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HOW LONG TO PARETO EFFICIENCY?

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How Long to Pareto Efficiency?[*][†]

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Abstract

We consider uncoupled dynamics (i.e., dynamics where each player knows only his own payoff function) that reach Pareto efficient and individually rational outcomes. We prove that the number of periods it takes is in the worst case exponential in the number of players.

1 Introduction

We are looking for "natural" dynamics that lead to "good" outcomes. By "good" outcome, in this paper, we mean *Pareto efficient*; i.e., an outcome such that there is no other feasible outcome that is better for all players. Clearly, efficiency is a prominent and desirable property. There are a few reasonable properties that we should require of a "natural" dynamic. One property is uncoupledness, which means that a player's strategy depends on his own payoff function only. Another reasonable property of a "natural" dynamic is an acceptable speed of convergence to "good" outcomes: we would like the speed of convergence *not* to be exponential. There are a

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few more resaonable properties of "natural" dynamic, but in this paper we focus only on these two. We will attempt to show that even with these two requirements of uncoupledness and acceptable speed of convergence a "natural" dymanic does not exist.

Conitzer and Sandholm [1] introduced the idea of providing lower bounds on the speed of convergence of uncoupled dynamics by considering the communication complexity of the problem (see also Kushilevitz and Nisan [3]). Then, Hart and Mansour [2] used this idea to prove that the communication complexity of the Nash equilibria problem (pure or mixed) is exponential in the number of players n. As a result, for every uncoupled dynamic there exists an n-person game where the time it takes to reach a Nash equilibrium is exponential in n.

In this paper we generalize the ideas from [2] in order to extend the result to the problem of convergence to a Pareto efficient payoff. We prove that the communication complexity of reaching an outcome that is individually rational and Pareto efficient is exponential in the number of players. Moreover, we show that the trivial procedure where each player reports his entire payoff function does not achieve much worse communication complexity than the lower bound that we present.

In addition, we show that without individual rationality the communication complexity of Pareto efficiency is polynomial in n.

2 Preliminaries

The notations are based on those in Hart and Mansour [2].

We use the standard notations for the strategic form game G. Let $n \geq 2$ be the number of players. A^i is the action set of player i. $A := A^1 \times A^2 \times ... \times A^n$ is the action profile set. Denote by $u^i : A \to \mathbb{R}$ the utility function of player i, and by $u = (u^1, u^2, ..., u^n) : A \to \mathbb{R}^n$ the mapping. As usual, $u : A \to \mathbb{R}^n$ could be multilinearly extended to $u : \Delta(A) \to \mathbb{R}^n$:

$$u(s) = \sum_{a \in A} s(a)u(a)$$

where $s(a)$ is the weight of s on a. Let $s \in \Delta(A)$, we denote $s(B) = \sum_{a \in B} s(a)$, for any subset $B \subset A$.

Let Γ_m^n be the set of all n-player games where each player has at most m actions.

For every game G let

$$F(G) := Conv\{u(a) | a \in A\} \subset \mathbb{R}^n$$

be the set of all the feasible payoffs, and let

$$\begin{aligned} PO(G) \quad &: \quad = \{s \in \Delta(A) | \text{ there is no } x = (x^i)_{i=1}^n \in F(G) \\ \text{such that } u^i(s) \quad &< \quad x^i \text{ for all } i = 1, 2, ..., n\} \subset \mathbb{R}^n \end{aligned}$$

be the set of *Pareto optimal* distributions.

The *individually rational level* of player i is defined by

$$v^i = \max_{a^i \in A^i} \left(\min_{a^{-i} \in A^{-i}} u^i(a^i, a^{-i}) \right)$$

and let

$$IR(G) := \{s \in \Delta(A) | \ u^i(s) \geq v^i \text{ for every } i = 1, 2, ..., n\}$$

be the set of all the *individually rational* distributions; i.e., every player gets a payoff that is not less than what he could guarantee by any pure action.

Finally let $PIR(G) := PO(G) \cap IR(G)$ be the set of Pareto optimal and individually rational distributions.

Of course, strengthening the IR condition (e.g., minmax, maximum over mixed strategies $s^i \in \Delta(A^i)$, etc.) can only decrease the resulting PIR set. For instance, IR includes the distributions from the folk Theorem. Moreover, PIR includes the core of the game.

The set of IR distributions is never empty; for example, take the pure $a \in A$ where for each i, the action $a^i \in A^i$ guarantees v^i. Therefore $PIR \neq \varnothing$, because Pareto optimal distributions of any non-empty set is non-empty.

2.1 Communication Complexity

We introduce here a brief sketch of the required definitions from the communication complexity theory. For more details see [2] and [3].

$x, y \in \{0,1\}^K$ are the inputs of players 1 and 2 respectively, where K is a finite set. $f(x, y)$ is the function that both players want to compute. The players send bits to one another. At the end of the communication both of them know the value of f. The rule by which both players send their bits is called a *protocol* and denoted by Π. The communication complexity of a function f for inputs x, y and protocol Π is the number of bits sent during the communication, and is denoted by $CC(\Pi, f, x, y)$. Finally, the communication complexity of a function f is defined by

$$CC(f) = \min_{\Pi} \left(\max_{x,y \in \{0,1\}^K} CC(\Pi, f, x, y) \right).$$

A well-studied function in communication complexity is the disjointedness function, which operates on two subsets of S (or $\{0,1\}^S \times \{0,1\}^S$ equivalently), where S is a finite set, and defined by $DISJ_S(S_1, S_2) = 1$ iff $S_1 \cap S_2 = \varnothing$.

In this paper we will use the following result: $CC(DISJ_S) = |S|$ (see [2] and [3]).

This setup could be generalized to dynamics in game theory. A *PO*-procedure for a family of games $\mathcal{G}$, with a fixed action space A, is defined as follows: each player i holds at the beginning of the procedure his own payoff function u^i (the uncoupledness assumption). At each step player i chooses an action $a^i \in A^i$, and observes the played action a (the "communication"). At the end of the procedure the players reach a distribution $s \in \Delta(A)$ that satisfies $s \in PO$.

Remark 1 *"Reach" should mean "play" in the game theoretical point of view, and it should mean "know" in communication complexity point of view. We will use the meaning of "know" but, as we will see it won't matter.*

The PIR-procedure is defined identically.

In this setup, the strategies of the players induce a protocol of the procedure. Let $tCC(\Pi, PO, G)$ be the number of steps till the termination of the procedure, and

$$tCC(PO, \mathcal{G}) = \min_{\Pi} \left(\max_{G \in \mathcal{G}} tCC(\Pi, PO, G) \right).$$

$tCC(PIR, \mathcal{G})$ is defined similarly.

The relation between tCC and CC is given by

$$\frac{1}{\log_2 |A|} CC \leq tCC \leq CC.$$

3 The Results

First, we show that the communication complexity of the PO-procedure is low (polynomial in the number of players):

Claim 2 $CC(PO, \Gamma_m^n) \leq \lceil n \log m \rceil$.

Proof. An example of a procedure that finds a PO distribution in $\lceil \log_2 |A| \rceil = \lceil n \log m \rceil$ steps is the following: Player 1 informs the other players which action $\overline{a} \in A$ maximizes his payoff (it can be done in $\log_2 |A|$ steps). At the end of this procedure all the players know the distribution $\delta_{\overline{a}}$, which is a Pareto optimal distribution. ■

But the procedure above can lead to an unreasonable payoff in terms of individual rationality. The payoff of player $i \neq 1$ could be less than what he could guarantee by some pure action.

Let us note that an IR distribution needs no communication:

Claim 3 $CC(IR, \Gamma_s^n) = 1$.

Proof. This follows from the fact that each player knows his action a_0^i that guarantees his individually rational level; therefore, after one step of communication, where each player plays a_0^i, the players know an action $a_0 \in A$ that satisfies $\delta_{a_0} \in IR$. ■

However, if we require both PO and IR, then the communication complexity becomes exponential.

We present first a weaker theorem that proves it for n player games with at least 3 action for every player. Above we will show that this is true also for binary games. The Theorem for $m \geq 3$ is still interesting because we can extend this result for approximated PIR and product PIR (see comments 1,2 in section 5) what we cannot do for binary games ($m = 2$).

Theorem 4 *Any PIR-procedure has exponential (in the number of players) communication complexity; i.e., for every $m \geq 3$*

$$CC(PIR, \Gamma_m^n) \geq CC(PIR, \Gamma_3^n) \geq 2^n.$$

Proof. We shall adapt the proof of Theorem 3 in Hart and Mansour [2] to the PIR problem.

We consider the following set of games:

$\mathcal{G} = \{G(T_1, T_2) | T_1, T_2 \subset \{0,1\}^n\}$ where the game $G(T_1, T_2)$ is defined as follows:

Let the set of players be[1] $\{(l, i) | l = 1, 2 \text{ and } i = 1, 2, ..., n/2\}$ and let the action set of each player be $\{0, 1, 2\}$. Clearly, if we prove the result for a specific family of games where $|A^i| = 3$, then the result follows for all Γ_m^n for $m \geq 3$.

We define $B := \{0,1\}^n$, that is an subset of the action set $A = \{0, 1, 2\}^n$.

The payoff function of player (l, i) is defined as follows:

$$u_{l,i}(a) = \begin{cases} 3 & \text{if } a \in B \text{ and } a \in T_l \\ 0 & \text{if } a \in B \text{ and } a \notin T_l \\ 2 & \quad\quad \text{if } a \notin B \end{cases}$$

The pure individually rational level of all the players is at least 2 (each player can guarantee it by playing $a^{(l,i)} = 2$).

If $T_1 \cap T_2 = \varnothing$, then $(u^{(1,1)}(a), u^{(2,1)}(a)) \in \{(0,0), (0,3), (3,0)\}$ for every action $a \in B$. Therefore, every PIR distribution $s \in \Delta(A)$ satisfies $s(B) =$

[1] We assume that n is even.

0, because positive weight on actions in B will decrease the payoff of one of the players $(1,1)$ or $(1,2)$ below the individually rational level 2.

If $T_1 \cap T_2 \neq \varnothing$, then there exists $a^* \in B$ such that $u(a^*) = (3,3,...,3)$. So every PIR distribution s satisfies $s(B) = 1$, because for actions $a \notin B$ the payoff is $u(a) = (2,2,...,2)$, which is not Pareto optimal.

At the beginning of the procedure each player (l,i) can construct his payoff function by knowing T_l only (without knowing T_k for $k \neq l$). At the end of the PIR procedure every player can calculate whether $T_1 \cap T_2 = \varnothing$ (by calculating whether $s(B) = 1$ or $s(B) = 0$). Therefore every PIR-procedure on the set of games $\mathcal{G}$ can induces a protocol of the $DISJ_{\{0,1\}^n}$ function as follows: player $l = 1,2$ how holds S_l constructs $n/2$ dummy-players (l,i) with the payoff function as described above, and then they simulate the PIR-procedure. At the end of the procedure the players know whether $T_1 \cap T_2 = \varnothing$.

Therefore, we have

$$CC(PIR, \Gamma_3^n) \geq CC(DISJ_{\{0,1\}^n}) = 2^n.$$

■

By the same ideas we can prove the exponential lower bound for binary games ($m = 2$).

Theorem 5 *Any PIR-procedure has exponential (in the number of players) communication complexity, i.e.,*

$$CC(PIR, \Gamma_2^n) \geq 2^n - 2^{n/2+1} + 1.$$

Proof. Let $W \subset \{0,1\}^n$ be defined by

$$W = \{(a_1, a_2 ... a_{\frac{n}{2}}, a_{\frac{n}{2}+1}, ..., a_n) | (a_1, ... a_{\frac{n}{2}}) \neq (\underbrace{1,1,...,1}_{n/2}) \text{ and } (a_{\frac{n}{2}+1}, ..., a_n) \neq (\underbrace{1,1,...,1}_{n/2})\}.$$

The size of W is $|W| = 2^n - 2^{n/2+1} + 1$.

We consider the following set of games:

$\mathcal{G} = \{G(T_1, T_2) | T_1, T_2 \subset W\}$, where the game $G(T_1, T_2)$ is defined as follows:

As in the previous proof the set of players will be $\{(l,i)|l = 1,2$ and $i = 1,2,...,n/2\}$ and the action set of each player be $\{0,1\}$.

The payoff function of player (l,i) is defined as follows:

$$u_{l,i}(a) = \begin{cases} 1+\frac{1}{n} & \text{if } a \in T_l \\ 1 & \text{if } a \notin T_l \text{ and } a^{(l,i)} = 1 \\ 0 & \text{if } a \notin T_l \text{ and } a^{(l,i)} = 0 \end{cases}$$

The pure individually rational level of all the players is 1 (each player can guarantee it by playing $a^{(l,i)} = 1$).

We claim that if $T_1 \cap T_2 = \varnothing$, then the only PIR distribution is the pure distribution $\delta_{(1,1,...,1)}$.

To simplify notations denote by $\Sigma u(x) := \sum_{i=1}^{n/2}(u^{(1,i)}(x) + u^{(2,i)}(x))$ the sum of payoffs of all the players for a distribution x.

First we prove that $\Sigma u(a) \leq n - 1/2$ for every $a \neq (1,1,...,1)$. If $(a^{(1,1)},...a^{(1,n/2)}) = (1,1,...,1)$, then $a \notin T_1, T_2$ because $T_1, T_2 \subset W$, so $\Sigma u(a) \leq n-1$. The same is true if $(a^{(2,1)},...a^{(2,n/2)}) = (1,1,...,1)$. In the remaining case where $(a^{(1,1)},...a^{(1,n/2)}) \neq (1,1,...,1)$ and $(a^{(2,1)},...a^{(2,n/2)}) \neq (1,1,...,1)$, there are at most $n/2$ players that get a payoff of $1+1/n$ (because $T_1 \cap T_2 = \varnothing$), and at most $n/2-1$ players that get a payoff of 1 (because $(a^{(l,1)},...a^{(l,n/2)}) \neq (1,1,...,1)$ for $l = 1,2$). Therefore,

$$\Sigma u(a) \leq \frac{n}{2}(1+\frac{1}{n}) + (\frac{n}{2}-1)\cdot 1 = n - \frac{1}{2}.$$

Clearly, $\Sigma u(1,1,...,1) = n$; now let $x \in PIR$; then $\Sigma u(x) \geq n$ because each player gets at least 1, and so $x_a = 0$ for $a \neq (1,1,...,1)$ because otherwise $\Sigma u(x) < n$.

On the other hand, if $T_1 \cap T_2 \neq \varnothing$, then there exists a profile where every player gets a payoff of $1+1/n$, and so for every $x \in PIR$ $x_{(1,1,...,1)} = 0$ because $u(1,1,...,1) = (1,1,...,1)$, which is not Pareto optimal.

Summarizing, we get that if $x \in PIR$ if $x_{(1,1,...,1)} = 1$, then $T_1 \cap T_2 = \varnothing$, and if $x_{(1,1,...,1)} = 0$, then $T_1 \cap T_2 \neq \varnothing$. As in the proof of Theorem 4, we do a reduction from the PIR problem on games in $\mathcal{G}$ to the $DISJ_W$ and we get that

$$CC(PIR, \Gamma_2^n) \geq CC(PIR, \mathcal{G}) \geq CC(DISJ_W) = 2^n - 2^{n/2+1} + 1.$$

■

4 Upper bound

In this section we present trivial procedure that achieves near optimal communication complexity to the lower bound of theorem 5. This demonstrates that for PIR problem naive procedure isn't far from the optimal.

Let U^i be a family of payoff functions of player i. For each $a \in A$, the *encoding* of the payoff of player i at a is $enc(U^i, a) := log|\{u^i(a)|u^i \in U^i\}|$; i.e., the number of bits required to encode the possible values of $u^i(a)$ as u^i varies over U^i; the *encoding of the family of games* U is $enc(U) := \max_{i=1,2,\ldots,n} \max_{a \in A} enc(U^i, a)$. For more details see [2] Section 5.

Proposition 6 ***Claim 7*** *For every $n \geq 2$ let $\mathcal{U}_r^n \subset \Gamma_2^n$ be a family of binary-action games whose encoding is at most r bits, i.e., $enc(\mathcal{U}_r^n) \leq r$. Then,*

$$CC(PIR, \mathcal{U}_r^n) \leq rn2^n.$$

Proof. Each player can send to others his whole payoff function in $r2^n$ bits (each payoff can be sent in r bits) after $rn2^n$ bits all the players will know the payoff function. Now they can calculate a PIR distribution and they all could select the same one by some selecting rule, for example: the players has some common order over the distributions (for example lexicographic order over the weights of the distribution), and they choose the first one[2].

■

[2] The "first one" is well defined because the set of PIR distributions is closed.

5 Comments

1. Approximated PIR Given $\varepsilon > 0$, let ε-PIR be the set of distributions that lead to outcome that is ε close[3] to $u(PIR)$. The exponential result of the ε-PIR problem for games in Γ_m^n where $m \geq 3$ can be derived by a proof similar to this of Theorem 4: Consider the same family of games, any ε-PIR distribution s will satisfy $s(B) \leq 2\varepsilon$ if $S_1 \cap S_2 = \varnothing$ and $s(B) \geq 1 - \varepsilon$ if $S_1 \cap S_2 \neq \varnothing$. Therefore, for $\varepsilon < \frac{1}{3}$, the players can deduce whether $S_1 \cap S_2 = \varnothing$ from the ε-PIR distribution.

While the Nash equilibria problem is known to be exponential and the related Nash ε-eqilibria problem remains an open question, in the PIR problems same proof solves both PIR and ε-PIR.

Note that the result of Theorem 5 for binary games cannot be generalized to ε-PIR in the same simple way.

2. Independent mixtures Consider the problem of finding a PIR product distribution, denoted by PIR_{pro}. This set could be empty. For example:

$3,0$	$0,0$	$1,1$
$0,0$	$0,3$	$1,1$
$1,1$	$1,1$	$1,1$

If we consider the class of games where $PIR_{prod} \neq \varnothing$ for games in Γ_m^n where $m \geq 3$, and we define the PIR_{prod}-procedure to be a procedure that terminates when every player knows his $s^i \in \Delta(A^i)$, such that $s = (s^1, s^2, ..., s^n) \in PIR_{prod}$, then the communication complexity of this problem is also exponential by the proof of Theorem 4, except that instead of checking whether $s(B) = 1$ or $s(B) = 0$, now player i should check whether $s^i(\{0,1\}) = 1$ or $s^i(\{2\}) = 1$.

[3] By "close" we mean close in $|| \, ||_\infty$ norm on $F(G) \subset \mathbb{R}^n$.

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