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TRUTH AND ENVY IN CAPACITATED ALLOCATION GAMES

By

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Truth and Envy in Capacitated Allocation Games *

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Abstract

We study auctions with additive valuations where agents have a limit on the number of items they may receive. We refer to this setting as *capacitated allocation games*. We seek truthful and envy free mechanisms that maximize the social welfare. *I.e.*, where agents have no incentive to lie and no agent seeks to exchange outcomes with another.

In 1983, Leonard showed that VCG with Clarke Pivot payments (which is known to be truthful, individually rational, and have no positive transfers), is also an envy free mechanism for the special case of n items and n unit capacity agents. We elaborate upon this problem and show that VCG with Clarke Pivot payments is envy free if agent capacities are all equal. When agent capacities are not identical, we show that there is no truthful and envy free mechanism that maximizes social welfare if one disallows positive transfers.

For the case of two agents (and arbitrary capacities) we show a VCG mechanism that is truthful, envy free, and individually rational, but has positive transfers. We conclude with a host of open problems that arise from our work.

1 Introduction

We consider *allocation problems* where a set of objects is to be allocated amongst m agents, where every agent has an additive and non negative valuation function. We study mechanisms that are truthful, envy free, and maximize the social welfare (sum of valuations). The utility of an agent i is the valuation of the bundle assigned to i, $v_i(\mathrm{OPT})$, minus any payment, p_i.

A mechanism is incentive compatible (or truthful) if it is a dominant strategy for every agent to report her private information truthfully [4]. A mechanism is envy-free if no agent wishes to switch her outcome with that of another [1, 2, 9, 6, 7, 10].

Any allocation that maximizes the social welfare has payments that make it truthful — in particular — any payment of the form

$$p_i = h_i(t^{-i}) - \sum_{j\neq i} v_j(\mathrm{OPT}) \quad (1)$$

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where OPT is an allocation maximizing the social welfare and t^{-i} are the types of all agents but agent i. Similarly, any allocation that maximizes the social welfare has payments that make it envy free, this follows from a characterization of envy free allocations (see [3]). Unfortunately, the set of payments that make the mechanism truthful, and the set of payments that make the mechanism envy free, need not intersect. In this paper we seek such payments, *i.e.*, payments that make the mechanism simultaneously truthful and envy free.

An example of a mechanism that is simultaneously truthful and envy free is the Vickrey 2nd price auction. Applying the 2nd price auction to an allocation problem assigns items successively, every item going to the agent with the highest valuation to the item at a price equal to the 2nd highest valuation. If, for example, for all items, agent i has maximal valuation, then agent i will receive all items.

Leonard [5] considered the problem of assigning people to jobs, n people to n positions, and called this problem the permutation game. The Vickrey 2nd price auction is irrelevant in this setting because no person can be assigned to more than one position. Leonard showed that VCG with Clarke Pivot payments is simultaneously truthful and envy free. Under Clarke Pivot payments, agents internalize their externalities, *i.e.*,

$$h_i(t^{-i}) = \sum_{j \neq i} v_j(\mathrm{OPT}^{-i}) \tag{2}$$

where OPT^{-i} is the optimal allocation if there was no agent i. By substituting $\sum_{j \neq i} v_j(\mathrm{OPT}^{-i})$ for $h_i(t^{-i})$ in Equation 1 one can interpret Clarke Pivot payments as though an agent pays for how much others lose by her presence, *i.e.*, the agent internalizes her externalities.

Motivated by the permutation game, we consider a more general capacitated allocation problem where agents have associated capacities. Agent i has capacity U_i and cannot be assigned more than U_i items. Like Leonard, we seek a mechanism that is simultaneously truthful and envy free. The private types we consider may include both the valuation and the capacity (private valuations and private capacities) or only the valuation (private valuations, public capacity). Leonard's proof uses LP duality and it is not obvious how to extend it to more general settings.

Before we address this question, one needs to ask what does it mean for one agent to envy another when they have different capacities? A lower capacity agent may be unable to switch allocations with a higher capacity agent. To deal with this issue, we allow agent i, with capacity less than that of agent i' to choose whatever items she desires from the i' bundle, up to her capacity. *I.e.*, we say that agent i envies agent i' if agent i prefers a subset of the allocation to agent i', along with the price set for agent i', over her own allocation and price.

The VCG mechanism (obey Equation 1) is always truthful. In fact, any truthful mechanisms that choose the socially optimal allocation in capacitated allocation problems must be VCG [8]. We obtain the following:

1. For agents with private valuations and either private or public capacities, under the VCG mechanism with Clarke Pivot payments, a higher capacity agent will never envy a lower capacity agent. In particular, if all capacities are equal then the mechanism is envy free. (See Section 3).

2. For agents with private valuations, and either private or public capacities, any envy free VCG payment must allow positive transfers. (See Section 4).

3. For two agents with private valuations and arbitrary public capacities, there exist VCG payments such that the mechanism is envy free. It follows that such payments must allow positive transfers. (See Section 5).

4. For two agents with private valuations and private capacities, and for two items, there exist VCG payments such that the mechanism is envy free. (See Section 6).

2 Preliminaries

Let U be a set of objects, and let v_i be a valuation function associated with agent i, $1 \leq i \leq m$, that maps sets of objects into $\Re$. We denote by v a sequence $< v_1, v_2, \ldots, v_m >$ of valuation functions one for each agent.

An allocation function[1] a maps a sequence of valuation functions $v =< v_1, v_2, \ldots, v_m >$ into a partition of U consisting of m parts, one for each agent. *I.e.*,

$$a(v) =< a_1(v), a_2(v), \ldots, a_m(v) >,$$

where $\cup_i a_i(v) \subseteq U$ and $a_i(v) \cap a_j(v) = \emptyset$ for $i \neq j$. A payment function[2] is a mapping from v to $\Re^m$, $p(v) =< p_1(v), p_2(v), \ldots, p_m(v) >$, $p_i(v) \in \Re$. We assume that payments are from the agent to the mechanism (if the payment is negative then this means that the transfer is from the mechanism to the agent).

A mechanism is a pair of functions, $M = \langle a, p \rangle$, where a is an allocation function, and p is a payment function. For a sequence of valuation functions $v = \langle v_1, v_2, \ldots, v_m \rangle$, the utility to agent i is defined as $v_i(a_i(v)) - p_i(v)$. Such a utility function is known as quasi-linear.

Let $v =< v_1, v_2, \ldots, v_m >$ be a sequence of valuations, we define (v'_i, v^{-i}) to be the sequence of valuation functions arrived by substituting v_i by v'_i, i.e.,

$$(v'_i, v^{-i}) =< v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_m > .$$

We next define mechanisms that are incentive compatible, envy-free, and both incentive compatible and envy-free.

- A mechanism is *incentive compatible* (IC) if it is a dominant strategy for every agent to reveal her true valuation function to the mechanism. *I.e.*, if for all i, v, and v'_i:

$$\begin{aligned} & v_i(a_i(v)) - p_i(v) \geq v_i(a_i(v'_i, v^{-i})) - p_i(v'_i, v^{-i}); \\ & \Leftrightarrow p_i(v) \leq p_i(v', v^{-i}) + \Big(v_i(a_i(v)) - v_i(a_i(v'_i, v^{-i}))\Big). \end{aligned} \tag{3}$$

- A mechanism is *envy-free* (EF) if no agent seeks to switch her allocation and payment with another. *I.e.*, if for all $1 \leq i, j \leq m$ and all v:

$$\begin{aligned} & v_i(a_i(v)) - p_i(v) \geq v_i(a_j(v)) - p_j(v); \\ & \Leftrightarrow p_i(v) \leq p_j(v) + \Big(v_i(a_i(v)) - v_i(a_j(v))\Big). \end{aligned} \tag{4}$$

- A mechanism (a, p) is *incentive compatible and envy-free* ($IC \cap EF$) if (a, p) is both incentive compatible and envy-free.

[1] Here we deal with indivisible allocations, although our results also extend to divisible allocations with appropriate modifications.

[2] In this paper we consider only deterministic mechanisms and can therefore omit the allocation as an argument to the payment function.

Vickrey-Clarke-Groves (VCG) mechanism: A mechanism $M = \langle a, p \rangle$ is called a VCG mechanism if:

- $a(v) \in \text{argmax}_{a \in A} \sum_{i=1}^{m} v_i(a_i(v))$, and
- $p_i(v) = h_i(v^{-i}) - \sum_{j \neq i} v_j(a_j(v))$, where h_i does not depend on v_i, $i = 1, \ldots, m$.

It is known that any mechanism whose allocation function a maximizes $\sum_{i=1}^{m} v_i(a_i(v))$ (social welfare) is incentive compatible if and only if it is a VCG mechanism (See, *e.g.*, [8], Theorem 9.37). In the following we will denote by *opt* an allocation a which maximizes $\sum_{i=1}^{m} v_i(a_i(v))$.

The *Clarke-pivot payment* for a VCG mechanism is defined by

$$h_i(v^{-i}) = \max_{a' \in A} \sum_{j \neq i} v_j(a').$$

3 VCG with Clarke-pivot payments

A *capacitated allocation game* has m agents and n items that need to be assigned to the agents. Agent i is associated with a capacity $U_i \geq 0$, denoting the limit on the number of items she can be assigned, and each item j is associated with a capacity $Q_j \geq 0$, denoting the number of available copies of item j. The valuation $v_i(j)$ denotes how much agent i values item j, and $\sum_{j \in S} v_i(j)$ is the valuation of agent i to the bundle S.

A capacitated allocation game has a corresponding bipartite graph G, where every agent $1 \leq i \leq m$ has a vertex i associated with it on the left side, and every item $1 \leq j \leq n$ has a vertex j associated with it on the right side. The weight of the edge (i, j) is $v_i(j)$. An assignment is a subgraph of G that satisfies the capacity constraints, i.e. agent i is assigned at most U_i items and item j is assigned to at most Q_j agents. Recall that we denote by *opt* an assignment of maximum value. We describe *opt* by a matrix M where M_{ij} is the number of copies of item j allocated to agent i in *opt*.

For player i, the graph G^{-i} is constructed by removing the vertex associated with agent i and its incident edges from G. The assignment with maximum value in G^{-i} is defined by a matrix M^{-i}.

Let M be an assignment (either in G or in G^{-i} for some i.). We denote by M_i r the i'th row of M, $(M_{i1}, M_{i2}, \ldots, M_{in})$ which gives the bundle that agent i gets. We define $v_k(M_i) = \sum_{j=1}^{n} M_{ij} v_k(j)$ and $v(M) = \sum_{i=1}^{m} v_i(M_i)$.

The Clarke-pivot payment of agent k is

$$p_k = v(M^{-k}) - v(M) + v_k(M_k) \ . \tag{5}$$

The main result of this section is that in a VCG mechanism with Clarke-pivot payments, no agent will ever envy a lower-capacity agent. In particular, this says that if all agents have the same capacity, the VCG mechanism with Clarke-pivot payments is both incentive compatible and envy-free.

The proof of our main result (Theorem 3.1) is given in terms of a factional assignment but also holds for integral assignments.

Special case of capacitated allocation games, in which there are n items and n agents, and each agent can get at most a single item was first introduced in a paper by Leonard [5], and was called a *permutation game*. Leonard proved Theorem 3.1 for this special case only, and its proof technique does not seem to generalize for larger capacities. Our proof is different.

Here is our main theorem.

Theorem 3.1. *Consider a VCG mechanism consisting of an optimal allocation M and Clarke-pivot payments (5). Then if $U_i \geq U_j$, agent i does not envy agent j.*

Let agent 1 and agent 2 be arbitrary two agents such that the capacity of agent 1 is $\geq$ that of agent 2, that is $U_1 \geq U_2$.

Let M be an optimal assignment, M^{-1} an optimal assignment without agent 1, and M^{-2} some optimal assignment without agent 2. Agent 1 does *not* envy agent 2 iff

$$v_1(M_1) - p_1 \geq v_1(M_2) - p_2$$

Based on Equation 5, this is true when:

$$\begin{aligned}
&v_1(M_1) - (v(M^{-1}) - v(M) + v_1(M_1)) = \\
&v(M) - v(M^{-1}) \geq \\
&v_1(M_2) - (v(M^{-2}) - v(M) + v_2(M_2)) = \\
&v_1(M_2) + v(M) - v(M^{-2}) - v_2(M_2)
\end{aligned}$$

Rearranging we obtain that agent 1 does not envy agent 2 iff

$$v(M^{-2}) \geq v(M^{-1}) + v_1(M_2) - v_2(M_2). \tag{6}$$

We prove the theorem by establishing (6). We use the assignments M and M^{-1} to construct an assignment D^{-2} on G^{-2} such that

$$v(D^{-2}) \geq v(M^{-1}) + v_1(M_2) - v_2(M_2)\ . \tag{7}$$

From the optimality of M^{-2}, $v(M^{-2}) \geq v(D^{-2})$, which combined with (7) implies (6).

Given assignments M and M^{-1}, we construct a flow f on an associated bipartite digraph, G_f, with vertices for every agent and item. We define arcs and flows on arcs in G_f for every agent i and item j:

- If $M_{ij} - M_{ij}^{-1} > 0$ then G_f includes an arc $i \to j$ with flow $f_{i\to j} = M_{ij} - M_{ij}^{-1}$.
- If $M_{ij} - M_{ij}^{-1} < 0$ then G_f includes an arc $j \to i$ with flow $f_{j\to i} = M_{ij}^{-1} - M_{ij}$.
- If $M_{ij} = M_{ij}^{-1}$ then G_f contains neither $i \to j$ not $j \to i$.

We define the *excess* of an agent i in G_f, and the *excess* of an item j in G_f, to be

$$\begin{aligned}
ex_i &= \sum_{(i\to j)\in G_f} f_{i\to j} - \sum_{(j\to i)\in G_f} f_{j\to i} = \sum_j \big(M_{ij} - M_{ij}^{-1}\big), \\
ex_j &= \sum_{(j\to i)\in G_f} f_{j\to i} - \sum_{(i\to j)\in G_f} f_{i\to j} = \sum_i \big(M_{ij}^{-1} - M_{ij}\big),
\end{aligned}$$

respectively.

In other words the excess is the difference between the amount flowing out of the vertex and the amount flowing into the vertex. Clearly the sum of all excesses is zero. We say that a node is a *source* if its excess is positive and we say that a node is a *target* if its excess is negative.

Observation 3.2. To summarize,

$$
\begin{aligned}
i \text{ is an agent and a source} &\Rightarrow \\
0 \le \sum_j M^{-1}_{ij} + |ex_i| &= \sum_j M_{ij} \le U_i; && (8)\\
i \text{ is an agent and a target} &\Rightarrow \\
0 \le \sum_j M_{ij} + |ex_i| &= \sum_j M^{-1}_{ij} \le U_i; && (9)\\
j \text{ is an item and a source} &\Rightarrow \\
0 \le \sum_i M_{ij} + |ex_j| &= \sum_i M^{-1}_{ij} \le Q_j; && (10)\\
j \text{ is an item and a target} &\Rightarrow \\
0 \le \sum_i M^{-1}_{ij} + |ex_j| &= \sum_i M_{ij} \le Q_j. && (11)
\end{aligned}
$$

By the standard flow decomposition theorem we can decompose f into simple paths and cycles where each path connects a source to a target. Each path and cycle T has a positive flow value $f(T) > 0$ associated with it. Given an arc $x \to y$, if we sum the values $f(T)$ of all paths and cycles T including $x \to y$ then we obtain $f_{x \to y}$.

Notice that $M^{-1}_{1j} = 0$ for all j and therefore $f_{1 \to j} \ge 0$ for all j. It follows that there are no arcs of the form $j \to 1$ in G_f.

Observation 3.3. For each path $P = u_1, u_2, \ldots, u_t$ in flow decomposition G_f, where u_1 is a source and u_t is a target, we have $f(P) \le \min\{ex_{u_1}, |ex_{u_t}|\}$.

We define the value of a path or a cycle $T = u_1, u_2, \ldots, u_t$ in G_f, to be

$$v(P) = \sum_{\substack{\text{agent } u_i, \\ \text{item } u_{i+1}}} v_{u_i}(u_{i+1}) - \sum_{\substack{\text{item } u_i, \\ \text{agent } u_{i+1}}} v_{u_{i+1}}(u_i).$$

It is easy to verify that the $\sum_T f(T) \cdot v(T)$ over all paths and cycles in our decomposition is $v(M) - v(M^{-1})$.

Lemma 3.4. *Without loss of generality, we can assume that M^{-1} is such that*

1. *There are no cycles of zero value in G_f.*
2. *There is no path $P = u_1, u_2, \ldots, u_t$ of zero value such that $u_1 \ne 1$ is a source and u_t is a target.*

Proof. Assume that there is a cycle or a path T in the flow decomposition of G_f such that $v(T) = 0$. Let x be the smallest flow along an arc e of T. We modify M^{-1} as follows: For every agent to item arc $i \to j \in T$ we increase M^{-1}_{ij} by x and for every item to agent arc $j \to i \in T$ we decrease M^{-1}_{ij} by x. Let the resulting flow be $\tilde{M}^{-1}$.

If T is a cycle then the capacity constraints are clearly preserved. If T is not a cycle, then the capacity constraints are trivially preserved for all nodes other than u_1 and u_t. From Equation (8) we know that

$$\sum_j M^{-1}_{u_1 j} \;\le\; U_{u_1} - |ex_{u_1}| \le U_{u_1} - x \text{ if } u_1 \text{ is an agent.}$$

Ergo, if u_1 is an agent we can increase the allocation of $M^{-1}_{u_1 u_2}$ by x, while not exceeding the capacity of agent u_1 (U_{u_1}). If u_1 is an item, agent u_2 can release x units of item u_1 without violating any capacity constraints.

We can similarly see that the capacities constraints of u_t are not violated (Equation (11)).

Furthermore $v(\tilde{M}^{-1}) = v(M^{-1}) - xv(T) = v(M^{-1})$ and if we replace M^{-1} by $\tilde{M}^{-1}$ then G_f changes by decreasing the flow along every arc of T by x, and removing arcs whose flow becomes zero (in particular at least one arc will be removed). This process does not introduce any new edges to G_f.

We repeat the process until G_f does not contain zero cycles or paths as defined. □

From now on we assume that M^{-1} is chosen according to Lemma 3.4 [3].

Lemma 3.5. *The flow f in G_f does not contain cycles.*

Proof. Assume that f contains a cycle C which carries $\epsilon > 0$ flow. Clearly C does not contain agent 1 since there is not any arc entering agent 1 in G_f.

Assume first that $v(C) < 0$. Create an assignment $\widehat{M}$ from M by decreasing M_{ij} by ϵ for each agent to item arc $i \to j \in C$ and increasing M_{ij} by ϵ for each item to agent arc $j \to i \in C$. This can be done because $M - M^{-1}$ has a flow of ϵ along the agent to item arc $i \to j$, so, it must be that $M_{ij} \geq \epsilon$. Similarly, $M - M^{-1}$ has a flow of ϵ along item to agent arcs $j \to i$ so it must be the $M_{ij} \leq U_i - \epsilon$. Since C is a cycle the assignment $\widehat{M}$ still satisfies the capacity constraints. Furthermore $v(\widehat{M}) = v(M) - \epsilon v(C) > v(M)$ which contradicts the maximality of M.

If $v(C) > 0$ we create assignment $\widehat{M}^{-1}$ from M^{-1} as follows. For every item to agent arc $j \to i \in C$ we decrease M^{-1}_{ij} by ϵ and for every agent to item arc $i \to j \in C$ we increase M^{-1}_{ij} by ϵ. This can be done because $M^{-1} - M$ has a flow of ϵ along the item to agent arc $j \to i$, so, it must be that $M^{-1}_{ij} \geq \epsilon$. Since C is a cycle $\widehat{M}^{-1}$ still satisfies the capacity constraints. Furthermore $v(\widehat{M}^{-1}) = v(M^{-1}) + \epsilon v(C) > v(M^{-1})$ which contradicts the maximality of M^{-1}.

We need to argue that $\widehat{M}^{-1}$ makes no assignment to agent 1, this follows because agent 1 has no incoming flow in G_f and cannot lie on any cycle.

By assumption, there no cycles of value zero in G_f. □

In particular Lemma 3.5 implies that there are no cycles in our flow decomposition.

Lemma 3.6. *Agent* 1 *is the only source node.*

Proof. We give a proof by contradiction, assume some other node, $u_1 \neq 1$, is a source. Then, there is a flow path $P = u_1, u_2, \ldots u_t$ from that node to a target node u_t. Since there are no arcs incoming into vertex 1, the path P cannot include agent 1.

Let ϵ be the flow along the path P in the flow decomposition.

If $v(P) > 0$ define $\widehat{M}^{-1}_{ij} = M^{-1}_{ij} + \epsilon$ for each agent to item arc $i \to j$ in P and $\widehat{M}^{-1}_{ij} = M^{-1}_{ij} - \epsilon$ for each item to agent arc $j \to i$ in P. For all other item/agent pairs (i, j), let $\widehat{M}^{-1}_{ij} = M^{-1}_{ij}$. We have that

$$v(\widehat{M}^{-1}) = v(M^{-1}) + \epsilon v(P) > v(M^{-1})$$

this would contradict the maximality of M^{-1} if $\widehat{M}^{-1}$ is a legal assignment.

[3] Since Equation (7) depends only on the value of M^{-1} it does not matter which M^{-1} we work with

If $v(P) < 0$ define $\widehat{M}_{ij} = M_{ij} - \epsilon$ for each agent to item arc $i \to j$ in P and $\widehat{M}_{ij} = M_{ij} + \epsilon$ for each item to agent arc $j \to i$ in P. For all other item/agent pairs (i, j), let $\widehat{M}_{ij} = M_{ij}$. We have that

$$v(\widehat{M}) = v(M) - \epsilon v(P) > v(M)$$

which contradicts the maximality of M.

We still need to argue that the assignment $\widehat{M}^{-1}$ (if $v(P) > 0$) and the assignment $\widehat{M}$ (if $v(P) < 0$) are legal. Because P has a flow of ϵ, $M^{-1}_{ij} \geq \epsilon$ for each item to agent arc $j \to i$ along P, and $M_{ij} \geq \epsilon$ for each agent to item arc $i \to j$ along P.

We also worry about exceeding capacities at the endpoints of P, since the size of assignments of agents/items that are internal to the path do not change.

We increase the capacity of u_1 while constructing M^{-1} only if u_1 is an agent, and increase the capacity of u_t while constructing M^{-1} only if it is an item. By Observation 3.2 this is legal. A similar argument shows that in $\widehat{M}$ the assignment of u_1 and u_t is smaller than their capacities.

According to the way we choose M^{-1}, it cannot be that $v(P) = 0$ and that P carries a flow in G_f. □

In particular Lemma 3.6 implies that all the paths in our flow decomposition start at agent 1.

We construct D^{-2} from M^{-1} as follows.

1. Stage I: Initially, $D^{-2} := M^{-1}$.

2. Stage II: For each item j let $x = \min\{M_{2j}, M^{-1}_{2j}\}$. Set $D^{-2}_{2j} := M^{-1}_{2j} - x$ and $D^{-2}_{1j} := x$.

3. Stage III: For each flow path P in the flow decomposition of G_f that contains agent 2 we consider the prefix of the path up to agent 2. For each agent to item arc $i \to j$ in this prefix we set $D^{-2}_{ij} := D^{-2}_{ij} + f(P)$, and for each item to agent arc $j \to i$ in this prefix we set $D^{-2}_{ij} := D^{-2}_{ij} - f(P)$.

It is easy to verify that D^{-2} indeed does not assign any item to agent 2. Also, the assignment to agent 1 in D^{-2} is of the same size as the assignment to agent 2 in M^{-1}. Since $U_1 \geq U_2$, D^{-2} is a legal assignment.

Lemma 3.7. *The assignment D^{-2} satisfies Equation (7).*

Proof. Rearranging Equation (7)

$$\begin{aligned} v(D^{-2}) \geq\ & v(M^{-1}) && (12)\\ & + \sum_{j=1}^{n} (v_1(j) - v_2(j)) \cdot \min(M_{2j}, M^{-1}_{2j}) && (13)\\ & + \sum_{j \mid M_{2j} > M^{-1}_{2j}} (v_1(j) - v_2(j)) \cdot (M_{2j} - M^{-1}_{2j}). && (14) \end{aligned}$$

At the end of stage I, we have $D^{-2} = M^{-1}$ and so the inequality above at line (12) (without adding (13) and (14)) holds trivially. It is also easy to verify that at the end of stage II, the inequality above that spans (12) and (13) but without (14) holds. Finally, at the end of stage III, the full inequality in (12), (13) and (14) will hold as we explain next.

Consider an item j such that $M_{2j} > M^{-1}_{2j}$. In G_f we have an arc $2 \to j$ such that $f_{2\to j} = M_{2j} - M^{-1}_{2j}$. Therefore in the flow decomposition we must have paths $P_1, \ldots, P_\ell$ all containing $2 \to j$ such that

$$\sum_{k=1}^{\ell} f(P_k) = f_{2\to j} = M_{2j} - M^{-1}_{2j} \tag{15}$$

Let $\widehat{P}_k$ be the prefix of P_k up to agent 2. Consider the cycle C consisting of $\widehat{P}_k$ followed by $2 \to j$ and $j \to 1$. It has to be that that value of this cycle is non-negative. (Otherwise, construct $\widehat{M}$ by decreasing each agent to item arc $i \to j$ on the cycle $\widehat{M}_{ij} = M_{ij} - \epsilon$ and increasing each item to agent arc $j \to i$ on the cycle $\widehat{M}_{ij} = M_{ij} + \epsilon$. It follows, that $v(\widehat{M}) = v(M) - \epsilon v(C) > v(M)$ in contradiction of maximality of M. The matching $v(\widehat{M})$ is legal since it preserves capacities and decreases assignment associated with arcs with flow on them.)

Therefore,

$$\begin{aligned} v(\widehat{P}_k) + v_2(j) - v_1(j) &\geq 0; \\ \Rightarrow v(\widehat{P}_k) &\geq (v_1(j) - v_2(j)); \\ \Rightarrow f(P_k)v(\widehat{P}_k) &\geq f(P_k)(v_1(j) - v_2(j)); \\ \Rightarrow \sum_{k=1}^{\ell} \big(f(P_k)v(\widehat{P}_k)\big) &\geq (v_1(j) - v_2(j)) \sum_{k=1}^{\ell} f(P_k). \end{aligned}$$

Substituting Equation (15) into the above gives us that

$$\sum_{k=1}^{\ell} \big(f(P_k)v(\widehat{P}_k)\big) \geq \big(v_1(j) - v_2(j)\big)\big(M_{2j} - M_{2j}^{-1}\big). \tag{16}$$

The left hand side of equation (16) is exactly the gain in value of the matching when applying stage III to the paths $\widehat{P}_1, \ldots, \widehat{P}_\ell$ during the construction of D^{-2} above. The right hand side is the term which we add in Equation (14).

To conclude the proof of Lemma 3.7, we note that stage III may also deal with other paths that start at agent 1 and terminate at agent 2. Such paths must have value ≥ 0 and thus can only increase the value of the matching D^{-2}. (Otherwise we can build assignment $\widehat{M}$, such that $v(\widehat{M}) > v(M)$ by decreasing M_{ij} by ϵ for each arc $i \to j \in P$ and increasing M_{ij} by ϵ for each arc $j \to i \in P$ as we did before. The matching $\widehat{M}$ is legal since it preserves capacities on inner nodes of the path, decreases only arcs with flow on them, $M_{ij} > \epsilon$. Capacity of a source agent node can be increased according to Observation 3.2.) □

Corollary 3.8. *If all agent capacities are equal then the VCG allocation with Clarke-pivot payments is envy-free.*

Do Clarke-pivot payments work also under heterogeneous capacities? The answer is no. This follows since in the next section we show that any mechanism that is both incentive compatible and envy-free must have positive transfers, and Clarke-pivot payments do not.

4 Heterogeneous capacities: IC∩EF payments imply positive transfers

Consider an arbitrary VCG mechanism. Let

$$opt =< opt_1, opt_2, \ldots, opt_n >$$

denote the allocation and let

$$p_i = h_i(v^{-i}) - v^{-i}(opt) \tag{17}$$

be the payments, where

$$v^{-i}(opt) = \sum_{\substack{1 \le j \le n \\ j \ne i}} v_j(opt_j).$$

Let $v(opt) = \sum_{j=1}^{n} v_j(opt_j)$ and let

$$opt^{-i} =< opt_1^{-1}, opt_2^{-1}, \ldots, opt_{i-1}^{-i}, \emptyset, opt_{i+1}^{-i}, \ldots, opt_n >,$$

be the allocation maximizing

$$v^{-i}(opt^{-i}) = \sum_{\substack{1 \le j \le n \\ j \ne i}} v_j(opt_j^{-i}).$$

We substitute the VCG payments (17) into the envy-free conditions (4) and obtain that i does not envy j if and only if

$$\begin{aligned}
& v_i(opt_j) - p_j \le v_i(opt_i) - p_i \\
& \Leftrightarrow p_i - p_j \le v_i(opt_i) - v_i(opt_j) \\
& \Leftrightarrow h_i(v^{-i}) - v^{-i}(opt) - \big(h_j(v^{-j}) - v^{-j}(opt)\big) \\
& \qquad \le v_i(opt_i) - v_i(opt_j) \\
& \Leftrightarrow h_i(v^{-i}) - h_j(v^{-j}) \\
& \qquad \le v^{-i}(opt) - v^{-j}(opt) + v_i(opt_i) - v_i(opt_j) \\
& \Leftrightarrow h_i(v^{-i}) - h_j(v^{-j}) \\
& \qquad \le v(opt) - (v(opt) - v_j(opt_j)) - v_i(opt_j) \\
& \Leftrightarrow h_i(v^{-i}) - h_j(v^{-j}) \le v_j(opt_j) - v_i(opt_j). \qquad (18)
\end{aligned}$$

Theorem 4.1. *Consider a capacitated allocation game with heterogeneous capacities such that the number of items exceeds the smallest agent capacity. There is no mechanism that simultaneously optimizes the social welfare, is IC∩EF, and has no positive transfers (the mechanism never pays the agents). That is, any IC∩EF mechanism has some valuations v for which the mechanism pays an agent.*

Note that the conditions on the capacities of the agents and the number of items are necessary – If capacities are homogeneous or the total supply of items is at most the minimum agent capacity then Clarke-pivot payments, that are known to be incentive compatible, individually rational, and have no positive transfers, are also envy-free.

In the rest of this section we prove Theorem 4.1. We start with a capacitated allocation game with two agents and two items where agent i has capacity i ($i = 1, 2$). We then generalize the proof to arbitrary heterogeneous games.

To ease the notation we abbreviate in the rest of the paper $v_i(j)$ to v_{ij}.

We partition the valuations into three sets A, B_1, and B_2 as follows (we omit cases with ties).[4]

[4] The optimal allocation that maximizes social welfare is uniquely defined when there are no ties. Valuations v's with ties form a lower dimensional measure 0 set. It suffices to consider valuations without ties for both existence or non-existence claims of IC or EF payments. This is clear for non-existence, for existence, the payments for a v with ties is defined as the limit when we approach this point through v's without ties that result in the same allocation. Clearly IC and EF properties carry over, also IR and nonnegativity of payments.

- (A) $v_{21} > v_{11}$ and $v_{22} > v_{12}$. For these valuations in an optimal allocation agent 2 obtains the bundle $\{1,2\}$ and agent 1 obtains the empty bundle.
- (B$_1$) $v_{11} - v_{21} > \max\{0, v_{12} - v_{22}\}$. For these valuations in an optimal allocation item 1 is assigned to agent 1 and item 2 to agent 2.
- (B$_2$) $v_{12} - v_{22} > \max\{0, v_{11} - v_{21}\}$. For these valuations in an optimal allocation item 1 is assigned to agent 2 and item 2 to agent 1.

Substituting the above in (18) we obtain that for $v \in B_1$, agent 1 does not envy agent 2 if and only if

$$h_1(v_2) - h_2(v_1) \le v_2(opt_2) - v_1(opt_2) = v_{22} - v_{12} \ .$$

Agent 2 does not envy agent 1 if and only if

$$h_2(v_1) - h_1(v_2) \le v_1(opt_1) - v_2(opt_1) = v_{11} - v_{21} \ .$$

Combining we obtain that there is no envy for $v \in B_1$, if and only if

$$v_{21} - v_{11} \le h_1(v_2) - h_2(v_1) \le v_{22} - v_{12} \ . \tag{19}$$

For a fixed $\epsilon > 0$, and $x > 5\epsilon$, the valuation v such that $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = v_{22} = 0$ is clearly in B_1. Substituting in (19) we obtain

$$-(x + 3\epsilon) \le h_1(0,0) - h_2(x + 3\epsilon, x + \epsilon) \le -(x + \epsilon) \tag{20}$$

.

The valuation v such that $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = x + \epsilon$, and $v_{22} = x$ is also clearly in B_1 and from (19) we obtain

$$x + \epsilon - (x + 3\epsilon) \le h_1(x + \epsilon, x) - h_2(x + 3\epsilon, x + \epsilon) \le x - (x + \epsilon)$$

hence

$$-2\epsilon \le h_1(x + \epsilon, x) - h_2(x + 3\epsilon, x + \epsilon) \le -\epsilon \ . \tag{21}$$

Combining (20) and (21) we obtain

$$h_1(x + \epsilon, x) \le h_2(x + 3\epsilon, x + \epsilon) - \epsilon \le h_1(0,0) + x + 3\epsilon \tag{22}$$

The no positive transfers requirement is that for any v,

$$h_1(v_2) \ge v_2(opt_2) \ . \tag{23}$$

Consider now the valuations v such that $v_{21} = x+\epsilon$, $v_{22} = x$, $v_{11} = v_{12} = x-\epsilon$. Clearly, $v \in A$ (agent 2 gets both items), hence $v_2(opt_2) = 2x-\epsilon$. Substituting this and (22) in (23) we obtain $2x-\epsilon \le h_1(0,0)+x+3\epsilon$, hence $h_1(0,0) \ge x - 4\epsilon$. Clearly, for valuations with large enough x we obtain a contradiction, that is, there exist valuations where the mechanism pays an agent.

Heterogeneous capacities, multiple agents and items: Let c be the smallest agent capacity and assume it is the capacity of agent 1. Let agent 2 be any agent with capacity $> c$. There are $\ge c + 1$ items. It suffices to consider restricted valuation matrices v where $v_{ij} = 0$ when $i > 2$ or when $j > c + 1$ and $v_{ij} \equiv v_{i2}$ for $i = 1, 2$ and $2 \le j \le c + 1$. We partition these valuations into four sets A, B_1, B_1^+, B_2, as follows (we omit cases with ties and only define the assignment of items $1, \ldots, c + 1$):

- (A) $v_{21} > v_{11}$ and $v_{22} > v_{12}$. For these valuations in an optimal allocation agent 2 obtains the bundle $\{1, \ldots, c+1\}$.
- (B_1) $v_{11} > v_{21}$ and $v_{12} < v_{22}$. For these valuations in an optimal allocation items 1 is assigned to agent 1 and items $2, \ldots, c+1$ to agent 2.
- (B_1^+) $v_{11} - v_{21} > v_{12} - v_{22}$ and $v_{12} > v_{22}$. For these valuations in an optimal allocation items $1, \ldots, c$ are assigned to agent 1 and item $c+1$ is assigned to agent 2.
- (B_2) $v_{12} - v_{22} > \max\{0, v_{11} - v_{21}\}$. For these valuations in an optimal allocation item 1 is assigned to agent 2 and items $2, \ldots, c+1$ to agent 1.

Substituting the above in (18) we obtain that for $v \in B_1^+$, agent 1 does not envy agent 2 if and only if

$$h_1(v_2) - h_2(v_1) \leq v_2(opt_2) - v_1(opt_2) = v_{22} - v_{12} \ .$$

Agent 2 does not envy agent 1 if and only if

$$\begin{aligned} h_2(v_1) - h_1(v_2) &\leq v_1(opt_1) - v_2(opt_1) \\ &= v_{11} + (c-1)v_{12} - v_{21} - (c-1)v_{22} \ . \end{aligned}$$

Combining we obtain that there is no envy for $v \in B_1^+$, if and only if

$$v_{21} + (c-1)v_{22} - v_{11} - (c-1)v_{12} \leq h_1(v_2) - h_2(v_1) \leq v_{22} - v_{12} \ . \tag{24}$$

For a fixed $\epsilon > 0$ and for $x > \epsilon$, the valuation v such that $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = v_{22} = 0$ is clearly in B_1^+. For such v the left hand side of (24) is

$$\begin{aligned} & v_{21} + (c-1)v_{22} - v_{11} - (c-1)v_{12} \\ &= -(x+3\epsilon) - (c-1)(x+\epsilon) \\ &= -cx - (c+2)\epsilon \end{aligned}$$

Substituting in (24) we obtain

$$-cx - (c+2)\epsilon \leq h_1(0,0) - h_2(x+3\epsilon, x+\epsilon) \leq -(x+\epsilon) \ . \tag{25}$$

The valuation v such that $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = x + \epsilon$, and $v_{22} = x$ is also clearly in B_1^+. For such v the left hand side of (24) is

$$\begin{aligned} & v_{21} + (c-1)v_{22} - v_{11} - (c-1)v_{12} \\ &= x + \epsilon + (c-1)x - (x+3\epsilon) - (c-1)(x+\epsilon) \\ &= -(c+1)\epsilon \end{aligned}$$

From (24) we obtain

$$-(c+1)\epsilon \leq h_1(x+\epsilon, x) - h_2(x+3\epsilon, x+\epsilon) \leq -\epsilon. \tag{26}$$

Combining (25) and (26) we obtain,

$$\begin{aligned} h_1(x+\epsilon, x) &\leq h_2(x+3\epsilon, x+\epsilon) - \epsilon \\ &\leq h_1(0,0) + cx + (c+2)\epsilon - \epsilon \\ &= h_1(0,0) + cx + (c+1)\epsilon \end{aligned}$$

For valuations $v_{21} = x + \epsilon$, $v_{22} = x$, $v_{11} = v_{12} = x - \epsilon$, we clearly have $v \in A$ (agent 2 gets all items), hence $v_2(opt_2) = (c+1)x - (c+1)\epsilon$.

For a sufficiently large x (relative to ϵ and $h_1(0,0)$), $h_1(v_2) = h_1(x+\epsilon, x) \leq h_1(0,0) + cx + (c+1)\epsilon < (c+1)x - (c+1)\epsilon = v_2(opt_2)$, which contradicts the no positive transfers requirement (23).

5 2 agents, Public Capacities

In this section we assume that capacities are public and derive $IC \cap EF$ payments for any game with two players.

Lemma 5.1. *Any 2-player capacitated allocation game with public capacities has an $IC \cap EF$ individually rational mechanism.*

Proof. Let c_i be the capacity of player i and assume without loss of generality that $c_1 \leq c_2$. For a vector $(x_1, x_2 \ldots)$ let $top_b\{x\}$ be the set of the b largest entries in x. We show that

$$h_1(v_2) = \sum_{j \in top_{c_1}\{v_2\}} v_{2j}$$

and

$$h_2(v_1) = \sum_{j \in top_{c_1}\{v_1\}} v_{1j}$$

give VCG payments which are envy-free.

It suffices to show that for $\{i, j\} = \{1, 2\}$,

$$h_i(v^{-i}) - h_j(v^{-j}) \leq v_j(opt_j) - v_i(opt_j) \ .$$

That is,

$$\sum_{j \in top_{c_1}\{v_2\}} v_{2j} - \sum_{j \in top_{c_1}\{v_1\}} v_{1j} \leq v_2(opt_2) - v_1(opt_2) \tag{27}$$

and

$$\sum_{j \in top_{c_1}\{v_1\}} v_{1j} - \sum_{j \in top_{c_1}\{v_2\}} v_{2j} \leq v_1(opt_1) - v_2(opt_1). \tag{28}$$

Assume first that the number of items is exactly $c_1 + c_2$. In the optimal solution, player 1 will get the c_1 items that maximize $v_{1j} - v_{2j}$ and player 2 will get the c_2 items that minimize this difference.

We establish (28) as follows

$$\begin{aligned}
&\sum_{j \in top_{c_1}\{v_1\}} v_{1j} - \sum_{j \in top_{c_1}\{v_2\}} v_{2j} \\
\leq &\sum_{j \in top_{c_1}\{v_1\}} (v_{1j} - v_{2j}) \\
\leq &\sum_{j \in top_{c_1}\{v_1 - v_2\}} (v_{1j} - v_{2j}) \\
= &\sum_{j \in opt_1} (v_{1j} - v_{2j}) = v_1(opt_1) - v_2(opt_1) \ .
\end{aligned}$$

We establish (27) as follows

$$\begin{aligned}
&\sum_{j\in top_{c_1}\{v_2\}} v_{2j} - \sum_{j\in top_{c_1}\{v_1\}} v_{1j} \\
&\leq \sum_{j\in top_{c_1}\{v_2\}} (v_{2j} - v_{1j}) \\
&\leq \sum_{j\in top_{c_1}\{v_2-v_1\}} (v_{2j} - v_{1j}) \\
&\leq \sum_{j\in opt_2} v_{2j} - \sum_{j\in top_{c_1}\{v_1(opt_2)\}} v_{1j}
\end{aligned}$$

where $v_1(opt_2)$ is the vector of the values of player 1 to the items player 2 gets in the optimal solution.

If there are fewer than $c_1 + c_2$ items, we add "dummy" items with valuations $v_{1j} = v_{2j} = 0$ and the lemma follows from the previous argument for the case with $c_1 + c_2$ items.

If there are more than $c_1 + c_2$ items then consider the set of $c_1 + c_2$ items than participate in the optimal solution. We now observe that (27) and (28) only involve items that participate in the optimal solution ($top_{c_2}\{v_2\}$ and $top_{c_1}\{v_1\}$ must both be included in the optimal solution). □

6 2 agents, 2 items, Private Capacities

In this section, valuations and capacities are private. We give VCG payments which are envy-free and individually rational for any game with two agents and two items. We specify the payments by giving the functions $h_1(v_2, c_2)$ and $h_2(v_1, c_1)$. Note that with two items, all $c_i \geq 2$ are equivalent, therefore we only need to consider capacities $\in \{0, 1, 2\}$.

We show that the following give envy-free payments

$$\begin{aligned}
h_1(v_2, c_2) &= \begin{cases} \max(v_{21}, v_{22}) & c_2 \in \{1, 2\} \\ 0 & c_2 = 0 \end{cases} \\
h_2(v_1, c_1) &= \begin{cases} \max(v_{11}, v_{12}) & c_1 \in \{1, 2\} \\ 0 & c_1 = 0 \end{cases}
\end{aligned}$$

The payments are envy-free if and only if

$$\begin{aligned}
\delta_{12} = h_1(v_2, c_2) - h_2(v_1, c_1) &\leq v_2(opt_2) - v_1(opt_2), \\
\delta_{21} = h_2(v_1, c_1) - h_1(v_2, c_2) &\leq v_1(opt_1) - v_2(opt_1).
\end{aligned}$$

The conditions when $\{c_1, c_2\} = \{1, 2\}$ were worked out in the previous section and the correctness for $h_1(v_2, 2)$ and $h_2(v_1, 1)$ carries over (and symmetrically, if we switch capacities of the agents). Consider the following remaining cases.

- $c_1 = c_2 = 2$**:** agent 1 does not envy agent 2 if and only if:

$$h_1(v_2, 2) - h_2(v_1, 2) \leq \begin{cases} v_{21} + v_{22} - v_{11} - v_{12} & \text{if } v_{21} > v_{11}, v_{22} > v_{12} \\ v_{22} - v_{12} & \text{if } v_{21} < v_{11}, v_{22} > v_{12} \\ v_{21} - v_{11} & \text{if } v_{21} > v_{11}, v_{22} < v_{12} \\ 0 & \text{if } v_{21} < v_{11}, v_{22} < v_{12} \end{cases}$$

Symmetrically, agent 2 does not envy agent 1 if and only if:

$$h_2(v_1,2) - h_1(v_2,2) \leq \begin{cases} v_{11}+v_{12}-v_{21}-v_{22} & \text{if } v_{11} > v_{21}, v_{12} > v_{22} \\ v_{12}-v_{22} & \text{if } v_{11} < v_{21}, v_{12} > v_{22} \\ v_{11}-v_{21} & \text{if } v_{11} > v_{21}, v_{12} < v_{22} \\ 0 & \text{if } v_{11} < v_{21}, v_{12} < v_{22} \end{cases}$$

Combining, we obtain the condition

$$\begin{aligned} &\min\{v_{21}-v_{11},0\} + \min\{v_{22}-v_{12},0\} \\ &\leq \quad h_1(v_2,2) - h_2(v_1,2) \\ &\leq \quad \max\{v_{21}-v_{11},0\} + \max\{v_{22}-v_{12},0\}. \end{aligned} \tag{29}$$

We now show that our particular h's satisfy (29). It suffices to establish one of the inequalities: We have

$$\begin{aligned} v_{21} &\leq \max\{v_{11},v_{12}\} + \max\{v_{21}-v_{11},0\} \\ v_{22} &\leq \max\{v_{11},v_{12}\} + \max\{v_{22}-v_{12},0\} \end{aligned}$$

Combining, we obtain the desired relation:

$$\begin{aligned} &\max\{v_{21},v_{22}\} \\ &\leq \quad \max\{v_{11},v_{12}\} + \max\{v_{21}-v_{11},0\} + \max\{v_{22}-v_{12},0\}\ . \end{aligned}$$

• $c_1 = c_2 = 1$**:** agent 1 does not envy agent 2 if and only if:

$$\begin{aligned} &h_1(v_2,1) - h_2(v_1,1) \\ &\leq \quad \begin{cases} v_{22}-v_{12} & v_{11}+v_{22} > v_{12}+v_{21} \\ v_{21}-v_{11} & v_{11}+v_{22} < v_{12}+v_{21} \end{cases} \end{aligned}$$

Symmetrically, agent 2 does not envy agent 1 if and only if:

$$\begin{aligned} &h_2(v_1,1) - h_1(v_2,1) \\ &\leq \quad \begin{cases} v_{12}-v_{22} & v_{21}+v_{12} > v_{22}+v_{11} \\ v_{11}-v_{21} & v_{21}+v_{12} < v_{22}+v_{11} \end{cases} \end{aligned}$$

Combining, we obtain

$$\begin{aligned} &\min\{v_{22}-v_{12}, v_{21}-v_{11}\} \\ &\leq \quad h_1(v_2,1) - h_2(v_1,1) \\ &\leq \quad \max\{v_{22}-v_{12}, v_{21}-v_{11}\} \end{aligned} \tag{30}$$

We now show that our particular h's satisfy (30). It suffices to establish one of the inequalities: We have

$$\begin{aligned} v_{21} &\leq \max\{v_{11},v_{12}\} + v_{21}-v_{11} \\ v_{22} &\leq \max\{v_{11},v_{12}\} + v_{22}-v_{12}\} \end{aligned}$$

Combining, we obtain the desired relation:

$$\max\{v_{21}, v_{22}\} \le \max\{v_{11}, v_{12}\} + \max\{v_{21} - v_{11}, v_{22} - v_{12}\} .$$

• $c_1 = 1, c_2 = 0$**:** No agent envies the other if and only if

$$\begin{aligned} h_1(v_2, 0) - h_2(v_1, 1) &\le 0 \\ h_2(v_1, 1) - h_1(v_2, 0) &\le \max\{v_{11}, v_{12}\} \end{aligned}$$

Combining, we obtain

$$-\max\{v_{11}, v_{12}\} \le h_1(v_2, 0) - h_2(v_1, 1) \le 0 \tag{31}$$

Symmetrically, when $c_1 = 0, c_2 = 1$:

$$-\max\{v_{21}, v_{22}\} \le h_2(v_1, 0) - h_1(v_2, 1) \le 0 \tag{32}$$

Our particular h's trivially satisfy (31) and (32).

• $c_1 = 2, c_2 = 0$**:** No agent envies the other if and only if

$$\begin{aligned} h_1(v_2, 0) - h_2(v_1, 2) &\le 0 \\ h_2(v_1, 2) - h_1(v_2, 0) &\le v_{11} + v_{12} \end{aligned}$$

Combining, we obtain

$$-v_{11} - v_{12} \le h_1(v_2, 0) - h_2(v_1, 2) \le 0 \tag{33}$$

Symmetrically, when $c_1 = 0, c_2 = 2$:

$$-v_{21} - v_{22} \le h_2(v_1, 0) - h_1(v_2, 2) \le 0 \tag{34}$$

Our particular h's trivially satisfy (33) and (34).

7 Conclusion and open problems

We have begun to study truthful and envy free mechanisms for maximizing social welfare for the capacitated allocation problem.

There is much left open, for example:

1. Is there a truthful and envy free mechanism (with positive transfers) for the capacitated allocation problem (arbitrary capacities):
   (a) With public capacities and more than two agents.
   (b) With private capacities for more than 2 agents and 2 items?
2. How well can we approximate the social welfare by a mechanism that is incentive-compatible, envy-free, invidually rational, and without positive transfers for capacitated allocations ?
3. Noam Nisan has observed that for superadditive valuations, there may be no mechanism that is both truthful and envy free. We conjecture that one can obtain mechanisms that are both truthful and envy free for subadditive valuations.

References

[1] L.E. Dubins and E.H. Spanier. How to cut a cake fairly. *American Mathematical Monthly*, 68:1–17, 1961.

[2] D. Foley. Resource allocation and the public sector. *Yale Economic Essays*, 7:45–98, 1967.

[3] Claus-Jochen Haake, Matthias G. Raith, and Francis Edward Su. Bidding for envyfreeness: A procedural approach to n-player fair-division problems. *Social Choice and Welfare*, 19:723–749, 2002.

[4] Leonard Hurwicz. Optimality and informational efficiency in resource allocation processes. In K.J. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical Methods in the Social Sciences*, pages 27–46. Stanford University Press, Stanford, CA, 1960.

[5] Herman B. Leonard. Elicitation of honest preferences for the assignment of individuals to positions. *The Journal of Political Economy*, 91:3:461–479, 1983.

[6] Eric S. Maskin. On the fair allocation of indivisible goods. In *G. Feiwel (ed.), Arrow and the Foundations of the Theory of Economic Policy (essays in honor of Kenneth Arrow)*, volume 59:4, pages 341–349, 1987.

[7] Herve Moulin. *Fair Division and Collective Welfare*. MIT Press, 2004.

[8] Noam Nisan. Introduction to mechanism design. In Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay Vazirani, editors, *Algorithmic Game Theory*, chapter 16. Cambridge University Press, 2007.

[9] L.-G. Svensson. On the existence of fair allocations. *Journal of Economics*, 43:301–308, 1983.

[10] H. Peyton Young. *Equity: In Theory and Practice*. Princeton University Press, 1995.