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A TWO-DIMENSIONAL PROBLEM OF REVENUE MAXIMIZATION

By

OMER LEV

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מרכז לחקר הרציונליות

CENTER FOR THE STUDY OF RATIONALITY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

A Two-Dimensional Problem of Revenue Maximization*

Omer Lev[†]

Abstract

We consider the problem of finding the mechanism that maximizes the revenue of a seller of multiple objects. This problem turns out to be significantly more complex than the case where there is only a single object (which was solved by Myerson [5]). The analysis is difficult even in the simplest case studied here, where there are two exclusive objects and a single buyer, with valuations uniformly distributed on triangular domains. We show that the optimal mechanisms are piecewise linear with either 2 or 3 pieces, and obtain explicit formulas for most cases of interest.

1 Introduction and Goals

The problem of building an optimal mechanism to maximize the revenue of an auction holder has been the focus of much research since the 1980s. Myerson [5] established

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[†]Department of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel. *E-mail:* omerl@math.huji.ac.il

some of the basic results for an auction of a single item. His paper considers the case where a seller wishes to sell a single object and several bidders wish to buy it. The value of the object may be different to each bidder, but we assume that the values are distributed according to a density function f, which gives the probability of each set of bidders' values. We seek a mechanism that will be, in a sense, an equilibrium: it should maximize the revenue for the seller even when the bidders know how it works (and change their behavior accordingly), and the seller knows the strategies of the bidders.

The first major principle that simplifies the analysis of this problem [2] is that we may focus on mechanisms which are based on the *revelation principle*: each bidder reveals truthfully the value of the object for him, i.e., he won't profit from lying about his preference. Furthermore, we wish our truthful bidder to have no desire to lie about the object's value for him, and such a mechanism is called *incentive compatible* (or IC). Another, more trivial, principle is that bidders should actually wish to participate in the auction; accordingly, we seek a mechanism that is *individually rational* (or IR), under which the value of taking part in the auction for the bidder - whatever his value of the auctioned object is - won't be negative.

In the case of only a single bidder, the optimal solution is that there be a "minimum price" (which is dependent on f, the distribution of bidder values for the object), below which the bidder won't get the object; above it, he will. Similarly, when there are multiple bidders, if all have the same probability distribution function f, the optimal solution is a "second-price auction" (which is IC) with a "minimum price." When the preferences of each bidder have a different distribution (but are independent of each other) the solution is slightly more complex, but still relatively straightforward. Thus, the object is either not sold, or definitely sold (there is no possibility of a value for which there is a possibility of both obtaining and not obtaining the object).

Trying to add dimensions to this problem by adding more objects to be auctioned seems, at first, to be no more difficult than holding several unconnected auctions. However, as soon as we assume that the seller can sell "bundles" (several objects together), or when we allow "inter-dependence" between the auctioned items' values, the issue becomes far more complex. As Rochet and Stole [6] show in their detailed survey of various methods to solve multi-dimensional problems, handling such problems is extremely problematic and there are very few helpful results.

Even in the case of only one seller ("monopolist") and two objects and their bundling, we have only a very general picture (for example, Manelli and Vincent [4] provided a method that can verify that a candidate optimal solution is an extreme point - when extreme points are a strict superset of optimal solutions). For better results, Manelli and Vincent [3, 4] add the assumption that the preference for each object is independent (and thus, for example, the probability of a certain value for a bundle is the product of the probabilities of all objects values). Even in these cases, specific solutions are rare.

Thus, the simple case of two mutually exclusive objects whose total valuation is less than 2 - even when we assume valuations are distributed uniformly - isn't straightforward. More formally, we consider the problem where the values for the two mutually exclusive options are found in the triangle with the vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$. The solution for this specific problem is shown in Corollary 31 (with a graphic representation shown in Appendix A.1).

In approaching this problem, we show that the optimal solution must be of the type for which the probability that no object is sold is either 0 or 1 (Theorem 10; we use a method adapted from Hart and Reny [1]). This is an obvious extension of the single-dimension solution. We prove that this is true for our "shape" (the triangle) if we assume a certain condition on the probability density function (a condition that holds for the uniform distribution). Dealing with these types of optimal solutions enables us to rewrite the problem as essentially an optimization problem with only one variable.

In seeking a solution to our problem, we first obtain a more general result - if the probability of winning is either 0 or 1, in many types of shapes and distributions the mechanism will be, in a sense, "piecewise linear." That is, it will be composed of regions, in each one of which the probabilities of receiving the objects are constant. Furthermore, for values uniformly distributed in "triangles," we find the optimal solution for many of the triangles (Corollary 30), and our method can be used to find the optimal solution for any such triangle.

2 Preliminaries

2.1 Basics

We begin with several definitions of the two-dimensional problem (all straightforward extensions of the single-dimension problem presented in [2]), and with several basic properties.

Definition 1. $v = (v_1, v_2) \in \mathbb{R}^2_+$ *are the values of the buyer for each of the two objects (v_1 for the first, v_2 for the second). v is distributed according to a distribution F, with density function f.*

Consider now a direct mechanism, with one buyer and one seller, with two mutually exclusive objects to sell. For every bid the buyer offers, there is a probability that he will receive one of the objects, and there is a cost - a payment to the seller. We will seek to maximize the seller's revenue - the expected payment from the buyer.

Definition 2. *$q : \mathbb{R}_+^2 \to [0,1]^2$ is a function representing the probabilities that the buyer will receive the first object (q_1) and the second one (q_2). Thus, $q(y_1, y_2) = (q_1(y_1, y_2), q_2(y_1, y_2))$ are the probabilities of receiving the objects if the buyer announces that his values of the objects are y_1 and y_2. Since we assume that the objects are mutually exclusive, $q_1(y_1, y_2) + q_2(y_1, y_2) \leq 1$.*

Definition 3. *$c : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a function representing the payment to the seller. $c(y_1, y_2)$ is the cost for the buyer if he declares values of y_1 for the first object and y_2 for the second.*

Definition 4. *We define the function $u : \mathbb{R}_+^2 \to \mathbb{R}$ as $u(y_1, y_2) = q(y_1, y_2) \cdot v - c(y_1, y_2)$, with v being the value of the objects for the buyer.*

By the IC principle, Definition 4 is actually $u(v_1, v_2) = q(v_1, v_2) \cdot (v_1, v_2) - c(v_1, v_2)$, which is the utility function of the buyer. In simple terms, it means that the utility for the buyer is the expected value he gains, less his cost.

As in the single-dimension case, we will use u as a variable of the optimization problem. It has several properties that enable us to better analyze it.

Lemma 5. *If $u(v)$ satisfies the IC constraint, then it is a convex function with the gradient $\nabla u(v)$ existing for almost every v, and $\frac{\partial}{\partial v_1} u(v_1, v_2) = q_1(v_1, v_2)$ and $\frac{\partial}{\partial v_2} u(v_1, v_2) = q_2(v_1, v_2)$ almost everywhere.*

Proof of Lemma See [4]. $\square$

2.2 Solution Characterization

Our optimization method works when the solution is of the type where the probability of getting some object, i.e., $q_1 + q_2$, equals either 0 or 1. We shall now show several conditions that ensure that there is always a solution of this type for a family of problems - including our specific problem where (v_1, v_2) is uniformly distributed on a triangle with the vertex $(0,0)$ and $(1,1)$ is in the opposite edge. Our constraints and conditions are based on those shown in [1].

Our goal is to maximize the seller's revenue, which is the expected payment from the buyer, i.e., $c(v_1, v_2)$. Thus we seek to maximize the expression $\int_{\mathbb{R}^2_+}(q(v) \cdot v - u(v))\,\mathrm{d}F = \int_{\mathbb{R}^2_+}(q(v) \cdot v - u(v))f(v)\,\mathrm{d}v$.

Assumption 6. *We assume that there is a convex, compact set $W \subset \mathbb{R}^2_+$ that includes $(0,0)$ and $(1,1)$, such that $(x,y) \notin W \Rightarrow f(x,y) = 0$.*

W is the "range" of the values.

Assumption 7. *We assume that W is a triangle, with one vertex at $(0,0)$. We shall use the notation ∂W for the edge of W for the side opposite $(0,0)$, and we shall assume $(1,1) \in \partial W$.*

This means for $w = (w_1, w_2) \in \partial W$, there is a constant s for which $w_2 = (1 - s)w_1 + s$.

Assumption 8. *f almost everywhere has the following property: for $t \in [0,1]$ and $v = (v_1, v_2)$:*

$$2f(tv) + \frac{d}{dt}tf(tv) \geq 0$$

.

For example, f uniform satisfies Assumption 8.

Definition 9. *Since W is convex and by Assumption 7, any $v = (v_1, v_2) \in W$ can be expressed as $(w - (t,t))$ for a unique $w \equiv w_v \in \partial W$. Thus, for any $u : W \to \mathbb{R}$ we* *can define a new function $\tilde{u} : W \to \mathbb{R}$ as follows:*

$$\tilde{u}(v_1, v_2) = \tilde{u}(w_v - (t,t)) = u(w_v) - t$$

By Assumption 7, $\tilde{u}$ is guaranteed to be well defined and convex.

Notice that the definition means that $\frac{d}{dt}\tilde{u}(v_0+(t,t)) = 1$, and since $\frac{d}{dt}\tilde{u}(v_0+(t,t)) = \tilde{q}_1(v_0) + \tilde{q}_2(v_0)$ (where $\tilde{q} = \nabla\tilde{u}$), at every point $v \in W$, $\tilde{q}_1(v) + \tilde{q}_2(v) = 1$.

Theorem 10. *By Assumptions 7 and 8, for any optimal u, $u = max(0, \tilde{u})$ a.e. Thus, the seller's revenue is maximized when $q_1(v_1, v_2) + q_2(v_1, v_2) \in \{0, 1\}$.*

Proof. Suppose u is an optimal solution. We now define the function $\hat{u} = max(0, \tilde{u})$. We shall now prove that a.e. $u = \hat{u}$, and that will prove the theorem, since $\hat{u}$ fulfills its requirements. Because our domain (W) is convex and by Assumption 7, we can change the coordinate system to (t, w) where $t \in [0, 1]$ and $w \in \partial W$, and every v in the domain may be expressed by tw. Since ∂W lies on the line $w_2 = (1-s)\cdot w_1 + s$, the transformation is $(tw_1, t((1-s)w_1+s))$, and the absolute value of the Jacobian for this transformation is $|s|t \equiv rt$. Since $q(v)\cdot v - u(v) = q(tw)\cdot tw - u(tw) = t\frac{d}{dt}u(tw) - u(tw)$, we now seek to maximize

$$\int_W (q(v)\cdot v - u(v))f(v)\,\mathrm{d}v = \int_{\partial W}\int_0^1 (t\frac{d}{dt}u(tw) - u(tw))rtf(tw)\,\mathrm{d}t\,\mathrm{d}w$$

Since $\int_0^1 t(\frac{d}{dt}u(tw))rtf(tw)\,\mathrm{d}t = rt^2 f(tw)u(tw)|_0^1 - \int_0^1 (2rt(tw) + rt^2\frac{d}{dt}f(tw))u(tw)\,\mathrm{d}t$, we now have

$$\int_{\partial W} ru(w)f(w) - \int_0^1 r(3tf(tw) + t^2\frac{d}{dt}f(tw))u(tw)\,\mathrm{d}t\,\mathrm{d}w$$

Notice that $u(tw) \geq 0$ (by IR) and $3tf(tw) + t^2\frac{d}{dt}f(tw) = t(3f(tw) + t\frac{d}{dt}f(tw)) =$

$t(2f(tw) + \frac{d}{dt}tf(tw)) \geq 0$ (by Assumption 8), This means that for any given u with values on ∂W, we wish to minimize the value of u on the interior of W. Since u and $\tilde{u}$ coincide on ∂W, and since $\frac{d}{dt}u(w-(t,t)) = -(q_1(w-(t,t)) + q_2(w-(t,t))) \geq -1$, it follows that $u(w-(t,t)) \geq u(w) - t = \tilde{u}(w-(t,t))$. Every point $v \in W$ can be represented as $w_v - (t,t)$ for $t \geq 0$ and $w_v \in \partial W$, and so $u(v_1, v_2) \geq \tilde{u}(v_1, v_2)$. By IR, $u \geq 0$, and therefore $u(v_1, v_2) \geq \hat{u}(v_1, v_2)$. Furthermore, if there is a measurable set where $0 < q_1 + q_2 < 1$, this inequality becomes strict (for a subset of W). Thus

$$u(w)f(w) - \int_0^1 r(3tf(tw) + t^2 \frac{d}{dt} f(tw))u(tw)\,\mathrm{d}t \leq \hat{u}(w)f(w) - \int_0^1 r(3tf(tw) + t^2 \frac{d}{dt} f(tw))\hat{u}(tw)\,\mathrm{d}t$$

and

$$\int_W (q(v) \cdot v - u(v))f(v)\,\mathrm{d}v \leq \int_W (\hat{q}(v) \cdot v - \hat{u}(v))f(v)\,\mathrm{d}v$$

Notice that if not a.e. $u = \hat{u}$, due to the fact that u is a.e. differentiable, then there is a measurable set where $0 < q_1 + q_2 < 1$, and therefore the inequalities above become strict. □

Note. *In the sequel, we don't rely on our assumptions on W and f; we only require that the optimal solution be of the sort described in Theorem 10, i.e., $q_1 + q_2 \in \{0, 1\}$.*

2.3 Reframing the Problem

We shall now reduce our two-dimensional problem to essentially one dimension, using the characterization in Theorem 10. To do so, we shall first change the coordinate system and then rewrite the equation we wish to optimize.

We will change our axis system from the regular (v_1, v_2) structure by turning it $45°$ counterclockwise. One axis will be the line $v_1 + v_2 = 0$, and the other will be the line $v_1 = v_2$. We shall use the letter x to denote the former, and t to denote the latter.

Definition 11. *$g : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $g(x,t) = (-\frac{x}{2}+t, \frac{x}{2}+t)$. Thus,*

$$Dg|_{(x,t)} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{pmatrix}$$

and $Jg|_{(x,t)} = -1$.

We shall now "move" the function $\tilde{u}$ to these axes as well.

Definition 12. *We define $\varphi : \mathbb{R} \to \mathbb{R}$ as $\varphi(x) = \tilde{u}(-\frac{x}{2}, \frac{x}{2})$.*

Lemma 13. *$\tilde{u}(v_1, v_2) = \varphi(v_2 - v_1) + \frac{v_1+v_2}{2}$. Therefore, $D_1\tilde{u}|_{(v_1,v_2)} = -\varphi'(v_2 - v_1) + \frac{1}{2}$, $D_2\tilde{u}|_{(v_1,v_2)} = \varphi'(v_2 - v_1) + \frac{1}{2}$, and $|\varphi'(x)| \leq \frac{1}{2}$.*

Proof of Lemma

$$\begin{aligned} \varphi(v_2 - v_1) + \frac{v_1 + v_2}{2} &= \tilde{u}(\frac{v_1 - v_2}{2}, \frac{v_2 - v_1}{2}) + \frac{v_1 + v_2}{2} = \\ &= \tilde{u}(\frac{v_1 - v_2}{2} + \frac{v_1 + v_2}{2}, \frac{v_2 - v_1}{2} + \frac{v_1 + v_2}{2}) = \tilde{u}(v_1, v_2) \end{aligned}$$

The derivatives are a result of simple arithmetic. □

Recall that we seek to maximize the seller's revenue, i.e., $\int_{\mathbb{R}^2_+} q \cdot v - u(v)\,\mathrm{d}F = \int_{\mathbb{R}^2_+} (q \cdot v - u(v)) f(v)\,\mathrm{d}v$. Starting with $\tilde{u}$ we get

$$\int_{\mathbb{R}^2} D\tilde{u} \cdot v - \tilde{u}(v)\,\mathrm{d}v = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_1\tilde{u} \cdot v_1 + D_2\tilde{u} \cdot v_2 - \tilde{u}(v_1, v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2$$

Next, we change variables to the (x,t) axes. We need to multiply the integrals by

$|Jg|_{(x,t)}|$. Since, as shown in Definition 11 , $Jg|_{(x,t)} = -1$, we multiply by 1

$$D_1\tilde{u}\cdot v_1 + D_2\tilde{u}\cdot v_2 - \tilde{u}(v_1,v_2) = ((-\varphi'(x)+\frac{1}{2})(t-\frac{x}{2}) + (\varphi'(x)+\frac{1}{2})(t+\frac{x}{2}) - (\varphi(x)+t)) =$$
$$= -\varphi'(x)t + \frac{t}{2} + \frac{\varphi'(x)x}{2} - \frac{x}{4} + \varphi'(x)t + \frac{t}{2} + \frac{\varphi'(x)x}{2} + \frac{x}{4} - \varphi(x) - t = \varphi'(x)x - \varphi(x)$$

So our equation for $\tilde{u}$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\varphi'(x)x - \varphi(x)) f(g(x,t))\,\mathrm{d}t\,\mathrm{d}x$$

Now we'll return to u. Since in areas in which $q(v_1,v_2) \neq 0$, $u(v_1,v_2) = \tilde{u}(v_1,v_2)$ (recall that Theorem 10 showed $u = max(0,\tilde{u})$), it follows that the optimized equation is

$$\int_{-\infty}^{\infty}\int_{t_0(x)}^{\infty} (\varphi'(x)x - \varphi(x)) f(g(x,t))\,\mathrm{d}t\,\mathrm{d}x = \int_{-\infty}^{\infty} (\varphi'(x)x - \varphi(x)) \int_{t_0(x)}^{\infty} f(g(x,t))\,\mathrm{d}t\,\mathrm{d}x$$

for $t_0(x) = inf\{t | u(-\frac{x}{2}+t, \frac{x}{2}+t) > 0\}$ (i.e., the point where u stops being 0).

Let's take a closer look at $t_0(x)$.

Lemma 14. *There is an interval $[b_1, b_2]$ (for $b_1 \leq 0$, $b_2 \geq 0$), for which $t_0(x) = -\varphi(x)$, and outside it $t_0(x)$ does not depend on φ, but only on the shape of W.*

Proof of Lemma $t_0(x)$ has two constraints:

Since $u \geq 0$, we get

$$t_0(x) + \varphi(x) \geq 0 \Rightarrow t_0(x) \geq -\varphi(x)$$

And since u is defined on $W \subset \mathbb{R}^2_+$, we get

$$(-\frac{x}{2}, \frac{x}{2}) + (t_0(x), t_0(x)) \in W$$

Finally, since $(0,0) \in W$, $t_0(0) = -\varphi(0)$. Due to the convexity of φ and W, if $t_0(x') = -\varphi(x')$, then $\forall x \in [0, x']$ (or $[x', 0]$): $t_0(x) = -\varphi(x)$. $\square$

Definition 15. *Put* $\tilde{f}(x) = \int\limits_{t_0(x)}^{\infty} f(g(x,t))\,\mathrm{d}t$

Therefore, the equation we wish to find an optimum for can be written as

$$\int\limits_{-\infty}^{\infty} (\varphi'(x)x - \varphi(x))\tilde{f}(x)\,\mathrm{d}x$$

Assumption 16. *We assume that* $\frac{d}{dx}\tilde{f}(x)$ *exists and is continuous.*

Using $\varphi'(x)x - \varphi(x) = (\frac{\varphi(x)}{x})'x^2$, we can view the problem as maximizing the equation:

$$-\int\limits_{-\infty}^{\infty} \varphi(x)(2\tilde{f}(x) + x\tilde{f}'(x))\,\mathrm{d}x$$

Using the definitions of b_1 and b_2 from Lemma 14 we can rewrite the equation as

$$\int\limits_{-\infty}^{b_1} (\varphi_0'(x)x - \varphi_0(x))\tilde{f}(x)\,\mathrm{d}x + \int\limits_{b_1}^{b_2} (\varphi_0'(x)x - \varphi_0(x))\tilde{f}_{\varphi_0}(x)\,\mathrm{d}x + \int\limits_{b_2}^{\infty} (\varphi_0'(x)x - \varphi_0(x))\tilde{f}(x)\,\mathrm{d}x$$

(we use the notation $\tilde{f}_{\varphi_0}$ to indicate that for $\{x|b_1 < x < b_2\}$, $\tilde{f}(x)$ depends on φ_0).

3 General Case

Since φ_0 is optimal, for every φ that fits our criteria (convex, $|\varphi'(x)| \leq \frac{1}{2}$) and that has the same "b"s as φ_0, we know that $(1-\epsilon)\varphi_0 + \epsilon\varphi$ (for $\epsilon > 0$) for small ϵ are very close to φ_0, but still aren't optimal:

$$\int_{-\infty}^{\infty} (\varphi_0'(x)x - \varphi_0(x))\tilde{f}(x)\,\mathrm{d}x \geq \int_{-\infty}^{\infty} (((1-\epsilon)\varphi_0'(x) + \epsilon\varphi'(x))x - ((1-\epsilon)\varphi_0(x) + \epsilon\varphi(x)))\tilde{f}(x)\,\mathrm{d}x$$

Thus,

$$\frac{\partial}{\partial\epsilon}\int_{-\infty}^{\infty} (((1-\epsilon)\varphi_0'(x) + \epsilon\varphi'(x))x - ((1-\epsilon)\varphi_0(x) + \epsilon\varphi(x)))\tilde{f}(x)\,\mathrm{d}x|_{\epsilon=0} \leq 0$$

We know that $\tilde{f}$ does not depend on φ_0 for $x < b_1$ and $x > b_2$, while it does for $b_1 < x < b_2$. Also, since W is compact, there is a $z > 0$ such that $\forall |x| > z : \tilde{f}(x) = 0$. Thus we get the following constraint:

$$\begin{aligned}
&\int_{-z}^{b_1} (\varphi_0(x) - \varphi(x))(2\tilde{f}(x) + x\tilde{f}'(x))\,\mathrm{d}x + \\
&+ \int_{b_1}^{b_2} (\varphi_0(x) - \varphi(x))(2\tilde{f}_{\varphi_0}(x) + x\tilde{f}'_{\varphi_0}(x) - (\varphi_0'(x)x - \varphi_0(x))f(x, -\varphi_0(x)))\,\mathrm{d}x + \\
&+ \int_{b}^{z} (\varphi_0(x) - \varphi(x))(2\tilde{f}(x) + x\tilde{f}'(x))\,\mathrm{d}x \leq 0
\end{aligned}$$

Lemma 17. *On intervals $[d_1, d_2]$ in which $\forall x \in [d_1, d_2]$, if $b_1 < x < b_2$: $2\tilde{f}(x) + x\tilde{f}'(x) - (\varphi_0'(x)x - \varphi_0(x))f(x, -\varphi_0(x)) \neq 0$, and if $x \leq b_1$ or $x \geq b_2$: $2\tilde{f}(x) + x\tilde{f}'(x) \neq 0$, then φ_0 is composed, at most, of two linear* [1] *parts on the interval.*

Proof of Lemma We shall build a φ such that $\varphi|_{x \leq d_1, x \geq d_2} = \varphi_0$. Since for all $x \in [d_1, d_2]$ the "multiplier" $(2\tilde{f}(x) + x\tilde{f}'(x) - (\varphi_0'(x)x - \varphi_0(x))f(x, -\varphi_0(x))$ or $2\tilde{f}(x) + x\tilde{f}'(x))$ isn't 0, then the sign of the multiplier throughout $[d_1, d_2]$ is the same (by

[1] When describing φ (or sections of it) as being linear, we actually mean that it is affine (since $\varphi(0)$ doesn't necessarily equal 0). In doing so, we chose to follow common usage.

Assumption 16).

If the multiplier is < 0 then if we define φ to be the straight line connecting $\varphi_0(d_1)$ with $\varphi_0(d_2)$, then for $x \in [d_1, d_2]$: $\varphi(x) \geq \varphi_0(x)$, and if $\varphi_0|_{[d_1,d_2]} \neq \varphi$, the inequality is strict for some interval. Since for $x \notin [d_1, d_2]$: $\varphi_0(x) = \varphi(x)$, the equation is positive - a contradiction.

Similarly, if the multiplier is > 0, we define $\varphi|_{[d_1,d_2]}$ to be the straight line from $\varphi_0(d_1)$ with the slope $\varphi_0'(d_1)$, until it changes to be the straight line going through $\varphi_0(d_2)$ with the slope $\varphi_0'(d_2)$. φ is, of course, still convex, and for $x \in [d_1, d_2]$: $\varphi(x) \leq \varphi_0(x)$, and if $\varphi_0|_{[d_1,d_2]} \neq \varphi$ the inequality is strict for some interval. Once again, since for $x \notin [d_1, d_2]$: $\varphi_0(x) = \varphi(x)$, the equation is positive, a contradiction. $\square$

Definition 18. *We shall define* $\ell(x) = 2\tilde{f}(x) + x\tilde{f}'(x)$. *We also define* $L(x)$ *thus:*

$$L(x) = \begin{cases} \int\limits_{-z}^{x} \ell(t)\,\mathrm{d}t & x < b_1 \\ \int\limits_{x}^{z} \ell(t)\,\mathrm{d}t & x > b_2 \end{cases}$$

Lemma 19. *On the intervals* $[-z, b_1]$ *and* $[b_2, z]$, φ *is piecewise linear, and the slope changes only at points* y *for which* $L(y) = 0$.

Proof of Lemma For intervals where $\ell(x) \neq 0$, we showed piecewise linearity in Lemma 17. If there is an interval $[d_1, d_2]$ in which $\ell(x) = 0$, the shape of φ_0 doesn't matter: it can be anything, even a straight line. One can see this by using the alternative representation of the equation we wish to optimize: $-\int\limits_{-z}^{b_1} \varphi_0\ell(x)\,\mathrm{d}x$ or $-\int\limits_{b_2}^{z} \varphi_0\ell(x)\,\mathrm{d}x$. On the interval $[d_1, d_2]$, where $\ell(x) = 0$, the value of φ_0 is irrelevant: only the values in the edges (d_1 and d_2) might matter.

We shall now prove that the change points of different line slopes are y such that $L(y) = 0$. Let's look first at $\{x|x \geq b_2\}$. Suppose that the line $a_1x + c$ changes to

$a_2x + a_1d_1 + c - a_2d_1$ at point $d_1 > b_2$ for which $L(d_1) > 0$; then let us take a φ that equals φ_0 up to d_1, but changes to a slope of a_2 at $d_2 > d_1$, and then continues to change slopes just like φ_0 (at the same points, to the same slopes). d_2 is close enough to d_1 so that $L(d_2) > 0$ and $L(d_2) - L(d_1) < L(d_2)$ (such a d_2 exists due to $\tilde{f}$ continuity). Thus

$$0 \geq \int_{d_1}^{d_2} (a_2x + a_1d_1 - a_2d_1 - a_1x)\ell(x)\,\mathrm{d}x + \int_{d_2}^{z} (a_1d_1 - a_2d_1 - a_1d_2 + a_2d_2)\ell(x)\,\mathrm{d}x =$$

$$=(a_2 - a_1)(\int_{d_1}^{d_2} (x - d_1)\ell(x)\,\mathrm{d}x + (d_2 - d_1)\int_{d_2}^{z} \ell(x)\,\mathrm{d}x)$$

But according to our definition of d_2, and since $(a_2 - a_1) > 0$ (due to convexity), our equation is larger than 0, a contradiction. A similar problem arises when $L(d_1) < 0$ (using $d_2 < d_1$).

For $\{x|x \leq b_1\}$ the problem is solved in the same manner. φ is built to be exactly like φ_0 from d_1 onward, and the changes are in the area between $-z$ and d_1.

Since $\tilde{f}$ isn't dependent on φ_0 for $[-z, b_1]$ and $[b_2, z]$, neither is $L(y)$, and thus we have an independent criterion for "slope change" points. □

Observation 20. *Lemmas 17 and 19 show that φ_0 is piecewise linear except for intervals $[s,t] \subseteq [b_1, b_2]$, for which $\forall x \in [s,t] : 2\tilde{f}_{\varphi_0}(x) + x\tilde{f}'_{\varphi_0}(x) - (\varphi_0'(x)x - \varphi_0(x))f(x, -\varphi_0(x)) = 0$.*

Theorem 21. *On the interval $[b_2, z]$, φ_0 is piecewise linear, with at most two linear pieces. Moreover, the slope of the second piece, if it exists, equals $\frac{1}{2}$. Similarly, on the interval $[-z, b_1]$, φ_0 is piecewise linear with at most two linear pieces, and the slope of the first piece (if it exists) is $-\frac{1}{2}$.*

Proof. First, we shall consider $[b_2, z]$. Let $y > b_2$ be the minimal point for which $L(y) = 0$ and $\int_y^z x\ell(x) < 0$. φ_0 on the interval $[b_2, z]$ is made of I linear parts, and for each $i \in I$ the part is from $[d_{i-1}, d_i]$ (obviously, $d_0 = b_2$), and its equation is $a_i + c_i$ (the c_is are arranged so that φ_0 is continuous). We now define φ to be

$$\varphi(x) = \begin{cases} \varphi_0 & -z \le x < d_1 \\ a_1 + c_1 & d_1 \le x < y \\ \frac{x}{2} + a_1 y + c_1 - \frac{y}{2} & y \le x \end{cases}$$

Let $I' \subset I$ be a set of parts for which for every $i \in I'$: $d_i < y$. We define $\hat{i} \notin I'$ as one for which $y \in [d_{\hat{i}-1}, d_{\hat{i}}]$. Finally, let $I'' = I \setminus (I' \cup \hat{i})$. Our requirement is that

$$0 \ge \sum_{i \in I'} \int_{d_{i-1}}^{d_i} (a_i x + c_i - a_1 x - c_1)\ell(x)\,\mathrm{d}x + \int_{d_{\hat{i}-1}}^{y} (a_{\hat{i}} x + c_{\hat{i}} - a_1 x - c_1)\ell(x)\,\mathrm{d}x +$$
$$+ \int_{y}^{d_{\hat{i}}} (a_{\hat{i}} x + c_{\hat{i}} - \frac{x}{2} - a_1 y - c_1 + \frac{y}{2})\ell(x)\,\mathrm{d}x + \sum_{i \in I''} \int_{d_{i-1}}^{d_i} (a_i x + c_i - \frac{x}{2} - a_1 y - c_1 + \frac{y}{2})\ell(x)\,\mathrm{d}x$$

However, according to Lemma 19, change points are such that $L(d_i) = 0$, and since $L(y) = 0$ as well, our requirement is actually

$$0 \ge \sum_{i \in I'} (a_i - a_1) \int_{d_{i-1}}^{d_i} x\ell(x)\,\mathrm{d}x + (a_{\hat{i}} - a_1) \int_{d_{\hat{i}-1}}^{y} x\ell(x)\,\mathrm{d}x +$$
$$+ (a_{\hat{i}} - \frac{1}{2}) \int_{y}^{d_{\hat{i}}} x\ell(x)\,\mathrm{d}x + \sum_{i \in I''} (a_i - \frac{1}{2}) \int_{d_{i-1}}^{d_i} x\ell(x)\,\mathrm{d}x \ge$$
$$\ge (+) \int_{d_2}^{y} x\ell(x)\,\mathrm{d}x + (-) \int_{y}^{z} x\ell(x)\,\mathrm{d}x$$

(+) and (−) represent positive and negative multipliers respectively (due to φ_0 convexity). $(+) = 0$ only if $\varphi_0|_{[b_2,y]} = \varphi$ and $(-) = 0$ only if $\varphi_0|_{[y,z]} = \varphi$. Since $\int_y^z x\ell(x)\,\mathrm{d}x < 0$, and y is the earliest point where this is true (as well as $L(y) = 0$), it follows that $\int_{d_2}^z x\ell(x)\,\mathrm{d}x \geq 0$, and $\int_{d_2}^y x\ell(x)\,\mathrm{d}x \geq 0$. Thus

$$(+)\int_{d_2}^{y} x\ell(x)\,\mathrm{d}x + (-)\int_{y}^{z} x\ell(x)\,\mathrm{d}x \geq 0$$

If the inequality is strict, we have a contradiction. If it isn't strict, since $\int_y^z x\ell(x)\,\mathrm{d}x < 0$, it follows that $(-) = 0$, and $\varphi_0|_{[y,z]} = \varphi$. If $\varphi_0 \neq \varphi$, then $(+) > 0$, and $\int_{d_1}^z x\ell(x)\,\mathrm{d}x = 0$. In this case we need to take a look at the alternative representation of our original equation, which we wish to maximize, namely, $-\int_{d_1}^z \varphi_0(x)\ell(x)\,\mathrm{d}x$. φ_0 is linear (and $L(d_1) = L(y) = 0$), and so $\int_{d_1}^y \varphi_0(x)\ell(x)\,\mathrm{d}x = 0$, and the values of φ_0 on $[d_1, y]$ don't matter (after y we know that $\varphi_0 = \varphi$). Thus, without loss of generality, $\varphi_0 = \varphi$.

The case of $[-z, b_1]$ is identical. □

4 Uniform Distribution

As we move closer to our objective of finding the optimal u for the uniform distribution on the triangle with the vertices $(0,0)$, $(2,0)$, and $(0,2)$, we decompose our problem along the $v_1 = v_2$ axis. In particular, we analyze the family of triangles that includes the triangle with the vertices $(0,0)$, $(0,2)$, and $(1,1)$ (with a simple transformation, this also includes the triangle with the vertices $(0,0)$, $(2,0)$, and $(1,1)$), and then we join them together.

Assumption 22. *We shall now work under the assumption that f is distributed uniformly on the set $\{(v_1, v_2)|v_2 \geq v_1 \geq 0, v_2 \leq a \cdot v_1 + 1 - a\}$, where $a < 1$. These are cases where ∂W lies on the line $v_2 = av_1 + 1 - a$, and W is the triangle with vertices $(0,0),(1,1), and (0,A)$. (We define $A = 1 - a$).*

For $x > A$ or $x < 0$: $\tilde{f}(x) = 0$. Therefore, there is no b_1, only b_2, which henceforth we shall refer to as b. Furthermore, the constraint on t_0 for this shape is simple, and for $\{x|x \geq b\}$, $t_0(x) = \frac{x}{2}$, and b is a point in which $-\varphi(b) = \frac{b}{2}$.

In the "interesting" area - where $\tilde{f} \neq 0$ - we can see $\tilde{f}(x) = \int\limits_{t_0(x)}^{\infty} f(x,t)\,\mathrm{d}t = \int\limits_{t_0(x)}^{-\frac{2-A}{A}\cdot\frac{x}{2}+1} 1\,\mathrm{d}t = 1 - \frac{2-A}{A} \cdot \frac{x}{2} - t_0(x)$.

Thus, we seek to maximize the expression

$$\int\limits_0^b (\varphi'(x)x - \varphi(x))(1 - \frac{2-A}{2A} \cdot x + \varphi(x))\,\mathrm{d}x + \int\limits_b^A (\varphi'(x)x - \varphi(x))(1 - \frac{2-A}{2A} \cdot x - \frac{x}{2})\,\mathrm{d}x$$

Using the identities $\varphi'(x)x - \varphi(x) = (\frac{\varphi(x)}{x})'x^2$ and $\varphi'(x)\varphi(x)x - \varphi^2(x) = (\frac{\varphi^2(x)}{2x^2})'x^3$ we get

$$-\frac{b^3}{8} - \frac{3}{2}\int\limits_0^b (\varphi(x) - \frac{2-A}{2A}x + \frac{2}{3})^2\,\mathrm{d}x + \frac{1}{6}\int\limits_0^b (2 - \frac{3(2-A)}{2A}x)^2\,\mathrm{d}x - \int\limits_b^A \varphi(x)(2 - \frac{3}{A}x)\,\mathrm{d}x$$

Using the same parameter variation method we used in Section 3, we get

$$\int\limits_0^b (\varphi_0(x) - \varphi(x))(2 - \frac{3(2-A)}{2A}x + 3\varphi_0(x))\,\mathrm{d}x + \int\limits_b^A (\varphi_0(x) - \varphi(x))(2 - \frac{3}{A}x)\,\mathrm{d}x \leq 0$$

Proposition 23. *φ_0 changes to a straight line with a slope of $\frac{1}{2}$ at $max(b, \frac{A}{3})$.*

Proof. In Section 3, we showed that for $\{x|x > b\}$, $\varphi_0(x)$ is made of two linear parts at most, with the slope changes occurring at y where $\int_y^z \ell(x)\,\mathrm{d}x = 0$. In our particular case, this means that

$$0 = \int_y^A 2 - \frac{3}{A}x\,\mathrm{d}x = 2(A-y) - \frac{3}{2A}x^2|_y^A = (\frac{A}{3} - y)(A-y)$$

So the point of the last change to φ_0 is at b or $\frac{A}{3}$. Since the last part of φ_0 has the slope $\frac{1}{2}$ (the maximal one), the proposition is proved. $\square$

Let's look at the different possibilities of b

Proposition 24. *If $b \geq \frac{2}{3}A$:*

(i) If $A < 1$ and $b \geq \frac{2}{3}A$, then

$$\varphi_0(x) = \frac{x}{2} - b$$

and

- *$b = A$ when $A < \frac{1}{3}$.*
- *$b = \sqrt{\frac{A}{3}}$ when $\frac{1}{3} \leq A < \frac{3}{4}$.*
- *$b = \frac{2}{3}A$ when $\frac{3}{4} \leq A < 1$.*

(ii) If $A \geq 1$ and $b \geq \frac{2}{3}A$, then

$$\varphi_0(x) = \begin{cases} (\frac{1}{b} - \frac{2-A}{4A} - \frac{3}{4})x + \frac{b}{2A} - 1 & 0 \leq x \leq b \\ \frac{x}{2} - b & b < x \leq A \end{cases}$$

for $b = \frac{2}{3}A$.

Proof. Due to convexity and minimality / maximality concerns, there is a straight line from $(0, \varphi_0(0))$ to $(b, -\frac{b}{2})$. This is because $b \geq \frac{2}{3}A$ means that $(b, -\frac{b}{2})$ is below $\frac{2-A}{2A}x - \frac{2}{3}$, and due to convexity, that line can't be crossed twice on $[0, b]$; it is crossed, at most, once, at point d. However, on the interval $[0, d]$, we seek to minimize φ_0, while on the interval $[d, b]$ (or $[0, b]$, if there is no d) we seek to maximize it. Minimality / maximality concerns mean that there is one straight line from 0 to b. Therefore, we're seeking a line of the form $mx + d$ that goes through the point $(b, -\frac{b}{2})$. Thus, our line is $mx - b(m + \frac{1}{2})$. At the point $(b, -\frac{b}{2})$, the line changes to $\frac{x}{2} - b$ (a slope of $\frac{1}{2}$). We wish to find, for a specific b (and A), the optimal m:

$$\int_0^b (mx - nx - bm + bn)(2 - \frac{3(2-A)}{2A}x + 3mx - 3bm - 1.5b)\,\mathrm{d}x \leq 0$$

(For any $-\frac{1}{2} \leq m \leq \frac{1}{2}$).

Simplifying this equation, we get

$$(m - n)(b^3 m + \frac{3}{4}b^3 + \frac{2-A}{4A}b^3 - b^2) \leq 0$$

If $m = \frac{-\frac{3}{4}b^3 - \frac{2-A}{4A}b^3 + b^2}{b^3} = \frac{1}{b} - \frac{2-A}{4A} - \frac{3}{4}$ the equation always equals 0, and for $b > \frac{2}{3}A$ and $A \geq 1$, m is in the required parameters (i.e., $|m| \leq \frac{1}{2}$). For $A \leq \frac{1}{2}$ the function $b^3 m + \frac{3}{4}b^3 + \frac{2-A}{4A}b^3 - b^2$ is always negative (for $\frac{2}{3}A < b \leq A$), and therefore m must be maximal, i.e., $m = \frac{1}{2}$. For $\frac{1}{2} < A < 1$, for some "b"s $m = \frac{1}{b} - \frac{2-A}{4A} - \frac{3}{4} \leq \frac{1}{2}$, and for the other "$b$"s $m = \frac{1}{2}$ (due to the negativity of the equation). So the equation looks like this:

$$\varphi_0(x) = \begin{cases} \frac{x}{2} - b & 0 < A < \frac{1}{2} \text{ or } \frac{b}{2-2b} < A < 1 \\ (\frac{1}{b} - \frac{2-A}{4A} - \frac{3}{4})x + \frac{b}{2A} - 1 & A \geq 1 \text{ or } A < \frac{b}{2-2b} \text{ and } 0 \leq x \leq b \\ \frac{x}{2} - b & A \geq 1 \text{ or } A < \frac{b}{2-2b} \text{ and } b < x \leq A \end{cases}$$

Now, the optimal b for this family of equations (i.e., $b \geq \frac{2}{3}A$) is found by differentiating the original equation (the one we wished to optimize). For the case where the equation is always $\frac{x}{2} - b$, we wish to maximize the following:

$$\int_0^b b(1 - \frac{2-A}{2A}x + \frac{x}{2} - b)\,\mathrm{d}x + \int_b^A b(1 - \frac{A}{2A}x - \frac{x}{2})\,\mathrm{d}x = -\frac{b^3}{2} + \frac{A}{2}b$$

The derivative of this equation is $-\frac{3}{2}b^2 + \frac{A}{2}$, which reaches 0 when $b = \sqrt{\frac{A}{3}}$. This expression is within our constraints ($\frac{2}{3}A < b \leq A$) for $\frac{1}{3} \leq A \leq \frac{3}{4}$. For $A < \frac{1}{3}$ it is always positive, so the maximum is reached at the largest b possible, $b = A$. For $A > \frac{3}{4}$ the derivative is always negative; that is, the maximum is reached at the smallest possible b (which is $\frac{2}{3}A$).

For the second type of equation, we wish to maximize the following equation:

$$\int_0^b (1 - \frac{b}{2A})(1 - \frac{2-A}{2A}x + \frac{1}{b}x - \frac{2-A}{4A}x - \frac{3}{4}x + \frac{b}{2A} - 1)\,\mathrm{d}x$$
$$+ \int_b^A b(1 - \frac{2-A}{2A}x - \frac{x}{2})\,\mathrm{d}x = \frac{1+4A}{8A^2}b^3 - \frac{1+2A}{2A}b^2 + \frac{1+A}{2}b$$

Differentiating that equation results in $\frac{3+12A}{8A^2}b^2 - \frac{1+2A}{A}b + \frac{1+A}{2}$, which is always negative for $\frac{2}{3}A \leq b < A$, so that for $A \geq 1$, the maximal value is reached at $b = \frac{2}{3}A$.

For $\frac{1}{2} < A < 1$ the smallest b converges with the case of $\frac{x}{2} - b$, and so the optimal value for $\frac{1}{2} < A < 1$ is reached at $b = max(\frac{2}{3}A, \sqrt{\frac{A}{3}})$. $\square$

Proposition 25. *If $\frac{A}{3} \leq b < \frac{2}{3}A$:*

(i) If $A < 1$ and $\frac{A}{3} \leq b < \frac{2}{3}A$, then

$$\varphi_0(x) = \frac{x}{2} - b$$

and $b = min(\frac{2}{3}A, \sqrt{\frac{A}{3}})$.

(ii) If $A \geq 1$ and $b < \frac{2}{3}$ and $\frac{A}{3} \leq b < \frac{2}{3}A$, then

$$\varphi_0(x) = \frac{x}{2} - b$$

(iii) If $A \geq 1$ and $b \geq \frac{2}{3}$ and $\frac{A}{3} \leq b < \frac{2}{3}A$:

$$\varphi_0(x) = \begin{cases} \frac{2-A}{2A}x - \frac{2}{3} & 0 \leq x \leq max(0, \frac{A}{A-1}(b - \frac{2}{3})) \\ \frac{x}{2} - b & max(0, \frac{A}{A-1}(b - \frac{2}{3})) < x \leq A \end{cases}$$

and

- $b = \sqrt{\frac{A}{3}}$ *when $A < 1\frac{1}{3}$.*
- $b = \frac{2A - \sqrt{A^2 - A}}{3}$ *when $A \geq 1\frac{1}{3}$.*

Proof. If $b \geq \frac{A}{3}$, the optimal value on the interval $[0, b]$ should be minimal, and thus should be as close as possible to $\frac{2-A}{2A}x - \frac{2}{3}$ as long as possible and then change to a line with the slope $\frac{1}{2}$. For $A \geq 1$ this isn't a problem, as $|\frac{2-A}{2A}| \leq \frac{1}{2}$; but for $0 < A < 1$, and other cases where $b < \frac{2}{3}$, there is no part where $\varphi_0(x) = \frac{2-A}{2A}x - \frac{2}{3}$, so there is only one part, with the slope $\frac{1}{2}$, namely:

$$\varphi_0(x) = \begin{cases} \frac{x}{2} - b & A < 1 \text{ or } \frac{A}{3} \leq b < \frac{2}{3} \\ \frac{2-A}{2A}x - \frac{2}{3} & A \geq 1 \text{ and } b \geq \frac{2}{3} \text{ and } 0 \leq x \leq max(0, \frac{A}{A-1}(b - \frac{2}{3})) \\ \frac{x}{2} - b & A \geq 1 \text{ and } b \geq \frac{2}{3} \text{ and } max(0, \frac{A}{A-1}(b - \frac{2}{3})) < x \leq A \end{cases}$$

In seeking the optimal b for each A, we already solved in Proposition 24 the case of one single line $\frac{x}{2} - b$. For $A > 1$ and $b \geq \frac{2}{3}$, we wish to maximize

$$\int_0^{\frac{A}{A-1}(b-\frac{2}{3})} \frac{2}{3}(1 - \frac{2-A}{2A}x + \frac{2-A}{2A}x - \frac{2}{3})\,\mathrm{d}x + \int_{\frac{A}{A-1}(b-\frac{2}{3})}^{b} b(1 - \frac{2-A}{2A}x + \frac{x}{2} - b)\,\mathrm{d}x$$

$$+ \int_b^A b(1 - \frac{2-A}{2A}x - \frac{x}{2})\,\mathrm{d}x =$$

$$= \frac{-1}{2-2A}b^3 + \frac{A}{1-A}b^2 + \frac{-3A^2-A}{6-6A}b + \frac{4A}{27-27A}$$

Differentiating this equation results in $\frac{-3}{2-2A}b^2 + \frac{2A)}{1-A}b + \frac{-3A^2-A}{6-6A}$, which means the optimal $b = \frac{2A-\sqrt{A^2-A}}{3}$. However, for $1 \leq A < 1\frac{1}{3}$ this is smaller than $\frac{2}{3}$, and so the optimum is reached at the optimal $b < \frac{2}{3}$, which is $b = \sqrt{\frac{A}{3}}$. Furthermore, for $A > 1\frac{1}{3}$, $\sqrt{\frac{A}{3}} > \frac{2}{3}$, and so the optimal b is $\frac{2A-\sqrt{A^2-A}}{3}$. $\square$

Proposition 26. *If $A > 1\frac{1}{3}$, the optimal b value is $\geq \frac{2}{3}$.*

Proof. Letting the slope m be a variable, the original equation (for $b \leq \frac{2}{3}$) is

$$\int_0^b b(m+\frac{1}{2})(1-\frac{2-A}{2A)}x + mx - b(m+\frac{1}{2}))\,\mathrm{d}x + \int_b^{\frac{A}{3}} b(m+\frac{1}{2})(1-\frac{2-A}{2A}x - \frac{x}{2})\,\mathrm{d}x$$

$$+\int_{\frac{A}{3}}^{A} (b(m+\frac{1}{2}) - \frac{A}{3}m + \frac{A}{6})(1-\frac{2-A}{2A}x-\frac{x}{2})\,\mathrm{d}x$$

Differentiating this, the optimal b is one that satisfies

$$(m+\frac{1}{2})(-b^2(\frac{3}{2}m+\frac{3}{4}) + \frac{A}{2}) = 0$$

For $A > 1\frac{1}{3}$ and $b < \frac{2}{3}$ this is always positive, so that the optimal $b \geq \frac{2}{3}$. $\square$

Theorem 27.

(i) If $0 < A < 1\frac{1}{3}$, the optimal φ_0 is

$$\varphi_0(x) = \frac{x}{2} - b$$

with $b = min(\sqrt{\frac{A}{3}}, A)$.

(ii) If $1\frac{1}{3} \leq A \leq 3$, the optimal φ_0 is

$$\varphi_0(x) = \begin{cases} \frac{2-A}{2A}x - \frac{2}{3} & 0 \leq x \leq \frac{A}{A-1}(b-\frac{2}{3}) \\ \frac{x}{2} - b & max(0, \frac{A}{A-1}(b-\frac{2}{3})) < x \leq A \end{cases}$$

with $b = \frac{2A-\sqrt{A^2-A}}{3}$.

Proof. Using the proofs from Proposition 25 (and for $A < 1$, from Proposition 24), what is left to prove is that the optimal b is larger than $\frac{A}{3}$. As seen in the previous proposition, when looking at φ_0 that is constructed of two parts - one line with a slope m (crossing the point $(b, -\frac{b}{2})$) up to $\frac{A}{3}$ where it changes to a slope of $\frac{1}{2}$, for each m the optimal b is one that satisfies

$$(m + \frac{1}{2})(-b^2(\frac{3}{2}m + \frac{3}{4}) + \frac{A}{2}) = 0$$

Therefore

$$b = \sqrt{\frac{A}{3(m + \frac{1}{2})}}$$

Notice that any possible solution is either of this sort (straight line of slope m, then changing somewhere to the slope $\frac{1}{2}$) or has a part where it equals $\frac{2-A}{2A}x - \frac{2}{3}$, and then it continues with slope m until changing to slope $\frac{1}{2}$. This "cutting off" (the line with slope m is "cut" by the line $\frac{2-A}{2A}x - \frac{2}{3}$) can only make the solution larger, as can easily be inferred from the relevant part of the alternative representation of the optimized equation (the part $-\frac{3}{2}\int\limits_0^b(\varphi(x) - \frac{2-A}{2A}x + \frac{2}{3})^2\,\mathrm{d}x$).

Furthermore, what we gain (with the "cutoff") is larger as b grows, as there is more to "cut off," and thus the optimal point may get larger, but not smaller. More formally, if $\hat{b}_m$ is the optimal b when φ_0 is made of two parts (without the "cutoff"), and $\bar{b}_m$ after the "cutoff," $\hat{b}_m \leq \bar{b}_m$. Also, notice that for all m, $\hat{b}_{\frac{1}{2}} \leq \hat{b}_m$. Therefore, if $\hat{b}_{\frac{1}{2}} \geq \frac{A}{3}$, then the optimal point is reached in the realm we dealt with in Proposition 25. A simple calculation shows that for $A \leq 3$, $\hat{b}_{\frac{1}{2}} \geq \frac{A}{3}$. $\square$

Corollary 28. *The optimal u is*

- *For $0 < A \leq \frac{1}{3}$:*

$$u_A(v_1, v_2) = max(0, v_2 - A)$$

- *For* $\frac{1}{3} \le A \le 1\frac{1}{3}$*:*

$$u_A(v_1, v_2) = max(0, v_2 - \sqrt{\frac{A}{3}})$$

Notice this means only v_2 *determines the value of* u_A*.*

- *For* $1\frac{1}{3} \le A \le 3$*:*

$$u_A(v_1, v_2) = max(0, \frac{1}{A}v_2 + \frac{A-1}{A}v_1 - \frac{2}{3}, v_2 - \frac{2A - \sqrt{A^2 - A}}{3})$$

Therefore, up to "above" a line parallel to $v_1 = v_2$*, only* v_2 *determines value, and after a certain point, the relationship between* v_1 *and* v_2 *has a slope of* $1-A$*, which is parallel to* ∂W*.*

Proof. Write u according to the definition of φ_0, using Lemma 13. □

4.1 Joining Triangles

Utilizing the results we have achieved, we can easily extend our solutions to the case of W (the range of object values) that is made of two joined "triangles" - one with vertices $(0,0)$, $(1,1)$, and $(0, A_1)$ and the other with vertices $(0,0)$, $(1,1)$, and $(A_2, 0)$ (in order for W to be convex, $(1 - A_1)(1 - A_2) \ge 1$). Obviously, optimal solutions for each triangle separately that form a valid solution when triangles are "joined" (e.g., the solution is still convex) are optimal for the complete polygon.

Corollary 29. *If* W *is a convex polygon with the vertices* $(0,0)$*,* $(0, A_1)$*,* $(1,1)$*, and*

$(A_2, 0)$*, and* $1\frac{1}{2} \leq A_1, A_2 \leq 3$*, then*

$$\varphi_0(x) = \begin{cases} \frac{x}{2} - b_1 & max(0, \frac{A_1}{A_1-1}(b_1 - \frac{2}{3})) < x \leq A_1 \\ \frac{2-A_1}{2A_1}x - \frac{2}{3} & 0 \leq x \leq \frac{A_1}{A_1-1}(b_1 - \frac{2}{3}) \\ \frac{A_2-2}{2A_2}x - \frac{2}{3} & 0 > x \geq -\frac{A_2}{A_2-1}(b_2 - \frac{2}{3}) \\ -\frac{x}{2} - b_2 & min(0, -\frac{A_2}{A_2-1}(b - \frac{2}{3})) > x \geq -A_2 \end{cases}$$

and $b_1 = \frac{2A_1 - \sqrt{A_1^2 - A_1}}{3}$ *and* $b_2 = \frac{2A_2 - \sqrt{A_2^2 - A_2}}{3}$.

Proof. The optimal solution for the triangle with the vertices $(0, 0)$, $(1, 1)$, and $(0, A_1)$ for $1\frac{1}{2} \leq A_1 \leq 3$ was proven above. The optimal solution for the triangle with the vertices $(0, 0)$, $(1, 1)$, and $(A_2, 0)$ is equivalent (by replacing v_1 with v_2 and vice versa) to the solution of the triangle with the vertices $(0, 0)$, $(1, 1)$, and $(0, A_2)$, which was shown above.

The solution is reached by using the optimal solution for each triangle. There is a small technical issue to notice: since the second triangle is denoted by negative "x" values, we must flip the sign of the coefficient in order to retain the values of φ_0. Due to the range of A_1 and A_2 we selected, the resulting φ_0 is continuous (since $\varphi_0(0) = \frac{2}{3}$ and doesn't depend on A), and due to the convexity of W (i.e., $(1-A_1)(1-A_2) \geq 1$), the resulting φ_0 is convex. $\square$

Corollary 30. *If* W *is a convex polygon with the vertices* $(0, 0)$*,* $(0, A_1)$*,* $(1, 1)$*, and* $(A_2, 0)$*, and* $1\frac{1}{2} \leq A_1, A_2 \leq 3$*, then*

$$u = max(0, v_2 - \frac{2A_1 - \sqrt{A_1^2 - A_1}}{3}, \frac{1}{A_1}v_2 + \frac{A_1 - 1}{A_1}v_1 - \frac{2}{3},$$
$$\frac{1}{A_2}v_1 + \frac{A_2 - 1}{A_2}v_2 - \frac{2}{3}, v_1 - \frac{2A_2 - \sqrt{A_2^2 - A_2}}{3})$$

Proof. Write u according to the definition of φ_0, using Lemma 13. □

Finally, we reach the solution for our original problem:

Corollary 31. *For W the triangle with the vertices $(0,0)$, $(2,0)$, and $(0,2)$ the optimal u is*

$$u = max(0, v_2 - \frac{4-\sqrt{2}}{3}, v_1 - \frac{4-\sqrt{2}}{3}, \frac{v_2+v_1}{2} - \frac{2}{3})$$

5 Conclusion and Discussion

In solving the specific problem that we addressed (where the values are uniformly distributed on the triangle with the vertices $(0,0)$, $(2,0)$, and $(0,2)$), we obtained several interesting results, without simplifying the two-dimensional problem (as others did) by requiring independence of between the two variables. We dealt with problems for which the optimal solution turns out to be of the form where the probability of getting the object is either 0 or 1, which is a fairly large family of problems that includes many common convex shapes with the uniform distribution.

For problems with these types of solutions, we showed that, whatever the distribution, from a certain point the optimal mechanism will have (at most) two sections (at the "edges" of the shape, i.e., the areas closer to the borders, but farther from $(0,0)$) for which there is a fixed probability for obtaining the objects. Furthermore, in many cases there will be a section for which there is, in effect, a "minimum price," just as in the single-dimension case. This result makes sense, as the areas near the axes (and hence, near the borders of W) are those for which there is a significant value for one object, but a small one for the other, indicating that it will be much more profitable for the owner to agree to sell only the object for which there is a high value. We were also able to characterize the points where the objects' distribution will no longer depend solely on the price for only one object (the point changes according to distribution and the "shape" of W).

Our results also show that for many shapes (of W) and distributions, the mechanism will be "piecewise linear," in the sense that it will be made by regions, in each one of which the probabilities of receiving the objects are constant. In the uniform distribution, and in our triangle-shaped W, we were able to show the solution for a family of problems (where $1\frac{1}{2} \leq A \leq 3$), and our method provides a fairly straightforward method to solve the optimal problem for the rest of the family of triangles.

Further work could concentrate on characterizing the types of problems for which the optimal mechanism is one in which either no object is given, or one is surely sold, which we believe encompasses more than the problems shown in Section 2.2. Another direction would be to further characterize the distributions for which the optimal solution is piecewise linear, which might help simplify the solution of this class of problems.

Extending our method to n-dimensions isn't straightforward, but we believe it may yield at least partial results (e.g., a "minimum price" for objects near the axes). While one seeks an elegant solution for all two-dimensional (and n-dimensional) auction problems, we believe that due to the complexity inherent in the problem (as described in [6]), seeking assumptions - beside variable independence - to simplify the problem is the way forward.

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A Graphic Representation of Selected Problems

A.1 The Original Problem - $A_1 = A_2 = 2$

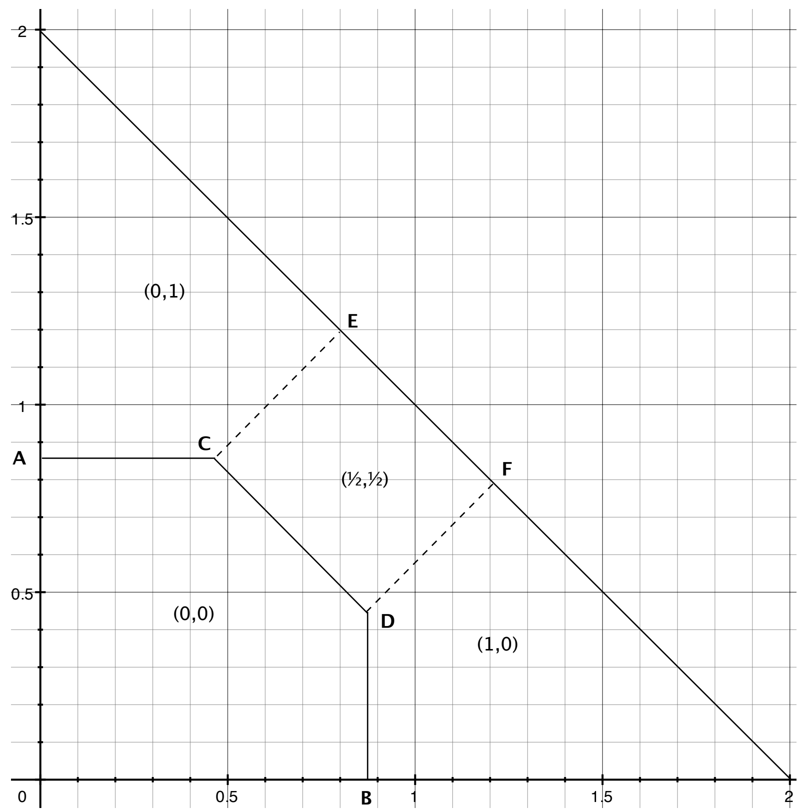

Figure 1: *Uniform distribution on the triangle with the vertices* $(0,0)$*,* $(2,0)$*,* $(0,2)$*.* Numbers indicate values of (q_1, q_2).
$A = (0, \frac{4-\sqrt{2}}{3}) \sim (0, 0.862)$
$B = (\frac{4-\sqrt{2}}{3}, 0) \sim (0.862, 0)$
$C = (\frac{\sqrt{2}}{3}, \frac{4-\sqrt{2}}{3}) \sim (0.471, 0.862)$
$D = (\frac{4-\sqrt{2}}{3}, \frac{\sqrt{2}}{3}) \sim (0.862, 0.471)$
$E = (\frac{1+\sqrt{2}}{3}, \frac{5-\sqrt{2}}{3}) \sim (0.805, 1.195)$
$F = (\frac{5-\sqrt{2}}{3}, \frac{1+\sqrt{2}}{3}) \sim (1.195, 0.805)$

A.2 When $A_1 = A_2 = 1$

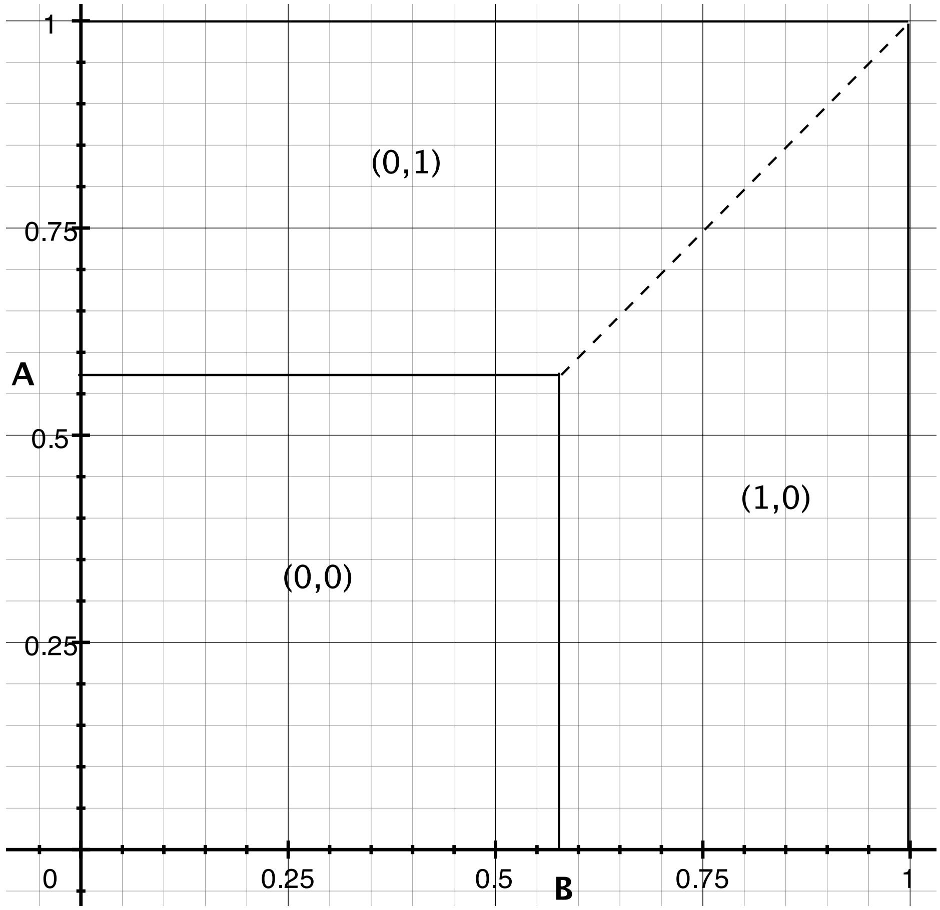

Figure 2: *Uniform distribution on the square with the vertices* $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$. Numbers indicate values of (q_1, q_2).
$A = (0, \frac{1}{\sqrt{3}}) \sim (0, 0.577)$
$B = (\frac{1}{\sqrt{3}}, 0) \sim (0.577, 0)$

A.3 When $A_1 = 1.5, A_2 = 3$

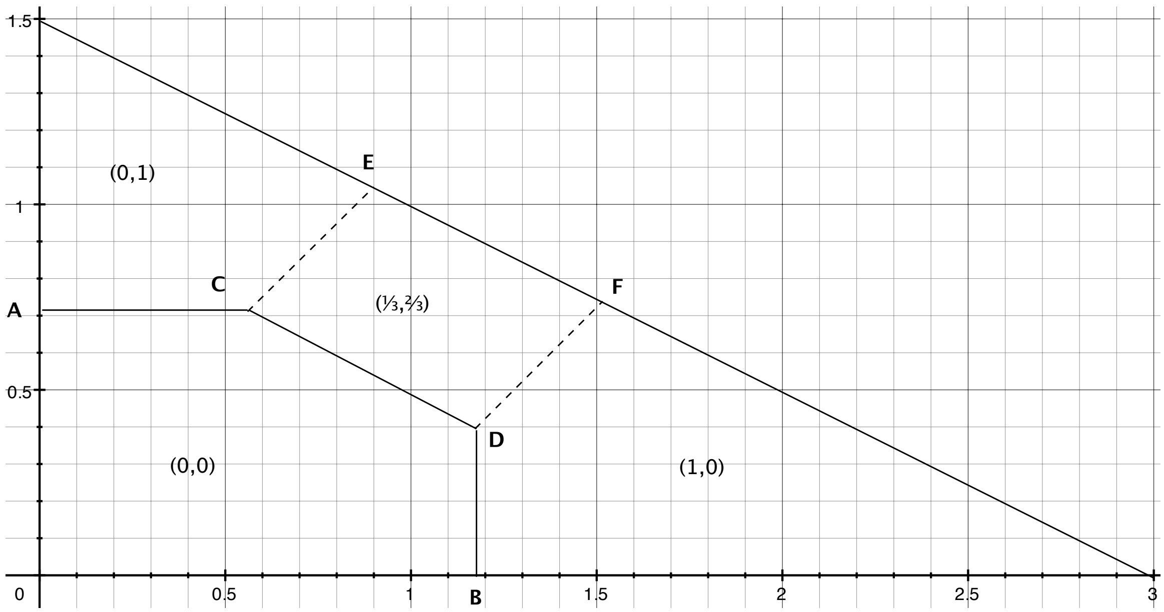

Figure 3: *Uniform distribution on the triangle with the vertices* $(0,0)$*,* $(0,1.5)$*,* $(3,0)$. Numbers indicate values of (q_1, q_2).
$A = (0, 1 - \frac{1}{2\sqrt{3}}) \sim (0, 0.711)$
$B = (2 - \sqrt{\frac{2}{3}}, 0) \sim (1.184, 0)$
$C = (\frac{1}{\sqrt{3}}, 1 - \frac{1}{2\sqrt{3}}) \sim (0.577, 0.711)$
$D = (2 - \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}) \sim (1.184, 0.408)$
$E = (\frac{1}{3} + \frac{1}{\sqrt{3}}, \frac{4}{3} - \frac{1}{2\sqrt{3}}) \sim (0.911, 1.045)$
$F = (\frac{7}{3} - \sqrt{\frac{2}{3}}, \frac{1}{3} + \frac{1}{\sqrt{6}}) \sim (1.517, 0.742)$

A.4 When $A_1 = 1.75, A_2 = 1.5$

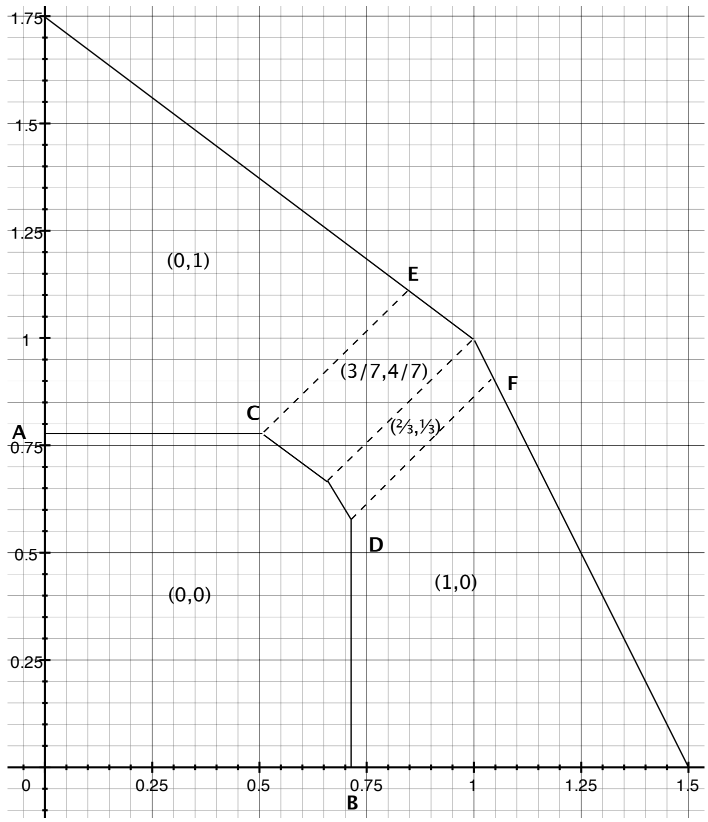

Figure 4: *Uniform distribution on the polygon with the vertices* $(0,0)$, $(0,1.75)$, $(1,1)$, $(1.5,0)$. Numbers indicate values of (q_1,q_2).
$A = (0, \frac{3.5-\frac{\sqrt{21}}{4}}{3}) \sim (0, 0.785)$
$B = (1 - \frac{1}{2\sqrt{3}}, 0) \sim (0.711, 0)$
$C = (\frac{\sqrt{21}}{9}, \frac{3.5-\frac{\sqrt{21}}{4}}{3}) \sim (0.509, 0.785)$
$D = (1 - \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}) \sim (0.711, 0.577)$
$E = (\frac{\sqrt{21}+3}{9}, \frac{4.5-\frac{\sqrt{21}}{4}}{3}) \sim (0.843, 1.118)$
$F = (\frac{4}{3} - \frac{1}{2\sqrt{3}}, \frac{1}{3} + \frac{1}{\sqrt{3}}) \sim (1.045, 0.911)$