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An infinitely repeated rental model with incomplete information

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Abstract

In an infinitely repeated rental model with two types of buyer and no discounting, the set of all Nash equilibrium payoffs for the seller and the buyer is characterized.

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1. Introduction

Hart and Tirole (1988) study a discounted finitely repeated rental model with two types of buyer. Their finding for the noncommitted short-term contract implies that the seller's expected profit, on average, converges to the buyer's lower evaluation $\underline{v}$ when the discount factor $\delta \to 1$ and the time horizon goes to infinity. In order to obtain some intuition about Hart and Tirole's result, Fudenberg and Tirole (1992) extend Hart and Tirole to an infinitely repeated rental model and construct a perfect Bayesian equilibrium in which the seller's expected profit on average *equals* $\underline{v}$ when $\delta \to 1$.

We follow Fudenberg and Tirole to study the infinitely repeated model but without discounting.[1] Our objective is to characterize the set of all Nash equilibrium payoffs for the seller and the buyer. By means of this characterization, we aim to understand why the equilibrium outcome $\underline{v}$ (that is consistent with the one of the Coasian conjecture[2] in the model of the durable good monopoly) prominently dominates the rest.

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[1] The result also covers the discounted case with $\delta \to 1$.

[2] See Fudenberg and Tirole (1992) for details.

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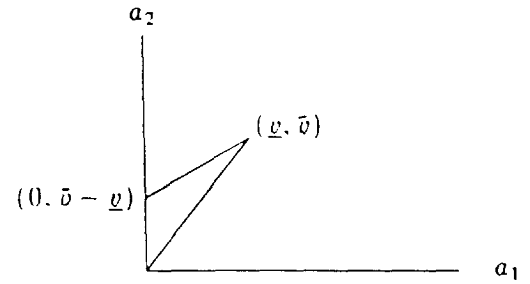

Fig. 1. (a_1, a_2).

This equilibrium does have a specific feature. It is the one that yields the seller the highest profit among all nonrevealing equilibria (assuming that both types have positive probability). For any other equilibria where the seller may obtain a higher profit than $\underline{v}$, the buyer must reveal that he is the higher evaluation type along some equilibrium paths. If the buyer attempts to conceal his private known information in order not to be identified by the seller, the best the seller can do who does not know the buyer's true type is to offer the price $\underline{v}$ under which both types of buyer rent the service in each period.

The set of Nash equilibrium payoffs for the seller and the buyer can be described simply (again, assuming that both types have positive probability). Denote the equilibrium payoffs to the two types of buyer by (a_1, a_2) and the seller β, then β satisfies $0 \leq \beta \leq \bar{v} - \alpha_2$, where (a_1, a_2) is in the triangle illustrated in Fig. 1.

This result is shown in Section 3. First, we would like to introduce the repeated rental model.

2. Repeated rental model

At each discrete time $n = 1, 2, \ldots, \infty$, a seller wishes to rent a service to a buyer by offering a price $r \in T := [0, M]$, where M is a large real number; the buyer then chooses an action in $S = \{0, 1\}$; $s = 1$ means accept and $s = 0$, reject. The buyer possesses two types, $K = \{1, 2\}$, with evaluations $0 \leq \underline{v} = v_1 < v_2 = \bar{v}$ with a common prior $p \in \Delta(K)$. Let $\mathbf{1}(z = i) = i$ for $i = 0, 1$. The buyer with type v_k has payoff $x_k(s, r) = (v_k - r)\mathbf{1}(z = s)$ and the seller has payoff $x_3(s, r) = r\mathbf{1}(z = s)$ if the seller's price offer is r and the buyer chooses s in a period.

The actions chosen by the buyer and the seller in period[3] n depend on the previous history, H_{n-1}, $H_{n-1} = (S \times T)^{n-1}$. A strategy σ for the buyer is an infinite sequence of acceptance $(\sigma_n)_{n=1}^{\infty}$ for each type; $\sigma_n : K \times H_{n-1} \times T \rightarrow \Delta(S)$. A strategy τ for the seller is an infinite sequence of price offers $(\tau_n)_{n=1}^{\infty}$; $\tau_n : H_{n-1} \rightarrow T$. Observe that we restrict the seller to the pure price offers. Later, this restriction is shown equivalently to assume that the seller chooses prices in a finite action set.[4]

[3] A period can be thought of in two stages. The seller proposes a price at stage 1 and the buyer makes a choice to accept or reject at stage 2.

[4] One can even think that the seller can only choose two prices, 0 and M, if she is allowed to carry out a mixture of actions in each period. This type of restriction will not affect the result.

Given a pair of strategies (σ, τ), denote $a_N^k = 1/N\ \Sigma_{n=1}^N x_k(s_n, r_n)$ and $\beta_N = 1/N\ \Sigma_{n=1}^N x_{3,\kappa}(s_n, r_n)$, where κ is a random variable. α_N^k is the average payoff up to stage N for the buyer of type k, and β_N the average payoff up to stage N for the seller.

Let $E_{\sigma,\tau,p}$ be the expectations operator associated with σ, τ and p. Let $E_{\sigma,\tau}^k$ be the expectations operator conditional on the true type k. Define[5] $a^k = \lim_{N\to\infty} E_{\sigma,\tau}^k(a_N^k)$ and $\beta = \lim_{N\to\infty} E_{\sigma,\tau,p}(\beta_N)$. Thus, a pair of strategies (σ, τ) is a *uniform equilibrium* if $\forall\epsilon > 0$, $\exists N_0$ such that for all $N > N_0$, $\forall k \in K$, $\forall\sigma'$, $\forall\tau'$: $E_{\sigma',\tau}^k(a_N^k) \leq a^k + \epsilon$ and $E_{\sigma,\tau',p}(\beta_N) \leq \beta + \epsilon$.

Hart (1985) has shown that the uniform equilibria are equivalent in payoffs to the Nash equilibria.

3. Result

Recall that (α_1, a_2) denotes an equilibrium payoff to the buyer and β to the seller. Let $X = \text{co}\{(0, 0), (\underline{v}, \bar{v}), (0, \bar{v} - \underline{v})\}$, the triangle in Fig. 1.

Theorem. For any $p \in \text{int}\,\Delta(K)$, (a_1, a_2, β) is an equilibrium payoff if and only if (a_1, a_2) is in the set X and β satisfies $0 \leq \beta \leq \bar{v} - a_2$.

Proof. Let $u_1(p)$ $(u_2(p))$ be the value of the zero-sum game $\Sigma_k\ p_k x_k(s, r)$ $(x_3(s, r))$ where the buyer (seller) is the maximizer. Clearly, both $u_1(p) = 0$ and $u_2(p) = 0$ for all $p \in \Delta(K)$. Thus, the individually rational payoffs are in $R_+^2 \times R_+$ by Blackwell's approachability theorem. Next, we specify the set of all nonrevealing feasible payoffs.

Suppose the buyer chooses $s = 0$ for any price $r \in T$, then the vector payoffs to the buyer and the seller will be $0 \in R^3$. Once the buyer chooses $s = 1$ for any price $r \in T$, then $x_k = v_k - r$ for the buyer with type $k \in K$, and $x_3 = r$ for the seller. Denote $\mathcal{F}_1 = \{(x_1, x_2, x_3): x_k = v_k - r$ for $k \in K$ and $x_3 = r$ for $r \in T\}$. Note that $\mathcal{F}_1$ is a convex compact set whenever T is.

Let $\tilde{T}$ be any finite set that consists of 0 and M. Define $\tilde{\mathcal{F}} = \{(x_1, x_2, x_3): x_k = v_k - r$ for $k \in K$ and $x_3 = r$ for $r \in \tilde{T}\}$. Observe that $\tilde{\mathcal{F}}$ is not a convex set. Denote the convex hull of a set A by co(A). Thus, $\text{co}(\tilde{\mathcal{F}}) = \{(x_1, x_2, x_3): x_k = v_k - r$ for $k \in K$ and $x_3 = r$ for $r \in \text{co}(\tilde{T})\}$ because the payoff functions z are linear in r. If one restricts the seller to choosing prices in the finite set $\tilde{T}$, the strategy τ of the seller should be replaced by $(\tau_n)_{n=1}^\infty$; $\tau_n: H_{n-1} \to \Delta(\tilde{T})$. Since $\text{co}(\tilde{T}) = T$, we have $\mathcal{F}_1 = \text{co}(\tilde{\mathcal{F}})$.

Let $\mathcal{F}$ be the set of all nonrevealing feasible payoffs for the buyer and the seller. Thus, $\mathcal{F} = \text{co}(\{0 \cup \mathcal{F}_1\})$ when the seller's action set is T. $\mathcal{F} = \text{co}(\{0 \cup \text{co}(\tilde{\mathcal{F}})\})$ when the seller's action set is $\tilde{T}$. They are the same set because $\mathcal{F}_1 = \text{co}(\tilde{\mathcal{F}})$. This shows that the seller's choice of pure behavior strategies among T is equivalent to the situation where the seller is restricted to choosing behavior strategies among the finite set $\tilde{T}$ in terms of all feasible payoffs (because of this, we can supply Hart, 1985, to our models).

[5] When the expected payoffs do not converge, the limit is defined for a given Banach limit. See Hart (1985) for details.

Next, we structure the set of equilibrium payoffs of our model for all $p \in \Delta(K)$ by using Hart's characterization theorem and theorems 4.7 and 4.8 in Aumann and Hart (1986).

Define $V_1 = \{(a, \beta) \in R^2_+ \times R_+;\ \exists (c, \beta) \in \mathcal{F}$ such that $\underline{v} \geq a_1 \geq c_1$ and $a_2 = c_2\}$; and $V_2 = \{(a, \beta) \in R^2_+ \times R_+;\ \exists (c, \beta) \in \mathcal{F}$ such that $\bar{v} \geq a_2 \geq c_2$ and $a_1 = c_1\}$. (See Fig. 2; $OCAB$ is the boundary of V_1 and OAB the boundary of V_2.)

Let $W = \{(a, \beta, p) \in R^2 \times R \times \Delta(K)$: (i) $p_1 = 0$, $(a, \beta) \in V_1$; (ii) $p_1 = 1$, $(a, \beta) \in V_2$; (iii) $p_1 \in (0, 1)$, $(a, \beta) \in V_1 \cap V_2\}$. Observe that $V_1 \cap V_2 = \mathcal{F} \cap R^2_+ \times R_+$, all individual rational and feasible payoffs, whose project on R^2_+ (a component) has been described by Fig. 1. Let W^* be the largest set that contains W such that no bounded real bi-convex function f defined on W^* and continuous on W can separate any point $z \in W^*$ from W, i.e. $f(z) \leq \sup\{f(w): w \in W\}$; see Forges (1992). To specify W^*: for all $(a, \beta) \in V_1 \cap V_2$, fix the a component and convexify the β component in V_1 and V_2; this leads to a set V^*, say. Because of the special structure of V_1 and V_2, in fact $V^* = V_1 \cup V_2$. We can now show that[6] $W^* = \{(a, \beta, p) \in R^2 \times R \times \Delta(K)$: (i) $p_1 = 0$, $(\alpha, \beta) \in V_1$; (ii) $p_1 = 1$, $(\alpha, \beta) \in V_2$; and (iii) $p_1 \in (0, 1)$, $(a, \beta) \in V^*\}$. To show this, define two functions f and g on $R^2 \times \Delta(K)$ (ignore the β component and $V_i(a)$ is the project of V_i on the a component):

$$f(a, p) = p_2 d(a, V_1(a)) , \tag{1}$$

$$g(a, p) = p_1 d(a, V_2(a)) . \tag{2}$$

Both functions are bi-convex, bounded, continuous, non-negative and vanishing on $V^*(a)$. Therefore, Aumann and Hart's theorems yield the result.

Now by Hart's characterization theorem, we have that

Lemma. For any $p \in \Delta(K)$, (a, β) is an equilibrium payoff if and only if $(a, \beta, p) \in W^*$.

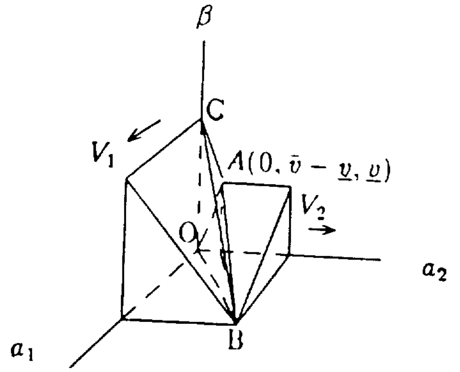

Fig. 2. The set of equilibrium payoffs.

[6] Also see Shalev (1993).

By this lemma and the definition of W^*, we have: (i) $p_1 = 0$, the set of equilibrium payoffs is V_1; (ii) $p_1 = 1$, the set of equilibrium payoffs is V_2; (iii) $p \in \text{int}\, \Delta(K)$, the set of equilibrium payoffs is V^*. By some computation (the equation of the plane ABC in Fig. 2 is $\beta = \bar{v} - a_2$), we can obtain the relation between β and (a_1, a_2) as given in the theorem. This completes the proof. □

The set of all equilibrium payoffs is illustrated in geometry in Fig. 2. Notice that point A is the equilibrium outcome specified by Hart and Tirole (1988) and Fudenberg and Tirole (1992). This outcome yields the seller the highest average profit among all nonrevealing equilibria (the area OAB in Fig. 2 when $p \in \text{int}\, \Delta(K)$ whose project to a components is the set X defined in the theorem).

From the theorem and Fig. 2, if $\beta > \underline{v}$, then $a_2 < \bar{v} - \underline{v}$, which is possible if and only if (a_1, a_2, β) is in V_1. Moreover, such a (a_1, a_2, β) is not in OAB because $\beta > \underline{v}$. This means that the posterior μ_1, say, from some stage onwards must equal 0. This in turn corresponds to the situation that there are some equilibrium paths where the buyer must reveal his higher evaluation type (assuming that the initial common prior p is in the interior of $\Delta(K)$). Hence, our conclusion in the introduction stands up.

4. Concluding remarks

By means of the characterization, we conclude that the infinitely repeated rental model does not have the property of the Coasian conjecture in all but one equilibrium that yields the seller the highest profit among all nonrevealing equilibria. This is the equilibrium specified in Hart and Tirole (1988) and Fudenberg and Tirole (1992).

Hart (1985) and Aumann and Hart (1986) form a powerful and elegant tool for us to handle many similar economical situations.[7] For example, one can also consider that the seller has two types of production costs for a service and the buyer offers the price, and the same argument, with minor changes, applies to this case. Another example is to study how the outside options for the seller and buyer may affect the equilibrium payoffs. These variant rental models are, however, left for future study.

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[7] See Ma (1994) for an application in industrial organization.

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