rh) \leq S(r) + S(h)$ for any security r and h.*

Recall that for a security r, $S(r) = R(\log r)$. Defining two gambles, $g = \ln r$ and $h = \ln k$, the proofs of Lemmas (3.4), (3.5), and (3.6) follow immediately from Section 4.H. in Aumann and Serrano (2008).

To summarize: if two securities are identically distributed and hence have the same riskiness s, then if the securities are "totally" positively correlated (i.e., equal), the product security has riskiness $2s$. If they are independent, the product has the same riskiness s as each of the securities separately. When they are "totally" negatively correlated, the risk is minimal but need not vanish.

3.5 Other Properties

Many other properties of our index, such as monotonicity in first- and second-order stochastic dominance and continuity, can be derived directly from the equivalent properties of the index of absolute riskiness, by using (3); see Aumann and Serrano (2008).

4 Comparing Risks by Acceptance and Rejection

Let s_1 and s_2 denote two risky assets, either two gambles or two securities. There are cases where it is clear that s_1 is less risky than s_2. This is certainly so when s_1 (second-degree) stochastically dominates s_2. In this case, all risk-averse decision makers prefer s_1 to s_2. Since stochastic dominance is only a partial (rather than complete) order on assets, we would expect that any complete order of riskiness of assets will be an extension of the stochastic dominance order.

In relation to gambles, Hart (2011) weakens the requirement of stochastic dominance, by asking only that gamble g be accepted more than gamble h. He says that *g acceptance dominates h* if "every time that g is rejected by a risk-averse decision-maker then so is h". Although the acceptance dominance extends the stochastic dominance and allows one to compare more pairs of gambles, it is still a partial order. Hence, Hart (2011) obtains two other orders on gambles that he calls "wealth uniform dominance" and "utility uniform dominance". The two new orders extend acceptance dominance and thus, a fortiori, stochastic dominance. He shows that wealth uniform dominance is equivalent to R and that utility uniform dominance is equivalent to the Foster-Hart measure of riskiness. In this section we extend the definitions of acceptance dominance and wealth uniform dominance, in a natural way, to include also securities (not only gambles). Moreover, we show that the order of wealth uniform dominance is equivalent to S, once it is applied to securities.[7] Another result of this section is that in some cases one can conclude that one security is acceptance dominates another security from the comparison of their relative riskiness, i.e., if the riskiness of one of them is "much higher" then the riskiness of the other.

In this section we will use two standard assumptions on utility functions set forth by Arrow. The first assumption is that the absolute risk aversion of any utility decreases with wealth and the second assumption is that the relative risk aversion of any utility increases with wealth. We will denote by U^* the resulting class of utilities.

[7] In appendix B we show that an extension of the utility uniform dominance order to securities is equivalent to the Foster-Hart measure of riskiness, defined on securities.

4.1 Acceptance Dominance

In the setup of gambles, Hart (2011) defines the (incomplete) order of *acceptance dominance*. We extend this definition to any risky asset, including securities, as follows.

Definition 4.1. *The risky asset s_1 acceptance dominates the risky asset s_2 ($s_1 >_A s_2$) if for every $u \in U^*$ and for every $w > 0$,*

$$u \text{ rejects } s_1 \text{ at } w \text{ implies that } u \text{ rejects } s_2 \text{ at } w.$$

While acceptance dominance allows one to compare more pairs of assets than does stochastic dominance, as Hart (2011) indicates, it is still a very partial order: for general assets s_1 and s_2, there are instances where s_1 is rejected and s_2 is accepted and other instances where s_2 is rejected and s_1 is accepted.

Given a risky asset s (where s is either a gamble or a security), we denote by s_{max} and s_{min} the maximal and the minimal values that s takes with positive probability. For convenience, we define $\widehat{S}$ as $\widehat{S}(r) \equiv S(r)/(1+S(r))$.[8] The following Theorem asserts that given two securities, r and k, if r is "much riskier" than k, then $k >_A r$.

Theorem 4.2. *Given two securities, r and k, if either*

$$\frac{\widehat{S}(r)}{\widehat{S}(k)} > \frac{k_{max}}{r_{min}}, \tag{6}$$

or

$$\frac{R(r-1)}{R(k-1)} > \frac{k_{max}}{r_{min}}, \tag{7}$$

then, $k \geq_A r$.

Note that conditions (6) and (7) are not equivalent: there are instances where (6) holds but not (7), and vice versa.

As the following theorem asserts, an equivalent result in relation to gambles is somewhat weaker and depends on the initial wealth w.

[8] $\widehat{S}(r)$ equals to the reciprocal of the parameter of the CRRA utility that is indifferent between accepting and rejecting r. It is easy to see that $\widehat{S}$ and S are ordinally equivalent.

Theorem 4.3. *Given two gambles, g and h, if either*

$$\frac{R(g)}{R(h)} > \frac{w + h_{max}}{w + g_{min}}, \tag{8}$$

or

$$\frac{\widehat{S}(1 + g/w)}{\widehat{S}(1 + h/w)} > \frac{w + h_{max}}{w + g_{min}}, \tag{9}$$

then, for all $u \in U^*$, if u rejects h at w then u rejects g at w.

Note that the expression in (9) , $\widehat{S}(1 + g/w)$, is not well defined for any value of $w > 0$. Hence, (9) should be interpreted as referring to values of w such that the expression is well defined. As before, conditions (8) and (9) are not equivalent and there are instances where (8) holds but not (9) and vice versa.

Recall that given a security r, $r(\alpha) = 1 + \alpha(r - 1)$. Investing w in $r(\alpha)$ is equivalent to investing only αw in r. The following corollary is of interest:

Corollary 4.4.

1. *For any two gambles g and h, if $R(g) > R(h)$ then there is a number w^* such that for any $w > w^*$, for all $u \in U^*$, if u rejects h at w then u rejects g at w.*

2. *For any two securities r and k, if $R(r - 1) > R(k - 1)$ then there is a number α^* such that for any $\alpha < \alpha^*$, $k(\alpha) \geq_A r(\alpha)$.*

Proof.

1. It follows from (8). If w goes to infinity, the right-hand side of (8) goes to one.

2. It follows from (7). Replace r by $r(\alpha)$ and k by $k(\alpha)$. If α goes to zero, the scaling axiom implies that the left-hand side of (7) is constant but the value of the right-hand side goes to one.

4.2 Riskiness In the Small

The Arrow-Pratt measures of risk aversion have simple behavioral interpretations. Arrow shows that the risk aversion measures the insistence of an individual for "more than fair" odds, when the bets are small; see Arrow (1965). A similar interpretation has been developed independently by Pratt (1964). A crucial assumption for the measures of risk aversion to be relevant for all decision makers is that the distribution is sufficiently concentrated.[9] Here we show a "dual" result in relation to riskiness: if the riskiness of the risky assets is small, for any utility in U^*, rejecting the less risky asset implies rejecting the riskier asset as well.

Theorem 4.5.

1. *For any two gambles g and h, if $R(g) > R(h)$ then for any $w > 0$ there is a number δ^* such that for any $\delta < \delta^*$, for any $u \in U^*$, if u rejects δh at w then u rejects δg at w.*

2. *For any two securities r and k, if $S(r) > S(k)$ then there is a number λ^* such that for any $\lambda < \lambda^*$, $k^\lambda \geq_A r^\lambda$.*

Proof. Follows from theorems (4.2) and (4.3) and the scaling axiom.

4.3 Wealth Uniform Dominance

Hart (2011) defines the stochastic order of wealth uniform dominance in the setup of gambles as follows.

Definition 4.6. *A gamble g wealth uniformly dominates a gamble h, denoted $g \geq_{WU} h$, whenever:*

$$\text{if } g \text{ is rejected by } u \text{ at all } w > 0$$
$$\text{then } h \text{ is rejected by } u \text{ at all } w > 0$$

for every utility $u \in U^*$.

Similarly, we define the stochastic order of wealth uniform dominance in the setup of securities as follows.

[9] Pratt (1964) calls it "risk in the small".

Definition 4.7. *A security r wealth uniformly dominates a security h, denoted $r \geq_{WU} h$, whenever:*

$$\text{if } r \text{ is rejected by } u \text{ at all } w > 0$$
$$\text{then } h \text{ is rejected by } u \text{ at all } w > 0$$

for every utility $u \in U^*$.

Definitions (4.6) and (4.7) are obviously extensions of the the acceptance dominance that we defined before. In other words, given two gambles or two securities, s_1 and s_2, $s_1 \geq_{WU} s_2$ implies that $s_1 \geq_A s_2$. On the other hand, $s_1 \geq_A s_2$ does not imply that $s_1 \geq_{WU} s_2$.

The following theorem cites the result of Hart (2011) in relation to gambles and adds our results in relation to securities:

Theorem 4.8.

1. *For any two gambles g and h, $g \geq_{WU} h$ if and only if $R(g) \leq R(h)$. (H)*
2. *For any two securities r and k, $r \geq_{WU} k$ if and only if $S(r) \leq S(k)$.*

For the proof of the first part of Theorem (4.8) see Hart (2011). The second part of the theorem follows from the first part and the insight that for any security r, $S(r) = R(\log(r))$.

5 Absolute and Relative Risks

Up to now, we have adopted the second approach to measuring risks by proposing an index of riskiness of securities that does not depend on the initial wealth or the distribution of wealth after the investment is made. In this section, we propose a way to measure the risks that arise from any investment, at a certain level of wealth. According to this approach, the risk of an investment is still objective in some sense, as it does not depend on the utility function. The risk that arises from making an investment depends only on two parameters: the initial level of wealth w, and the randomly distributed wealth $\widetilde{w}$ that the individual will have after making the investment.

Let w be a positive real number, denotes a certain level of wealth, and let $\widetilde{w}$ be a real-valued random variable having only positive numbers. For simplicity we assume that $\widetilde{w}$ has finitely many values, say $x_1, x_2 ... \ x_m$ with respective probabilities $p_1, p_1, ... \ p_m$.[10] We call the pair $(\widetilde{w}, w)$ a *deal*. We say that an agent with utility function u accepts a deal $(\widetilde{w}, w)$ if $Eu(\widetilde{w}) > u(w)$. Otherwise she rejects it.

We proceed now to define the concepts absolute risk of a deal and the relative risk of a deal. Given a deal $d = (\widetilde{w}, w)$, we denote the absolute risk of a deal by $AR(d)$ and the relative risk of the deal by $RR(d)$. There are situations in which we consider the two risks to be zero. That happens if $w < \min(\widetilde{w})$, as in this case any risk-averse agent would accept the deal. On the other hand, since any risk-averse decision maker will reject the deal if $E\widetilde{w} < w$, we consider the absolute riskiness of this deal to be infinity. In addition, if the the geometric mean of $\widetilde{w}$ is lower than w, then we consider the relative risk of this deal to be also infinity. In all the other cases, we measure the two risks as follows:

$$\begin{aligned} \mathrm{AR}(\widetilde{w}, w) &= R(\widetilde{w} - w), \\ \mathrm{RR}(\widetilde{w}, w) &= \widehat{S}(\widetilde{w}/w). \end{aligned} \tag{10}$$

where $\widehat{S}(r) \equiv S(r)/(1 + S(r))$.[11]

The definitions of AR and RR imply that accepting a gamble exposes the agent to absolute risk that is exactly equal to the absolute riskiness of the gamble. Similarly, accepting a security exposes the agent to a relative risk that is exactly equal to the (adjusted) relative riskiness of the security. These risks depend only on the risky asset, not on the initial wealth. On the other hand, accepting a gamble exposes the agent to a relative risk that depends on the initial wealth, while accepting a security exposes the agent to an absolute risk that depends on the initial wealth. It is important to note that given two deals with the same initial wealth, it might be that one deal is absolutely riskier while the second deal is relatively riskier. In other words, the two aspects of risks that arises from making an invested, namely the absolute risk and the relative risk, are not ordinally equivalent, and hence

[10]We denote by $\max(\widetilde{w}) = \max_i x_i$ the maximal value of $\widetilde{w}$ and by $\min(\widetilde{w}) = \min_i x_i$ the minimal value of $\widetilde{w}$.

[11]$\widehat{S}$ is an adjusted version of our index S. $\widehat{S}$ satisfies the duality axiom but not the scaling axiom. It is easy to see that S and $\widehat{S}$ are ordinally equivalent.

they capture two different aspects of the risk that arises from making an investment.

In Section 4.2 we showed that just as the Arrow–Pratt concept of risk aversion is relevant for all decision makers when the risks are small (in some sense), the concepts of absolute and relative riskiness are also relevant for all decision makers when the risks are small. The same idea is relevant also to the absolute risk and relative risk that arises from any investment. Given two deals, $d_1 = (\widetilde{w}_1, w)$ and $d_2 = (\widetilde{w}_2, w)$, we define similar deals that depend on a parameter δ: $d_1(\delta) = (w+\delta(\widetilde{w}_1-w), w)$ and $d_2(\delta) = (w+\delta(\widetilde{w}_2-w), w)$, where by definition, $d(1) = d$. It follows from Section 4.2 that if $AR(d_1) > AR(d_2)$ then there is δ^*, s.t., for any $\delta < \delta^*$, $d_2(\delta) \geq_A d_1(\delta)$, where the relation "$\geq_A$" is defined as follows.

Definition 5.1. *Let $d_1 = (\widetilde{w}_1, w)$ and $d_2 = (\widetilde{w}_2, w)$ be two deals with the same initial wealth w. We say that d_1 acceptance dominates d_2, denoted $d_1 \geq_A d_2$, if any utility function $u \in U^*$ who rejects d_1 rejects also d_2.*

A similar analysis can be done in relation to securities.

The definition of acceptance dominance in relation to deals is an extension of the Hart (2011) definition of acceptance dominance, originally defined in relation to gambles.

If $\widetilde{w}_1$ stochastically dominates $\widetilde{w}_2$, then $(\widetilde{w}_1, w)$ also acceptance dominates $(\widetilde{w}_2, w)$, no matter what w is. Here, we consider situations in which $\widetilde{w}_1$ does not necessarily stochastically dominate $\widetilde{w}_2$, but still $(\widetilde{w}_1, w) \geq_A (\widetilde{w}_2, w)$. The following theorem bounds the ratio between the absolute and the relative risk that arises from a fair deal.

Theorem 5.2.

$$\min(\widetilde{w}) < \frac{AR(\widetilde{w}, w)}{RR(\widetilde{w}, w)} < \max(\widetilde{w}), \tag{11}$$

Let $(\widetilde{w}_1, w)$ and $(\widetilde{w}_2, w)$ be two deals whose absolute riskiness and relative riskiness are both between zero and infinity. We denote the minimal value of $\widetilde{w}_1$ and $\widetilde{w}_2$ by $\min(\widetilde{w}) = \min(\min(\widetilde{w}_1), \min(\widetilde{w}_2))$, and the maximal value of $\widetilde{w}_1$ and $\widetilde{w}_2$ by $\max(\widetilde{w}) = \max(\max(\widetilde{w}_1), \max(\widetilde{w}_2))$.

Theorem 5.3. *If either*

$$\frac{AR(\widetilde{w}_1, w)}{RR(\widetilde{w}_2, w)} < \min(\widetilde{w}), \tag{12}$$

or

$$\max(\widetilde{w}) < \frac{AR(\widetilde{w}_2, w)}{RR(\widetilde{w}_1, w)}, \tag{13}$$

then $d_1 \geq_A d_2$.

Corollary 5.4. *If either*

$$\frac{AR(\widetilde{w}_1, w)}{AR(\widetilde{w}_2, w)} < \frac{\max(\widetilde{w})}{\min(\widetilde{w})}, \tag{14}$$

or

$$\frac{RR(\widetilde{w}_1, w)}{RR(\widetilde{w}_2, w)} < \frac{\max(\widetilde{w})}{\min(\widetilde{w})}, \tag{15}$$

then $d_1 \geq_A d_2$.

6 Conclusions

We proposed an index of relative riskiness that induces a complete order of riskiness on the set of securities. The concept of relative riskiness corresponds to the Arrow–Pratt concept of relative risk aversion: individuals who are more averse to risk prefer less risky securities. Our concept of relative riskiness thus complements the (absolute) riskiness of Aumann and Serrano (2008).

In addition to the proposed index, this paper proposes to measure the absolute aspect of the risk that arises from any investment by the Aumann–Serrano index of absolute riskiness and to measure the relative aspect of the risk by our proposed index of relative riskiness. The absolute and relative risk are correspond to the concepts of absolute risk aversion and relative risk aversion of Arrow and Pratt. The two concepts of risk, i.e., absolute risk (AR) and relative risk (RR), induce different orders of risk on investments.

Appendix

A An Alternative Axiomatic Characterization

Foster and Hart (2011) provide an axiomatic characterization of the measure of riskiness of gambles introduced by Foster and Hart (2009) and other axiomatic characterization of the Aumann-Serrano index of riskiness of gambles. Here, we show that translating the axiomatic characterization of the Aumann–Serrano index, from absolute terms (gambles) to relative terms (securities) leads to our index of riskiness. For each axiom, we present the original axiom followed by our translation to absolute terms. Let g and h be two gambles and let r and k be two securities. We denote by Q an index of riskiness defined on either gambles or securities. The axioms are the following:

Axiom. *Distribution.*

1. *If g and h have the same distribution then $Q(g) = Q(h)$.*
2. *If r and k have the same distribution then $Q(r) = Q(k)$.*

Axiom. *Scaling.*

1. *$Q(\lambda r) = \lambda Q(r)$ for every $\alpha > 0$.*
2. *$Q(r^{\lambda}) = \lambda Q(r)$ for every $\alpha > 0$.*

Axiom. *Monotonicity.*

1. *If $g \geq h$ and $r \neq h$ then $Q(g) < Q(h)$.*
2. *If $r \geq k$ and $r \neq h$ then $Q(r) < Q(k)$.*

Axiom. *Wealth Independent Compound Asset.*

1. *Let $f = g + \mathbf{1}_A h$ be a compound gamble, where A is an event such that g is constant on A, i.e., $g|_A \equiv s$ for some x and h is independent of A. If $Q(h) = Q(g)$ then $Q(f) = Q(g)$.*
2. *Let $f = r \times (1 + \mathbf{1}_A(k - 1))$ be a compound security, where A is an event such that r is constant on A, i.e., $r|_A \equiv x$ for some x, and k is independent of A. If $Q(k) = Q(r)$ then $Q(f) = Q(r)$.*

Theorem A.1. *A function Q satisfies the four axioms if and only if it is a positive multiplication of R or S.*

The proof of the theorem in relation to R appears in Foster and Hart (2011). The translation to relative terms is based on the simple observation that for any security r, $S(r) = R(\log(r))$.

B Utility Uniform Dominance

Foster and Hart (2009) propose another measure of riskiness of gambles. The riskiness of gamble g, denoted by $R^{FH}(g)$, is defined implicitly by the formula

$$E\left[\log\left(1+\frac{1}{R^{FH}(g)}g\right)\right]=0. \tag{16}$$

Their measure is based on the critical wealth level below which becomes "risky"[12] to accept the gamble; see Foster and Hart (2009). Although Foster and Hart (2009) focuses on the setting of gambles, they propose also a measure of riskiness of securities, defined as follows: if r is a security then

$$S^{FH}(r)\equiv R^{FH}(r-1). \tag{17}$$

In relation to gambles, Hart (2011) defines two orders that extend the stochastic dominance order to be a complete order: "wealth uniform dominance" and "utility uniform dominance". In Section 3 we extended the definition of utility uniform dominance to include also securities and we showed that the utility uniform dominance on securities is equivalent to the order that our proposed index of relative riskiness, S, induces. Here, we extend the definition of the second order, i.e., utility uniform dominance, to securities and show that it is equivalent to the order that S^{FH} induces.

Following Hart (2011), we will use one more assumption on utilities, in addition to the two assumptions that we already used. In Section 4, we used the assumptions that all utility functions are DARA and IRRA. Here, we add the assumption that as the value of the wealth goes to zero, the utility of the agent goes to minus infinity. Formally, the assumption asserts that $\lim_{w\to 0} u(w)=-\infty$. We will denote by U^{**} the resulting class of utilities; i.e., U^{**} is the class of utilities that are DARA and IRRA and that satisfy the last assumption. Hart (2011) defines utility uniform dominance as follows:

Definition B.1. *A gamble g utility uniformly dominates a gamble h, denoted $g \geq_{UU} h$ whenever:*

if g is rejected by all $u\in U^{**}$ at w
then h is rejected by all $u\in U^{**}$ at w

[12] In terms of going bankrupt.

Given a security r, we define a new security $r(\alpha) = 1 + \alpha(r - 1)$, for $0 < \alpha < 1$.

Definition B.2. *A security r utility uniformly dominates a security k, denoted $r \geq_{UU} k$, whenever:*

$$\text{if } r(\alpha) \text{ is rejected by all } u \in U^{**} \text{ and all } w, \text{ at } \alpha$$
$$\text{then } k(\alpha) \text{ is rejected by all } u \in U^{**} \text{ and all } w, \text{ at } \alpha$$

According to this definition, r utility uniformly dominates k, if and only if, the investment of αw in r is rejected by all utilities and wealth implies that investing αw in k is rejected by all utilities and wealth. The following theorem cites the result of Hart (2011) in relation to gambles and adds our result in relation to securities:

Theorem B.3.

1. *For any two gambles g and h, $g \geq_{UU} h$ if and only if $R^{FH}(g) \leq R^{FH}(h)$. (FH)*
2. *For any two securities r and k, $r \geq_{UU} k$ if and only if $S^{FH}(r) \leq S^{FH}(k)$*

C Proofs

In this section, investors i and j have utility functions u_i and u_j and Arrow–Pratt coefficients ϱ_i and ϱ_j of relative risk aversion. Since utilities may be modified by additive and positive multiplicative constants, we may – and do – assume throughout the following:

$$u_i(1) = u_j(1) = 0 \text{ and } u_i'(1) = u_j'(1) = 1. \tag{18}$$

Lemma C.1. *For some $\delta > 1$, suppose that $\varrho_i(w) > \varrho_j(w)$ at each w with $1/\delta < w < \delta$. Then $u_i(w) < u_j(w)$ whenever $1/\delta < w < \delta$ and $w \neq 1$.*

Proof. Let $1/\delta < y < \delta$. If $y > 1$, then by (18),

$$\log u_i'(y) = \log u_i'(y) - \log u_i'(1) = \int_1^y [\log u_i'(z)]' dz = \int_1^y \frac{u_i''(z)}{u'(z)} dz$$

$$= \int_1^y -(\varrho_i(z)/z)dz < \int_1^y -(\varrho_j(z)/z)dz = \log u_j'(y)$$

if $y < 1$ the reasoning is similar but the inequality is reversed. So if $w > 1$, then by (18),

$$u_i(w) = \int_1^w u_i'(y)dy < \int_1^w u_j'(y)dy = u_j(w);$$

and if $w < 1$, then

$$u_i(w) = \int_w^1 u_i'(y)dy < \int_w^1 u_j'(y)dy = u_j(w).$$

Corollary C.2. *If $\varrho_i(w) \leq \varrho_j(w)$ for all $w > 0$, then $u_i(w) \geq u_j(w)$ for all $w > 0$.*

Lemma C.3. *If r is a security, its riskiness $S(r)$ is well defined.*

Proof. Let r be a security and define a function f_r as follows:

$$f_r(\beta) \equiv Er^\beta = \Sigma p_i r_i^\beta, \tag{19}$$

where the first and second derivatives of f_r are

$$f_r'(\beta) = \Sigma p_i r_i^\beta \log r_i \tag{20}$$

$$, f_r''(\beta) = \Sigma p_i r_i^\beta (\log r_i)^2. \tag{21}$$

Since, by definition, r has at least one value is greater than one and at least one value is lower than one,

$$\lim_{\beta \to \pm\infty} f_r(\beta) = \infty. \tag{22}$$

In addition, since f_r'' is positive for all β, f_r' increases in β which implies that f_r has a single minimum point. From (19) it follows that $f_r'(0) = 1$. If $f_r'(0) \neq 0$, there should be another value of β, for which $f_r(\beta) = 1$. Based on this insight, we define β^* as follows:

1. If $f_r'(0) > 0$, then there is only one additional value of β, $\beta = \beta^*$, in which $f_r(\beta^*) = 1$ and $\beta^* < 0$.

2. If $f_r'(0) < 0$, then there is only one additional value of β, $\beta = \beta^*$, in which $f_r(\beta^*) = 1$ and $\beta^* > 0$.

3. If $f_r'(0) = 0$, then there is no other value of β, $\beta \neq 0$, in which $f_r(\beta) = 1$. In this case we set $\beta^* = 0$.

Since we assumed that the weighted geometric mean of any security is greater than one, $f_r'(0) = \Sigma p_i \log r_i > 0$ and we are in the first case where $\beta^* < 0$. Defining $S(r) = -1/\beta^*$ ends the proof.

Two notes:

1. The analysis shows that for any security r, $S(r) > 0$.

2. If we consider securities whose geometric mean can be lower than one, then $\beta^* \geq 0$. It can be shown that if the expectation of a security is positive, then $\beta^* < 1$.

Lemma C.4. *For any two portfolios r and k,*

$$S(k) > S(r) \Leftrightarrow f_r(-1/S(k)) < 1.$$

Proof. It follows from the proof of Lemma (C.3). Since $f_r'(0) > 0$, $\beta^* < 0$ and the minimum point of f_r is between $\beta^* < 0$ and 0 (scenario 1 in the proof of (C.3)). This, together with the continuity of f_r, imply that for any $\beta^* < \beta < 0$, $f_r(\beta) < 1$. Define $\beta = -1/S(k)$ and $\beta^* = -1/S(r)$ completes the proof.

Lemma C.5. *For any value of α and $\delta > 1$ there is a security $r = r(\alpha, \delta)$, such that, $u_\alpha(r) = 0$ and $\forall i$, $1/\delta < r_i < \delta$, where r_is are the values the r takes.*

Proof. Let $f(\epsilon)$ be defined as: $f(\epsilon) = \epsilon u_\alpha(\sqrt{1/\delta}) + (1-\epsilon)u_\alpha(\sqrt{\delta})$. It is easy to see that if $\epsilon = 0$ $f(\epsilon) > 0$, and if $\epsilon = 1$ $f(\epsilon) < 0$. Since f is continuous in ϵ, $f(\epsilon^*) = 0$ for some ϵ^* between zero and one. The desired security is $r(\alpha, \delta) = [\epsilon^*, 1-\epsilon^*; \sqrt{1/\delta}, \sqrt{\delta}]$.

The following lemma is equivalent to Lemma 4 in Aumann and Serrano (2008); however, another proof is needed.

Lemma C.6. *If $\varrho_i(w_i) > \varrho_j(w_j)$, then there is a security r that j accepts at w_j and i rejects at w_i.*

Proof. Without loss of generality, $w_i = w_j = 1$, so $\varrho_i(1) > \varrho_j(1)$.[13] Let ϱ be a number between $\varrho_i(w)$ and $\varrho_j(w)$, $\varrho_i(w) > \varrho > \varrho_j(w)$. Since u_i and u_j are twice continuously differentiable, it follows that there is a number $h > 1$ such that $\varrho_i(w) > \varrho > \varrho_j(w)$ at each w with $1/h < w < h$. By Lemma (C.5), there is a security $r(\varrho, h)$ such that u_ϱ is indifferent to. So, by Lemma (C.1),

$$u_i(w) < u_\varrho(w) < u_j(w) \text{ whenever } 1/\delta < w < \delta \text{ and } w \neq 1, \tag{23}$$

implies that $u_i(r(\varrho, h)) < 0 < u_j(r(\varrho, h))$. Hence i rejects the security but j accepts it.

Proof of Lemma (2.3)**.** We have to show that $\varrho_i(w) \geq \varrho_j(w)$ for all wealth levels w if and only if i is no less relative risk averse than j.
"If": Assume that there is a w with $\varrho_i(w) < \varrho_j(w)$. So by Lemma (C.6), there is a security that i accepts at w and j rejects at w, thereby contradicting i being less risk averse than j.

"Only if": Assuming that $\varrho_i(w) \geq \varrho_j(w)$ for all wealth levels w, we must show that for each wealth level w and security r, if i accepts r at w, then j accepts r at w. Without loss of generality, $w = 1$, so we must show that

if i accepts r at 1, then j accepts r at 1.

From Corollary (C.2) (with i and j reversed), we conclude that $u_j(w) \geq u_i(w)$ for each w, so $Eu_j(r) \geq Eu_i(r)$, which yields the above claim.

Proof of Theorem (2.2). For $\alpha > 0$, let $u_\alpha(x)$ be the CRRA utility function with parameter α. The functions u_α satisfy (18), so by Lemma (C.1) (with δ arbitrarily large) their graphs are nested, that is,

$$\text{if } \alpha > \beta, \text{ then } u_\alpha(x) < u_\beta(x) \text{ for all } x > 0. \tag{24}$$

The existence of $S(r)$ is proved in (C.3).

To see that S satisfies the duality axiom, let i, j, r, h, and w be as in the hypothesis of that axiom; without loss of generality, $w = 1$. Set $\gamma \equiv 1 + 1/S(r)$,$\eta \equiv 1 + 1/S(h)$, $\alpha_i = \inf \varrho_i$ and $\alpha_j = \sup \varrho_j$. Thus

$$Eu_\gamma(r) = 0 \text{ and } Eu_\eta(h) = 0. \tag{25}$$

[13] For arbitrary w_i and w_j, define $u_i^*(x) = [u_i(xw_i) - u_i(w_i)]/(w_i u_i'(w_i))$ and u_j^* similarly, and apply the current reasoning to u_i^* and u_j^*. u_i^* and u_j^* accept or reject securities at $w = 1$, just as u_i and u_j accept or reject securities at w_i and w_j, respectively. In addition, ${u^*}'_i(1) = {u^*}'_j(1) = 1$.

By hypothesis, $S(r) > S(h)$, so $\eta > \gamma$. By Corollary (C.2)

$$u_i(x) \leq u_{\alpha_i}(x) \text{ and } u_{\alpha_j}(x) \leq u_j(x) \text{ for all x.} \tag{26}$$

Now assume $Eu_i(r) > 0$; we must prove that $Eu_j(h) > 0$. From $Eu_i(r) > 0$ and (26), it follows that $Eu_{\alpha_i}(r) > 0$. So by (25), $E_\gamma(r) = 0 < Eu_{\alpha_i}(r)$. So by (24), $\gamma > \alpha_i$. By (2.3) $\alpha_i \geq \alpha_j$ so $\eta > \gamma$ yields $\alpha_j < \eta$. Since (25),(24) and (26) yielded $0 < Eu_\eta(h) < Eu_{\alpha_j}(h) < Eu_j(h)$, it follows that S satisfies the duality axiom.

That S satisfies the scaling axiom is immediate, so indeed S satisfies the axioms.

In the opposite direction, let Q be an index that satisfies the axioms. We first show that

$$\text{Q is ordinally equivalent to S.} \tag{27}$$

If this is not true, then there must exist r and h that are ordered differently by Q and R. This means either that the respective orderings are reversed, that is,

$$Q(r) > Q(h) \text{ and } S(r) < S(h), \tag{28}$$

or that the equality holds for exactly one of the two indices, that is,

$$Q(r) > Q(h) \text{ and } S(r) = S(h) \tag{29}$$

or

$$Q(r) = Q(h) \text{ and } S(r) > S(h). \tag{30}$$

If either (29) or (30) holds, then by scaling, replacing r by r^δ for sufficiently small $\delta > 1$ leads to reversed inequalities. So without loss of generality we may assume (28).

Now let $\gamma \equiv 1 + 1/S(r)$ and $\eta \equiv 1 + 1/S(h)$; then (25) holds. By (28), $\gamma > \eta$. Choose μ and ν so that $\gamma > \mu > \nu > \eta$. Then $u_\gamma(x) < u_\mu(x) < u_\nu(x) < u_\eta(x)$ for all $x \neq 0$. So by (25) $Eu_\mu(r) > Eu_\gamma(r) = 0$ and $Eu_\nu(h) < Eu_\eta(h) = 0$. So if i and j have utility functions u_μ and u_ν, respectively, then i accepts r and j rejects h. But from $\mu > \nu$ and (2.3), it follows that $i \succ j$, contradicting the duality axiom for Q. So (27) is proved.

To see that Q is a positive multiple of R, let r_0 be an arbitrary but fixed security and set $\lambda \equiv Q(r_0)/S(r_0)$. If r is any security and $t \equiv Q(r)/Q(r_0)$, then $Q(r_0^t) = tQ(r_0) = Q(r)$, so $tS(r_0) = S(r_0^t) = S(r)$ by the ordinal equivalence between Q and S, so $S(r)/S(r_0) = t = Q(r)/Q(r_0)$, so $Q(r)/S(r) = Q(r_0)/S(r_0) = \lambda$, so $Q(r) = \lambda S(r)$. This completes the proof of Theorem A.

Needless to say, both duality and scaling are essential to Theorem A. Thus the mean log $E \log r$ satisfies scaling but violates duality; and the index $[S(r)]$, where $[x]$ denotes the integer part of x, satisfies duality but violates the scaling axiom. Neither $E \log r$ nor $[S(r)]$ is even ordinally equivalent to S.

Proof of Lemma (2.4)**.** Recall that all CRRA utility functions have the form

$$u_\alpha(x) = \begin{cases} \frac{(x^{1-\alpha}-1)}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(x) & \text{if } \alpha = 1 \end{cases} \tag{31}$$

for $\alpha > 0$.

"Only if": Let $u_\alpha(x)$ be a CRRA utility with parameter α. u_α accepts r at w if and only if $Eu_\alpha(wr) > u_\alpha(w)$, that is, if and only if $Eu_\alpha(r) > u_\alpha(1)$.

"If": It follows from Lemma (C.6); just take $j = i$.

Proof of Lemma (2.5)**.** Based on the function $f_r(\beta)$ (defined at (19)) we define the utility function u by,

$$u_\alpha(r) = \frac{f_r(\beta) - 1}{\beta},$$

where $\alpha = 1/\beta$, and $\alpha^* = 1/\beta^*$. It follows from the analysis of the behavior of $f_r(\beta)$ in the proof of (C.3), that if $\alpha > \alpha^*$ then $u_\alpha(r) > 1$ and if $\alpha < \alpha^*$ then $u_\alpha(r) < 1$.

Proof of Lemma (2.7)**.** Let u_i be i's utility and assume that $\varrho_i(x) < 1/S(r) + 1$ for all x between $w \min r$ and $w \max r$. Define a utility u_j as follows: when x is between $w \min r$ and $w \max r$, define $u_j(x) \equiv u_i(x)$; when $x \leq w \min r$, define $u_j(x)$ to equal a CRRA utility with parameter $\varrho_i(w \min r)$ and $u_j(w \min r) = u_i(w \min r)$ and $u_j'(w \min r) = u_i'(w \min r)$; when $x \geq w \max r$, define $u_j(x)$ to equal a CRRA utility with parameter $\varrho_i(w \max r)$ and $u_j(w \max r) = u_i(w \max r)$ and $u_j'(w \max r) = u_i'(w \max r)$. Let u_k be a CRRA utility with parameter $[1/S(r) + 1] - \epsilon$. Then

$$\min_x \varrho_k(x) > \max_x \varrho_j(x)$$

for positive ϵ sufficiently small. By theorem (2.6), a CRRA person with parameter $[1/S(r)+1]$ is indifferent between taking and not taking g. Therefore, k who is less risk averse, accepts g, so by (2.5), j also accepts r. But between the minimum and maximum of wr, the utilities of i and j are the same. So i accepts r at w. The proof of the second part of (2.7) is similar.

Proof of Lemma (3.2)***.*** Let $r = [x_1, p_1; x_2, p_2; ...; x_n, p_n]$ be a security. For simplicity we denote $\epsilon_i = x_i - 1$. Recall that $S(r)$ is defined implicitly by $\Sigma p_i(1+\epsilon_i)^{-1/S(r)} = 1$ and $S(r(\alpha))$ is defined implicitly by $\Sigma p_i(1+\alpha\epsilon)^{-1/S(r(\alpha))} = 1$. We must show that for any $0 < \alpha < 1$, $S(r) > S(r(\alpha))$. Define a new function

$$K_r(\alpha) = \Sigma p_i(1 + \alpha\epsilon_i)^{-1/S(r)},$$

whose first and second derivatives are:

$$K'_r(\alpha) = -1/S(r)\Sigma p_i(1 + \alpha\epsilon_i)^{-(1/S(r)+1)}\epsilon_i$$

$$K''_r(\alpha) = (1/S(r))((1/S(r) + 1)\Sigma p_i(1 + \alpha\epsilon_i)^{-(1/S(r)+2)}\epsilon_i^2.$$

Note that $K(0) = 1$ and $K(1) = 1$. Since the second derivative is positive for any $0 < \alpha < 1$, it follows that for any $0 < \alpha < 1$, $K(\alpha) < 1$. According to (19), $K_r(\alpha) = f_{r(\alpha)}(-1/S(r)) < 1$. From Lemma (C.4) it follows that $S(r) > S(r(\alpha))$.

Proof of Lemma (3.3)***.*** The proof is based on Theorem (5.2). Given a security r, let $w = 1/\alpha$ and $\widetilde{w} = 1/\alpha + r - 1$. Replacing w and $\widetilde{w}$ by their values in (11) we get:

$$\begin{aligned} 1/\alpha + (r_{min} - 1) &< \tfrac{R(r-1)}{\widehat{S}(r(\alpha))} < 1/\alpha + (r_{max} - 1) \\ 1 + \alpha(r_{min} - 1) &< \alpha\tfrac{R(r-1)}{\widehat{S}(r(\alpha))} < 1 + \alpha(r_{max} - 1) \\ \frac{R(r-1)}{1 + \alpha(r_{max} - 1)} &< \tfrac{\widehat{S}(r(\alpha))}{\alpha} < \frac{R(r-1)}{1 + \alpha(r_{min} - 1)}. \end{aligned} \quad (32)$$

As α goes to zero, the values on both sides of expression (32) go to $R(r-1)$.

Proof of Theorems (4.2) ***and*** (4.3)***.*** Let l be a gamble and let the value of w be such that $1 + l/w$ is a security. Lemma (2.7) plus our assumptions that all utilities in U^* have decreasing absolute risk aversion (DARA) and increasing relative risk aversion (IRRA), imply the following:

1. i accepts l at $w \Rightarrow \rho_i(w + l_{max}) < 1/R(l)$.

2. i rejects l at $w \Rightarrow \rho_i(w + l_{min}) > 1/R(l)$.

3. i accepts $1 + l/w$ at $w \Rightarrow \varrho_i(w + l_{min}) < 1/\widehat{S}(1 + l/w) \Rightarrow$
   $\rho_i(w + l_{min}) < 1/(\widehat{S}(1 + l/w)(w + l_{min}))$.

4. i rejects $1 + l/w$ at $w \Rightarrow \varrho_i(w + l_{max}) > 1/\widehat{S}(1 + l/w) \Rightarrow$
   $\rho_i(w + l_{max}) > 1/\big(\widehat{S}(1 + l/w)(w + l_{max})\big)$.

Assume that $\frac{\widehat{S}(r)}{\widehat{S}(k)} > \frac{k_{max}}{r_{min}}$ and by contradiction assume that there exists an agent i, who at w accepts r but rejects k. It follows from observation 3 (by replacing l=w(r-1)) that $\rho_i(wr_{min}) < 1/(\widehat{S}(r)(wr_{min}))$ and from observation 4 (by replacing l=w(k-1)) that $\rho_i(wk_{max}) > 1/\big(\widehat{S}(k)(wk_{max})\big)$. The DARA property of the utility function of agent i implies that $\rho_i(wk_{max}) < \rho_i(wr_{min})$ which implies that $1/(\widehat{S}(r)(wr_{min})) > 1/\big(\widehat{S}(k)(wk_{max})\big)$, in contradiction to the first assumption. A similar reasoning holds for the other claims in Theorems (4.2) and (4.3).

Proof of Theorem (5.2). Denote $r = \widetilde{w}/w$ and $g = \widetilde{w} - w$. The theorem says that

$$\min(\widetilde{w}) < \frac{R(g)}{\widehat{S}(r)} < \max(\widetilde{w}).$$

Assume by contradiction that $1/(\min(\widetilde{w})\widehat{S}(r)) < 1/R(g)$. Now consider a CARA agent i whose parameter ρ satisfies $1/(\min(\widetilde{w})\widehat{S}(r)) < \rho_i < 1/R(g)$. From Lemma (2.7) we get: i rejects g since $\rho_i < 1/R(g)$, but i accepts r since $\min(\widetilde{w})\rho_i > 1/\widehat{S}(r)$, a contradiction. If $1/(\max(\widetilde{w})\widehat{S}(r)) > 1/R(g)$, the reasoning is similar.

Theorem (5.2) implies two corollaries:

Corollary C.7. *Let g be a gamble and let the value of w be such that $1+g/w$ has the properties of securities.*

$$w + \min(g) < \frac{R(g)}{\widehat{S}(1 + g/w)} < w + \max(g) \tag{33}$$

Corollary C.8. *Let r be a security; then*

$$\min(r) < \frac{R(r-1)}{\widehat{S}(r)} < \max(r), \tag{34}$$

In (34) we used the scaling axiom for gambles.

Proof of Theorem (5.3)**.** Assume that

$$\frac{AR(\widetilde{w}_1, w)}{RR(\widetilde{w}_2, w)} < \min(\widetilde{w}). \tag{35}$$

By replacing $\widetilde{w}_1 - w = g$ and $\widetilde{w}_2 - w = h$ in (35), it can be rewritten as

$$\frac{R(g)}{\widehat{S}(1 + h/w)} < \min(\widetilde{w}), \tag{36}$$

where $\min(\widetilde{w})$ is the minimum of $\{w + g, w + h\}$. Now, assume by contradiction that there is an agent i who accepts h at w but rejects g at w (violating the claim that $d_1 \geq_A d_2$). From observation 2 of the proof of (4.2) we conclude that $\rho_i(w + g_{min}) > 1/R(g)$ which implies that $\rho_i(\min(\widetilde{w})) > 1/R(g)$ (the DARA property). From observation 3 we conclude that $\rho_i(w + h_{min}) < 1/(\widehat{S}(1+h/w)(w+h_{min}))$ which implies that $\rho_i(\min(\widetilde{w})) < 1/(\widehat{S}(1+h/w)(\min(\widetilde{w})))$ (the IRRA property). So for holding this it must be that $1/(\widehat{S}(1 + h/w)(\min(\widetilde{w}))) > 1/R(g)$, a contradiction. The same reasoning holds for the second part of Theorem (5.3).

Proof of Theorem (4.8). For the proof of the first part of Theorem (4.8), see Hart (2011). The proof of the second part can be based on the first part of the Theorem. Given a utility function u_i, we define the utility function $\widehat{u}_i$ as follows:

$$\widehat{u}_i(x) = u_i(e^x). \tag{37}$$

It follows that if u_i accepts a security r for all w, then $\widehat{u}_i$ accepts the gamble $\ln r$ for all w. So two securities r and k satisfy $r \geq_{WU} k$ if and only if the two gambles $\ln r$ and $\ln k$ satisfy $\ln r \geq_{WU} \ln k$, and that happens if and only if $R(\ln k) \geq R(\ln r)$, which, by definition, is equivalent to $S(k) \geq S(r)$.

Proof of Theorem (B.3). For the proof of the first part of Theorem (B.3), see Hart (2011). Here we prove the second part.

1. Assume $r \geq_{UU} k$. Recall that r and k are securities but $r-1$ and $k-1$ are gambles. The first step is to show that the two gambles $r-1$ and $k-1$ satisfy $(r-1) \geq_{UU} (k-1)$. To this end, we have to show that if the gamble $(r-1)$ is rejected by all u at w then $(k-1)$ is also rejected by all u at w. Let w be such a level of wealth that $(r-1)$ is rejected by all u at w. For any agent, taking the gamble $(r-1)$ at w results in the same distribution of wealth as investing in a security $r(\alpha)$, where $\alpha = 1/w$. Hence, the security $r(\alpha)$ is rejected by all at w. From the assumption, the security $k(\alpha)$ is rejected by all at w. It follows that the gamble $k-1$ is rejected by all at w, which completes the proof that $(r-1) \geq_{UU} (k-1)$. But since, $S^{FH}(r) = R^{FH}(r-1)$, $R^{FH}(k-1) \geq R^{FH}(r-1) \Rightarrow S^{FH}(k) \geq S^{FH}(r)$.

2. Assume $S^{FH}(r) \leq S^{FH}(k)$. From the assumption and the definition of S^{FH} it follows that for any (positive) α and w, $R^{FH}(w\alpha(r-1)) \leq R^{FH}(w\alpha(k-1))$. It is easy to see that, for any α and w, the gamble $w\alpha(r-1)$ is rejected by all u at w if and only if the security $r(\alpha)$ is rejected by all $u \in U^*$ at w. But if the gamble $w\alpha(r-1)$ is rejected by all at w, it follows from the first part of the theorem that also $w\alpha(k-1)$ is rejected by all at w; hence, $k(\alpha)$ is also rejected by all at w. That completes the proof.

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