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REMARKS ON BARGAINING AND COOPERATION IN STRATEGIC FORM GAMES

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Remarks on Bargaining and Cooperation in Strategic Form Games*

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Abstract. Although possessing many beautiful features, the Hart and Mas-Colell bargaining model is not flawless: the concept of threat in this model may behave quite counter-intuitive, and its SP equilibrium expected payoff vector may not be the same as the min-max solution payoff vector in zero-sum games. If we postpone realizations of all threats to the end of the game, the two problems can be solved simultaneously. This is exactly the 2(a) model suggested by Hart and Mas-Colell in the last section of their paper. I show that the new model, unfortunately, can only guarantee the existence of an SP equilibrium in the two player case. For the original model, I reduce the computation of an SP equilibrium to a system of linear inequalities. Quantitative efficiency and symmetric SP equilibria are also discussed.

1 Introduction

Inspired directly by Hart and Mas-Colell(1996,[10]) and Nash (1953, [18]), Hart and Mas-Colell (2010,[11]) introduce a strategic-form-game-based bargaining model. In their model, correlated action profiles are allowed, hence players are able to cooperate to some extent. In the sequel, we will refer to this model as the HM model.

As usual, we use $G = (N, (A^i)_{i\in N}, (u^i)_{i\in N})$ to denote a strategic form game: N is a finite set of players; For each player $i \in N$, A^i is her finite action set, and $u^i : A \to \mathbb{R}$ her utility function, where $A = \prod_{i\in N} A^i$. For each $i \in N$, we use $a^i \in A^i$ to denote a pure action, and $x^i \in \Delta(A^i)$ a mixed action, where $\Delta(A^i) = \{(x^i(a^i))_{a^i\in A^i} \in \mathbb{R}_+^{A^i} : \sum_{a^i\in A^i} x^i(a^i) = 1\}$ is the probability simplex on A^i. For any subset of players $S \subseteq N$, $A^S = \prod_{i\in S} A^i$ is the set of pure action profiles of S, and $\Delta(A^S)$ is the set of correlated action profiles of S. We also use $b^S \in A^S$ to denote a pure action profile, and $z^S \in \Delta(A^S)$ a correlated action profile.

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1.1 Model description

The game may have multiple rounds, and each round is characterized by a *state* $\omega = (S, b^{N \setminus S})$, where S is the set of *active* players, and $b^{N \setminus S} \in A^{N \setminus S}$ is a pure action profile of $N \setminus S$. Initially, $\omega = (N, \emptyset)$. In each round $\omega = (S, b^{N \setminus S})$, a player is chosen uniformly from S. Say k is chosen. Then k chooses an *announcement* $\sigma_\omega^k = (z_k^S, x^k)$, where $z_k^S \in \Delta(A^S)$ is called a *proposal*, and $x^k \in \Delta(A^k)$ is called a *threat*. If all the other players in S accept this proposal, then S will play the proposal and the game is ended. Otherwise, that is at least one active player rejects the proposal, the game goes to the next round: With probability ρ, the set of active players stays unchanged, thus the state is updated as $\omega := \omega$; And with probability $1 - \rho$, the proposer is kicked out of the set of active players and her threat is realized, say as b^k, and revealed to all, thus the state is updated as $\omega := (S \setminus k, b^{(N \setminus S) \cup k})$[1].

The procedure is described in Fig. 1.

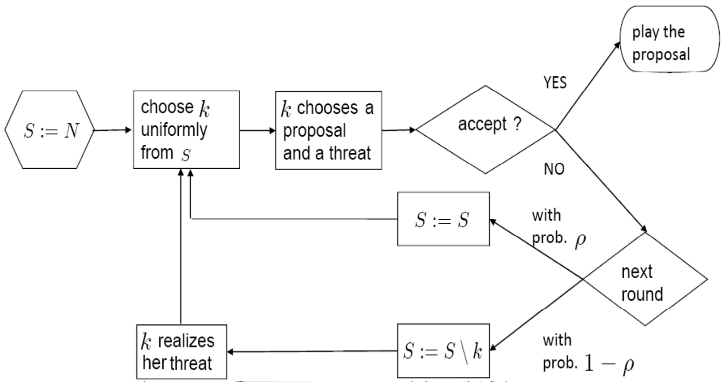

Fig. 1. The HM Model

1.2 Main results of Hart and Mas-Colell

The equilibrium used to study the HM model is the *stationary (subgame-) perfect equilibrium* (SP equilibrium for short), where each player chooses a stationary

[1] For simplicity, I use k instead of $\{k\}$ in the set operations.

strategy, i.e. in any round $\omega = (S, b^{N\setminus S})$, proposer k's announcement $\sigma_\omega^k = (z_k^S, x^k)$ depends only on the set of active players S and the action profile of the inactive players $b^{N\setminus S}$, and the decision of any other active player depends only on the set of active players S, the action profile of the inactive players $b^{N\setminus S}$, the identity of the proposer k, the proposal z^S, and the threat x^k.

Assuming that *(A1) each player is an expected utility rational person and use the tie-breaking rule of accepting any proposal when accepting and rejecting give the same expected utility,* Hart and Mas-Colell ([11]) show that (i) SP equilibrium always exists; (ii) when ρ is close to 1, the SP equilibrium is almost Pareto efficient; (iii) in two-player games, as $\rho \to 1$, the SP equilibrium converges to the Nash bargaining solution; (iv) in the transferable utility case (see [11] for exact definition), the SP equilibrium is an extension of the Shapley value.

The stationary strategy restriction can be viewed either as exogenous or as endogenous. The former, which excludes history-dependent or calender-dependent strategies at the very beginning, is reasonable in many aspects. First of all, people do use this kind of stationary strategy a lot in the real world, because it is simple and easy to apply. Second, without this restriction, it's very likely that we would have a folk-like theorem, which is quite uninteresting. Third, the idea of stationary restriction is widespread in the literature of game theory. The latter, which means that we are studying a special subgame perfect equilibrium, is supported by the observation from the proof of Proposition 1 of [11] that an SP equilibrium is indeed a subgame-perfect equilibrium, i.e. in any SP equilibrium, each player's (stationary) strategy is a best response to the other players' strategies not only in the set of stationary strategies, but also in the set of all strategies, which may be history-dependent or calender-dependent.

The HM model has many interesting features:

(i) It is based on the strategic form game, and therefore can be potentially applied to much wider scenarios than the classical bargaining models which are based on the coalitional form game, as the strategic form game is usually viewed as a more basic model than the coalitional form game.

(ii) The Shapley-value-like formula gives a new interesting connection to strategic game theory and coalitional one.

(iii) Correlated action profiles are allowed, and this enables players to cooperate to some extent. Take Prisoners' Dilemma for instance, the only Nash equilibrium, which is also the only correlated equilibrium, is Defy-Defy, while in the only SP equilibrium of the HM model, Cooperate-Cooperate can occur with a positive probability, and larger ρ leads to higher degree of cooperation [2].

(iv) The SP equilibrium of the HM model can be used as a selection or a modification of the correlated equilibrium. A "problem" of the correlated equilibrium is that it's usually not unique. To solve this problem, one approach is to refine it through perturbations[3]. The other approach is to simply select an "op-

[2] See Example 3 in Subsection 2.1 for details.

[3] There is very little literature on this topic in contrast with numerous refinement concepts of Nash equilibrium. See Dhillon and Mertens (1996,[5]) for perfect cor-

timal" one which maximizes the total utility of all the players, and this selection is done by a third party [4].

The HM model provides a third approach: it serves as a micro procedure for selecting a correlated equilibrium, and this selection is endogenous. Since there may not be any SP equilibrium which is also a correlated equilibrium[5], a natural idea is to modify the HM model by requiring any proposal be a correlated equilibrium (w.r.t. ω). We call it the HM' model. As the set of correlated equilibria is a nonempty polytope, the existence proof of the SP equilibrium carries over easily to the HM' model. [6]. Another perspective is that we can view the HM model as a way to modify the correlated equilibria, since the concept of correlated equilibrium is sometimes not completely satisfactory (again, consider the Prisoners' Dilemma).

1.3 My contributions

Without considering the implementability problem, the HM model is still not flawless. One flaw is that the threat used in the HM model might behave very counter-intuitive. Assuming $\rho = 0$, consider Matching Pennies. Say player 1 is the "lucky" proposer, then player 2 will reject any proposal which gives her less than 1, because according to the procedure of the HM model, the threat of player 1 should be realized if player 2 rejects, and this will give player 2 a great advantage which guarantees her a payoff of 1. This is still the case for general ρ[7]. This "odd" property is not rare. In fact, the proposer has a disadvantage in any zero-sum game[8]. Similar things occur in non-zero sum games, although it may not be so extreme.

The second flaw of the HM model is that in zero-sum games the expected payoff vector of the SP equilibrium, which is unique and not dependent on ρ, may not be the same as the min-max value payoff vector of von Neumann ([16]) [9], although it is indeed so in many interesting cases[10]. It's well known that the min-max value is almost perfect, most of the important later solution concepts generalize it: Nash equilibrium coincides with it in zero-sum games, the payoff

related equilibrium, and Kim and Wong (2005, [14]) for a more recent research on evolutionary stable correlated equilibrium.

[4] This treatment is of special interest to mechanism design, and how to compute such optimal correlated equilibria in succinct games is a topic of the field of algorithmic game theory, cf. Papadimitriou and Roughgarden (2008, [20]).

[5] See Example 6 in Subsection 2.1.

[6] The HM' model also solves a problem of the HM model: it's not implementable in the sense that an agreed proposal which is not a correlated equilibrium needs a very powerful third party to enforce it. In contrast, when the agreed proposal is a correlated equilibrium, it can be implemented by a much less powerful third party.

[7] See Example 4 in Subsection 2.1.

[8] See part (c) of Theorem 1.

[9] See Example 5 in Subsection 2.1.

[10] See part (b) of Theorem 1.

vector in the correlated equilibrium is also $(v, -v)$ for zero-sum games[11]. A more recent concept, *the coco value*, suggested by Kalai and Kalai (2009, [12]), generalizes the min-max value too.

The first flaw lies in the assumption in the HM model that threats should be realized if the corresponding proposal is rejected and then, most importantly, revealed to the others. To conquer this flaw, a natural idea is to postpone the realizations of threats to the end of the game, which is equivalent to realizing a threat immediately after the corresponding proposal is rejected but keeping it not revealed to the others until the end of the game. This is the 2(a) model proposed by Hart and Mas-Colell in the last section of [11]. We refer to this model, which is described in Fig. 2., as the HM* model in the sequel.

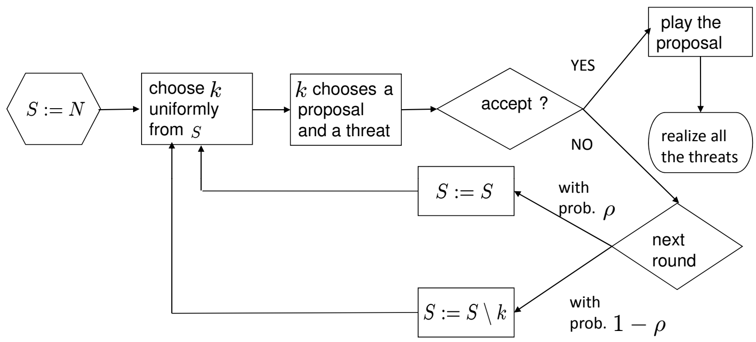

Fig. 2. The HM* Model

The above two flaws are eliminated simultaneously in the HM* model: the counter-intuitive threat disappears: in fact, no one cares about whether she is a proposer or responder, as in either case she has an expected payoff as in the min-max solution[12].

[11] In fact, it can be easily shown that this is also true for a wider equilibrium concept, *the Hannan set* (sometimes referred to as “coarse correlated equilibrium”; see Hannan (1957, [6]), Moulin and Vial (1978, [15]), Hart (2005, [9]).

[12] See Theorem 7.

So our main concern is whether the HM* model preserves all the virtues of the HM model. First of all, can the existence of the SP equilibrium be guaranteed in the HM* model?

Unfortunately, the answer is universally "yes" only in the two player special case. The two player positive result (Theorem 6,Subsection 5.1) is proved quite routinely following the proof of Proposition 2 of [11], using Kakutani's Fixed Point Theorem (cf. Hildenbrand, 1974,[7], C.III(14)) and Berge's Maximum Theorem (Berge, 1959, [2]), after extending Lemma 2 of [11] from linear functions to concave ones (Lemma 3,Subsection 5.1). A counter example of the three player case is constructed after observing a discontinuity property of the SP equilibrium in the two player case (Theorem 8,Subsection 5.2).

For the HM model, I get the following results:

(i) I show that the set of SP equilibria can be determined by systems of linear inequalities through the complementary slackness property between a linear programming and its dual programming. This algorithm is efficient when n, the number of players, is small (Section 3). This is the 8-th problem suggested in the last section of [11]. It's well known that generally a fixed point is hard to compute, and computing an arbitrary Nash equilibrium is hard even when there are only two players (c.f.[19]). The HM model, however, has a special structure which allows efficient computation.

(ii) Quantitative efficiency is considered. Although the SP equilibrium is weakly Pareto efficient, and almost Pareto efficient when ρ is close to 1, it can be demonstrated easily that its PoA, which is defined as the ratio between the total utility of the worst SP equilibrium and the optimal total utility [13], and PoS, which is defined as the ratio between the total utility of the best SP equilibrium and the optimal total utility [14], are both 0. That is, the SP equilibrium can be as inefficient as possible. For a special case which I name *weakly symmetric*, it's a little better: PoA and PoS are both $1/n$, where n is the number of players (Theorem 2, Subsection 2.3).

(iii) Motivated directly by Nash's result on the existence of symmetric Nash equilibrium ([17]), existence of "symmetric" SP equilibria are considered. I distinguish three different symmetric concepts: *strategy symmetric, distribution symmetric* and *payoff symmetric*, and give a basic result about the *characteristic group* of strategic form games with identical strategy sets, which says that when $m \geq n$, then any subgroup of the symmetric group $\mathcal{S}_n$ can be a characteristic group of some game, where m is the number of strategies, and n the number of

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[13] This concept is due to Koutsoupias and Papadimitriou, (1999, [13]). Its exact definition for the HM model should be $PoA = \inf\limits_{G} \inf\limits_{\pi \in SP(G)} \frac{t(\pi)}{OPT(G)}$, where G denotes an instance of the HM model, $SP(G)$ the set of its SP equilibria, $t(\pi)$ the total utility of any SP equilibrium π, and $OPT(G)$ the largest total utility among all action profiles.

[14] This concept is due to Schulz and Moses, (2003,[22]). Its exact definition for the HM model should be $PoA = \inf\limits_{G} \sup\limits_{\pi \in SP(G)} \frac{t(\pi)}{OPT(G)}$, all the notations following the last footnote.

players. When $m < n$, this is not true. In particular, the alternating group $\mathcal{A}_n$ can not be the characteristic group of any game.

The rest of this paper is organized as follows: the next subsection gives the needed notations and preliminaries. Section 2 examines several examples (Subsection 2.1) and two simple properties of the HM model: one on the zero-sum game (Subsection 2.2), and the other on quantitative efficiencies. Section 3 devotes to the computation of SP equilibria of the HM model, Section 4 to symmetric SP equilibria. Section 5 is completely for the HM* model, and Section 6 concludes this paper with some further remarks.

1.4 Notations and preliminaries

For convenience of the readers and the completeness of this paper, I move the main body of Subsection 2.2 in [11] here. Readers who are familiar with [11] can skip this subsection.

Let σ be a stationary strategy profile of all the players. All the following notations are defined under a fixed σ.

For any state $\omega = (S, b^{N\setminus S})$, we denote $\zeta^S_{\omega,k} \in \Delta(A^S)$ as the probability distribution of the final action profiles of S after $k \in S$ is selected as the proposer, and $\zeta^S_\omega \in \Delta(A^S)$ the *outcome* of state ω, i.e. the the probability distribution of the final action profiles of S. Since each player in S has the same probability $1/|S|$[15] being chosen as the proposer, we know that

$$\zeta^S_\omega = \frac{1}{|S|} \sum_{k \in S} \zeta^S_{\omega,k}. \tag{1}$$

For every $k \in S$ and every $b^k \in A^k$, let $(\omega||b^k) := (S \setminus k, (b^{N\setminus S}, b^k))$ denote the state obtained from ω when k is kicked out of the active player set and her action is realized as b^k. For any threat $x^k \in \Delta(A^k)$, let $\eta^S_{\omega,k}(x^k) \in \Delta(S)$ be the outcome of S if the proposal of k is rejected and her threat is x^k, then obviously

$$\eta^S_{\omega,k}(x^k) = \sum_{b^k \in A^k} x^k(b^k) \left(\zeta^{S\setminus k}_{(\omega||b^k)} \times \{b^k\} \right), \tag{2}$$

where $x^k(b^k)$ is the probability assigned to b^k by x^k, $\{b^k\}$ is the probability distribution in $\Delta(A^k)$ which assigns 1 to b^k.

With the above notations, Assumption (A1) can be rewritten as: for each state $\omega = (S, b^{N\setminus S})$, any active player $i \in S, i \neq k$, accepts k's announcement (z^S, x^k), iff

$$u^i(z^S, b^{N\setminus S}) \geq \rho u^i(\zeta^S_\omega, b^{N\setminus S}) + (1-\rho) u^i(\eta^S_{\omega,k}(x^k), b^{N\setminus S}). \tag{3}$$

It's valuable to note that $\eta^S_{\omega,k}(x^k)$ is linear in x^k and thus (3) is a linear inequality in (z^S, x^k), which is not the case in the HM* model. This is also the

[15] $|\cdot|$ denotes the cardinality of a set.

main barrier for the HM* model to guarantee an SP equilibrium. See Section 5 for details.

Since the response of any player is quite automated as in (3), her strategy is completely determined by her proposal and threat in any state. Thus σ, the stationary strategy profile of all the players, can be denoted as $\big((\sigma^k_\omega)_{k\in S}\big)_{\omega=(S,b^k)\in\Omega}$, where Ω is the set of all states.

Let $\boldsymbol{\zeta} := \Big((\zeta^S_{\omega,k})_{k\in S}\Big)_{\omega=(S,b^k)\in\Omega}$ denote the outcome configuration obtained from σ. The sets of k's acceptable announcements Y, optimal announcements Y^*, acceptable proposals Z, optimal proposals Z^* are defined respectively as follows:

$$
\begin{aligned}
&Y \equiv Y_{\omega,k}(\boldsymbol{\zeta}) := \{(z^S, x^k) \in \Delta(A^S) \times \Delta(A^k) : (3) \text{ holds for all } i \in S \setminus k\};\\
&Y^* \equiv Y^*_{\omega,k}(\boldsymbol{\zeta}) := \operatorname*{arg\,max}_{(z^S,x^k)\in Y} u^k(z^S, b^{N\setminus S});\\
&Z \equiv Z_{\omega,k}(\boldsymbol{\zeta}) := \{z^S \in \Delta(A^S) : (z^S, x^k) \in Y \text{ for some } x^k \in \Delta(A^k)\};\\
&Z^* \equiv Z^*_{\omega,k}(\boldsymbol{\zeta}) := \{z^S \in \Delta(A^S) : (z^S, x^k) \in Y^* \text{ for some } x^k \in \Delta(A^k)\}.
\end{aligned}
$$

Proposition 1 of [11] says that σ is an SP equilibrium iff the following fixed-point-type condition holds for every state $\omega \in \Omega$ and every player $k \in S$:

$$\zeta^S_{\omega,k} \in Z^*_{\omega,k}(\boldsymbol{\zeta}).$$

2 The HM Model

In this subsection, we shall study some properties of the SP equilibria for the HM model. We shall first discuss several examples. The not-so-easy calculations of two of them, which are based on the method described in the next section, will be postponed to the appendices. Above all, we give a basic observation.

Lemma 1. *In any SP equilibrium: an action profile that is strictly Pareto dominated can never be in the support of any player's proposal.*

The proof is trivial, since whenever a strictly Pareto dominated action profile is in the support of any player's proposal, she can shift the weight to a better one and get strictly better off. This observation is also true for the HM* model. Weakly Pareto dominated action profiles, however, may be used.

2.1 Several examples

In the two player HM model, as in [11], we use q^i to denote the payoff of player i that another player j can hold i to, by using pure strategies, where $\{i, j\} = N = \{1, 2\}$, i.e.,

$$q^i = \min_{a^j\in A^j} \max_{a^i\in A^i} u^i(a^i, a^j). \tag{4}$$

Example 1. Suppose $a > 0, b > 0$,we define a two-player game as follows:

	L	R
T	$a, 0$	$0, 0$
D	$0, 0$	$0, b$

.

It's easy to see that $q^1 = q^2 = 0$, the row player's optimal proposal when she is chosen as the proposer is (T, L), and the column player's optimal proposal is (D, R).

Although this game is not symmetric when $a \neq b$, the two players' "bargaining powers" are identical in the HM model. For any $0 \leq \rho < 1$, the expected payoff vector $(a/2, b/2)$ in the SP equilibrium also coincides with that in the Nash bargaining solution (the disagreement point is (0,0)).

Example 2. Suppose $\epsilon > 0$ is a small number, we define a two-player game as follows:

	L	R
T	ϵ, ϵ	$\epsilon, 0$
D	$0, \epsilon$	$100, 0$

.

It's easy to see that $q^1 = q^2 = \epsilon$, and the optimal proposal of either player is (T, L). This shows that SP equilibrium can be quite inefficient (in the PoA sense). See Theorem 2 for more discussions.

Example 3 (Prisoners' Dilemma). The payoffs are set as follows:

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

.

We know that the unique Nash equilibrium, which is also the unique correlated equilibrium, is (D,D). But since (D,D) is strictly dominated by (C,C), it will never occur in any SP equilibrium. Suppose $\alpha_1^*, \alpha_2^*, \alpha_3^*$ and β_1^*, β_2^*, β_3^* are the probabilities assigned to (D,C), (C,C) and (C,D), in the optimal proposals by the two players, respectively. Calculation (See Appendix A) shows that the unique SP equilibrium is:

$$(\alpha_1^*, \alpha_2^*, \alpha_3^*) = \left(\frac{2-2\rho}{3-\rho}, \frac{1+\rho}{3-\rho}, 0\right), (\beta_1^*, \beta_2^*, \beta_3^*) = \left(0, \frac{1+\rho}{3-\rho}, \frac{2-2\rho}{3-\rho}\right).$$

Hence in the HM model, players do cooperate to some extent. For the Prisoners' Dilemma, larger ρ leads to larger degree of cooperation, and as ρ approaches 1, players can fully cooperate. In this example, the only SP equilibrium is symmetric. See Section 4 for more discussions.

We can also observe that the proposer has an advantage over the responder, and the smaller ρ is, the more advantage she has. This is no surprising, because smaller ρ means bigger probability that her threat will be realized. This is not the case, however, for zero-sum games. See the example of Matching Pennies below.

Example 4 (Matching Pennies). The payoff matrix is as follows:

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

.

It's easy to see that $q^1 = q^2 = 1$. Let α_1^*, α_2^* and β_1^*, β_2^* be the probabilities assigned to $(1,-1)$ and $(-1,1)$, by the two players in any optimal proposals, respectively[16]. Simple calculation gives:

$$(\alpha_1^*, \alpha_2^*) = (\frac{1}{4}\rho, 1 - \frac{1}{4}\rho), (\beta_1^*, \beta_2^*) = (1 - \frac{1}{4}\rho, \frac{1}{4}\rho).$$

This equilibrium is also symmetric, although the game is not symmetric[17]. The proposer has a big disadvantage over the responder, and the smaller ρ is, the more disadvantage she has. The expected payoff vector, $(0,0)$, however, is the same as the min-max payoff vector.

Example 5. Two other zero-sum games:

	L	R
T	$1,-1$	$0,\ \ 0$
D	$0,\ \ 0$	$1,-1$

,

	L	R
T	$2,-2$	$0,\ \ 0$
D	$0,\ \ 0$	$1,-1$

.

In both games, $q^1 = 1, q^2 = 0$, thus the expected payoff vector in the SP equilibrium are both $(1/2, -1/2)$, which is also the min-max payoff vector to the left game. The min-max payoff vector to the right game, however, becomes $(2/3, -2/3)$. Therefore, the right game illustrates that the SP equilibrium payoff vector may not be the same as the min-max payoff vector in zero-sum games. Compared with the left game, the row player (player 1) has a more beneficial payoff structure in the right game, and this change is respected in the min-max value. In the HM model, however, this change is not respected.

Example 6 (Example(2.7) in Aumann, 1974 ([1])). There are two players, the payoff matrix is:

	L	R
T	$6,6$	$2,7$
D	$7,2$	$0,0$

.

It's easy to calculate that there are three Nash equilibria in total, (D, L), (T, R), and $(\frac{2}{3}T + \frac{1}{3}D, \frac{2}{3}L + \frac{1}{3}R)$.

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[16] We combine the two points (H,T) and (T,H) as $(1,-1)$, and combine (H,H) and (T,T) as $(-1,1)$.

[17] However, it is *self-anonymous*, in the sense that all players have identical action sets, and for each player her utility on any action profile is determined only by the numbers of every actions in this action profile. This concept, due to Brandt et. al. (2009, [3]), characterizes also a kind of "symmetry".

Let x, y, z, w be the probabilities assigned to $(D, L), (T, L), (T, R)$ and (D, R), then the set of correlated equilibria is characterized by the following two inequalities:

$$\min\{x, z\} \geq \max\{\frac{1}{2}y, 2w\},$$

$$x + y + z + w = 1, x, y, z, w \geq 0.$$

The unique SP equilibrium of this example (see Appendix B) is

$$(\alpha_1^*, \alpha_2^*, \alpha_3^*) = \left(\frac{8-8\rho}{8-3\rho}, \frac{5\rho}{8-3\rho}, 0\right), (\beta_1^*, \beta_2^*, \beta_3^*) = \left(0, \frac{5\rho}{8-3\rho}, \frac{8-8\rho}{8-3\rho}\right).$$

Since $\frac{1}{2}(\alpha_1^*, \alpha_2^*, \alpha_3^*) + \frac{1}{2}(\beta_1^*, \beta_2^*, \beta_3^*) = \left(\frac{4-4\rho}{8-3\rho}, \frac{5\rho}{8-3\rho}, \frac{4-4\rho}{8-3\rho}\right)$, we know that the SP equilibrium is never a Nash equilibrium, and it is a correlated equilibrium if and only if $0 \leq \rho \leq \frac{8}{13}$.

Notice also that the game is symmetric, and the unique SP equilibrium is symmetric too.

2.2 Two person zero sum games

For zero-sum games, remember that the von Neumann min-max value is defined as:

$$v = \max_{x^1 \in \Delta(A^1)} \min_{a^2 \in A^2} u^1(x^1, a^2) = \min_{x^2 \in \Delta(A^2)} \max_{a^1 \in A^1} u^1(a^1, x^2).$$

Theorem 1. *For zero-sum games, we have:*

(a) Each SP equilibrium has an expected payoff vector of:

$$\left(\frac{1}{2}(q^1 - q^2), \frac{1}{2}(q^2 - q^1)\right).$$

(b) The expected payoff vector is the min-max payoff vector $(v, -v)$ if either (i) the game has a pure Nash equilibrium, i.e. a saddle point, or (ii) the game is symmetric.

(c) The proposer always has a disadvantage, i.e. any player's expected payoff as a proposer is no more than her expected payoff as a responder.

Proof. Suppose x^* is the expected payoff of player 1 when she is the proposer, and y^* the payoff of player 2 when she is the proposer, then the optimization problems they face are respectively $\begin{cases} \max \quad x \quad s.t. \\ -x \geq \frac{1}{2}\rho(-x^* + y^*) + (1-\rho)q^2 \end{cases}$ and $\begin{cases} \max \quad y \quad s.t. \\ -y \geq \frac{1}{2}\rho(x^* - y^*) + (1-\rho)q^1 \end{cases}$. It's easy to see that the inequalities in the above programs should both hold as equalities. Combining them gives $x^* = (\frac{1}{2}\rho - 1)q^2 + \frac{1}{2}\rho q^1$, $y^* = (\frac{1}{2}\rho - 1)q^1 + \frac{1}{2}\rho q^2$, and $x^* - y^* = q^1 - q^2$, and hence part (a) of the theorem.

For part (b), since $-q^2 \leq v \leq q^1$, and the existence of a saddle point means $q^1 = -q^2$, the sufficiency of condition (i) is obvious. Symmetry means $q^1 = q^2$ and $v = 0$, thus the payoff vector of the SP equilibria is the same as the min-max value payoff vector. Since in zero-sum games, we always have $q^1 + q^2 \geq 0$, part (c) is obvious. □

Remark 1. Since zero-sum game is a special case of strategic TU game, the Shapley-value-like formula gives part (a) of the above theorem immediately. See Subsection 4.2 of [11] for the formula.

Remark 2. Neither of the two conditions in part (b) of the above theorem is necessary. Just think about Matching Pennies.

2.3 Quantitative efficiency

Intuitively, a weakly symmetric game is a game where the feasible set of payoff vectors is symmetric.

Definition 1. *Suppose $G = (N, (A^i)_{i\in N}, (u^i)_{i\in N})$ is a strategic form game. If $\forall a^N \in A^N, \forall$ permutation π of N, there exists $b^N \in A^N$ such that $u^N(b^N) = \pi\big(u^N(a^N)\big)$, we call G a weakly symmetric game.*

Obviously, a symmetric game is a weakly symmetric game.

Theorem 2. *For the HM model, PoA and PoS are both 0. For the HM model restricted on weakly symmetric games, PoA and PoS are both $1/n$, where n is the number of players.*

Proof. The former part of the theorem is immediately from Example 2. For any weakly symmetric game, suppose $a^{*N} \in A^N$ is a pure action profile with the largest total payoff, that is, $\sum_{i\in N} u^i(a^{*N}) = \max\{\sum_{i\in N} u^i(a^N) : a^N \in A^N\}$. Then for any pure action profile a^N that is not strictly dominated, there is $i \in N$ such that $u^i(a^N) \geq \frac{1}{n}\sum_{i\in N} u^i(a^{*N})$. Since otherwise $u^N(a^N)$ will be strictly dominated by $\frac{1}{n!}\sum_{\pi\in\Pi} \pi(u^N(a^{*N}))$, where Π is the set of all permutations of N. Because any action profile in the support of any player's proposal is not strictly dominated, we know that PoA and PoS are both no less than $1/n$ on weakly symmetric games. To show that they are exactly $1/n$, the example below is sufficient.

We take $\rho = 0$, i.e. the threat will surely be executed if the corresponding proposal is rejected. There are n players in total, the common action set is $\{U, M, D\}$. $\forall i \in N$, her payoff function is:

$$u^i(a^1, a^2, \cdots, a^n) = \begin{cases} 1+\epsilon & if\ a^i = L, \forall j \neq i, a^j = M \\ 1 & if\ \forall j \in N, a^j = M \\ 0 & if\ \exists j \in N, a^j = D \end{cases}.$$

It's easy to check that this game is symmetric, and thus weakly symmetric. For each player, a threat of D will make all the other players get a payoff of 0. Suppose a_i^* is the action profile where i's action is U and the actions of the other players are all M, then each player i will assign probability 1 to a_i^* when she is chosen as the proposer, and she will get a payoff of $1+\epsilon$, while all the others getting 0. However, action profile $(M, M, \cdots, M)$ will give every player a payoff of 1. Hence for this game $PoA = PoS = \frac{1+\epsilon}{n} \to \frac{1}{n}(\epsilon \to 0)$. □

3 Computing the SP equilibria

We use $\left(\sigma_\omega^{*i} = \left(z_i^{*S}, x^{*i}\right)\right)_{\omega=(S,b^{N\setminus S})\in\Omega, i\in S}$ to denote any SP equilibrium. We need to calculate it recursively. It's trivial when $|S| = 1$. Suppose now the stationary strategies (z_i^{*S}, x^{*i}) have been computed for all $|S| < n$ and $i \in S$, we consider the $S = N$ case.

So now $\omega = (N, \emptyset)$. We also let $((q_j(i, a^i))_{j\in N\setminus i}$ be the payoff vector of $N \setminus i$ when player i is selected as the proposer, her proposal is rejected and her threat is realized as $a^i \in A^i$, i.e.

$$q_j(i, a^i) = \frac{1}{n-1} \sum_{k\in N\setminus i} \sum_{a^{N\setminus i}\in A^{N\setminus i}} z^{*k}(a^{N\setminus i}) u^j(a^{N\setminus i}, a^i).$$

Notice that currently $(q_j(i, a^i))_{j\in N\setminus i}$ have been calculated, and when $n = 2$, we need only to calculate q^1 and q^2 defined in the last section.

Since for any strictly Pareto dominated action profile a^N, we have $z^{*i}(a^N) = 0, \forall i \in N$, we let A^{*N} be the set of all pure action profiles that are not strictly dominated. The calculation of A^{*N} is closely related with *the convex hull computation problem* in the field of computational geometry. From[21], we know that A^{*N} can be computed in time $O\big(|A^N| \log^{n+2} |A^{*N}| + (|A^N|\cdot|A^{*N}|)^{1-1/(\lfloor n/2\rfloor+1)} \log^{O(1)} |A^N|\big)$. And when $n = 2, 3$, there are also algorithms with time complexity $O(|A^N| \log |A^{*N}|)$.

$\forall j \in N$, let t_j^* be player j's overall expected payoff, i.e.

$$t_j^* := \frac{1}{n} \sum_{1\le k\le n} \sum_{a^N\in A^{*N}} z_k^*(a^N) u^j(a^N).$$

Player i's stationary strategy (z_i^*, x_i^*) must be an optimal solution to the following problem:

$$\begin{cases} \max \sum_{a^N\in A^{*N}} z_i(a^N) u^i(a^N) \quad s.t. \\ \sum_{a^N\in A^{*N}} z_i(a^N) u^j(a^N) \ge \rho t_j^* + (1-\rho) \sum_{a^i\in A^i} x^i(a^i) q_j(i, a^i), \forall j \in N \setminus i \\ \sum_{a^N\in A^{*N}} z_i(a^N) = 1 \\ \sum_{a^i\in A^i} x^i(a^i) = 1 \\ z_i(a^N) \ge 0, \forall a^N \in A^{*N} \\ x^i(a^i) \ge 0, \forall a^i \in A^i \end{cases},$$

whose dual programming is:

$$(5)\qquad \begin{cases} \min \sum_{j\in N\setminus i} \rho t_j^* x_{ij} + y_i + w_i \quad s.t. \\ \sum_{j\in N\setminus i} u^j(a^N)x_{ij} + y_i \geq u^i(a^N), \forall a^N \in A^{*N} \\ -(1-\rho)\sum_{j\in N\setminus i} q_j(i,a^i)x_{ij} + w_i \geq 0, \forall a^i \in A^i \\ x_{ij} \leq 0, \forall j \in N\setminus i \end{cases}.$$

Note that the n primal programmings are all self-referring and inter-referring, which embodies the spirit of the SP equilibrium. Each dual programming (5), however, has a feasible region unrelated with (z_i^*, x_i^*). This is the critical reason that we can jump out of the self and inter referrings and express the SP equilibria explicitly through the complementary slackness property. The details are as follows.

Let P_i be the set of all vertices of the feasible region of (5). Computing P_i is called *the vertex enumeration problem*. From [4] we know that P_i can be computed in time $O(n|A^{*N}|^{n+1})$, which is polynomial in the number of pure action profiles when n is a constant.

For any $((x_{ij})_{j\in N\setminus i}, y_i, w_i) \in P_i$, define E_i and F_i as the sets of tight constraints for the first type and second type of inequalities, respectively, in (5), i.e.:

$$E_i \equiv E_i\Big((x_{ij})_{j\in N\setminus i}, y_i, w_i\Big) := \left\{ a^N \in A^{*N} : \sum_{j\in N\setminus i} u^j(a^N)x_{ij} + y_i = u^i(a^N) \right\},$$

$$F_i \equiv F_i\Big((x_{ij})_{j\in N\setminus i}, y_i, w_i\Big) := \left\{ a^i \in A^i : -(1-\rho)\sum_{j\in N\setminus i} q_j(i,a^i)x_{ij} + w_i = 0 \right\}.$$

Note that when $n = 2$, the definition of F_i is not needed because optimal threats are easy to be calculated directly. Using the complementary slackness of the duality theorem for linear programmings, it's easy to get:

Theorem 3. *Suppose the stationary strategies (z_i^{*S}, x^{*i}) have been calculated for all $|S| < n$ and $i \in S$, then the stationary strategy profile $(z_i^*, x^{i*})_{i\in N}$ for state $(N, \emptyset)$, combined with the above stationary strategies, is an SP equilibrium, iff for all $i \in N$, there is $((x_{ij})_{j\in N\setminus i}, y_i, w_i) \in P_i$ such that $(z_i^*, x^{i*})_{i\in N}$ is a* *solution to the following system of linear inequalities:*

$$\begin{cases} \sum\limits_{a^N \in A^{*N}} z_i(a^N) u^j(a^N) \geq_{x_{ij}} \frac{1}{n}\rho \sum\limits_{1\leq k\leq n} \sum\limits_{a^N \in A^{*N}} z_k(a^N) u^j(a^N) \\ \qquad\qquad + (1-\rho) \sum\limits_{a^i \in A^i} x^i(a^i) q_j(i, a^i), \forall j \in N \setminus i, \forall i \in N \\ \sum\limits_{a^N \in A^{*N}} z_i(a^N) = 1, \forall i \in N \\ \sum\limits_{a^i \in A^i} x^i(a^i) = 1, \forall i \in N \\ z_i(a^N) \geq 0, \forall a^N \in A^{*N}, \forall i \in N \\ x^i(a^i) \geq 0, \forall a^i \in A^i, \forall i \in N \\ z^i(a^N) = 0, \forall i \in N, \forall a^N \in A^{*N} \setminus E_i \\ x^i(a^i) = 0, \forall i \in N, \forall a^i \in A^i \setminus F_i \end{cases}$$

where $\geq_{x_{ij}} = \begin{cases} \geq & if\ x_{ij} = 0 \\ = & otherwise \end{cases}.$

So the algorithm is simply to first enumerate all the vertex profiles in $\prod_{i\in N} P_i$, then determine the corresponding E_is and F_is, and finally solve a system of linear inequalities for each profile. Each solution to any system of linear inequalities is a qualified strategy profile for state $(N, \emptyset)$. This also shows that the set of $(z_i^*, x^{i*})_{i\in N}$ is a union of convex sets.

4 Symmetric SP equilibria

This section is motivated by Nash's result on the existence of symmetric Nash equilibrium ([17]), as well as the symmetric solutions to Examples 3,6 in Subsection 2.1.

4.1 Symmetries

Definition 2 (Nash, 1951). *A symmetry θ of game G is a permutation on $\bigcup_{i\in N} A^i$ such that:*

(i) if $\theta(a^i) \in A^j$, then $\theta(A^i) = A^j$; Hence, θ implies a natural permutation on the player set N, which is denoted as π_θ.

(ii) $u^{\pi_\theta(i)}(\theta(a^N)) = u^i(a^N), \forall a^N \in A^N$ and $i \in N$, where

$$\theta(a^N) := \left(\theta\left(a^{\pi_\theta^{-1}(1)}\right), \theta\left(a^{\pi_\theta^{-1}(2)}\right), \cdots, \theta\left(a^{\pi_\theta^{-1}(n)}\right)\right).$$

Let the set of all θ be $\Theta(G)$, and the set of π_θ be $\Pi(G)$. $\Theta(G)$ is never empty, since the identity I, which maps any strategy into itself, is always a member. For any $\theta \in \Theta(G)$, its inverse θ^{-1} is also in $\Theta(G)$, since it's trivial that $\pi_\theta^{-1} = \pi_{\theta^{-1}}$, and $\forall a^N \in A$, $\forall i \in N$:

$$\begin{aligned} & u^{\pi_{\theta_1 \circ \theta_2}(i)}\big((\theta_1 \circ \theta_2)(a^N)\big) \\ = & u^{\pi_{\theta_1}(\pi_{\theta_2}(i))}\big(\theta_1(\theta_2(a))\big) \\ = & u^{\pi_{\theta_2}(i)}\big(\theta_2(a^N)\big) \\ = & u^i(a^N). \end{aligned}$$

Still easily, $\Theta(G)$ is *close*, i.e., for any $\theta_1, \theta_2 \in \Pi(G)$, we have $\theta_1 \circ \theta_2 \in \Theta(G)$, since $\pi_{\theta_1} \circ \pi_{\theta_2} = \pi_{\theta_1 \circ \theta_2}$, and $\forall a^N \in A^N$, $\forall i \in N$:

$$\begin{aligned} & u^{\pi_{\theta_1 \circ \theta_2}(i)}\big((\theta_1 \circ \theta_2)(a^N)\big) \\ = & u^{\pi_{\theta_1}(\pi_{\theta_2}(i))}\big(\theta_1(\theta_2(a))\big) \\ = & u^{\pi_{\theta_2}(i)}\big(\theta_2(a^N)\big) \\ = & u^i(a^N). \end{aligned}$$

Therefore, $\Theta(G)$ is a subgroup of the symmetric group $\mathcal{S}_{\sum_{i\in N}|A^i|}$[18], and $\Pi(G)$ a subgroup of $\mathcal{S}_n$.

Definition 3. *$\Pi(G)$ is called the characteristic group of G.*

θ can be naturally extended to mixed strategies. For all $i \in N$ and $x^i \in \Delta(A^i)$, $\theta(x^i) \in \Delta(A^{\pi_\theta(i)})$ is defined as

$$(\theta(x^i))(\theta(a^i)) := x^i(a^i), \forall a^i \in A^i, \text{or equivalently}$$

$$(\theta(x^i))(a^j) := x^i(\theta^{-1}(a^j)), \forall a^j \in A^j, \text{where} j = \pi_\theta(i).$$

Similarly, for any $x^N \in \Delta(A^N)$, we define

$$\theta(x^N) := \left(\theta\left(x^{\pi_\theta^{-1}(1)}\right), \theta\left(x^{\pi_\theta^{-1}(2)}\right), \cdots, \theta\left(x^{\pi_\theta^{-1}(n)}\right)\right).$$

For any $i \in N$ and $a_1^i, a_2^i \in A^i$, if $u^N(a_1^i, a^{-i}) = u^N(a_2^i, a^{-1})$ holds for all $a^{-i} \in \prod_{j\neq i} A^j$, then a_1^i and a_2^i are identical strategies. By duplicating this kind of identical strategies, we can assume w.l.o.g. that all players have the same number of strategies[19]. Let this number be m.

For any two strategies $s_1, s_2 \in \bigcup_{i\in N} A^i$, if there exists $\theta \in \Theta(G)$ such that $\theta(s_1) = s_2$, we say that $s_1 \sim s_2$. It's easy to show that $\sim$ is an equivalence relation, and hence it gives a partition of $\bigcup_{i\in N} A^i$.

We are especially interested in the special case where this partition has exactly m sets, $\{B_1, B_2, \cdots, B_m\}$, and thus $|A_i \cap B_j| = 1$ for all $1 \le i \le n$ and $1 \le j \le m$. This special case can be taken as that all players have the same strategy sets. For simplicity, we further assume that all players have identical strategy set $A_0 = \{a_1, a_2, \cdots, a_m\}$.

For any $a^S \in A_0^S$, let $\#a^S := (\#(a_1, a^S), \#(a_2, a^S), \#(a_m, a^S))$, where $\#(a_i, a^S)$ is the number of players that use a_i in the strategy profile a^S, $1 \le i \le m$. Below is the usual definition for symmetric games (cf. Brandt et. al. [3]).

Definition 4. *$G = (N, (A^i)_{i\in N}, (u^i)_{i\in N})$ is a symmetric game iff*[20]*:*

(i) All the players have identical strategy sets A_0.

[18] The set of all permutations on $\{1, 2, \cdots, \sum_{i\in N}|A^i|\}$.
[19] At least no generality is lost in this paper.
[20] Notice that conditions (ii) and (iii) can be combined as one.

(ii) In the eyes of any player, the other players are indistinguishable (but she can distinguish herself from the others), i.e., $u^i(a^i, a_{-i}) = u^i(b^i, b^{-i})$ *if* $a^i = b^i$ *and* $\#(a_{-i}) = \#(b_{-i})$.

(iii) Utility functions are identical, i.e., $u^i(a^i, a_{-i}) = u^j(b^j, b^{-j})$ *if* $a^i = b^j$ *and* $\#(a_{-i}) = \#(b_{-j})$.

Using the notion of characteristic group, we can give a more concise definition of symmetric games. In fact, the following theorem is almost trivial.

Theorem 4. *G is a symmetric game iff* $\Pi(G) = \mathcal{S}_n$.[21]

It's intuitive that the larger $\Pi(G)$ is, the "more symmetric" the game G is. A natural question is, given m, for any subgroup of $\mathcal{S}_n$, does there exist a game G such that $\Pi(G)$ is exactly this subgroup?

For a trivial counter-example, let $m = 1$ and $n = 4$, then there exists no game G with $\Pi(G) = \{I, (1,2)(3,4)\}$, because when $(1,2)(3,4)$[22] is a symmetry of G with $m = 1, n = 4$, so are $(1,2)$ and $(3,4)$. Generally, we have:

Theorem 5. *Suppose that all players have an identical strategy set, n is the number of players and m the number of strategies, then:*

(a) when $m \geq n$, *for any subgroup* $\mathcal{T}$ *of* $\mathcal{S}_n$, *there exists a game G such that* $\Pi(G) = \mathcal{T}$;

(b) when $m < n$, *any game G with* $\Pi(G) \supseteq \mathcal{A}_n$ *must satisfy* $\Pi(G) = \mathcal{S}_n$, *where* $\mathcal{S}_n$ *is the symmetric group, and* $\mathcal{A}_n$ *the alternating group. Hence there exists no game G with* $\Pi(G) = \mathcal{A}_n$.

Proof. Let $A = \{a_1, a_2, \cdots, a_m\}$ be the common strategy set, and A^N be the (pure) strategy profile space. Fix any subgroup $\mathcal{T}$ of $\mathcal{S}_n$, for any $a^N, b^N \in A^N$, we define $a^N \sim_{\mathcal{T}} b^N$, if there is a $\pi \in \mathcal{T}$ such that $\pi(a^N) = b^N$. It's easy to see that $\sim_{\mathcal{T}}$ is an equivalence relation, and therefore there is a partition of A^N associated with it, which is denoted as $\mathcal{T}(A^N)$. We also define $\sim_{\mathcal{T}} (a^N)$ as the set of strategy profiles that are equivalent to a^N in the sense of $\sim_{\mathcal{T}}$. Let $\mathcal{T}_1, \mathcal{T}_2$ be two subgroups of $\mathcal{S}_n$ and $\mathcal{T}_1 \subset \mathcal{T}_2$, then $\mathcal{T}_2(A^N)$ is coarser than $\mathcal{T}_1(A^N)$, that is, for any $X_1 \in \mathcal{T}_1(G)$, there exists a $X_2 \in \mathcal{T}_2(G)$ such that $X_1 \subseteq X_2$, since $a^N \sim_{\mathcal{T}_1} b^N$ implies $a^N \sim_{\mathcal{T}_2} b^N$.

It's obvious that if there exists a game G with $\Pi(G) = \mathcal{T}$, then there must exist a special one such that for any a^N, b^N that are partitioned into different sets in $\mathcal{T}(A^N)$, $u^i(a^N) \neq u^j(b^N)$ holds for all $i, j \in N, i \neq j$. So to prove (a), we only have to show that when $m \geq n$, if $\mathcal{T}_1, \mathcal{T}_2$ are subgroups of $\mathcal{S}_n$ with $\mathcal{T}_1 \subset \mathcal{T}_2$, then $\mathcal{T}_2(A^N)$ is strictly coarser than $\mathcal{T}_1(A^N)$, i.e., $\mathcal{T}_1(A^N) \neq \mathcal{T}_2(A^N)$. This is true because for $a^N = (a_1, a_2, \cdots, a_n)$ (notice that $m \geq n$ is used here), we have $\pi(a^N) \notin \sim_{\mathcal{T}_1} (a^N)$ for any $\pi \in \mathcal{T}_2 \setminus \mathcal{T}_1$, and therefore $\sim_{\mathcal{T}_1} (a^N) \subset \sim_{\mathcal{T}_2} (a^N)$.

To prove (b), we will show that when $m < n$, $\mathcal{A}_n(A^N) = \mathcal{S}_n(A^N)$. It suffices to show that for any $a^N, b^N \in A^N$, if there exists a permutation $\pi \in \mathcal{S}_n$ such

[21] Notice that $\Pi(G) = \mathcal{S}_n$ implies that all players have identical strategy sets.

[22] I is the identity, $(1,2)(3,4)$ is the permutation which exchanges 1 and 2, as well as 3 and 4.

that $\pi(a^N) = b^N$, then there must exist an even permutation $\pi_0 \in \mathcal{A}_n$ such that $\pi_0(a^N) = b^N$. Let k_i be the number of times that a_i is used in a^N, $1 \leq i \leq m$. W.o.l.g., we assume that all the k_i's are positive, and $m < n$ tells us that there exists at lease one i_0 such that $k_{i_0} \geq 2$. For simplicity, we assume that $k_1 \geq 2$. Let $a^{*N} = (\overbrace{a_1, \cdots, a_1}^{k_1}, \overbrace{a_2, \cdots, a_2}^{k_2}, \cdots, \overbrace{a_m, \cdots, a_m}^{k_m})$, and π_{10} be any even permutation such that the first two dimensions of $\pi_{10}(a^N)$ are both a_1. We also let $v(\pi_{10}(a^N)) := \{1 \leq i \leq n : (\pi_{10}(a^N))^i = a^{\pi_{10}^{-1}(i)} \neq a^{*i}\}$, $w_1(\pi_{10}(a^N))$ be the largest number of $v(\pi_{10}(a^N))$, and $w_2(\pi_{10}(a^N))$ be any i such that $i < w_1(\pi_{10}(a^N))$ and $(\pi_{10}(a^N))^i = a^{*w_1(\pi_{10}(a^N))}$. We want to construct an even permutation π_1 such that $\pi_1(a^N) = a^{*N}$. When $v(\pi_{10}(a^N)) = \emptyset$, we are done since we can simply let $\pi_1 := \pi_{10}$. Otherwise, we can continue the loop of $\pi_{11} := (w_1(\pi_1(a^N)), w_2(\pi_1(a^N))) \circ (1, 2)$ and $\pi_1 := \pi_{11} \circ \pi_{10} \circ \pi_1$, until $v(\pi_1(a^N)) = \emptyset$. By the same way, we can construct an even permutation π_2 such that $\pi_2(b^N) = a^{*N}$. Let $\pi_0 = \pi_2^{-1} \circ \pi_1$, then $\pi_0(a^N) = b^N$, which completes the proof to (b). $\square$

4.2 Symmetric solutions

Intuitively, a symmetry θ says that i and $\pi_\theta(i)$ are symmetric for all $i \in N$ in the sense that in the eyes of player $\pi_\theta(i)$, its strategy $\theta(a^i)$ plays exactly the same role as player i's strategy a^i does, as long as she sees $\theta(a^j)$ as i sees a^j,$\forall j \in N \setminus \{i, \pi_\theta(i)\}$. Very reasonably, for any arbitrage solution, $\pi_\theta(i)$ will expect to get at least the same amount of utility as i will get, otherwise she will ask to re-name all the players and strategies by re-naming j as $\pi_\theta(j)$ and a^j as $\theta(a^j)$, for all $j \in N$. Accordingly, an arbitrage solution should respect this kind of symmetry, i.e., it should remain unchanged after the above re-naming.

We call a strategy profile $x^N \in \prod_{i\in N} \Delta(A^i)$ symmetric if $\theta(x^N) = x^N$, i.e., $x^i = \theta(x^{\pi_{\theta^{-1}}(i)})$ holds for all $i \in N$, $\theta \in \Theta(G)$.

For any distribution $z \in \Delta(A^N)$, define $\theta(z) \in \Delta(A^N)$ as

$$(\theta(z))(\theta(a^N)) = z(a^N), \forall a^N \in A^N, \text{or equivalently}$$

$$(\theta(z))(a^N) = z(\theta^{-1}(a^N)), \forall a^N \in A^N.$$

We call a distribution $z \in \Delta(A^N)$ symmetric if $\theta(z) = z$, i.e., $z(\theta(a^N)) = z(a^N)$ for any $a^N \in A^N$.

We call a payoff vector $u^N \in \mathbb{R}^N$ symmetric if $u^{\pi(i)} = u^i$ for any $i \in N$ and $\pi \in \Pi(G)$. Accordingly:

Definition 5. *We call a solution (equilibrium) strategy-symmetric if the strategy profile used in this solution is symmetric, distribution-symmetric if the corresponding distribution of action profiles in this solution is symmetric, and payoff-symmetric if the corresponding expected payoff vector is symmetric.*

Let x^N be a symmetric strategy profile, then its associated action profile distribution z_{x^N}, defined as $z_{x^N}(a^N) = \prod_{i\in N} x^i(a^i)$, is also symmetric, because $z_{x^N}(\theta(a^N)) = \prod_{i\in N} x^{\pi_\theta(i)}(\theta(a^i)) = \prod_{i\in N} x^i(a^i) = z_{x^N}(a^N)$. Thus strategy-symmetric implies distribution-symmetric, which obviously further implies payoff-symmetric.

The HM model is kind of a means of arbitrage, so it's meaningful to study its symmetric SP equilibria. Since correlation is involved in the HM model, it's not surprising that, as shown by Example 6 in Subsection 2.1, the HM model may not possess any strategy-symmetric SP equilibrium, since it may not even have any solution in $\prod_{i\in N} \Delta(A^i)$.

Whether the HM model always has a distribution-symmetric SP equilibrium, or a payoff-symmetric, are interesting problems for further research. In the end of this section, we illustrate through Table 1. that games with very different characteristic groups may have very similar distribution-symmetric solutions, that is, a lot of symmetry information may be lost in the distribution-symmetric solution. Of course, information loss in payoff-symmetric solutions is even larger.

Table 1. Structures of distribution-symmetric solutions.

G_1

$\star$	A	B	$\triangle$	D	E	$\otimes$	G	H
A	$*$	C	D	$\diamondsuit$	F	G	$\square$	I
B	C	$\circ$	E	F	$\bigtriangledown$	H	I	$\oplus$

G_2

$\star$	A	B	A	D	E	B	C	F
A	D	C	D	$*$	G	E	G	H
B	E	F	C	G	H	F	H	$\circ$

G_3

$\star$	A	B	A	D	C	B	C	F
A	D	C	D	$*$	G	C	G	H
B	C	F	C	G	H	F	H	$\circ$

G_1, G_2, G_3 are three 3×3 games, with different characteristic groups: $\Pi(G_1)=\{I,(1,2)\}$, $\Pi(G_2)=\{I,(1,2,3),(1,3,2)\}$, $\Pi(G_3) = \mathcal{S}_3$. Each capital letter or symbol stands for the probability assigned to the corresponding action profile by a distribution-symmetric equilibrium. G_3 is much more symmetric than G_2, but the structures of their distribution-symmetric solutions are very similar, in fact, the only difference is that the three E's in the distribution-symmetric equilibrium of G_2 are replaced by C's in that of G_3.

5 The HM* Model

In the HM* model, a state should be defined as $\omega = (S, x^{N\setminus S})$, thus there are infinitely many possible states, while there are only finitely many states in the HM model. A second difference is, we should define

$$(\omega||x^k) := (S \setminus k, x^k, x^{N\setminus S}).$$

A third difference is $\eta^S_{\omega,k}(x^k)$ does not satisfy formula (2) any more, instead we only have

$$\eta^S_{\omega,k}(x^k) = \zeta^{S\setminus k}_{(\omega||x^k)},$$

which is not a linear function of x^k.

Parallel to (3), in the HM* model Assumption (A1) can be rewritten as: player $i \in S \setminus k$ accepts k's proposal iff

$$u^i(z^S, x^{N\setminus S}) \geq \rho u^i(\zeta^S_\omega, x^{N\setminus S}) + (1-\rho) u^i\big(\zeta^{S\setminus k}_{(\omega||x^k)}, x^{N\setminus S}\big), \tag{6}$$

which is not linear any longer. Similar to the HM model, we define:

$Y \equiv Y_{\omega,k}(\boldsymbol{\zeta}) := \big\{(z^S, x^k) \in \Delta(A^S) \times \Delta(A^k) : \text{(6) holds for all } i \in S \setminus k\big\}$;

$Y^* \equiv Y^*_{\omega,k}(\boldsymbol{\zeta}) := \underset{(z^S,x^k)\in Y_{\omega,k}(\boldsymbol{\zeta})}{\arg\max} \; u^k(z^S, x^{N\setminus S})$;

$Z \equiv Z_{\omega,k}(\boldsymbol{\zeta}) := \big\{z^S \in \Delta(A^S) : \exists x^k \in \Delta(A^k) \text{ such that } (z^S, x^k) \in Y_{\omega,k}(\boldsymbol{\zeta})\big\}$;

$Z^* \equiv Z^*_{\omega,k}(\boldsymbol{\zeta}) := \Big\{z^S \in \Delta(A^S) : \exists x^k \in \Delta(A^k) \text{ such that } (z^S, x^k) \in Y^*_{\omega,k}(\boldsymbol{\zeta})\Big\}$.

Parallel to Proposition 1 of [11], we have:

Lemma 2. *σ is an SP equilibrium of the HM* model, iff for each state $\omega = (z^S, x^{N\setminus S})$ and each proposer $k \in S$, the following condition is satisfied:*

$$\zeta^S_{\omega,k} \in Z^*_{\omega,k}(\boldsymbol{\zeta}),$$

where $\boldsymbol{\zeta}$ is the outcome configuration derived by σ.

5.1 Positive result for the two player case

Although (6) is not linear, its concavity can be guaranteed when $n = 2$. To prove the existence of SP equilibrium in this case, the key point is to generalize Lemma 2 of [11] from linear functions to concave ones.

Lemma 3. *Suppose s and t are positive integers, $X \subseteq \mathbb{R}^s, Y \subseteq \mathbb{R}^t$. $f : X \to \mathbb{R}$ is a continuous function, and $g : Y \to \mathbb{R}$ is concave. $F(x) = \{y \in Y : g(y) \geq f(x)\}$. Let $X_0 = \{x \in X : F(x) \neq \emptyset\}$, then $F(x)$ is a continuous correspondence on X_0.*

Proof. Upper hemi-continuity[23] is immediate, we only have to show lower hemi-continuity.

Let $x_n, x_0 \in X_0, x_n \to x_0 (n \to \infty), y_0 \in F(x_0)$.

And define $y_n \in \arg \min_{g(y)\geq f(x_n)} ||y_n - y_0||$. We prove $y_n \to y_0$.

Suppose not, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $\delta > 0$ such that $||y_{n_k} - y_0|| > \delta$. For any $y \in Y$ with $||y - y_0|| \leq \delta$, we have $g(y) < f(x_{n_k})$ (otherwise y contradicts the definition of y_{n_k}). Let $k \to \infty$, we have $g(y) \leq f(x_0)$ (f is continuous). $y_0 \in f(x_0)$ implies that $g(y_0) \geq f(x_0)$. Since $||y_0 - y_0|| \leq \delta$, we have $g(y_0) \leq f(x_0)$. Therefore $f(x_0) = g(y_0)$ and $g(y) \leq g(y_0)$. By concavity, we know that y_0 is the maximum point of g. $x_n \in X_0$ implies $f(x_n) \leq g(y_n) \leq g(y_0)$. Hence by definition $y_n = y_0$, a contradiction with y_n not converging to y_0. □

Theorem 6. *SP equilibrium exists in the HM* model when $n = 2$.*

[23] The author shares the viewpoint that we use the terminology "hemi-" to refer to the concept defined through sequence, and use "semi-" to refer to the other concept defined through open sets, though the two concepts are equivalent on compact and convex sets.

Proof. We only need to consider the case $\omega = (N, \emptyset)$, where $N = \{1, 2\}$. We first have $\boldsymbol{\zeta} = (\zeta^N_{\omega,1}, \zeta^N_{\omega,2})$. For $\{i, j\} = \{1, 2\}$, we have

$$Y_{\omega,i}(\boldsymbol{\zeta}) = \left\{(z^N, x^i) \in \Delta(A^N) \times \Delta(A^i) : u^j(z^N) - (1-\rho) \max_{a^j \in A^j} u^j(x^i, a^j) \geq \rho u^j(\zeta^N)\right\},$$

where $\zeta^N := \frac{1}{2}(\zeta^N_{\omega,1} + \zeta^N_{\omega,2})$.

Let $Z^*(\boldsymbol{\zeta}) := (Z^*_1(\boldsymbol{\zeta}), Z^*_2(\boldsymbol{\zeta}))$. It is a correspondence on $\Delta(A^N) \times \Delta(A^N)$ to itself. Due to Lemma 2, we only need to prove that $Z^*(\cdot)$ has a fixed point on $\Delta(A^N) \times \Delta(A^N)$. Because $\Delta(A^N) \times \Delta(A^N)$ is nonempty, compact and convex, from Kakutani's Fixed Point Theorem, it suffices to prove that $Z^*(\boldsymbol{\zeta})$ has a nonempty, convex set of value, and is upper hemi-continuous, which is satisfied if we can prove $Y^*(\boldsymbol{\zeta}) := (Y^*_1(\boldsymbol{\zeta}), Y^*_2(\boldsymbol{\zeta}))$ has these properties, since $Z^*(\boldsymbol{\zeta})$ is a projection of $Y^*(\boldsymbol{\zeta})$. Further, it's done if $Y^*_1(\boldsymbol{\zeta})$ and $Y^*_2(\boldsymbol{\zeta})$ satisfy these properties. By symmetry, we prove only for $Y^*_1(\boldsymbol{\zeta})$.

As $u^1(.)$ is a linear function, for any fixed $\boldsymbol{\zeta}$, $Y_1(\boldsymbol{\zeta})$ is a convex set, we know that $Y^*_1(\boldsymbol{\zeta})$ is also a convex set. For any $x^1 \in \Delta(A^1)$, let $a^{2*} \in A^2$ satisfy $u^2(x^1, a^{2*}) = \max_{a^2 \in A^2} u^2(x^1, a^2)$, then it's easy to see that $z^N := \rho\zeta^N + (1-\rho)(x^1, a^{2^*}) \in Y_1(\boldsymbol{\zeta})$. Hence $Y_1(\boldsymbol{\zeta})$ is nonempty. The upper hemi-continuity of $Y_1(\boldsymbol{\zeta})$ is immediate from Lemma 3 and Berge's Maximum Theorem. ◻

Similar to Theorem 1, the following is also easy.

Theorem 7. *For zero-sum games, the expected payoff vector in the SP equilibrium of the HM* model is always the min-max value payoff vector $(v, -v)$, and no player cares about whether she is a proposer or a responder.*

5.2 Negative result for the general case

In the 3-person game, the main observation is that when one player is kicked out of the game with a threat, then the left problem is reduced to a 2-person game with payoffs equal to the corresponding expected utilities [24]. However, the SP equilibrium of the 2-player game, may not be continuous in the threat, and hence the payoff function of the first-selected player, conditioning that her proposal is accepted, may not have an maximum value. That is, a player may not be able to choose an optimal threat.

Example 7. There are three players in total, players 1, 2 and 3, whose strategy sets are respectively $\{T, D\}, \{L, R\}$ and $\{M1, M2\}$. The payoff table is:

$M1$:

	L	R
T	0,3,3	0,0,0
D	0,0,0	4,1,1

$M2$:

	L	R
T	0,3,3	4,2,0
D	0,0,0	0,1,1

.

Theorem 8. *If $\frac{2}{3} \leq \rho < 1$, then the game in Example 7 has no SP equilibrium in the HM* model.*

[24] The expectation is taken over the distribution of the threat of the kicked out player.

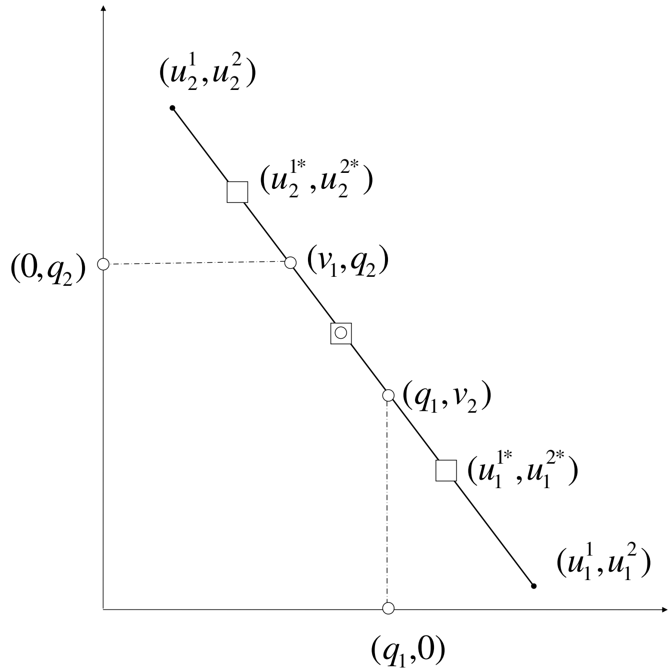

Fig. 3. Illustration of the small trick.

Before proof, we give a useful small trick for the 2-player case[25]. If a two player normal form game satisfies the following three conditions:

(i) the Pareto boundary is on a line. (Suppose (u_1^1, u_1^2) and (u_2^1, u_2^2) are the two end points.)

(ii) $u_1^1 \neq u_2^1$ and $u_1^2 \neq u_2^2$. (W.o.l.g., suppose $u_1^1 < u_2^1$. Let q_1 be the payoff level that player 2 can hold player 1 to by mixed strategy, that is, $q_1 = \min_{x^2 \in \Delta(A^2)} \max_{a_1 \in A^1} u^1(a^1, x^2)$. Similarly, we let $q_2 = \min_{x^1 \in \Delta(A^1)} \max_{a_2 \in A^2} u^2(a^2, x^1)$.)

(iii) $u_1^1 \leq q_1 \leq u_2^1$ and $u_2^1 \leq q_2 \leq u_2^2$. (Let (q_1, v_2) and (v_1, q_2) be the projection points of $(q_1, 0)$ and $(0, q_2)$ on the boundary.)

then the trick is: if (u_1^{1*}, u_1^{2*}) and (u_2^{1*}, u_2^{2*}) are the two proposal points by player 1 and player 2, respectively, in the SP equilibrium, then the middle point of (u_1^{1*}, u_1^{2*}) and (u_2^{1*}, u_2^{*2}) is exactly the middle point of (q_1, v_2) and (v_1, q_2). Therefore, the SP equilibrium payoff point can be derived easily by calculating v_1 and v_2. This is illustrated in Figure 3.

This trick can be proved easily as follows. (u_1^{*1}, u_1^{*2}) being the optimal proposal of player 1 tells us that $u_1^{2*} = \frac{1}{2}\rho(u_1^{2*} + u_2^{2*}) + (1-\rho)q_2$, which is equivalent

[25] This is similar to a well known trick, cf. Hart (2004, [8]).

to

$$(7) \qquad u_1^{1*} = \frac{1}{2}\rho(u_1^{1*} + u_2^{1*}) + (1-\rho)v_1.$$

Analogously, (u_2^{*1}, u_2^{*2}) being the optimal proposal of player 2 tells us that

$$(8) \qquad u_2^{*1} = \frac{1}{2}\rho(u_1^{1*} + u_2^{1*}) + (1-\rho)q_1.$$

Adding (7) and (8) gives $u_1^{1*} + u_2^{1*} = v_1 + q_1$ and hence the trick.

Proof. Notice that there are three points $(4,2,0), (4,1,1)$ and $(0,3,3)$ in total that are not strictly dominated ((0,1,1) is strictly dominated by a mixture of half (4,1,1) and half (0,3,3); we combined the two $(0,3,3)$ into one point). Suppose that the SP equilibrium (payoff) set, which is denoted as SPE, is non-empty, and in one of the SPE points, player 1 assigns probabilities α_1^* and α_2^* and $1-\alpha_1^*-\alpha_2^*$ respectively to the three points $(4,2,0), (4,1,1)$ and $(0,3,3)$. Similarly, β_1^*, β_2^* and $1-\beta_1^*-\beta_2^*$ are the probabilities assigned by player 2, γ_1^*, γ_2^* and $1-\gamma_1^*-\gamma_2^*$ by player 3. We also let $\lambda_i^* := \frac{1}{3}(\alpha_i^* + \beta_i^* + \gamma_i^*)$ for $i \in \{1,2\}$.

• Suppose now player 1 is selected as the first proposer. Let $(x, 1-x)$ be her threat, i.e. the probabilities assigned to T and D in the threat are respectively x and $1-x$. We show in the sequel that there is no optimal x.

If player 1's proposal is rejected and she is out of the next round of bargaining, then player 2 and player 3 face a problem of:

	M1	M2
L	3x,3x	3x,3x
R	1-x,1-x	1+x,1-x

For each $i \in \{2,3\}$, let $q_1^i(x)$ be the payoff level that the other player $j \neq i$ can hold i to, i.e.,

$$q_1^i(x) := \min_{x^j \in \Delta(A^j)} \max_{a_i \in A^i} u^i(a^i, x^j),$$

then $q_1^2(x) = \begin{cases} 1-x & if\ 0 \le x \le \frac{1}{4} \\ 3x & if\ \frac{1}{4} < x \le 1 \end{cases}$, and $q_1^3(x) = \begin{cases} 3x & if\ 0 \le x \le \frac{1}{4} \\ 1-x & if\ \frac{1}{4} < x \le 1 \end{cases}$.

We discuss in five cases to compute $SPE^{2,3}(x)$, the SP-equilibrium payoff vector set of players 2 and 3, when they face the above problem.

⋆ Case 1. $x = 0$. The only point that is not strictly dominated is $(1,1)$, so $SPE^{2,3}(x) = \{(1,1)\}$.

⋆ Case 2. $0 < x \le \frac{1}{4}$.

There are two points that are not strictly dominated, $(1+x, 1-x)$ and $(1-x, 1-x)$. Let ϱ^* and σ^* be the probabilities assigned to $(1+x, 1-x)$ in some SPE point by player 2 and 3, respectively. Obviously, $\varrho^* = 1$. So the optimization problem of player 3 is:

$$\begin{cases} \max \quad 1-x \quad s.t. \\ (1+x)\sigma + (1-x)(1-\sigma) \geq \frac{1}{2}\rho\big(2x(\sigma^*+1)+2-2x\big) + (1-\rho)(1-x) \\ 0 \leq \sigma \leq 1 \end{cases}$$

$$= \begin{cases} \max \quad 1-x \quad s.t. \\ \sigma \geq \frac{1}{2}\rho(\sigma^*+1) \\ 0 \leq \sigma \leq 1 \end{cases} \text{, which gives } \frac{\rho}{2-\rho} \leq \sigma^* \leq 1.$$

So $SPE^{2,3}(x) = \big\{(1+x\sigma^*, 1-x) : \frac{\rho}{2-\rho} \leq \sigma^* \leq 1\big\}$.

⋆ Case 3. $\frac{1}{4} < x < \frac{1}{2}$.

There are two points that are not strictly dominated, $(3x, 3x)$ and $(1+x, 1-x)$. The projection of $(3x, 0)$ and $(0, 1-x)$ are $(3x, 3x)$ and $(1+x, 1-x)$, respectively, thus by the small trick described before this proof, $SPE^{2,3}(x) = \big\{(2x+\frac{1}{2}, x+\frac{1}{2})\big\}$.

⋆ Case 4. $x = \frac{1}{2}$.

There are two points that are not strictly dominated, $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{3}{2}, \frac{1}{2})$. Let ϱ^* and σ^* be the probabilities assigned to $(\frac{3}{2}, \frac{3}{2})$ in some SPE point by player 2 and 3, respectively. Obviously, $\sigma^* = 1$. Then player 2 faces a problem of:

$$\begin{cases} \max \quad \frac{3}{2} \quad s.t. \\ \frac{3}{2}\varrho + \frac{1}{2}(1-\varrho) \geq \frac{1}{2}\rho\big((\varrho^*+1)+1\big) + (1-\rho)\cdot\frac{1}{2} \\ 0 \leq \varrho \leq 1 \end{cases}$$

$$= \begin{cases} \max \quad \frac{3}{2} \quad s.t. \\ \varrho \geq \frac{1}{2}\rho(\varrho^*+1) \\ 0 \leq \varrho \leq 1 \end{cases} \text{, which gives } \frac{\rho}{2-\rho} \leq \varrho^* \leq 1.$$

So $SPE^{2,3}(x) = \big\{(\frac{3}{2}, 1+\frac{1}{2}\varrho^*) : \frac{\rho}{2-\rho} \leq \varrho^* \leq 1\big\}$

⋆ Case 5. $\frac{1}{2} < x \leq 1$.

In this case, the only point which is strictly dominated is $(3x, 3x)$, and therefore:

$$SPE^{2,3}(x) = \big\{(3x, 3x)\big\}.$$

In sum, we have

$$SPE^{2,3}(x) = \begin{cases} \big\{(1,1)\big\} & if\ x = 0 \\ \big\{(1+xt, 1-x) : \frac{\rho}{2-\rho} \leq t \leq 1\big\} & if\ 0 < x \leq \frac{1}{4} \\ \big\{(2x+\frac{1}{2}, x+\frac{1}{2})\big\} & if\ \frac{1}{4} < x < \frac{1}{2} \\ \big\{(\frac{3}{2}, 1+\frac{1}{2}t) : \frac{\rho}{2-\rho} \leq t \leq 1\big\} & if\ x = \frac{1}{2} \\ \big\{(3x, 3x)\big\} & if\ \frac{1}{2} < x \leq 1 \end{cases}.$$

Let $\big(e_{12}(x, t_x), e_{13}(x, t_x)\big)$ be an arbitrary point in $SPE^{2,3}(x)$, where the parameter t (if needed) is chosen as t_x, then the optimization problem that player 1 faces when she is selected as the first proposer is:

$$\begin{cases} \max \quad 4\alpha_1 + 4\alpha_2 \quad s.t. \\ 2\alpha_1 + \alpha_2 + 3(1-\alpha_1-\alpha_2) \geq \rho\big(3 - \lambda_1^* - 2\lambda_2^*\big) + (1-\rho)e_{12}(x,t_x) \\ \alpha_2 + 3(1-\alpha_1-\alpha_2) \geq \rho\big(3 - 3\lambda_1^* - 2\lambda_2^*\big) + (1-\rho)e_{13}(x,t_x) \\ \alpha_1 + \alpha_2 \leq 1 \\ \alpha_1 \geq 0, \alpha_2 \geq 0 \\ 0 \leq x \leq 1 \end{cases}$$

$$= \begin{cases} \max \quad 4\alpha_1 + 4\alpha_2 \quad s.t. \\ \alpha_1 + 2\alpha_2 \leq 3(1-\rho) + \rho(\lambda_1^* + 2\lambda_2^*) - (1-\rho)e_{12}(x,t_x) \\ 3\alpha_1 + 2\alpha_2 \leq 3(1-\rho) + \rho(3\lambda_1^* + 2\lambda_2^*) - (1-\rho)e_{13}(x,t_x) \\ \alpha_1 + \alpha_2 \leq 1 \\ \alpha_1 \geq 0, \alpha_2 \geq 0 \\ 0 \leq x \leq 1 \end{cases},$$

Adding the first two inequalities in the above programming we have

$$\alpha_1 + \alpha_2 \leq \frac{3}{2}(1-\rho) + \rho(\lambda_1^* + \lambda_2^*) - \frac{1}{4}(1-\rho)(e_{12}(x,t_x) + e_{13}(x,t_x)).$$

Notice that the left side of the above inequality is exactly a quarter of the objective function.

Since $e_{12}(x,t_x)+e_{13}(x,t_x) = \begin{cases} 2 & if\ x=0 \\ 2-(1-t_x)x & if\ 0 \leq x \leq \frac{1}{4} \\ 3x+1 & if\ \frac{1}{4} < x < \frac{1}{2} \\ \frac{5}{2}+\frac{1}{2}t_x & if\ x = \frac{1}{2} \\ 6x & if\ \frac{1}{2} < x \leq 1 \end{cases}$ has an infimum of $\frac{7}{4}$ (when $x \to \frac{1}{4}^+$), which is not attainable, we know that

$$\alpha_1^* + \alpha_2^* < \frac{17}{16}(1-\rho) + \rho(\lambda_1^* + \lambda_2^*).$$

If $\frac{17}{16}(1-\rho) + \rho(\lambda_1^* + \lambda_2^*) \leq 1$[26], then the proof is done, which can be easily demonstrated as follows: for any arbitrarily small $\epsilon > 0$, letting $x = \frac{1}{4} + \epsilon$ and letting the first two inequalities in the above programming hold as equality, we would have

$$\begin{cases} \alpha_1 = \rho\lambda_1^* + \frac{1}{2}(1-\rho)(\frac{1}{4}+\epsilon) \geq 0 \\ \alpha_2 = \rho\lambda_2^* + (\frac{15}{16} - \frac{5}{4}\epsilon)(1-\rho) \geq 0 \end{cases},$$

[26] This condition is equivalent to $[\rho \cdot t_2^* + (1-\rho)\cdot 1] + [\rho \cdot t_3^* + (1-\rho)\cdot \frac{3}{4}] \geq 2$, where $t_2^* = 2\lambda_1^* + \lambda_2^* + 3(1-\lambda_1^*-\lambda_2^*), t_3^* = \lambda_2^* + 3(1-\lambda_1^*-\lambda_2^*)$ are the overall expected payoff of player 1 and player 2, respectively. Notice that $(1, \frac{3}{4})$ is the limit equilibrium payoff point of player 2 and player 3 when x approaches $\frac{1}{4}$ from above. So it means that the total "bargaining power" of players 1 and 2 is relatively large, in the sense that to get the nodding of player 2 and 3, player 1 needs to give them a total payoff of at least 2.

and $\alpha_1 + \alpha_2 = \frac{17}{16}(1-\rho) + \rho(\lambda_1^* + \lambda_2^*) - \frac{3}{4}(1-\rho)\epsilon < 1$, which tell us that the optimization problem of player 1 is unsolvable.

So we assume in the rest that

$$\frac{17}{16}(1-\rho) + \rho(\lambda_1^* + \lambda_2^*) > 1,$$

which gives

$$\lambda_1^* + \lambda_2^* > \frac{17\rho - 1}{16\rho}. \tag{9}$$

It's also straightforward that

$$\alpha_1^* + \alpha_2^* = 1. \tag{10}$$

• Suppose now player 2 is selected as the first proposer, whose threat is $(y, 1-y)$. Let $\big(e_{21}(y), e_{23}(y)\big)$ be an arbitrary point in $SPE^{1,3}(y)$, then the optimization problem that player 2 faces is:

$$\begin{cases} \max \quad 2\beta_1 + \beta_2 + 3(1-\beta_1-\beta_2) \quad s.t. \\ 4(\beta_1+\beta_2) \geq \rho(4\lambda_1^* + 4\lambda_2^*) + (1-\rho)e_{21}(y) \\ \beta_2 + 3(1-\beta_1-\beta_2) \geq \rho(3 - 3\lambda_1^* - 2\lambda_2^*) + (1-\rho)e_{23}(y) \\ \beta_1 + \beta_2 \leq 1 \\ \beta_1 \geq 0, \beta_2 \geq 0 \\ 0 \leq y \leq 1 \end{cases}.$$

Let y^* be one of the optimal y, it's easy to observe that

$$\beta_1^* + \beta_2^* = \rho(\lambda_1^* + \lambda_2^*) + \frac{1}{4}(1-\rho)e_{21}(y^*),$$

since otherwise we may decrease the value of β_1 or β_2, whichever is greater than 0, by a small enough amount, keeping all the constraints being satisfied and strictly increasing the objective function value. It's obvious that $e_{21}(y^*) \leq 4$, so we have

$$\beta_1^* + \beta_2^* \leq \rho(\lambda_1^* + \lambda_2^*) + 1 - \rho. \tag{11}$$

• Suppose now player 3 is selected as the first proposer, and her threat is $(z, 1-z)$, i.e. the probabilities assigned to $M1$ and $M2$ are z and $1-z$, respectively. Then players 1 and 2 face a problem of:

	R	L
T	0,3	4(1-z),2(1-z)
D	0,0	4z,1

.

For each $i \in \{1,2\}$, let $q_3^i(z)$ be the payoff level that the other player $j \neq i$ can hold i to.

Then $q_3^1(z) = 0$, $q_3^2(z) = \begin{cases} \min \quad u \quad s.t. \\ u \geq 3t \\ u \geq 2(1-z)t + 1 - t \\ 0 \leq t \leq 1 \end{cases} = \frac{3}{2(z+1)}$.

We discuss in two cases to compute the SPE set.

$\star$ Case 1. $0 \leq z \leq \frac{1}{2}$.

There are two points that are not strictly dominated in this case, $(4(1-z), 2(1-z))$ and $(0,3)$.

Let α^*, β^* be the probabilities assigned to $(4(1-z), 2(1-z))$ by players 1 and 2 respectively in some SPE point, then player 1 faces a problem of :

$$\begin{cases} \max \quad 4(1-z)\alpha \quad s.t. \\ 2(1-z)\alpha + 3(1-\alpha) \geq \frac{1}{2}\rho\big(6 - (1+2z)(\alpha^* + \beta^*)\big) + (1-\rho)\frac{3}{2(z+1)} \\ 0 \leq \alpha \leq 1 \end{cases}$$

$$= \begin{cases} \max \quad 4(1-z)\alpha \quad s.t. \\ \alpha \leq \frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3(1-\rho)}{2(1+z)} \\ 0 \leq \alpha \leq 1 \end{cases}.$$

There are two possibilities: 1) If $\frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3(1-\rho)}{2(1+z)} > 1$, then

$$\alpha^* = 1; \tag{12}$$

2) If $\frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3(1-\rho)}{2(1+z)} \leq 1$, then

$$\alpha^* = \frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3(1-\rho)}{2(1+z)}. \tag{13}$$

Player 2 faces a problem of:

$$\begin{cases} \max \quad 2(1-z)\beta + 3(1-\beta) \quad s.t. \\ 4(1-z)\beta \geq \frac{1}{2}\rho \cdot 4(1-z)(\alpha^* + \beta^*) \\ 0 \leq \beta \leq 1 \end{cases} = \begin{cases} \max \quad 3 - (1+2z)\beta \quad s.t. \\ \beta \geq \frac{1}{2}\rho(\alpha^* + \beta^*) \\ 0 \leq \beta \leq 1 \end{cases}$$, whose solution is

$$\beta^* = \frac{1}{2}\rho(\alpha^* + \beta^*). \tag{14}$$

1.1) Combining (12) and (14), we have $\alpha^* + \beta^* = \frac{1}{1-\frac{1}{2}\rho}$. However, $\frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3(1-\rho)}{2(1+z)} > 1$, which can be simplified as $3\rho^2 + (4z-5)\rho + 2 - 4z = (\rho - 1)(3\rho - 2 + 4z) > 0$, requires that $\rho > 1$ or $\rho < \frac{2-4z}{3}$, neither of which is true. So this case will not happen.

1.2) Combining (13) and (14), we have $\alpha^* + \beta^* = \frac{3}{2(1+z)}$. And $\frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3(1-\rho)}{2(1+z)} \leq 1$ requires that $\frac{2}{3}(1-2z) \leq \rho \leq 1$, which is always true. Therefore, $SPE^{1,2}(z) = \big\{\big(\frac{3(1-z)}{1+z}, \frac{9+6z}{4(1+z)}\big)\big\}$.

$\star$ Case 2. $\frac{1}{2} < z \leq 1$.

There are two points in this case that are not strictly dominated, $(4z, 1)$ and $(0, 3)$. Let α^*, β^* be the probabilities assigned to $(4z, 1)$ by players 1 and 2 respectively in some SPE point. Then player 1 faces a problem of:

$$\begin{cases} \max \quad 4z\alpha \quad s.t. \\ \alpha + 3(1-\alpha) \geq \frac{1}{2}\rho\big(6 - 2(\alpha^* + \beta^*)\big) + (1-\rho)\frac{3}{2(z+1)} \\ 0 \leq \alpha \leq 1 \end{cases}$$

$$= \begin{cases} \max \quad 4z\alpha \quad s.t. \\ \alpha \leq \frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3}{2}(1-\rho) - \frac{3(1-\rho)}{4(z+1)} \\ 0 \leq \alpha \leq 1 \end{cases} .$$

There are two possibilities: 1) If $\frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3}{2}(1-\rho) - \frac{3(1-\rho)}{4(z+1)} > 1$, then

$$\alpha^* = 1; \tag{15}$$

2) If $\frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3}{2}(1-\rho) - \frac{3(1-\rho)}{4(z+1)} \leq 1$, then

$$\alpha^* = \frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3}{2}(1-\rho) - \frac{3(1-\rho)}{4(z+1)}. \tag{16}$$

Player 2 faces a problem of:

$$\begin{cases} \max \quad \beta + 3(1-\beta) \quad s.t. \\ 4z\beta \geq \frac{1}{2}\rho \cdot 4z(\alpha^* + \beta^*) \\ 0 \leq \beta \leq 1 \end{cases} = \begin{cases} \max \quad 3 - 2\beta \quad s.t. \\ \beta \geq \frac{1}{2}\rho(\alpha^* + \beta^*) \\ 0 \leq \beta \leq 1 \end{cases} \text{, which gives}$$

$$\beta^* = \frac{1}{2}\rho(\alpha^* + \beta^*). \tag{17}$$

2.1) Combining (15) and (17), we have $\alpha^* + \beta^* = \frac{1}{1-\frac{1}{2}\rho}$. However, $\frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3}{2}(1-\rho) - \frac{3(1-\rho)}{4(z+1)} > 1$ can be simplified as $3(2z+1)\rho^2 - (10z+1)\rho + 4z - 2 = (\rho - 1)((6z+3)\rho - 4z + 2) > 0$, which require $\rho > 1$ or $\rho < \frac{4z-2}{6z+3}$. And so this case will not happen.

2.2) Combining (16) and (17), we have $\alpha^* + \beta^* = \frac{3}{2} - \frac{3}{4(z+1)}$. And $\frac{1}{2}\rho(\alpha^* + \beta^*) + \frac{3}{2}(1-\rho) - \frac{3(1-\rho)}{4(z+1)} \leq 1$ requires that $\rho \geq \frac{2}{3}$, which is always true.

Therefore, $SPE^{1,2}(z) = \big\{(3z - \frac{3}{2} + \frac{3}{2(z+1)}, \frac{3}{2} + \frac{3}{4(z+1)})\big\}$.

In sum, we have

$$SPE^{1,2}(z) = \begin{cases} \big\{(\frac{3(1-z)}{1+z}, \frac{9+6z}{4(1+z)})\big\} & if\ 0 \leq z \leq \frac{1}{2} \\ \big\{(3z - \frac{3}{2} + \frac{3}{2(z+1)}, \frac{3}{2} + \frac{3}{4(z+1)})\big\} & if\ \frac{1}{2} < z \leq 1 \end{cases} .$$

Let $\big(e_{31}(z), e_{32}(z)\big)$ be the only point in $SPE^{1,2}(z)$, then the optimization problem that player 3 faces is:

$$\begin{cases}\max \quad \gamma_2 + 3(1-\gamma_1-\gamma_2) \;\; s.t. \\ 4(\gamma_1+\gamma_2) \geq \rho(4\lambda_1^* + 4\lambda_2^*) + (1-\rho)e_{31}(z) \\ 2\gamma_1 + \gamma_2 + 3(1-\gamma_1-\gamma_2) \geq \rho(3-\lambda_1^*-2\lambda_2^*) + (1-\rho)e_{32}(z) \\ \gamma_1+\gamma_2 \leq 1 \\ \gamma_1, \gamma_2 \geq 0 \\ 0 \leq z \leq 1 \end{cases}.$$

Let z^* be one of the optimal z, then $\gamma_1^* + \gamma_2^* = \rho(\lambda_1^* + \lambda_2^*) + \frac{1}{4}(1-\rho)e_{31}(z^*)$. Since it's easy to see that $e_{31}(z^*) \leq 3$, we have

$$\gamma_1^* + \gamma_2^* \leq \rho(\lambda_1^* + \lambda_2^*) + \frac{3}{4}(1-\rho). \tag{18}$$

Combining (10) (11) and (18), we have

$$\lambda_1^* + \lambda_2^* \leq \frac{11-7\rho}{12-8\rho}. \tag{19}$$

As

$$\frac{11-7\rho}{12-8\rho} - \frac{17\rho-1}{16\rho} = \frac{3(2\rho-1)(\rho-1)}{16\rho(3-2\rho)} \leq 0,$$

a contradiction occurs between (9) and (19). □

Remark 3. It can be observed through the proof that this counter example is robust in the sense that any small permutation of it is still a counter example. Hence it is impossible for us to get a generic existence result.

6 Further Remarks

An obvious problem for future research is to prove or disprove the existence of a distribution-symmetric or payoff-symmetric SP equilibrium for the HM model. This is also a working subject of the author. The difficulty, as felt by the author, lies in the complicated structure of the SP equilibrium, to be more specific, the relations of all the states. Other interesting directions include: (i) design a bargaining procedure which maintains all the virtues of the HM model and conquers its flaws, (ii) compute all the possible characteristic groups for $m < n$, (iii) design more practical algorithms for computing an SP equilibrium, or an approximate SP equilibrium.

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Appendix A: Calculation of Example 3.

Let $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ be probabilities assigned to $(4,0), (3,3)$ and $(0,4)$ by player 1 and player 2, respectively. It's easy to see $q^1 = q^2 = 1$.

Suppose $((\alpha_i^*)_{i=1,2,3}, (\beta_i^*)_{i=1,2,3})$ is an SP-equilibrium, then:

$7\alpha_1^* + 6\alpha_2^* + 2\alpha_3^* =$

$$\begin{cases} \max \quad 4\alpha_1 + 3\alpha_2 \quad s.t. \\ 3\alpha_2 + 4\alpha_3 \geq \frac{1}{2}\rho(3\alpha_2^* + 4\alpha_3^* + 3\beta_2^* + 4\beta_3^*) + (1-\rho) \\ \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \alpha_1, \alpha_2, \alpha_3 \geq 0 \end{cases},$$

$3\beta_2^* + 4\beta_3^* =$

$$\begin{cases} \max \quad 3\beta_2 + 4\beta_3 \quad s.t. \\ 4\beta_1 + 3\beta_2 \geq \frac{1}{2}\rho(4\alpha_1^* + 3\alpha_2^* + 4\beta_1^* + 3\beta_2^*) + (1-\rho) \\ \beta_1 + \beta_2 + \beta_3 = 1 \\ \beta_1, \beta_2, \beta_3 \geq 0 \end{cases}.$$

It's easy to calculate that:

$$\begin{cases} P_1 = \{(0,4), (-1/3,4), (-3,12)\} \\ E_1(0,4) = \{(4,0)\} \\ E_1(-1/3,4) = \{(4,0), (3,3)\} \\ E_1(-3,12) = \{(3,3), (0,4)\} \\ P_2 = \{(0,4), (-1/3,4), (-3,12)\} \\ E_2(0,4) = \{(0,4)\}, \\ E_2(-1/3,4) = \{(3,3), (0,4)\} \\ E_2(-3,12) = \{(4,0), (3,3)\} \end{cases}.$$

Since there is obviously no solution when we choose $(0,4)$ from P_1 or P_2,we discuss in 4 cases:

• Case 1. Points selected from P_1 and P_2 are respectively $(-1/3,4)$ and $(-1/3,4)$, then the only possible non-zero variables are : $\alpha_1, \alpha_2, \beta_2, \beta_3$. Solving

$$\begin{cases} 3\alpha_2 = \frac{1}{2}\rho(3\alpha_2 + 3\beta_2 + 4\beta_3) + (1-\rho) \\ 3\beta_2 = \frac{1}{2}\rho(4\alpha_1 + 3\alpha_2 + 3\beta_2) + (1-\rho) \\ \alpha_1 + \alpha_2 = 1 \\ \beta_2 + \beta_3 = 1 \end{cases},$$

we get: $\begin{cases} \alpha_2 = \beta_2 = \frac{1+\rho}{3-\rho} \\ \alpha_1 = \beta_3 = \frac{2-2\rho}{3-\rho} \end{cases}.$

• Case 2. Points selected from P_1 and P_2 are respectively $(-1/3,4)$ and $(-3,12)$, then the only possible non-zero variables are: $\alpha_1, \alpha_2, \beta_1, \beta_2$. Solving

$$\begin{cases} 3\alpha_2 = \frac{1}{2}\rho(3\alpha_2 + 3\beta_2) + (1-\rho) \\ 4\beta_1 + 3\beta_2 = \frac{1}{2}\rho(4\alpha_1 + 3\alpha_2 + 4\beta_1 + 3\beta_2) + (1-\rho) \\ \alpha_1 + \alpha_2 = 1 \\ \beta_1 + \beta_2 = 1 \end{cases}, \tag{20}$$

we get: $\begin{cases} \alpha_2 = \frac{4}{3}\rho + \frac{1}{3} \\ \beta_2 = 3 - \frac{4}{3}\rho > 1 \end{cases}$, thus there is no solution in this case.

• Case 3. Points selected from P_1 and P_2 are respectively $(-3,12)$ and $(-1/3,4)$. There is no solution in this case since it's symmetric to Case 2.

• Case 4. Points selected from P_1 and P_2 are respectively $(-3,12)$ and $(-3,12)$, then the only possible non-zero variables are: $\alpha_2, \alpha_3, \beta_1, \beta_2$. Solving

$$\begin{cases} 3\alpha_2 + 4\alpha_3 = \frac{1}{2}\rho(3\alpha_2 + 4\alpha_3 + 3\beta_2) + (1-\rho) \\ 4\beta_1 + 3\beta_2 = \frac{1}{2}\rho(3\alpha_2 + 4\beta_1 + 3\beta_2) + (1+\rho) \\ \alpha_2 + \alpha_3 = 1 \\ \beta_1 + \beta_2 = 1 \end{cases},$$

we get: $\alpha_2 = \beta_2 = \frac{6-2\rho}{1+\rho} > 1$, thus there is no solution in this case either.

In sum, the only SP-equilibrium is:

$$(\alpha_1^*, \alpha_2^*, \alpha_3^*) = \left(\frac{2-2\rho}{3-\rho}, \frac{1+\rho}{3-\rho}, 0\right), (\beta_1^*, \beta_2^*, \beta_3^*) = \left(0, \frac{1+\rho}{3-\rho}, \frac{2-2\rho}{3-\rho}\right).$$

Appendix B: Calculation of Example 6

Let $\alpha_1, \alpha_2, \alpha_3$ be probabilities assigned to $(7,2), (6,6)$ and $(2,7)$ by player 1, and β_1, β_2, β_3 be probabilities assigned to $(7,2), (6,6)$ and $(2,7)$ by player 2. It's easy to get $q^1 = q^2 = 2$.

Suppose $((\alpha_i^*)_{i=1,2,3}, (\beta_i^*)_{i=1,2,3})$ is an SP-equilibrium, then:

$7\alpha_1^* + 6\alpha_2^* + 2\alpha_3^* =$

$$\begin{cases} \max \quad 7\alpha_1 + 6\alpha_2 + 2\alpha_3 \quad s.t. \\ 2\alpha_1 + 6\alpha_2 + 7\alpha_3 \geq \frac{1}{2}\rho(2\alpha_1^* + 6\alpha_2^* + 7\alpha_3^* + 2\beta_1^* + 6\beta_2^* + 7\beta_3^*) + 2(1-\rho) \\ \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \alpha_1, \alpha_2, \alpha_3 \geq 0 \end{cases},$$

$2\beta_1^* + 6\beta_2^* + 7\beta_3^* =$

$$\begin{cases} \max \quad 2\beta_1 + 6\beta_2 + 2\beta_3 \quad s.t. \\ 7\beta_1 + 6\beta_2 + 2\beta_3 \geq \frac{1}{2}\rho(7\alpha_1^* + 6\alpha_2^* + 2\alpha_3^* + 7\beta_1^* + 6\beta_2^* + 2\beta_3^*) + 2(1-\rho) \\ \beta_1 + \beta_2 + \beta_3 = 1 \\ \beta_1, \beta_2, \beta_3 \geq 0 \end{cases}.$$

It's easy to calculate that:

$$\begin{cases} P_1 = \{(-0.25, 7.5), (-4, 30), (0, 7)\} \\ E_1(-0.25, 7.5) = \{(7,2), (6,6)\} \\ E_1(-4, 30) = \{(6,6), (2,7)\} \\ E_1(0,7) = \{(7,2)\}, \\ P_2 = \{(-4, 30), (-0.25, 7.5), (0, 7)\} \\ E_2(-4, 30) = \{(7,2), (6,6)\} \\ E_2(-0.25, 7.5) = \{(6,6), (2,7)\} \\ E_2(0,7) = \{(2,7)\} \end{cases}.$$

Since $|P_1| \times |P_2| = 9$, we discuss in 9 cases:

• Case 1. Points selected from P_1 and P_2 are respectively $(-0.25, 7.5)$ and $(-4, 30)$, then the only possible non-zero variables are: $\alpha_1, \alpha_2, \beta_1, \beta_2$. Solving

$$\begin{cases} 2\alpha_1 + 6\alpha_2 = \frac{1}{2}\rho(2\alpha_1 + 6\alpha_2 + 2\beta_1 + 6\beta_2) + 2(1-\rho) \\ 7\beta_1 + 6\beta_2 = \frac{1}{2}\rho(7\alpha_1 + 6\alpha_2 + 7\beta_1 + 6\beta_2) + 2(1-\rho) \\ \alpha_1 + \alpha_2 = 1 \\ \beta_1 + \beta_2 = 1 \end{cases}$$

we get: $\begin{cases} \alpha_1 = 1 - \frac{5}{2}\rho \\ \beta_1 = \frac{5}{2}\rho - 4 < 0 \end{cases}$, thus there is no solution in this case.

• Case 2. Points selected from P_1 and P_2 are respectively $(-0.25, 7.5)$ and $(-0.25, 7.5)$, then the only possible non-zero variables are: $\alpha_1, \alpha_2, \beta_2, \beta_3$. Solving

$$\begin{cases} 2\alpha_1 + 6\alpha_2 = \frac{1}{2}\rho(2\alpha_1 + 6\alpha_2 + 6\beta_2 + 7\beta_3) + 2(1-\rho) \\ 6\beta_2 + 2\beta_3 = \frac{1}{2}\rho(7\alpha_1 + 6\alpha_2 + 6\beta_2 + 2\beta_3) + 2(1-\rho) \\ \alpha_1 + \alpha_2 = 1 \\ \beta_2 + \beta_3 = 1 \end{cases}$$

we get: $\begin{cases} \alpha_1 = \frac{8-8\rho}{8-3\rho} \\ \beta_2 = \frac{5\rho}{8-3\rho} \end{cases}$.

• Case 3. Points selected from P_1 and P_2 are respectively $(-0.25, 7.5)$ and $(0, 7)$, then the only possible non-zero variables are: $\alpha_1, \alpha_2, \beta_3$, and $\beta_3 = 1$. Solving

$$\begin{cases} 2\alpha_1 + 6\alpha_2 = \frac{1}{2}\rho(2\alpha_1 + 6\alpha_2 + 7) + 2(1-\rho) \\ 2 \geq \frac{1}{2}\rho(7\alpha_1 + 6\alpha_2 + 2) + 2(1-\rho) \\ \alpha_1 + \alpha_2 = 1 \end{cases},$$

we get: if $\rho = 0$, then $\alpha_1 = 1$, otherwise there is no solution.

• Case 4. Points selected from P_1 and P_2 are respectively $(-4, 30)$ and $(-4, 30)$, then the only possible non-zero variables are: $\alpha_2, \alpha_3, \beta_1, \beta_2$. Solving

$$\begin{cases} 6\alpha_2 + 7\alpha_3 = \frac{1}{2}\rho(6\alpha_2 + 7\alpha_3 + 2\beta_1 + 6\beta_2) + 2(1-\rho) \\ 7\beta_1 + 6\beta_2 = \frac{1}{2}\rho(6\alpha_2 + 2\alpha_3 + 7\beta_1 + 6\beta_2) + 2(1-\rho) \\ \alpha_2 + \alpha_3 = 1 \\ \beta_1 + \beta_2 = 1 \end{cases},$$

we get: $\begin{cases} \alpha_2 = \frac{10-5\rho}{2+3\rho} \\ \beta_1 = \frac{8\rho-8}{2+3\rho} < 0 \end{cases}$, thus there is no solution in this case.

• Case 5. Points selected from P_1 and P_2 are respectively $(-4, 30)$ and $(-0.25, 7.5)$, then the only possible non-zero variables are: $\alpha_2, \alpha_3, \beta_2, \beta_3$. Solving

$$\begin{cases} 6\alpha_2 + 7\alpha_3 = \frac{1}{2}\rho(6\alpha_2 + 7\alpha_3 + 6\beta_2 + 7\beta_3) + 2(1-\rho) \\ 6\beta_2 + 2\beta_3 = \frac{1}{2}\rho(6\alpha_2 + 2\alpha_3 + 6\beta_2 + 2\beta_3) + 2(1-\rho) \\ \alpha_2 + \alpha_3 = 1 \\ \beta_2 + \beta_3 = 1 \end{cases},$$

we get: $\begin{cases} \alpha_2 = 5 - \frac{5}{2}\rho > 1 \\ \beta_2 = \frac{5}{2}\rho \end{cases}$, thus there is no solution in this case.

• Case 6. Points selected from P_1 and P_2 are respectively $(-4, 30)$ and $(0, 7)$, then the only possible non-zero variables are: $\alpha_2, \alpha_3, \beta_3$, and $\beta_3 = 1$. Solving

$$\begin{cases} 6\alpha_2 + 7\alpha_3 = \frac{1}{2}\rho(6\alpha_2 + 7\alpha_3 + 7) + 2(1-\rho) \\ 2 \geq \frac{1}{2}\rho(6\alpha_2 + 2\alpha_3 + 2) + 2(1-\rho) \\ \alpha_2 + \alpha_3 = 1 \end{cases},$$

we get: there is no solution in this case.

• Case 7. Points selected from P_1 and P_2 are respectively $(0, 7)$ and $(-4, 30)$, then the only possible non-zero variables are: $\alpha_1 = 1, \beta_1, \beta_2$. Solving

$$\begin{cases} 2 \geq \frac{1}{2}\rho(2 + 2\beta_1 + 6\beta_2) + 2(1-\rho) \\ 7\beta_1 + 6\beta_2 = \frac{1}{2}\rho(7 + 7\beta_1 + 6\beta_2) + 2(1-\rho) \\ \beta_1 + \beta_2 = 1 \end{cases},$$

we get: there is no solution in this case.

• Case 8. Points selected from P_1 and P_2 are respectively $(0, 7)$ and $(-0.25, 7.5)$, then the only possible non-zero variables are: $\alpha_1 = 1, \beta_2, \beta_3$. Solving

$$\begin{cases} 2 \geq \frac{1}{2}\rho(2 + 6\beta_2 + 7\beta_3) + 2(1-\rho) \\ 6\beta_2 + 2\beta_3 = \frac{1}{2}\rho(7 + 6\beta_2 + 2\beta_3) + 2(1-\rho) \\ \beta_2 + \beta_3 = 1 \end{cases},$$

we get: if $\rho = 0$ then $\beta_2 = 0$, otherwise there is no solution.

• Case 9. Points selected from P_1 and P_2 are respectively $(0, 7)$ and $(0, 7)$, then the only possible non-zero variables are: α_1, β_3, and $\alpha_1 = 1, \beta_3 = 1$. Solving

$$\begin{cases} 2 \geq \frac{1}{2}\rho(2 + 7) + 2(1-\rho) \\ 2 \geq \frac{1}{2}\rho(7 + 2) + 2(1-\rho) \end{cases},$$

we get: if $\rho = 0$, then this case is equivalent to Case 8, otherwise there is no solution.

In sum, this game has a unique SP-equilibrium:

$$(\alpha_1^*, \alpha_2^*, \alpha_3^*) = \left(\frac{8-8\rho}{8-3\rho}, \frac{5\rho}{8-3\rho}, 0\right), (\beta_1^*, \beta_2^*, \beta_3^*) = \left(0, \frac{5\rho}{8-3\rho}, \frac{8-8\rho}{8-3\rho}\right).$$