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by

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The Maximal Variation of Martingales of Probabilities and Repeated Games with Incomplete Information

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Abstract

The variation of a martingale $p_0^k = p_0, \dots, p_k$ of probabilities on a finite (or countable) set X is denoted $V(p_0^k)$ and defined by $V(p_0^k) = E\left(\sum_{t=1}^k \|p_t - p_{t-1}\|_1\right)$. It is shown that $V(p_0^k) \leq \sqrt{2kH(p_0)}$, where $H(p)$ is the entropy function $H(p) = -\sum_x p(x) \log p(x)$ and log stands for the natural logarithm. Therefore, if d is the number of elements of X, then $V(p_0^k) \leq \sqrt{2k \log d}$. It is shown that the order of magnitude of the bound $\sqrt{2k \log d}$ is tight for $d \leq 2^k$: there is $C > 0$ such that for every k and $d \leq 2^k$ there is a martingale $p_0^k = p_0, \dots, p_k$ of probabilities on a set X with d elements, and with variation $V(p_0^k) \geq C\sqrt{2k \log d}$. It follows that the difference between v_k and $\lim_k v_k$, where v_k is the value of the k-stage repeated game with incomplete information on one side with d states, is bounded by $\|G\|\sqrt{2k^{-1} \log d}$ (where $\|G\|$ is the maximal absolute value of a stage payoff), and it is shown that the order of magnitude of this bound is tight.

1 Introduction

Let $p_0^k = p_0, \dots, p_k$ be a martingale of probabilities on a finite (or countable) set X with $d = |X|$ elements. The variation of the martingale p_0^k is denoted

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$V(p_0^k)$ and is defined by $V(p_0^k) = E\left(\sum_{t=1}^k \|p_t - p_{t-1}\|_1\right)$, where $\|p-q\|_1 := \sum_x |p(x) - q(x)|$ for any $p, q \in \Delta(X)$, where $\Delta(X)$ denotes all probabilities on X. We are interested in bounding the variation $V(p_0^k)$ as a function of k and d and likewise as a function of k and $p = p_0$. For $p \in \Delta(X)$ and a positive integer k we denote by $\mathcal{M}_k(X,p)$ the set of all martingales p_0^k with $p_t \in \Delta(X)$ and $p_0 = p$. Set

$$V(k,p) := \sup\{V(p_0^k) : p_0^k \in \mathcal{M}_k(X,p)\} \tag{1}$$

and

$$V(k,d) := \sup\{V(k,p) : p \in \Delta(X) \text{ and } |X| = d\} \tag{2}$$

A trivial inequality is $V(k,p) \leq k$. A classical[1] bound of $V(k,d)$ is

$$V(k,d) \leq \sqrt{k(d-1)}$$

This classical bound improves the trivial bound only for $d \leq k$. Our objective is to derive a meaningful bound that is applicable also for $d > k$ and such that its order of magnitude is the best possible one for large d. We have

Theorem 1

$$V(k,p) \leq \sqrt{2kH(p)}$$

and thus

$$V(k,d) \leq \sqrt{2k \log d}$$

where $H(p) = -\sum_x p(x) \log p(x)$ *and* log *stands for the natural logarithm.*

As $V(k,p) \leq k$, the results of the theorem are of interest for $H(p) \leq k/2$ and for $d \leq e^{k/2}$. For large values of $d \leq e^{k/2}$, the bound $\sqrt{2k \log d}$ is a significant improvement over the classical bound $\sqrt{k(d-1)}$. Moreover, as there are probabilities p over a countable set X with finite entropy, the bound $\sqrt{2kH(p)}$ is applicable independently of the size of the set X.

One may wonder if the order of magnitude of the bounds, $\sqrt{2k \log d}$ and $\sqrt{2kH(p)}$, are the best possible. For $X = \{0,1\}$ and $p(\alpha) = (\alpha, 1-\alpha) \in \Delta(X)$ we have $V(k, p(\alpha)) \leq \sqrt{k\alpha(1-\alpha)}$. As $\alpha(1-\alpha) = o(H(p(\alpha)))$ as $\alpha \to 0+$, the order of magnitude of the bound $\sqrt{kH(p)}$ is not tight. The next

[1]This classical bound is used in the theory of repeated games with incomplete information. See [1].

result demonstrates the tightness of the order of magnitude of the bound[2] $\sqrt{2k \log d}$ for large values of d. We have

Theorem 2 *There is a positive constant $C > 0$ such that for every k and d with $d \leq 2^k$ there is $p_0^k \in \mathcal{M}_k(X, p_0)$ with $|X| = d$ such that*

$$V(p_0^k) \geq C\sqrt{k \log d}$$

Bounds of the variation of martingales of probabilities are useful in the study of repeated games with incomplete information [1][3]. In a two-person zero-sum repeated game with incomplete information on one side (henceforth, RGIIOS) the players play repeatedly the same stage game G. However, the game depends on a state $x \in X$ known only to player 1 (P1) and x is chosen according to a probability $p \in \Delta(X)$ that is commonly known. In the course of the game player 2 (P2) may learn information about x only from past actions of player 1.

Formally, a RGIIOS Γ is defined by a state space X, a probability $p \in \Delta(X)$, finite sets of stage actions, I for P1 and J for P2, and for every $x \in X$ we have a two-person zero-sum $I \times J$ matrix game G^x. We write $\Gamma = \langle X, p, I, J, G \rangle$, where G stands for the list of matrix games $(G^x)_{x \in X}$. The (i, j)-th entry of G^x, denoted $G^x_{i,j}$, is the payoff from P2 to P1 when in state x the players play the action pair (i, j).

The k-stage repeated game, denoted $\Gamma_k(p)$, or Γ_k for short, is played as follows. Nature chooses $x \in X$ according to the probability p. P1 is informed of nature's choice x, P2 is not. At stage $1 \leq t \leq k$, P1 chooses $i_t \in I$ and simultaneously P2 chooses $j_t \in J$ (and these choices are observed by the players following the play in stage t). The choice of i_t may depend

[2] I wish to thank Benjamin Weiss for raising the question of tightness of the factor $\sqrt{\log d}$ in the bound, and demonstrating for each positive ℓ a simple martingale of probabilities p_0^ℓ on a set with 2^ℓ elements and with variation ℓ. Specifically, starting with the uniform probability, in each stage half of the non-zero probabilities (each half equally likely) move to zero, and the other half doubles its probabilities. Therefore for each fixed $\alpha > 0$ there is a positive constant $0 < C(\alpha)$ $(\to_{\alpha \to 0+} 0)$ such that for k and d with $\alpha \leq \frac{\log_2 d}{k} \leq 1$, $V(k, d) \geq [\log_2 d] \geq C(\alpha)\sqrt{k \log d}$.

[3] This book is based on reports by Robert J. Aumann and Michael Maschler that appeared in the sixties in *Report of the U.S. Arms Control and Disarmament Agency*. See "Game theoretic aspects of gradual disarmament" (1966, ST–80, Chapter V, pp. V1–V55), "Repeated games with incomplete information: a survey of recent results" (1967, ST–116, Chapter III, pp. 287–403), and "Repeated games with incomplete information: the zero-sum extensive case" (1968, ST–143, Chapter III, pp. 37–116).

on $x, i_1, j_1, \ldots, i_{t-1}, j_{t-1}$ (which is the information of P1 before the play at stage t) and the choice of j_t may depend on $i_1, j_1, \ldots, i_{t-1}, j_{t-1}$ (which is the information of P2 before the play at stage t).

A pair of strategies σ of P1 and τ of P2 (together with the initial probability p) define a probability distribution $P^p_{\sigma,\tau}$, or $P_{\sigma,\tau}$ for short, on the space of plays $x, i_1, j_1, \ldots, i_k, j_k$, and thus on the stream of payoffs $g_t := G^x_{i_t,j_t}$. The (normalized) payoff of the k-stage repeated game is the average of the payoffs in the k-stages of the game, namely, $\bar{g}_k = \frac{1}{k}\sum_{t=1}^k g_t$. The minmax value of $\Gamma_k(p)$ is $v_k(p) := \max_\sigma \min_\tau E_{\sigma,\tau}\bar{g}_k$, where $E_{\sigma,\tau}$ stands for the expectation w.r.t. the probability $P^p_{\sigma,\tau}$, the max is over all mixed (or behavioral) strategies σ of P1, and the min is over all mixed (or behavioral) strategies τ of P2.

For fixed components $\langle X, I, J, G\rangle$, the minmax value of the matrix game $\sum_x q(x)G^x$ is a function of $q \in \Delta(X)$ and is denoted $u(q)$. The least concave function on $\Delta(X)$ that is $\geq u$ is denoted $\operatorname{cav} u$. Aumann and Maschler [1] prove that $v_k(p) \geq (\operatorname{cav} u)(p)$ and that $v_k(p)$ converges to $(\operatorname{cav} u)(p)$ as $k \to \infty$. Moreover, [1] shows that the bound of the variation of the martingale of probabilities bounds the (nonnegative) difference $v_k(p) - (\operatorname{cav} u)(p)$. Explicitly, if $\|G\| := \max_{x,i,j} |G^x_{i,j}|$, we have

$$v_k(p) - (\operatorname{cav} u)(p) \leq \|G\| V(k,p)/k \tag{3}$$

Equation (3) yields on the one hand a rate of convergence of $v_k(p)$, and on the other hand enables us to approximate the value $v_k(p)$ for a specific k and a specific game. The classical bound of $V(k,p)$ that is used in [1] and in subsequent works is

$$V(k,p) \leq V(k,d) \leq \sqrt{k(d-1)}$$

For $d > k$ this bound is not useful. Our bound (Theorem 1) provides an effective bound when d is subexponential in k, namely, when $\log d = o(k)$, or more generally, when $H(p)/k$ is small. Applying the bound in Theorem 1 to the inequality (3) implies that

$$v_k(p) - (\operatorname{cav} u)(p) \leq \|G\| \sqrt{\frac{2\log d}{k}} \tag{4}$$

One may wonder if the order of magnitude of the bound in (4) is tight. We have

Theorem 3 *There is a positive constant C such that for every k and d with $d \leq 2^k$ there is a repeated game with incomplete information on one side Γ with ($\|G\| > 0$ and) d states such that*

$$v_k(p) - (cav\, u)(p) \geq C\|G\|V(k,d)/k \tag{5}$$

2 Proofs

2.1 Proof of Theorem 1

Let p_0^k be $(\mathcal{H}_t)_t$-adapted, that is, p_t is measurable w.r.t. the σ-algebra $\mathcal{H}_t \subset \mathcal{H}_{t+1}$. W.l.o.g. we can assume[4] that $\mathcal{H}_t$ are finite (namely, algebras). In that case we can assume that: 1) P is a probability on the product $X \times (\times_{t=0}^k A_t)$, where A_t are finite sets (e.g., the atoms of the algebra $\mathcal{H}_t$); 2) $(x, a_0, a_1, \ldots, a_k)$ is a vector of random variables having distribution P; and 3) p_t is the conditional distribution of x given $a_0, \ldots, a_t$. Let P_t be the conditional (joint) distribution of (x, a_t) given $(a_0, \ldots, a_{t-1})$, P_{tX} its marginal on X, and P_{tA_t} its marginal on A_t. By Pinsker's inequality (see, e.g., [2, p. 300]), we have[5]

$$\|P_t - P_{tX} \otimes P_{tA_t}\| \leq \sqrt{2}\sqrt{D(P_t\|P_{tX} \otimes P_{tA_t})}$$

where for two probabilities P and Q on a finite (or countable) set Y, $D(P\|Q) = \sum_{y \in Y} P(y) \log \frac{P(y)}{Q(y)}$ (where log denotes the natural logarithm and $0 \log 0 = 0$).

Let $H_{P_t}(x) := -\sum_x P_t(x) \log P_t(x)$, $H_{P_t}(a_t)$, and $H_{P_t}(x, a_t)$, denote the entropy of the random variables x, a_t, and (x, a_t), where (x, a_t) has distribution P_t, and $H_{P_t}(x \mid a_t) := H_{P_t}(x, a_t) - H_{P_t}(a_t)$. A straight forward computation yields $D(P_t\|P_{tX} \otimes P_{tA_t}) = H_{P_t}(x) - H_{P_t}(x \mid a_t)$. Therefore,

$$\|P_t - P_{tX} \otimes P_{tA_t}\| \leq \sqrt{2}\sqrt{H_{P_t}(x) - H_{P_t}(x \mid a_t)} \tag{6}$$

Note that P_t is a random variable, which is a function of $a_0, \ldots, a_{t-1}$, and therefore, by the properties of conditional entropy, $E_P H_{P_t}(x) = H_P(x \mid$

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[4] If p_t is measurable w.r.t. the σ-algebra $\mathcal{H}_t \subset \mathcal{H}_{t+1}$, one replaces $\mathcal{H}_t$ with an algebra $\mathcal{H}_t^* \subset \mathcal{H}_{t+1}^*$ such that $\|E(p_t \mid \mathcal{H}_t^*) - p_t\| \leq \varepsilon/k$, and replaces p_t with $\hat{p}_t := E(p_t \mid \mathcal{H}_t^*)$ $(= E(p_k \mid \mathcal{H}_t^*))$. Note that $\sum_{t=1}^k \|\hat{p}_t - \hat{p}_{t-1}\| + 2\varepsilon \geq \sum_{t=1}^k \|p_t - p_{t-1}\|$.

[5] The constant 2 appearing in the equality is actually an improvement on the one obtained by Pinsker.

$a_0, \ldots, a_{t-1})$ (where E_P denotes the expectation w.r.t. the probability distribution P) and $E_P H_{P_t}(x \mid a_t) = H_P(x \mid a_0, \ldots, a_{t-1}, a_t)$. Therefore,

$$E_P \left(H_{P_t}(x) - H_{P_t}(x \mid a_t)\right) = H_P(x \mid a_0, \ldots, a_{t-1}) - H_P(x \mid a_0, \ldots, a_{t-1}, a_t)$$

As the square root is a concave function we have, by Jensen's inequality,

$$E_P \|P_t - P_{tX} \otimes P_{tA_t}\| \leq \sqrt{2}\sqrt{H_P(x \mid a_0, \ldots, a_{t-1}) - H_P(x \mid a_0, \ldots, a_{t-1}, a_t)}$$

As $E_P(\|p_t - p_{t-1}\| \mid \mathcal{H}_{t-1})$ equals $\sum_{a \in A_t} P_{tA_t}(a) \sum_x |\frac{P_t(x,a)}{P_{tA_t}(a)} - P_{tX}(x)| = \sum_{a \in A_t} \sum_x |P_t(x,a) - P_{tA_t}(a) P_{tX}(x)| = \|P_t - P_{tX} \otimes P_{tA_t}\|$, we deduce that $E_P\|p_t - p_{t-1}\| = E_P\|P_t - P_{tX} \otimes P_{tA_t}\|$ and therefore by substituting $E_P\|p_t - p_{t-1}\|$ for $E_P\|P_t - P_{tX} \otimes P_{tA_t}\|$ we get

$$E_P\|p_t - p_{t-1}\| \leq \sqrt{2}\sqrt{H_P(x \mid a_0, \ldots, a_{t-1}) - H_P(x \mid a_0, \ldots, a_{t-1}, a_t)}$$

As the square root is a concave function, using Jensen's inequality and the equality and inequality $\sum_{t=1}^{k} \left(H(x \mid a_0, \ldots, a_{t-1}) - H(x \mid a_0, \ldots, a_t)\right) = H(x) - H(x \mid a_0, \ldots, a_k) \leq H(x)$, we have

$$E \sum_{t=1}^{k} \|p_t - p_{t-1}\| \leq \sqrt{2k}\sqrt{H(x)} \leq \sqrt{2k}\sqrt{\log d}$$

This completes the proof of Theorem 1. □

2.2 Proof of Theorem 2

Note that $V(k,d)$ is monotonic increasing in d and k, and that there is a positive constant $C_1 > 0$ such that $V(k,2) \geq C_1\sqrt{k}$.

If $p_0^{k_1}$ and $q_0^{k_2}$ are two martingales with total variation V_1 and V_2 respectively, then $p_0 \otimes q_0, \ldots, p_{k_1} \otimes q_0$ is a martingale with total variation V_1 and $p_{k_1} \otimes q_0, p_{k_1} \otimes q_1, \ldots, p_k \otimes q_{k_2}$ is a martingale with total variation V_2 and therefore $p_0 \otimes q_0, \ldots, p_{k_1} \otimes q_0, p_{k_1} \otimes q_1, \ldots, p_k \otimes q_{k_2}$ is a martingale with total variation $V_1 + V_2$. Therefore,

$$V(k_1, p) + V(k_2, q) \leq V(k_1 + k_2, p \otimes q) \tag{7}$$

from which it follows that

$$V(k_1, d_1) + V(k_2, d_2) \leq V(k_1 + k_2, d_1 d_2) \tag{8}$$

Inequality (8) implies that if k is a multiple of ℓ we have $V(k, 2^\ell) \geq \ell V(k/\ell, 2) \geq \ell C_1\sqrt{k/\ell} = C_1\sqrt{k\ell}$. Note that for every k and $2 \leq d \leq 2^k$ there is $k \geq k_1 > k/2$ that is a multiple of $\ell = [\log_2 d] \geq (\log_2 d)/2$ (where $[x]$ is the largest integer $\leq x$), and therefore $V(k, d) \geq V(k_1, 2^\ell) \geq C_1\sqrt{k_1\ell} \geq C_1/2\sqrt{k \log_2 d}$. This completes the proof of Theorem 2. $\square$

2.3 Proof of Theorem 3

Given two repeated games with incomplete information on one side, $\Gamma^1 = \langle X_1, p_1, I_1, J_1, G^1\rangle$ and $\Gamma^2 = \langle X_2, p_2, I_2, J_2, G^2\rangle$, we define the game $\Gamma = \Gamma_1 \otimes \Gamma_2$ by

$$\Gamma = \langle X = X_1 \times X_2, p = p_1 \otimes p_2, I = I_1 \times I_2 \times \{1, 2\}, J = J_1 \times J_2, G\rangle$$

where for $x = (x_1, x_2)$, $i = (i^1, i^2, b)$, and $j = (j^1, j^2) \in J$,

$$G^x_{i,j} = G^{x_b}_{i^b,j^b}$$

where G^{x_b} stands for the more explicit G^{b,x_b}. Note that $\|G\| = \max(\|G^1\|, \|G^2\|)$.

A possible helpful interpretation of Γ is that nature chooses a pair $x_1 \in X_1$ and $x_2 \in X_2$, equivalently a pair of games G^{x_1} and G^{x_2}, according to the product probability $p_1 \otimes p_2$. P1 is informed of the choice (G^{x_1}, G^{x_2}) of nature, P2 is not. In each stage of the repeated game, both players select strategies for the first and for the second game, and P1 chooses in addition which one of the two games determines the stage payoff.

As a function of $i = (i^1, i^2, b)$, for each fixed $b = 1, 2$, the payoff function $G^x_{i,j}$ does not depend on the coordinate i^c for $c \neq b$. Therefore we can replace the set I (which has $2|I_1||I_2|$ elements) of stage actions of P1 in the repeated game Γ with the disjoint union of I_1 and I_2.

Note that if v^b_k and v_k stand for the (normalized) values of the k-stage repeated games Γ^b and Γ, then

$$v_{k_1+k_2} \geq \frac{k_1 v^1_{k_1} + k_2 v^2_{k_2}}{k_1 + k_2}$$

Indeed, P1 can play $b_t = 1$ in stages $t = 1, \ldots, k_1$ and $b_t = 2$ in stages $t = k_1 + 1, \ldots, k_1 + k_2$, and the first coordinates i^1_t of i_t follow, in stages $t = 1, \ldots, k_1$, an optimal strategy of P1 in $\Gamma^1_{k_1}(p_1)$, and the second coordinates i^2_t of i_t follow, in stages $t = k_1 + 1, \ldots, k_1 + k_2$, an optimal strategy of P1 in $\Gamma^2_{k_2}(p_2)$.

For $\ell > 2$ and a sequence $\Gamma^1 = \langle X_1, p_1, I_1, J_1, G^1 \rangle, \ldots, \Gamma^\ell = \langle X_\ell, p_\ell, I_\ell, J_\ell, G^\ell \rangle$ of RGIIOS, we define by induction on ℓ the game $\Gamma = \otimes_{b=1}^{\ell} \Gamma^b$ by $\Gamma = \left(\otimes_{b=1}^{\ell-1} \Gamma^b\right) \otimes \Gamma^\ell$.

If v_k^b, respectively v_k, denotes the normalized value of the k-stage repeated game $\Gamma_k^b(p_b)$, respectively $\Gamma_k(\otimes_{b=1}^{\ell} p_b)$, and $k = k_1 + \ldots + k_\ell$, then

$$v_k \geq \frac{\sum_{b=1}^{\ell} k_b v_{k_b}^b}{k}$$

Note that a stage action of P1 in Γ is a list of stage actions $i^1, \ldots, i^\ell$ (with $i^b \in I_b$) and a number b (with $1 \leq b \leq \ell$). However, given b, the payoff depends only on the coordinate i^b of the stage actions. Therefore we can replace the stage actions of P1 in Γ with the disjoint union of the action sets I_b, and so with a set of size $\sum_b |I_b|$.

Consider the example of the RGIIOS $\Gamma^z = \langle X = \{0, 1\}, (1/2, 1/2), I, J, G \rangle$, introduced by Zamir [4, Section 3]. The set of states is $X = \{0, 1\}$, and players' action sets are $I = \{0, 1\}$ for P1, and $J = \{0, 1\}$ for P2. The two payoff matrices are G^0 and G^1: $G^0_{0,0} = 3$, $G^0_{0,1} = -1$, $G^1_{0,0} = 2 = -G^1_{0,1}$, and $G^*_{i,j} = -G^*_{1-i,j}$. Let v_k^z denote the normalized value of the k-stage repeated game Γ^z. Zamir [4] shows that $\lim_n v_n^z = 0$ and $v_k^z \geq C_1/\sqrt{k}$, where $C_1 > 0$ is a positive constant.

Consider the RGIIOS $\Gamma = \otimes_{b=1}^{\ell} \Gamma^z$, and let v_k denote the normalized value of the k-stage repeated game Γ. It follows that

$$v_k \geq \max \left\{ \frac{\sum_{b=1}^{\ell} k_b v_{k_b}^z}{k} : k_b \geq 0 \text{ and } \sum_{b=1}^{\ell} k_b = k \right\}$$

and therefore if k is a multiple of ℓ we can take $k_b = k/\ell$ and therefore

$$v_k \geq v_{k/\ell}^z \geq C_1 \sqrt{\ell/k}$$

For an arbitrary k and $d \leq 2^k$, there is $k \geq k_1 > k/2$ that is a multiple of $\ell := [\log_2 d]$ $(\geq (\log_2 d)/2)$. As P1 can play $(1/2, 1/2)$ in the last $k - k_1$ stages, $kv_k \geq k_1 v_{k_1}$, and thus $kv_k \geq C_1 \sqrt{k_1}\sqrt{\ell}$. Therefore, $v_k \geq \frac{C_1}{2\sqrt{k}} \sqrt{\log d}$.

Finally, the existence of an optimal strategy of P2 in the infinitely repeated game Γ^z (or a direct computation of the function $u(p)$, the minmax value of the game $\sum_x p(x) G^x$, for the game Γ) yields $\lim_n v_n = 0$. Note that the stage payoffs of the RGIIOS Γ are bounded by 3 (independent of the number of factors ℓ). Altogether, we have constructed for each k and $d \le 2^k$ a repeated game $\Gamma = \langle X, p, I, J, G \rangle$ with $|X| \le d$, equivalently $|X| = d$ ($|I| \le 2 \log d$) and $\|G\| = 3$, and

$$v_k - \lim_{n\to\infty} v_n \ge C_1/2\sqrt{\log d}/\sqrt{k}$$

This completes the proof of Theorem 3. □

3 Remarks

1. The asymptotic behavior of $V(k,p)$ deserves further study. For example, it is of interest to find a necessary and sufficient condition for a distribution p on a countable set X for $\sup_k \frac{1}{\sqrt{k}} V(k,p) < \infty$. We remark here on the sufficient conditions derived from the classical method and our method of bounding the variation of martingales of probabilities.

The classical bound of the variation of a martingale p_0^k is obtained by bounding, for each fixed $x \in X$, the expectation variation $E\|y(x)\|_1$, where $y(x) \in \mathbb{R}^k$ is the vector of martingale differences $(p_1(x) - p_0(x), \ldots, p_k(x) - p_{k-1}(x))$ (thus $\|y(x)\|_1 = \sum_{t=1}^k |p_t(x) - p_{t-1}(x)|$), and summing over all $x \in X$. Assuming w.l.o.g. that p_0 is a constant $p \in \Delta(X)$, we have (by the Cauchy-Schwartz inequality) $\|y(x)\|_1 \le \sqrt{k}\|y(x)\|_2$, and therefore, by Jensen's inequality, $E\|y(x)\|_1 \le \sqrt{k}\sqrt{E\|y(x)\|_2^2}$, which by the martingale property is $\le \sqrt{k}\sqrt{E((p_k(x))^2 - (p_0(x))^2)} \le \sqrt{k}\sqrt{E((p_k(x)) - (p_0(x))^2)} = \sqrt{k}\sqrt{p_0(x) - (p_0(x))^2}$. Therefore, if $p \in \Delta(X)$ and X is countable, the classical method yields that $\sup_k \frac{1}{\sqrt{k}} V(k,p) < \infty$ whenever $\sum_x \sqrt{p(x)} < \infty$. As $-q \log q = o(\sqrt{q})$ as $q \to 0+$, the condition $\sum_x \sqrt{p(x)} < \infty$ implies that $H(p) = -\sum_x p(x) \log p(x) < \infty$. Obviously, there are probabilities p over a countable set X such that $H(p) < \infty$ but $\sum_x \sqrt{p(x)} = \infty$. Therefore our bound provides a strictly sharper sufficient condition, $H(p) < \infty$, for $\sup_k \frac{1}{\sqrt{k}} V(k,p) < \infty$, compared to the one derived by using the classical method.

2. The asymptotic behavior of $V(k,d)$ deserves further study. [3] proves that $V(k,2)/\sqrt{k}$ converges as $k \to \infty$ to $\sqrt{\frac{2}{\pi}}$. It is of interest to find a corresponding limit theorem for $V(k,d)/\sqrt{k \log d}$ as $2^k \ge d \to \infty$. The above-mentioned result of [3] together with our construction in the proof of Theorem

2 yields that the lim inf of $V(k,d)/\sqrt{k \log_2 d}$ is $\geq \sqrt{\frac{2}{\pi}}$ as $\frac{\log d}{k} + 1/d \to 0$ (namely, as $\log d = o(k)$ and $d \to \infty$).

3. The proof of Theorem 3 constructs for each $d \leq 2^k$ a RGIIOS $\Gamma = \langle X, p, I, J, G \rangle$ with $|X| = d$ and $v_k \geq \lim_n v_n + C\sqrt{k^{-1} \log d}$ and, in addition, $|I| = O(\log d)$ and $|J| = O(d)$. We have not tried to minimize the order of magnitude of the number of elements of I and J.

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