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PROBABILITY INEQUALITIES FOR A GLADIATOR GAME

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Probability Inequalities for a Gladiator Game

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Abstract

Based on a model introduced by Kaminsky, Luks, and Nelson (1984), we consider a zero-sum allocation game called the Gladiator Game, where two teams of gladiators engage in a sequence of one-to-one fights in which the probability of winning is a function of the gladiators' strengths. Each team's strategy consist the allocation of its total strength among its gladiators. We find the Nash equilibria of the game and compute its value. To do this, we study interesting majorization-type probability inequalities concerning linear combinations of Gamma random variables.

Keywords and phrases: Allocation game, Colonel Blotto game, David and Goliath, exponential distribution, Nash equilibrium, probability inequalities, unimodal distribution.

MSC 2000 Classification. Primary 60E15, 91A05; secondary 91A60.

1 Introduction

The following Gladiator Game was proposed by Kaminsky et al. (1984). Two teams of gladiators engage in a sequence of one-to-one fights. Each gladiator has a strength parameter. When two gladiators fight, the ratio of their strengths determines the odds of winning. The loser dies and the winner retains his strength and is ready for a new duel. The team that is wiped out loses. Each team chooses the order in which gladiators go to the arena.

We construct a zero-sum two-team game where each team also has to allocate a fixed total strength among its players. The payoff is linear in the probability of winning. We find the Nash equilibria and compute the value of the game under two regimes. The main results are: (i) when the two teams have roughly equal total strengths, the optimal strategy for each team is to divide its total strength equally among all its members; (ii) when the total strength of one team is much larger than that of the other, the stronger team should divide its total strength equally while the weaker team should concentrate all the strength on a single member.

As the payoffs in the game involve the probability of winning, finding Nash equilibria amounts to proving interesting and hard probability inequalities, using techniques which are of independent interest. Much of the paper consists of these proofs.

The Gladiator game is a special case of allocation games and is very close to the classical Colonel Blotto game, whose original formulation goes back to Borel (1921), translated in Borel (1953). The problem was taken up by Borel and Ville (1938), then in a short note by Tukey (1949) and then by Gross and Wagner (1950); Gross (1950); Blackett (1954, 1958); Sion and Wolfe (1957); Friedman (1958); Cooper and Restrepo (1967); Bellman (1969); Penn (1971); Shubik and Weber (1981); Heuer (2001); Roberson (2006); Kvasov (2007); Hart (2008); Adamo and Matros (2009);

Powell (2009); Golman and Page (2009), among others. In the various versions of the Colonel Blotto game the players need to simultaneously distribute limited resources over several objects (or battlefields). In the classic version of the game, a battlefield is won by the player who assigns the most resources to it, and the payoff of a player is equal to the total number of battlefields won. We refer to Kovenock and Roberson (2010); Chowdhury, Kovenock, and Sheremeta (2010) for some history of the Colonel Blotto game and a good list of references.

Ours can be seen as a stochastic version of Colonel Blotto game, where assigning a larger amount of resources to a battlefield increases the probability of winning it, but does not guarantee victory. The model described below for the probability that gladiator i defeats j, is equivalent, with different parametrization, to the well-known Rasch model in educational statistics, (Rasch, 1960), in which the probability of correct response of subject i to item j is $\mathrm{e}^{\alpha_i - \beta_j} / (1 + \mathrm{e}^{\alpha_i - \beta_j})$ (see Lauritzen, 2008, for a recent study of Rasch models). The model presents also some analogy with the classical probability of ruin problem (see, e.g., Feller, 1968).

De Schuymer, De Meyer, and De Baets (2006) consider a dice game that has some analogies with ours. Both players can choose one of many dice having n faces and such that the total number of pips on the faces of the die is σ. The two dice are tossed and the player with the highest score wins a dollar.

The inequalities studied here are related to majorization type inequalities for linear combinations of Gamma variables that appear in Bock, Diaconis, Huffer, and Perlman (1987); Diaconis and Perlman (1990); Székely and Bakirov (2003); Yu (2008) and references therein.

The proof of the main theorems turned out to be more complicated than expected. We rely on Székely and Bakirov (2003) for some of the technical machinery. The problem is cast as a minimization problem involving convolutions of Gamma variables and is solved by perturbation arguments. A key identity, derived using Laplace transforms, directs our perturbation arguments to the analysis of the modal location of such Gamma convolutions, which are unimodal by self-decomposability. The analysis also involves applied probability notions such as the likelihood ratio order.

In Section 2 we describe the model. In Section 3 we compute the Nash equilibria and the value of the game. Section 4 contains related probability inequalities and formulations in terms of a variety of distributions, which follow from our main result and have some interest *per se*. Section 5 is devoted to proofs.

2 The model

We formalize the model described in the Introduction. Two teams of gladiators fight each other according to the following rules. Team A has m gladiators and total strength c_A. Team B has n gladiators and a total strength c_B. The numbers m, n, c_A, c_B are exogenously given. The coach of each team can decide how to allocate the total strength to the gladiators of the team. Let $\boldsymbol{a} = (a_1, \ldots, a_m)$ and $\boldsymbol{b} =$

$(b_1, \ldots, b_n)$ be the strength vectors of team A and B, respectively. This means that for team A the first gladiator who goes to fight has strength a_1, the second a_2, and so on. Analogously for team B. Note that the game is anonymous in the gladiators and each one of them is characterized only by his strength.

When a gladiator with strength a_i fights a gladiator with strength b_j, the first defeats the second with probability $a_i/(a_i + b_j)$, all fights being independent. The loser dies and cannot fight again, whereas the winner retains his whole strength and can be sent again to fight immediately or later. The fights go on until one team has lost all its gladiators. In this case the opposing team is declared the winner. Call $G_{m,n}(\boldsymbol{a}, \boldsymbol{b})$ the probability that team A with strength vector $\boldsymbol{a}$ wins over team B with strength vector $\boldsymbol{b}$.

We say that $X \sim \text{Exp}(1)$ if X has a standard exponential distribution, i.e., $\mathbb{P}(X > x) = \mathrm{e}^{-x}$ for $x > 0$.

Proposition 2.1. *[Kaminsky et al. (1984)] The probability $G_{m,n}(\boldsymbol{a}, \boldsymbol{b})$ of team A defeating B is*

$$G_{m,n}(\boldsymbol{a}, \boldsymbol{b}) = \mathbb{P}\left(\sum_{i=1}^{m} a_i X_i > \sum_{j=1}^{n} b_j Y_j \right), \tag{2.1}$$

where $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ are i.i.d. random variables, with $X_1 \sim \text{Exp}(1)$.

The implication of Proposition 2.1 is that the order in which the gladiators should be sent to fight is irrelevant to their teams' chances of winning. In particular, knowing the opposing team's order is irrelevant. Two vectors of strengths that are equal up to a permutation produce the same probability of victory.

We consider now the following zero-sum two-person game

$$\mathscr{G}(m, n, c_A, c_B) = \langle \mathscr{A}(m, c_A), \mathscr{B}(n, c_B), H_{m,n} \rangle \tag{2.2}$$

in which team A chooses $\boldsymbol{a} \in \mathscr{A}(m, c_A)$ and B chooses $\boldsymbol{b} \in \mathscr{B}(n, c_B)$, where

$$\mathscr{A}(m, c_A) = \left\{ (a_1, \ldots, a_m) \in \mathbb{R}_+^m : \sum_{i=1}^{m} a_i = c_A \right\}, \tag{2.3}$$

$$\mathscr{B}(n, c_B) = \left\{ (b_1, \ldots, b_n) \in \mathbb{R}_+^n : \sum_{i=1}^{n} b_i = c_B \right\}, \tag{2.4}$$

$$H_{m,n} = G_{m,n} - \frac{1}{2}. \tag{2.5}$$

The payoff of team A is then its probability of victory $G_{m,n}(\boldsymbol{a}, \boldsymbol{b})$ minus $1/2$. We subtracted $1/2$ to make the game zero-sum.

3 Main results

Consider the game $\mathscr{G}$ defined in (2.2). The action $\boldsymbol{a}^*$ is a best response against $\boldsymbol{b}$ if

$$\boldsymbol{a}^* \in \arg\max_{\boldsymbol{a}\in\mathscr{A}} H_{m,n}(\boldsymbol{a}, \boldsymbol{b}).$$

A pair of actions $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is a *Nash equilibrium* of the game $\mathscr{G}$ if

$$H_{m,n}(\boldsymbol{a}, \boldsymbol{b}^*) \le H_{m,n}(\boldsymbol{a}^*, \boldsymbol{b}^*) \le H_{m,n}(\boldsymbol{a}^*, \boldsymbol{b}), \quad \text{for all } \boldsymbol{a} \in \mathscr{A}(m, c_A) \text{ and } \boldsymbol{b} \in \mathscr{B}(n, c_B).$$

A pair of actions $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is a *minmax solution* of the game $\mathscr{G}$ if

$$\max_{\boldsymbol{a}\in\mathscr{A}(m,c_A)} \min_{\boldsymbol{b}\in\mathscr{B}(n,c_B)} H_{m,n}(\boldsymbol{a}, \boldsymbol{b}) = \min_{\boldsymbol{b}\in\mathscr{B}(n,c_B)} \max_{\boldsymbol{a}\in\mathscr{A}(m,c_A)} H_{m,n}(\boldsymbol{a}, \boldsymbol{b}) = H_{m,n}(\boldsymbol{a}^*, \boldsymbol{b}^*).$$

Since we are dealing with a zero-sum game, Nash equilibria and minmax solutions coincide (see, e.g., Osborne and Rubinstein, 1994, Proposition 22.2). The quantity $H_{m,n}(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is called the *value* of the game $\mathscr{G}$.

The next theorem shows that when the total strengths of the two teams are not too different, the game has a unique Nash equilibrium where each gladiator in a team gets the same strength.

Theorem 3.1. *Consider the game* $\mathscr{G}(m, n, c_A, c_B)$ *defined in* (2.2), *with*

$$\frac{m-1}{m} c_A \le c_B \le \frac{n}{n-1} c_A. \tag{3.1}$$

Then the unique Nash equilibrium of the game $\mathscr{G}$ *consists of the pair of strategies* $\boldsymbol{a}^*, \boldsymbol{b}^*$ *satisfying* $a_1^* = \cdots = a_m^*$ *and* $b_1^* = \cdots = b_n^*$.

The next result states that when one team is much stronger than the other, then the stronger team's Nash strategy is to divide its total strength equally between its members, while the team with the smaller total strength should assign all its strength to a single gladiator. Thus, the strategies proposed by the Philistine Goliath to the Israelites to send out a champion of their own to decide the outcome in a single combat with him (Samuel 1, chapter 17) was not optimal for the Philistines, as being the stronger side they should have divided their strength equally rather than concentrate it in one Goliath; it was optimal for the weaker Israelites.

Theorem 3.2. *Consider the game* $\mathscr{G}(m, n, c_A, c_B)$ *defined in* (2.2), *with*

$$c_B > \frac{2n}{n-1} c_A > 0. \tag{3.2}$$

If $\boldsymbol{a}^*, \boldsymbol{b}^*$ *are such that* $a_i^* = c_A$ *for some* $i \in \{1, \dots, m\}$ *and* $a_j^* = 0$ *for all* $j \ne i$, *and* $\boldsymbol{b}^*$ *has equal components, that is,* $b_1^* = \cdots = b_n^* = c_B/n$, *then* $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ *is Nash equilibrium of* $\mathscr{G}$, *and all Nash equilibria are of this form.*

In order to compute the value of the game $\mathscr{G}(m, n, c_A, c_B)$, we need the regularized incomplete beta function

$$I(x, \alpha, \beta) = \frac{1}{\mathrm{B}(\alpha, \beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1} \, \mathrm{d}t, \tag{3.3}$$

where

$$\mathrm{B}(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, \mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

When α and β are integers, then

$$I(x, \alpha, \beta) = \sum_{j=\alpha}^{\alpha+\beta-1} \binom{\alpha+\beta-1}{j} x^j (1-x)^{\alpha+\beta-1-j}. \tag{3.4}$$

For properties of incomplete beta functions see, for instance, Olver, Lozier, Boisvert, and Clark (2010).

Theorem 3.3. *Consider the game $\mathscr{G}(m, n, c_A, c_B)$.*

(i) *If* (3.1) *holds, then the value of the game is*

$$\frac{1}{2} - I\left(\frac{mc_B}{mc_B + nc_A}, m, n\right). \tag{3.5}$$

(ii) *If* (3.2) *holds, then the value of the game is*

$$\frac{1}{2} - I\left(\frac{c_B}{c_B + nc_A}, 1, n\right). \tag{3.6}$$

FIGURE 1 ABOUT HERE

We mention the following consequence of Theorem 3.1.

Corollary 3.4. *In the game $\mathscr{G}(m, n, c_A, c_B)$, if the two teams have equal strength (i.e., $c_A = c_B$), then the value is positive if $m > n$, namely, the team with more players has an advantage over the other team. Moreover, the value of the game is increasing in m and decreasing in n (see Figure 1).*

FIGURE 2 ABOUT HERE

Figure 2 shows an interesting implication of Theorem 3.3: team A may be at a disadvantage even if $c_A > c_B$, and this happens if the number m of its gladiators is much smaller than the number n of gladiators in B. As the relative difference in strength between the two teams increases, it takes a larger number of gladiators to compensate for the lower strength.

FIGURES 3 AND 4 ABOUT HERE

As Figures 3 and 4 show, if condition (3.2) holds, then team A is at a strong disadvantage. The disadvantage increases with the total strength c_B and the number n of gladiators of team B. The number m of gladiators of team A is totally irrelevant, since, in equilibrium, the whole strength c_A is assigned to only one gladiator.

The following theorem is the main tool to prove Theorems 3.1 and 3.2.

Theorem 3.5. *Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$, $m, n \geq 1$, be i.i.d. random variables with $X_1 \sim \mathrm{Exp}(1)$. For fixed $b > 0$, let $\mathscr{A}$ be as in (2.3) and let*

$$(a_1^*, \dots, a_m^*) \in \arg \min_{\boldsymbol{a} \in \mathscr{A}(m,m)} \mathbb{P}\left(\sum_{i=1}^{m} a_i X_i \leq b \sum_{j=1}^{n} Y_j \right).$$

(i) *If $0 < b < m/(n-1)$, then $a_1^* = \cdots = a_m^* = 1$.*

(ii) *If $b \geq 2m/(n-1)$, then $a_i^* = m$ for some $1 \leq i \leq m$ and $a_j^* = 0$, for $j \neq i$.*

For $n = 1$, there is no restriction on b, i.e., equal strength is team A's the best strategy against any team B with a single player. The same holds when team B has $n \geq 2$ equal strength players with less than $n/(n-1)$ times team A's total strength. When the opponent team B has $n \geq 2$ players with more than $2n/(n-1)$ times A's total strength, however, then A's best strategy is to allocate all strength to one player.

Note that the teams have equal total strength when $\sum_{i=1}^{m} a_i = m = bn$, in which case part (i) above applies. Therefore, the best response to a team of equal total strength and equal gladiators, is a team of equal gladiators.

The above results concern an egalitarian best response and one that concentrates all the strength on a single gladiator. In some situations these two extreme strategies are both inferior to assigning all the strength to some (more than one) but not all players, as the following example shows.

Example 3.6. Consider $\mathscr{G}$ with $c_A = 0.7$, $c_B = 1$, $m = 4$, $n = 5$. Then the best response to $\boldsymbol{b} = (1/5, 1/5, 1/5, 1/5, 1/5)$ is neither $\boldsymbol{a} = (0.7, 0, 0, 0)$ nor $\boldsymbol{a} = (0.7/4, 0.7/4, 0.7/4, 0.7/4)$; a better strategy is $\boldsymbol{a} = (0.7/3, 0.7/3, 0.7/3, 0)$.

It is not clear whether the best response against an arbitrary opponent is always in the form of equally dividing the total strength to some (not necessarily all) players.

4 Related probability inequalities

If $X_1, \dots, X_m$, and $Y_1, \dots, Y_n$ are i.i.d. random variables with $X_1 \sim \mathrm{Exp}(1)$, and

$$\bar{X} = \frac{1}{m} \sum_{i=1}^{m} X_i, \quad \bar{Y} = \frac{1}{n} \sum_{j=1}^{n} Y_j, \quad Z = \frac{m\bar{X}}{m\bar{X} + n\bar{Y}},$$

then Z has a Beta(m,n) distribution. Hence

$$\mathbb{P}(\bar{X} < \bar{Y}) = \mathbb{P}\left(Z < \frac{m}{m+n}\right) = I\left(\frac{m}{m+n}, m, n\right). \tag{4.1}$$

For $m > n$, by Corollary 3.4, we have

$$\mathbb{P}(\bar{X} < \bar{Y}) < \frac{1}{2}. \tag{4.2}$$

Since $\mathbb{E}[Z] = m/(m+n)$, (4.2) is equivalent to $\mathbb{P}\left(Z < \mathbb{E}[Z]\right) < 1/2$, that is, $\mathbb{E}[Z] < \mathrm{Med}[Z]$. This is a well known mean-median inequality for beta distributions (see Groeneveld and Meeden, 1977).

The inequality (4.2) has the following interesting statistical implication. If two statisticians estimate the mean of exponential variables, and use the sample mean as their unbiased estimate, then the statistician with the larger sample tends to have a larger (unbiased) estimate. If the two of them bet on who has a larger estimate, the one with the larger sample tends to win. For normal variables, or any symmetric variables, this clearly cannot happen and $\mathbb{P}(\bar{X} < \bar{Y}) = 1/2$.

Suppose now that the two statisticians share the first n variables, that is, for $i = 1, \ldots, n$ we have $X_i = Y_i$, and the remaining variables $X_{n+1}, \ldots, X_m$ are independent of the previous ones. Then

$$\begin{aligned}\mathbb{P}(\bar{X} < \bar{Y}) &= \mathbb{P}\left(\frac{1}{m}\left[\sum_{j=1}^{n} Y_j + \sum_{i=n+1}^{m} X_i\right] < \frac{1}{n}\sum_{j=1}^{n} Y_j\right) \\ &= \mathbb{P}\left(\frac{1}{m-n}\sum_{i=n+1}^{m} X_i < \frac{1}{n}\sum_{j=1}^{n} Y_j\right). \end{aligned} \tag{4.3}$$

By (4.2) the last expression in (4.3) is less than $1/2$ if and only if $m - n > n$, that is, $m > 2n$. It equals $1/2$ if $m = 2n$, and it is larger than $1/2$ if $m < 2n$, in which case (4.2) is reversed. Thus in the bet between the statisticians, if most of the variables are in common, the odds are against the one with the larger sample, contrary to the previous situation. This was noted by Abram Kagan.

Our main results can be presented in terms of various other distributional inequalities or monotonicity. Using (3.4) and Corollary 3.4 we obtain further results that cannot easily be proved more directly. We say that $X \sim \mathrm{Gamma}(\alpha, \beta)$ if X has a density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, \mathrm{e}^{-\beta x}\, x^{\alpha-1},\ \ x > 0.$$

Corollary 4.1. (i) *The function*

$$I\left(\frac{m}{m+n}, m, n\right)$$

is decreasing in m for fixed n, and increasing in n for fixed m.

(ii) *Let* $T \sim \text{Binom}(m+n-1, m/(m+n))$*. Then* $\mathbb{P}(T \geq m)$ *is decreasing in* m *and increasing in* n.

(iii) $\mathbb{P}(S \geq m)$ *is decreasing in* m*, where* $S \sim \text{Poisson}(m)$.

(iv) *Let* $R \sim \text{Gamma}(m, 1)$*. Then* $\mathbb{P}(R \leq m)$ *is decreasing in* m.

We say that a random variable $Q \sim \text{Geom}(p)$ if $\mathbb{P}(Q_1 = k) = (1-p)^k p$, $k = 0, 1, 2, \ldots$.

Proposition 4.2. *Let* $Q_1, \ldots, Q_m$ *be independent random variables such that* $Q_i \sim \text{Geom}(1/(1+a_i))$*. Define* $Q = \sum_{i=1}^m Q_i$.

(i) *We have*

$$1 - G_{m,n}(\boldsymbol{a}, \mathbf{1}_n) = \mathbb{P}\left(Q \leq n-1\right), \tag{4.4}$$

where $\boldsymbol{a} = (a_1, \ldots, a_m)$ *and* $\mathbf{1}_n$ *denotes the* n*-dimensional vector of ones.*

(ii) *If* $\sum_{i=1}^m a_i = n$*, then the probability in (4.4) is minimized when all* a_i*'s are equal. In this case* Q_i *are i.i.d. and* Q *has a negative binomial distribution.*

(iii) *If* $\mathbb{E}[Q] = m$*, then* $\mathbb{E}[Q] > \text{Med}[Q]$.

5 Proofs

Proof of Proposition 2.1. First note that if X, Y are i.i.d. random variables with $X \sim \text{Exp}(1)$, then $\mathbb{P}(aX > bY) = a/(a+b)$. Therefore, one can see a duel between gladiators i and j as a competition in which the probability of winning is the probability of living longer, when their lifetimes are $a_i X$ and $b_j Y$, respectively. It is then easy to understand that the teams' total lives are $\sum_{i=1}^m a_i X_i$ and $\sum_{j=1}^n b_j Y_j$, and the probability that team A wins is that it lives longer, which is $G_{m,n}(\boldsymbol{a}, \boldsymbol{b})$, so (2.1) follows. □

In order to prove Theorem 3.5 we need several preliminary results. Let G_1, G_2, Z_1, Z_2 be independent with $G_i \sim \text{Gamma}(u_i, 1)$, $Z_i \sim \text{Exp}(1)$, for $i = 1, 2$. For $u_i = 0$ we define $G_i = 0$ with probability 1.

Lemma 5.1. *Given* a_1^*, a_2^**, set* $a_1 = a_1^* + \delta/u_1$ *and* $a_2 = a_2^* - \delta/u_2$*. Then*

$$\frac{\partial}{\partial \delta}\mathbb{P}(a_1 G_1 + a_2 G_2 \leq x) = (a_1 - a_2)\frac{\partial^2}{\partial x^2}\mathbb{P}(a_1(G_1 + Z_1) + a_2(G_2 + Z_2) \leq x). \tag{5.1}$$

Proof. Let

$$F(x) = \mathbb{P}(a_1 G_1 + a_2 G_2 \leq x)$$
$$H(x) = \mathbb{P}(a_1 G_1 + a_2 G_2 + a_1 Z_1 + a_2 Z_2 \leq x)$$

and let f and h denote the corresponding densities. Let $\mathscr{L}$ denote the Laplace transform, that is,

$$\mathscr{L}(F) = \int_0^\infty \mathrm{e}^{-tx}\, F(x)\, \mathrm{d}x.$$

Note that (5.1) is equivalent to

$$\mathscr{L}\left(\frac{\partial}{\partial\delta}F(x)\right) = (a_1 - a_2)\mathscr{L}\left(\frac{\partial^2}{\partial x^2}H(x)\right). \tag{5.2}$$

Using integration by parts we get

$$\mathscr{L}\left(\frac{\partial^2}{\partial x^2}H(x)\right) = t\int_0^\infty \mathrm{e}^{-tx}\, h(x)\, \mathrm{d}x = t\; \mathbb{E}[\exp\{-t(a_1G_1 + a_2G_2 + a_1Z_1 + a_2Z_2)\}].$$

For the left hand side of (5.2) note that we can interchange differentiation and integration, and also that

$$\frac{\partial}{\partial\delta}\mathscr{L}(F(x)) = \mathscr{L}(F(x))\frac{\partial}{\partial\delta}\log\mathscr{L}(F(x)).$$

Again by integration by parts we have

$$\mathscr{L}(F(x)) = \frac{1}{t}\mathscr{L}(f(x)) = \frac{1}{t}\mathbb{E}[\exp\{-t(a_1G_1 + a_2G_2)\}].$$

It follows that (5.2) is equivalent to

$$\frac{1}{t}\frac{\partial}{\partial\delta}\log\mathscr{L}(f(x)) = (a_1 - a_2)\; t\; \mathbb{E}[\exp\{-t(a_1Z_1 + a_2Z_2)\}]. \tag{5.3}$$

Explicitly this becomes

$$\frac{1}{t}\frac{\partial}{\partial\delta}\log[(1 + a_1t)^{-u_1}(1 + a_2t)^{-u_2}] = (a_1 - a_2)t(1 + a_1t)^{-1}(1 + a_2t)^{-1}. \tag{5.4}$$

Using $a_1 = a_1^* + \delta/u_1$, and $a_2 = a_2^* - \delta/u_2$, (5.4) is verified by a straightforward calculation. □

A related result to Lemma 5.1, with a similar type of proof, appears in Székely and Bakirov (2003).

Lemma 5.2. *Given a nonnegative vector* $(a_1^*, \ldots, a_m^*)$*, let*

$$a_1 = a_1^* + \delta/u_1, \quad a_2 = a_2^* - \delta/u_2, \quad a_i = a_i^* \text{ for } 3 \le i \le m.$$

Define

$$Q(\boldsymbol{a}, \boldsymbol{u}) = \sum_{i=1}^m a_iG_i - b\sum_{j=1}^n Y_j, \tag{5.5}$$

where $(\boldsymbol{a}, \boldsymbol{u}) := (a_1, \ldots, a_m, u_1, \ldots, u_m)$, $G_1, \ldots, G_m, Y_1, \ldots, Y_n$ *are independent random variables with* $G_i \sim \text{Gamma}(u_i, 1)$, *for* $i = 1, \ldots, m$ *and* $Y_j \sim \text{Exp}(1)$, *for* $j = 1, \ldots, n$. *Let* $Z_i \sim \text{Exp}(1)$, *for* $i = 1, 2$ *be independent of all other variables. Then*

$$\frac{\partial}{\partial \delta} \mathbb{P}(Q(\boldsymbol{a}, \boldsymbol{u}) \le x) = (a_1 - a_2) \frac{\partial^2}{\partial x^2} \mathbb{P}(Q(\boldsymbol{a}, \boldsymbol{u}) + a_1 Z_1 + a_2 Z_2 \le x). \tag{5.6}$$

Proof. Set $T = \sum_{i=3}^m a_i G_i - b \sum_{j=1}^n Y_j$. Then

$$\frac{\partial}{\partial \delta} \mathbb{P}(Q(\boldsymbol{a}, \boldsymbol{u}) \le x | T) = (a_1 - a_2) \frac{\partial^2}{\partial x^2} \mathbb{P}(Q(\boldsymbol{a}, \boldsymbol{u}) + a_1 Z_1 + a_2 Z_2 \le x | T), \tag{5.7}$$

which is equivalent to (5.1) with a different x. Taking the expectation in (5.7) over T yields (5.6). □

Lemma 5.3. *Let* X *and* Y *be independent random variables where* $Y \sim \text{Exp}(1)$ *and* X *has a density* $f(x)$ *such that*

(i) $f(x)$ *is continuously differentiable with a bounded derivative on* $(-\infty, \infty)$,

(ii) $f(x) > 0$ *for sufficiently small* $x \in (-\infty, \infty)$,

(iii) $f(x)$ *is unimodal, i.e., there exists* $a \in (-\infty, \infty)$ *such that* $f'(x) \ge 0$ *if* $x < a$ *and* $f'(x) \le 0$ *if* $x > a$.

For $\lambda > 0$, *denote the density of* $X + \lambda Y$ *by* $f_\lambda(x)$. *Then* $f_\lambda(x)$ *is unimodal and if* $f'_\lambda(x_0) = 0$ *then* x_0 *is a mode of* f_λ. *Moreover, if* $\lambda > \lambda_0 > 0$, *then any mode of* $f_\lambda(x)$ *is strictly larger than any mode of* $f_{\lambda_0}(x)$.

Proof. This result is similar to Székely and Bakirov (2003, Lemma 1). We provide a quick proof using variation diminishing properties of sign regular kernels (see Karlin, 1968). First, since the density of λY is log-concave (a.k.a. strongly unimodal) its convolution with the unimodal $f(x)$ is also unimodal, that is, the pdf of $X + \lambda Y$ is unimodal (see Ibragimov, 1956; Karlin, 1968).

Differentiating yields

$$\begin{aligned} f'_\lambda(x) &= \int_0^\infty f'(x - z) \frac{1}{\lambda} \mathrm{e}^{-z/\lambda} \, \mathrm{d}z \\ &= \int_{-\infty}^x f'(z) \frac{1}{\lambda} \mathrm{e}^{(z-x)/\lambda} \, \mathrm{d}z \\ &= \frac{\mathrm{e}^{-x/\lambda}}{\lambda} \int 1_{(-\infty, x)}(z) f'(z) \mathrm{e}^{z/\lambda} \, \mathrm{d}z. \end{aligned}$$

Suppose $f'_\lambda(x_0) = 0$. Since $f'(z) \ge 0$ for $z \le a$, we know $f'_\lambda(x) > 0$ if $x \le a$, and hence $x_0 > a$. The function $\mathrm{e}^{x/\lambda} f'_\lambda(x)$ is nonincreasing in $x \in (a, \infty)$. Therefore $f'_\lambda(x) \ge 0$ if $x \in (a, x_0)$ and $f'_\lambda(x) \le 0$ if $x > x_0$. It follows that x_0 is a mode of $f_\lambda(x)$.

For fixed x, the function $1_{(-\infty,x)}(z)f'(z)$ as a function of z has at most one sign change from positive to negative, and the kernel $\mathrm{e}^{z/\lambda}$ is strictly reverse rule (see Karlin, 1968). It follows that $\int 1_{(-\infty,x)}(z)f'(z)\,\mathrm{e}^{z/\lambda}\;\mathrm{d}z$ has at most one sign change from negative to positive, as a function of λ. Thus, if for a given x, $f'_{\lambda_0}(x) = 0$ and $\lambda > \lambda_0$, then $f'_\lambda(x) > 0$, and the result follows. □

Lemma 5.4. *Let $\lambda_i, \alpha_i > 0,\ i = 1, 2$, and let X and Y be independent with $X \sim \text{Gamma}(\alpha_1, 1)$ and $Y \sim \text{Gamma}(\alpha_2, 1)$. Let $f(x)$ denote the pdf of $\lambda_1 X - \lambda_2 Y$. If $\alpha_1 \geq 2$, then $f'(0)$ and $\lambda_1(\alpha_1 - 1) - \lambda_2(\alpha_2 - 1)$ have the same sign.*

Proof. We have

$$f(x) = C \int_0^\infty (x+y)^{\alpha_1 - 1}\,\mathrm{e}^{-(x+y)/\lambda_1}\,y^{\alpha_2 - 1}\,\mathrm{e}^{-y/\lambda_2}\;\mathrm{d}y, \quad x \geq 0,$$

where C denotes a positive constant that changes in the proof. Since $\alpha_1 \geq 2$, differentiating under the integral sign is permitted. We get

$$f'(x) = C \int_0^\infty (x+y)^{\alpha_1 - 2}\left(\alpha_1 - 1 - \frac{x+y}{\lambda_1}\right)\mathrm{e}^{-(x+y)/\lambda_1}\,y^{\alpha_2 - 1}\,\mathrm{e}^{-y/\lambda_2}\;\mathrm{d}y, \quad x \geq 0.$$

In particular, easy calculations lead to $f'(0) = C[\lambda_1(\alpha_1 - 1) - \lambda_2(\alpha_2 - 1)]$, thus proving the lemma. □

Proof of Theorem 3.5. Let $Q(\boldsymbol{a}, \boldsymbol{u})$ be as in (5.5). Let

$$\Omega = \left\{\boldsymbol{a}, \boldsymbol{u} : 0 \leq a_i \leq m,\ u_i \geq 0,\ \sum_{i=1}^m u_i \leq m,\ \sum_{i=1}^m a_i u_i = m\right\}$$

and consider the minimization problem

$$\min_{\boldsymbol{a}, \boldsymbol{u} \in \Omega} \mathbb{P}(Q(\boldsymbol{a}, \boldsymbol{u}) \leq 0). \tag{5.8}$$

This is slightly more general than Theorem 3.5, and is more amenable to perturbation arguments. Since Ω is compact, and $\mathbb{P}(Q \leq 0)$ is continuous in $(\boldsymbol{a}, \boldsymbol{u})$, the minimum is attained, say, at $(\boldsymbol{a}^*, \boldsymbol{u}^*) \in \Omega$.

Claim 5.5. *We can assume $a_i^* u_i^* > 0$ for all $1 \leq i \leq m$.*

Proof. Otherwise suppose $a_1^* u_1^* > 0$ but $a_2^* u_2^* = 0$. Then we may modify $(\boldsymbol{a}^*, \boldsymbol{u}^*) \to (\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{u}})$ without changing the distribution of Q, by setting $\tilde{a}_2 = a_1^*,\ \tilde{u}_1 = \tilde{u}_2 = u_1^*/2$, and copying the rest of the entries of $(\boldsymbol{a}^*, \boldsymbol{u}^*)$. Note that we still have $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{u}}) \in \Omega$. □

Claim 5.6. *All a_i^* in any minimizing point $(\boldsymbol{a}^*, \boldsymbol{u}^*)$ of $\mathbb{P}(Q \leq 0)$ are equal.*

Proof. Assume the contrary, say $a_1^* \le \cdots \le a_m^*$ and $a_1^* < a_m^*$. We first show that for $m \ge 3$ we may assume $a_1^* \le a_2^* < a_m^*$; for $m = 2$ we have to consider only Case 1 below, and a separate argument will be given. Otherwise, if all $a_i^* = a_m^*,\ 2 \le i \le m$, then we may modify $(\boldsymbol{a}^*, \boldsymbol{u}^*) \to (\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{u}})$ without changing the distribution of Q by setting $\tilde{a}_1 = \tilde{a}_2 = a_1^*$, $\tilde{u}_1 = \tilde{u}_2 = u_1^*/2$, $\tilde{a}_i = a_m^*$, $\tilde{u}_i = \sum_{j=2}^m u_j^*/(m-2)$, $3 \le i \le m$. Hence we may assume the existence of a minimizer satisfying $a_1^* \le a_2^* < a_m^*$. Note that we need $m \ge 3$ for this construction.

Let $a_1 = a_1^* + \delta/u_1^*,\ a_2 = a_2^* - \delta/u_2^*,\ a_i = a_i^*,\ 3 \le i \le m$, and $\boldsymbol{u} = \boldsymbol{u}^*$. Then by (5.6) we have

$$\frac{\partial}{\partial \delta}\mathbb{P}(Q(\boldsymbol{a},\boldsymbol{u}) \le x) = (a_1 - a_2)\frac{\partial^2}{\partial x^2}\mathbb{P}(Q(\boldsymbol{a},\boldsymbol{u}) + a_1 Z_1 + a_2 Z_2 \le x), \tag{5.9}$$

where Z_1 and Z_2 are i.i.d. random variables with $Z_1 \sim \mathrm{Exp}(1)$, independent of Q. We can focus on $x = 0$.

Case 1. $a_1^* < a_2^*$. Since $\delta = 0$ achieves the minimum, both sides of (5.9) with $x = 0$ vanish at $\delta = 0$. The density function of $Q(\boldsymbol{a}^*, \boldsymbol{u}^*) + a_1^* Z_1$ is positive everywhere and is unimodal by self-decomposability (see Steutel and van Harn, 2004, Chapter V). By Lemma 5.3, $S = Q(\boldsymbol{a}^*, \boldsymbol{u}^*) + a_1^* Z_1 + a_2^* Z_2$ has a mode at zero.

Case 2. $a_1^* = a_2^*$. Then (5.9) gives

$$\lim_{\delta \downarrow 0} \frac{\partial \mathbb{P}(Q(\boldsymbol{a},\boldsymbol{u}) \le 0)}{\partial \delta} = 0$$

and

$$\frac{\partial^2}{\partial \delta^2}\mathbb{P}(Q(\boldsymbol{a},\boldsymbol{u}) \le 0)\bigg|_{\delta=0} = \left(\frac{1}{u_1^*} + \frac{1}{u_2^*}\right) \lim_{\delta \to 0} \frac{\partial^2}{\partial x^2}\mathbb{P}(Q(\boldsymbol{a},\boldsymbol{u}) + a_1 Z_1 + a_2 Z_2 \le x)\bigg|_{x=0}.$$

A minimum at $\delta = 0$ entails

$$\frac{\partial^2}{\partial x^2}\mathbb{P}(Q(\boldsymbol{a}^*,\boldsymbol{u}^*) + a_1^* Z_1 + a_2^* Z_2 \le x)\bigg|_{x=0} \ge 0,$$

showing that $S = Q(\boldsymbol{a}^*, \boldsymbol{u}^*) + a_1^* Z_1 + a_2^* Z_2$ has a mode that is nonnegative.

Thus S has a nonnegative mode in either case.

In order to prove Claim 5.6, consider first the case $m \ge 3$. By Lemma 5.3, any mode of $Q(\boldsymbol{a}^*, \boldsymbol{u}^*) + a_1^* Z_1 + a_m^* Z_2$ is strictly positive, i.e.,

$$\frac{\partial^2}{\partial x^2}\mathbb{P}(Q(\boldsymbol{a}^*,\boldsymbol{u}^*) + a_1^* Z_1 + a_m^* Z_2 \le x)\bigg|_{x=0} > 0.$$

The latter expression, multiplied by $(a_1 - a_m)$ is negative. Using (5.9) with (a_m^*, u_m^*) in place of (a_2^*, u_2^*), however, this implies that $\mathbb{P}(Q(\boldsymbol{a},\boldsymbol{u}) \le 0)$ strictly decreases under the perturbation $(a_1^*, a_m^*) \to (a_1^* + \delta/u_1^*, a_m^* - \delta/u_m^*)$ for small $\delta > 0$, which is a contradiction to the minimality at $\delta = 0$. Note that the crux of the proof is in comparing two perturbations.

When $m = 2$, we can assume by Claim 5.5 that $a_i^* u_i^* > 0$ $i = 1, 2$. Fix now u_i^*, $i = 1, 2$ and set $u_1 = u_2 = u_1^*/2$ and $u_3 = u_2^*$. Consider the problem

$$\min_{\boldsymbol{a}} \mathbb{P}(Q(\boldsymbol{a}) \le 0) \tag{5.10}$$

$$\text{s.t. } 0 \le a_i \le 2, \ \sum_{i=1}^{3} u_i \le 2, \ \sum_{i=1}^{3} a_i u_i = 2, \tag{5.11}$$

where

$$Q(\boldsymbol{a}) = \sum_{i=1}^{3} a_i G_i - b \sum_{j=1}^{n} Y_j,$$

and $G_1, G_2, G_3, Y_1, \ldots, Y_n$ are all independent with $G_i \sim \text{Gamma}(u_i, 1)$, $Y_i \sim \text{Exp}(1)$. If the solution is attained for $a_1 = a_2 = a_3$ then Claim 5.6 follows readily. Otherwise, when the a_i's are not all equal, we can repeat the above argument of comparing two perturbations, and again arrive at a contradiction to $a_1 = a_2 = a_3$. □

Define $\eta := \sum_{i=1}^{m} u_i^*$, and henceforth assume $a_1^* = \cdots = a_m^*$.

Claim 5.7. *If* $b \ge 2m/(n-1)$ *then* $\eta = 1$.

Proof. Assume the contrary, i.e., $\eta > 1$, and therefore, by definition of Ω, $a_1^* < m$. As shown above, $S = Q(\boldsymbol{a}^*, \boldsymbol{u}^*) + a_1^* Z_1 + a_2^* Z_2$ has a nonnegative mode. However, since all a_i^* are equal, $S \sim a_1^* G - bY$ where here $G \sim \text{Gamma}(\eta + 2, 1)$ and $Y \sim \text{Gamma}(n, 1)$ independently. Noting $(\eta+1)a_1^* = m + a_1^* < 2m \le (n-1)b$, by Lemma 5.4, any mode of S is strictly negative, a contradiction. □

Based on Claim 5.7 and the definition of Ω, if $b \ge 2m/(n-1)$, then a solution of the minimization problem (5.8) is $a_1^* = m$, $u_1^* = 1$, $a_i^* u_i^* = 0$, $2 \le i \le m$, and any solution satisfies $\sum_{i=1}^{m} a_i G_i \sim m\text{Gamma}(1, 1)$.

This proves part (ii) of Theorem 3.5, that is, allocating all strength to one player is the unique optimal strategy for the original problem.

Claim 5.8. *If* $0 < b < m/(n-1)$ *then* $\eta = m$.

Proof. Assume the contrary, i.e., $\eta < m$, and consider $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{u}}) \in \Omega$ defined by $\tilde{u}_1 = \eta$, $\tilde{a}_1 = a_1^* - \delta/\eta$, $\tilde{u}_2 = m - \eta$, $\tilde{a}_2 = \delta/(m-\eta)$, and $\tilde{a}_i = \tilde{u}_i = 0$, $3 \le i \le m$. Note that since the a_i^*'s are all equal, $(\tilde{\mathbf{a}}, \tilde{\mathbf{u}})$ is in Ω and it is again a minimizer if $\delta = 0$. With $\delta > 0$, $\mathbb{P}(Q(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{u}}) \le 0)$ can only increase; therefore, by (5.6) with reversed signs of the perturbations, this implies

$$\liminf_{\delta \downarrow 0} \frac{\partial^2}{\partial x^2} \mathbb{P}(Q(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{u}}) + \tilde{a}_1 Z_1 + \tilde{a}_2 Z_2 \le x) \Big|_{x=0} \le 0. \tag{5.12}$$

Note that now $\delta = 0$ means that the minimizer is on the boundary of Ω, so the derivative at $\delta = 0$ may not vanish.

Let now $G \sim \text{Gamma}(\eta+1, 1)$, $H \sim \text{Gamma}(m-\eta+1, 1)$, and $Y \sim \text{Gamma}(n, 1)$ independently. Since G, H, Y are all log-concave, $Q(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{u}}) + \tilde{a}_1 Z_1 + \tilde{a}_2 Z_2$, which is distributed as $\tilde{a}_1 G - bY + \tilde{a}_2 H$, dominates $\tilde{a}_1 G - bY$ in the likelihood ratio order, which itself increases in the likelihood ratio order as $\delta \downarrow 0$. It is easy to see that if f and g are unimodal densities and f dominates g in the likelihood ratio order, then the mode of g is not larger than the mode of f. Thus, for small enough $\delta > 0$,

$$\frac{\partial^2}{\partial x^2} \mathbb{P}(\tilde{a}_1 G - bY \le x) \bigg|_{x=0} \le 0.$$

The latter expression equals $f'(0)$ where now f is the density of $\tilde{a}_1 G - bY$. This contradicts Lemma 5.4, however, noting that $\eta \ge 1$ follows from the definition of Ω, and $\eta a_1^* = m > (n-1)b$. □

Based on Claim 5.8, if $0 < b < m/(n-1)$, then a solution of the problem (5.8) is $a_1^* = \cdots = a_m^* = 1 = u_1^* = \cdots = u_m^*$. Moreover, any optimal solution satisfies $\sum_{i=1}^m a_i G_i \sim \text{Gamma}(m, 1)$.

Thus $a_1^* = \cdots = a_m^* = 1$ is the unique optimal solution of the original problem of part (i) of Theorem 3.5. □

Proof of Theorem 3.1. Theorem 3.5(i) guarantees that $\boldsymbol{a}^*$ is the unique best response to $\boldsymbol{b}^*$ and vice versa. This proves that $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is a Nash equilibrium of the game. Assume now that there exists a different Nash equilibrium $(\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}})$. Because the game is zero-sum, the payoff for team A decreases along the path $(\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}) \to (\boldsymbol{a}^*, \widetilde{\boldsymbol{b}}) \to (\boldsymbol{a}^*, \boldsymbol{b}^*)$. The payoff for team A, however, increases along the path $(\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}) \to (\widetilde{\boldsymbol{a}}, \boldsymbol{b}^*) \to (\boldsymbol{a}^*, \boldsymbol{b}^*)$, which is a contradiction. Hence the equilibrium $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is unique. □

Proof of Theorem 3.2. Theorem 3.5(ii) guarantees that $\boldsymbol{a}^*$ is a best response to $\boldsymbol{b}^*$ and Theorem 3.5(i) guarantees that $\boldsymbol{b}^*$ is the unique best response to $\boldsymbol{a}^*$. This proves that $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is a Nash equilibrium of the game. By Proposition 2.1, this remains true if we replace $\boldsymbol{a}^*$ with any permutation. □

Proof of Theorem 3.3. (i) Using Theorem 3.1 and noting that in this case $a_1^* = \cdots = a_m^*$ and $b_1^* = \cdots = b_n^*$, we have

$$\sum_{i=1}^m a_i^* X_i \sim \text{Gamma}(m, m/c_A), \quad \sum_{j=1}^n b_j^* Y_j \sim \text{Gamma}(n, n/c_B).$$

Hence, see (4.1),

$$\mathbb{P}\left(\sum_{i=1}^m a_i^* X_i > \sum_{j=1}^n b_j^* Y_j\right) = 1 - I\left(\frac{mc_B}{mc_B + nc_A}, m, n\right),$$

where I is the regularized incomplete beta function defined in (3.3).

(ii) Using Theorem 3.2 we see that in this case

$$\sum_{i=1}^{m} a_i^* X_i \sim \mathrm{Gamma}(1, 1/c_A), \quad \sum_{j=1}^{n} b_j^* Y_j \sim \mathrm{Gamma}(n, n/c_B).$$

Hence

$$\mathbb{P}\left(\sum_{i=1}^{m} a_i^* X_i > \sum_{j=1}^{n} b_j^* Y_j\right) = 1 - I\left(\frac{c_B}{c_B + n c_A}, 1, n\right).$$

□

Proof of Corollary 3.4. The team with more players always has the option of not using them all. Therefore it cannot be worse off than the team with fewer players. However, since equal allocation is the unique best response, using them all is strictly better. The same argument proves the monotonicity in m and n. Note that directly verifying this from the properties of the incomplete beta function appears nontrivial.

□

Proof of Corollary 4.1. (i) is a restatement of the last part of Corollary 3.4.

(ii) follows from (i) and (3.4).

(iii) follows from (ii) by letting $n \to \infty$.

(iv) follows from (iii) and the identity

$$\mathbb{P}(S \geq m) = \frac{1}{\Gamma(m)} \int_0^m \mathrm{e}^{-t}\, t^{m-1}\, \mathrm{d}t.$$

□

Proof of Proposition 4.2. (i) The relation (4.4) can be explained directly: team A loses if all its gladiators together defeat at most $n-1$ opponents. Gladiator i from team A defeats a geometric random number, Q_i, of gladiators of strength 1 from team B since he fights until he loses, and he loses a fight with probability $1/(1+a_i)$. Thus if $\sum_{i=1}^m Q_i \leq n-1$, then team A defeats at most $n-1$ gladiators altogether, and loses.

(ii) This follows directly from Theorem 3.5.

(iii) Note that $\mathbb{E}[Q] = \sum_{i=1}^m a_i$. Letting $n = m$, and using (4.4) and part (ii), we conclude that $\mathbb{P}(Q \leq n-1) \geq 1 - G_{m,n}(\mathbf{1}_m, \mathbf{1}_n) = 1/2$. We obtain $\mathbb{P}(Q \leq \mathbb{E}[Q]) = \mathbb{P}(Q \leq n) > 1/2$, and therefore $\mathbb{E}[Q] > \mathrm{Med}[Q]$.

□

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Figures

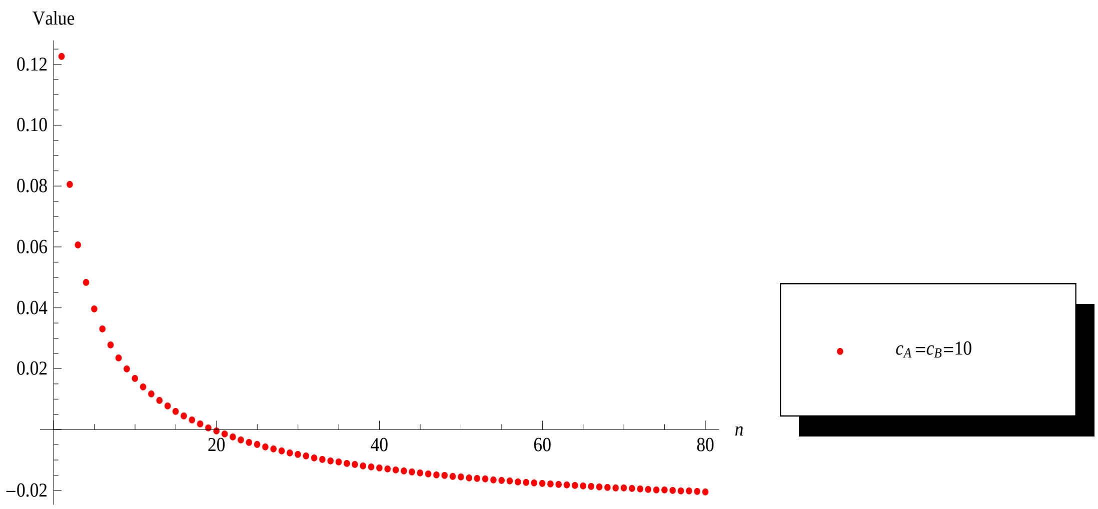

Figure 1: Value of $\mathscr{G}$ as a function of n for $m = 20$ and $(c_A = c_B)$.

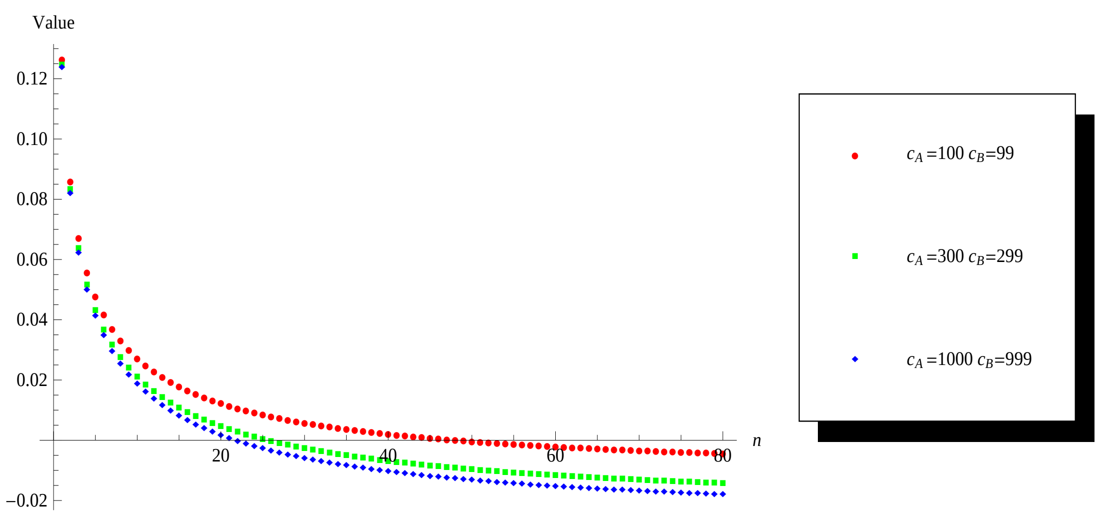

Figure 2: Value of $\mathscr{G}$ as a function of n for $m = 20$ and different pairs (c_A, c_B).

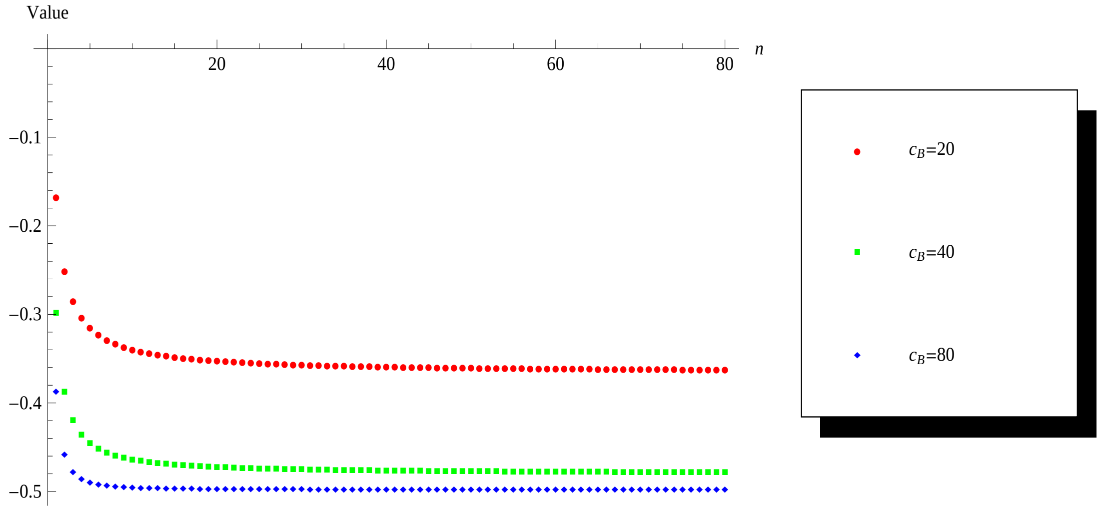

Figure 3: Value of $\mathscr{G}$ as a function of n for $c_A = 10$ and various c_B.

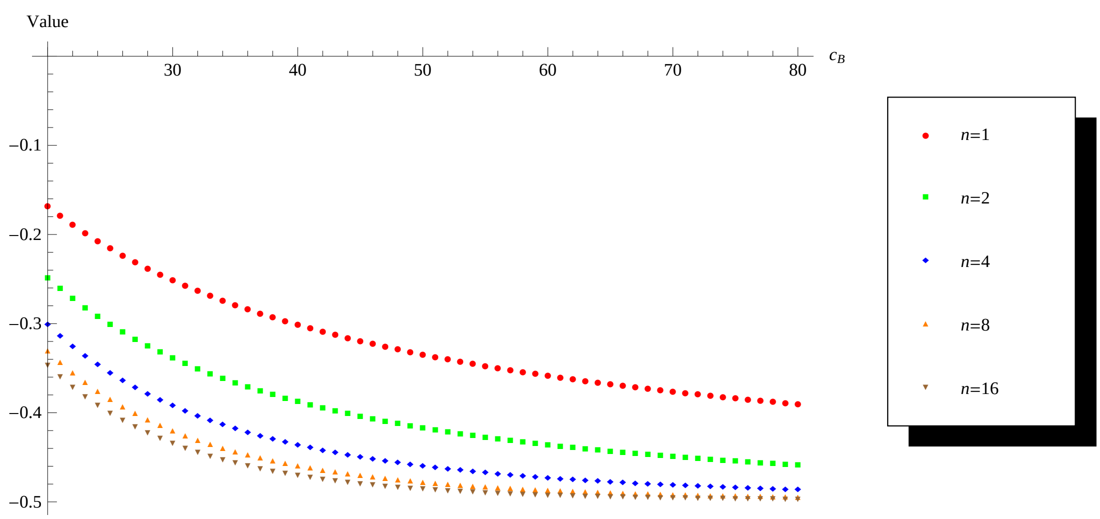

Figure 4: Value of $\mathscr{G}$ as a function of c_B for $c_A = 10$ and various n.