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TOWARDS A CHARACTERIZATION OF RATIONAL EXPECTATIONS

by

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Abstract

R. J. Aumann and J. H. Drèze (2008) define a *rational expectation* of a player i in a game G as the expected payoff of some type of i in some belief system for G in which common knowledge of rationality and common priors obtain. Our goal is to characterize the set of rational expectations in terms of the game's payoff matrix. We provide such a characterization for a specific class of strategic games, called *semi-elementary*, which includes Myerson's "elementary" games.

Introduction

In a recent paper, Aumann and Drèze [3] (henceforth A&D) define a *rational expectation* of a player in a game G as her expected payoff in a situation in which G is played, where common knowledge of rationality (CKR) and common priors (CP) obtain. More precisely, a *game situation* based on G is

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a belief hierarchy of a specific player i in G, which comprises her belief about what the others play, about what they believe she and the others play, about what they believe about that, and so on. A *rational expectation* of i is the expected payoff in a game situation obeying (CKR) and (CP).

A&D characterize the rational expectations (henceforth RE's) of i in G in terms of the correlated equilibrium payoffs in the doubled game $2G$, in which each strategy of i in G is listed twice. They do not, however, characterize the RE's explicitly.

This paper studies the structure of the RE set, and provides an explicit characterization when G is "elementary" in the sense of Myerson [4].

We prove two main results.

Theorem 1. The set of RE's of a player in a game is the union of finitely many closed intervals.

Recall that Myerson [4] calls a game *elementary* if it has a correlated equilibrium in which the defining inequalities are satisfied strictly.[1] Let us call G *semi-elementary* for i if it has a correlated equilibrium in which every profile of strategies appears with positive probability, and the inequalities pertaining to i are satisfied strictly.

Theorem 2. In a game G that is semi-elementary for i, the RE's of i constitute a closed interval whose lower endpoint is the maximin payoff for i and whose upper endpoint is the highest possible payoff in G for i.

Our last result concerns the connectedness of the RE set. A&D adduce an example of a game G with a player who has precisely two RE's. But the example is in a sense degenerate, in that G has no correlated equilibrium with full support. Here we construct an example of a disconnected RE set in

[1]Except, of course, for the condition that the probabilities sum to 1.

a game that has a correlated equilibrium with full support. In particular, we show that the convexity in Theorem 2 cannot be generalized to full-support games.

Definitions and Notations

Definitions

Let G be a strategic n-person game, S_i the strategy set of Player i, and U_i the payoff functions from $S_1 \times \ldots \times S_n$ to $\mathbb{R}$. A *belief system* B for G consists of:

1. For each player i, a finite set T_i, whose members t_i are called *types* of i.

2. For each type t_i of each player i,

    a. a strategy of i in G, denoted $s_i(t_i)$, and

    b. a probability distribution on $(n-1)$-tuples of types of the other players, called t_i's *theory*.

A *common prior* (CP) is a probability distribution π on $T_1 \times \ldots \times T_n$ that assigns positive probability to each type of each player, such that the theory of each type of each player is the conditional of π given that the player is of that type. A type of a player is *rational* if the strategy it prescribes maximizes her expected payoff given her theory. Rationality is *commonly known* (CKR) if this is so for all types of all players.

We analyze G from the viewpoint of Player 1. A *rational expectation* in G is the expected payoff of some type of Player 1 in some belief system for G in which CKR and CP obtain. We wish to characterize the set of rational expectations.

The *doubled* game $2G$ is the n-person game in which Player 1's strategy set is $S_1 \times \{1, 2\}$. That is, there are two copies of each of Player 1's strategies, while the payoff functions are identical to the original game functions and do not depend on which copy is used.

Given a strategic game G, we say that it is:

Definition 1. *Elementary*, if it has a correlated equilibrium that assigns positive probability to each strategy of each player, and all the inequalities associated with this equilibrium are strict.

Definition 2. *Full*, if it has a correlated equilibrium that assigns positive probability to each profile of strategies.

Definition 3. *Semi-elementary*, if it is a *full* game and it has a correlated equilibrium s.t. all the inequalities related only to Player 1 are strict. That is, if there exists a correlated equilibrium μ s.t. $\mu(s) > 0$ for every $s \in \prod_{i \in N} S_i$, and

$$\sum_{s_1 \in S_{-1}} \mu(s_1, s_{-1})(U_1(s_1, s_{-1}) - U_1(s_1', s_{-1})) > 0 \quad \text{for every } s_1, s_1' \in S_1, s_1 \neq s_1'.$$

For simplicity, we adopt the following notations, to be used throughout the paper.

Notations

Let G be a strategic game and μ a correlated equilibrium of G.

1. For every $s_1 \in S_1$ such that $\mu(s_1) := \sum_{s_{-1} \in S_{-1}} \mu(s_1, s_{-1}) > 0$, let $(\mu \mid s_1)$ be the conditional probability distribution vector over S_{-1}, given μ. That is, $(\mu \mid s_1) := \sum_{s_{-1} \in S_{-1}} [\mu(s_1, s_{-1})/\mu(s_1)] e_{s_{-1}}$, where $e_{s_{-1}}$ is the appropriate unit vector in $\mathbb{R}^{|S_{-1}|}$.

2. Let v be a probability distribution vector over S_{-1}. For every $s_1 \in S_1$ we define $H_{s_1}(v)$ to be the payoff on s_1, given v. That is, $H_{s_1}(v) := \sum_{s_{-1} \in S_{-1}} v_{s_{-1}} U_1(s_1, s_{-1})$.

3. For every strategy $s_1 \in S_1$ we define a set $C(s_1) \subseteq \mathbb{R}$ as follows: $\alpha \in C(s_1)$ if α is a conditional correlated equilibrium payoff for the strategy s_1; i.e., $\alpha \in C(s_1)$ iff there exists a correlated equilibrium μ of G s.t. $H_{s_1}(\mu \mid s_1) = \alpha$.

4. We denote the set of conditional correlated equilibrium payoffs of Player 1 by $C(G)$. Note that $C(G) = \bigcup_{s_1 \in S_1} C(s_1)$.

Before proceeding to our main results, we start with a preliminary observation:

Observation. Every elementary game is also a semi-elementary game.

Proof. Let G be an elementary game. By definition, G has a correlated equilibrium μ that assigns positive probability to each strategy of each player, and in which the associated inequalities are strict. Let S be the set of strategy profiles in G and let θ be a correlated strategy that assigns equal probabilities to all strategy profiles. Then, for sufficiently small $\varepsilon > 0$, $\lambda := (1-\varepsilon)\mu + \varepsilon\theta$ assigns positive probabilities to each strategy profile, and the associated inequalities are still strict. So G is a semi-elementary game.

□

Therefore, semi-elementary games are a larger class of games than elementary games, and all the results related to semi-elementary games are also valid for elementary games.

The Main Results

Theorem 1. For every game G, the set of rational expectations is closed.

Theorem 2. For every semi-elementary game G, the set of rational expectations is the closed interval

$$[\max_{v\in\Delta(S_1)} \min_{s_{-1}\in S_{-1}} U_1(s_1, v), \max_{s\in S_1\times\ldots\times S_n} U_1(s)].$$

We will also show that Theorem 2 does not extend to all full games.

To prove our results, we rely on the main Theorem in [3], which states the following:

Theorem (A&D)**.** The rational expectations in a game G are precisely the conditional payoffs to correlated equilibria in the doubled game $2G$.

That is, α is a rational expectation in G if and only if there exists a correlated equilibrium of the game $2G$ s.t. α is a conditional payoff for some strategy of Player 1. Using our notations we can write A&D's result as follows: α is a rational expectation if and only if $\alpha \in C(2G)$.

Proofs

Lemma 1. Let G be a game and let $s_1 \in S_1$ be a strategy of Player 1. The set $C(s_1)$ is a set of feasible solution values for a particular linear program, and hence a closed interval.[2]

[2] Every set of feasible solution values for a linear program problem is closed.

Proof. Look at the following linear program for $\psi \in \mathbb{R}^{|S_1|\times\ldots\times|S_n|}$:

$$max \sum_{s_{-1}\in S_{-1}} \psi(s_1, s_{-1})U_1(s_1, s_{-1}) \tag{1}$$

$$s.t.$$

$$\sum_{s_{-i}\in S_{-i}} \psi(s_i, s_{-i})(U_i(s_i, s_{-i}) - U_i(t_i, s_{-i}) \geq 0 \ \forall \ i \in N \ \forall s_i, t_i \in S_i \tag{2}$$

$$\sum_{s_{-1}\in S_{-1}} \psi(s_1, s_{-1}) = 1 \tag{3}$$

$$\psi(s) \geq 0 \ \forall s \in S \tag{4}$$

The linear operator in 1 defines a set of feasible solution values for the above linear program. It remains to show that every such value defines a conditional correlated equilibrium payoff, given s_1. Let ψ be a feasible solution corresponding to the value α, i.e.,

$$\sum_{s_{-1}\in S_{-1}} \psi(s_1, s_{-1})U_1(s_1, s_{-1}) = \alpha.$$

Let $\beta = \sum_{s\in S} \psi(s)$; note that from (3) and (4), $\beta \geq 1$. Define the correlated strategy μ to be

$$\mu(s) = \frac{\psi(s)}{\beta}.$$

That is, μ is a normalization of ψ. Yet from (2) it follows that μ is also a correlated equilibrium and, by the definition of ψ, $\mu(s_1) = \frac{1}{\beta} > 0$, so $H_{s_1}(\mu|s_1) = \alpha$. In particular, $C(s_1)$– the set of conditional correlated equilibrium payoffs, given s_1– is a set of feasible solution values for a linear program, and hence closed interval.

To compute the closed interval $C(s_1)$, it is sufficient to solve two linear programs. The right-hand interval end point is computed using the linear program defined above. The left-hand interval end point is obtained by minimizing (instead of maximizing) in (1). □

Corollaries from Lemma 1

Corollary 1. The set $C(G)$ of conditional correlated equilibrium payoffs is closed.

Proof. $C(G) = \bigcup_{s_1 \in S_1} C(s_1)$ is a finite union of closed sets, and hence is closed. □

Proof of Theorem 1. From A&D's corresponding Theorem, α is an RE of Player 1 iff $\alpha \in C(2G)$. In particular, we get the RE set as a finite union of closed intervals.

Moreover, using algorithms of linear programming (e.g., the simplex algorithm) we can compute the RE set in the same way described at the end of the Proof of Lemma 1.

□

Theorem 2

We will divide the Proof of Theorem 2 into two parts. In Part a we will prove the convexity of the RE set for a semi-elementary game. In Part b we will show that the RE set is the interval

$$[\max_{v \in \Delta(S_1)} \min_{s_{-1} \in S_{-1}} U_1(s_1, v), \max_{s \in S_1 \times \ldots \times S_n} U_1(s)].$$

Definition 4. For a strategic game G, the strategy $s_1 \in S_1$ is a best reply for $v \in \Delta(S_{-1})$ if for every $s'_1 \in S_1 : H_{s_1}(v) \geq H_{s'_1}(v)$.

Let G be a semi-elementary game. We will show that the RE set is a convex (closed) set.

We will first prove the following proposition:

Proposition 1. For every semi-elementary game G and best-reply distribution vector $v \in \Delta(S_{-1})$ for some strategy s_1, there exists a correlated equilibrium μ of $2G$ s.t. $(\mu \mid s_1^*) = v$, where s_1^* , s_1^{**} are the two copies of the strategy s_1 in $2G$.

Proof. Let π be the correlated equilibrium obtained when G is semi-elementary. We will define a correlated equilibrium μ on $2G$ s.t. $(\mu \mid s_1^*) = v$.

Let δ be s.t. $0 < \delta < \min_{s_{-1} \in S_{-1}} \{\pi(s_1, s_{-1}) \mid \pi(s_1, s_{-1}) > 0\}$. First we show that there exists a small enough $0 < \epsilon \leq \delta$ s.t. for every $s_1' \neq s_1$:

$$(*) \sum_{s_{-1} \in S_{-1}} (\pi(s_1, s_{-1}) - \epsilon v_{s_{-1}}) \cdot U_1(s_1, s_{-1}) \geq \sum_{s_{-1} \in S_{-1}} (\pi(s_1', s_{-1}) - \epsilon v_{s_{-1}}) \cdot U_1(s_1', s_{-1})$$

Now both sides of $(*)$ are continuous functions of ϵ. For $\epsilon = 0$ the inequality in $(*)$ is strict and both sides of $(*)$ are monotonic in ϵ. As a result, for every $s_1^{'} \neq s_1$ we can choose $0 < \epsilon\,(s_1^{'})$ s.t. the inequality in $(*)$ holds for every $0 \leq \varepsilon \leq \epsilon(s_1')$. If we define

$$\epsilon = \min\{\epsilon(s_1^{'}) \mid s_1' \in S_1, s_1' \neq s_1 \}$$

we will get the desired ϵ.

We define μ as follows:

For every $s_1' \neq s_1$ and for every $s_{-1} \in S_{-1}$

$$\begin{aligned} \mu(s_1'^{*}, s_{-1}) &= \pi(s_1', s_{-1}) \\ \mu(s_1'^{**}, s_{-1}) &= 0 \end{aligned}$$

and for s_1

$$\begin{aligned} \mu(s_1^{**}, s_{-1}) &= \pi(s_1, s_{-1}) - \epsilon v_{s_{-1}} \\ \mu(s_1^{*}, s_{-1}) &= \epsilon v_{s_{-1}}. \end{aligned}$$

□

Lemma 2. The above μ is a correlated equilibrium of $2G$.

Proof. For any player other than Player 1 all the required inequalities hold, because π is a correlated equilibrium. Now we have the same argument for $s_1' \neq s_1$, for the relevant $s_1'^*$. It remains to show that the inequalities hold for the two copies of s_1.

From the definition of ϵ and the fact that the inequalities in $(*)$ hold, it follows that for every $s_1' \neq s_1$ we have

$$H_{s_1}(\mu \mid s_1^{**}) \geq H_{s_1'}(\mu \mid s_1^{**}).$$

From the fact that v is a best reply to s_1 we may deduce directly from the definition that

$$H_{s_1^*}(\mu \mid s_1^*) = H_{s_1^*}(v) \geq H_{s_1'}(v) = H_{s_1'}(\mu \mid s_1^*).$$

We get μ as a correlated equilibrium of $2G$, and so we have proved Lemma 2. But $(\mu \mid s_1^*) = v$, and so we have also proved Proposition 1.

□

Part a of Theorem 2:

Proof. Let G be a semi-elementary game and let α, β be an RE of G, $\alpha \leq \beta$. We aim to show that the interval $[\alpha, \beta]$ is included in the RE set of G.

From the fact that α, β are RE's we got μ_α and μ_β correlated equilibria of $2G$ and $s_1, s_1' \in S_1$ s.t.

$$\begin{aligned} H_{s_1}(\mu_\alpha \mid s_1) &= \alpha \text{ and} \\ H_{s_1'}(\mu_\beta \mid s_1') &= \beta. \end{aligned}$$

We define functions $v(t):[0,1]\to\mathbf{\Delta}^{|S_{-1}|-1}$ and $f(t):[0,1]\to\mathbf{R}$ as follows:

$$\begin{aligned} v(t) &= t(\mu_\beta \mid s_1^{'}) + (1-t)(\mu_\alpha \mid s_1) \\ f(t) &= \max_{s_1\in S_1} H_{s_1}(v(t)). \end{aligned}$$

$H_{s_1}(v(t))$ is a continuous function for every $s_1 \in S_1$. Therefore $f(t)$ is a continuous function as a maximum over a finite set of continuous functions. Now for every $0 \le t \le 1$, $v(t)$ is a best-reply distribution vector for some $s_1 \in S_1$.

From Proposition 1 we got a correlated equilibrium $\lambda(t)$ of $2G$ s.t.

$$(\lambda(t) \mid s_1^*) = v(t).$$

So $f(t)$ is an RE for every $0 \le t \le 1$, that is, $f(t) \in C(2G)$. But $f(0) = \alpha$ and $f(1) = \beta$, and so we can deduce from the continuity of $f(t)$ that $[\alpha,\beta] \subseteq C(2G)$. □

Part b of Theorem 2:

Let $\widetilde{G}$ be the two-person zero-sum game derived from G where the strategy set of the row player is S_1 and the strategy set of the column player is S_{-1}. The payoff function is $g(s_1,s_{-1}) = U_1(s_1,s_{-1})$. Let

$$a = \max\min \widetilde{G},\ \ b = \max\{U_1(s) : s \in S_1 \times \ldots \times S_N\}$$

Proof of Part b. A&D showed that for every game G, $C(2G)$ is bounded from below by a, and in an elementary game $b \in C(2G)$. Using Proposition 1 it will be easy to generalize this to semi-elementary games.

Lemma 3. For every semi-elementary game G, $b \in R(G)$.

Proof. Let G be a semi-elementary game and let $s_1 \in S_1$, $s_{-1} \in S_{-1}$ s.t. $b = U_1(s_1,s_{-1})$. Now let $v \in \Delta(S_{-1})$ be defined by

$$v_{s_{-1}} = 1 \text{ and } v_{s_{-1}^{'}} = 0, \text{for } s_{-1}^{'} \neq s_{-1}.$$

By the definition of v we get

$$b = H_{s_1}(v) \geq H_{s_1^{'}}(v) \text{ for every } s_1^{'} \in S_1.$$

Therefore v is a best reply to s_1. Therefore, by Proposition 1, there exists a correlated equilibrium π of $2G$ s.t. $(\pi \mid s_1^*) = v$, and so we get b as a rational expectation of Player 1, $b \in C(2G)$.

□

Lemma 4. For a semi-elementary game G, $a \in C(2G)$.

Proof. a is the value of the game $\widetilde{G}$ defined above. Let $\overline{y}^* = \{y^*_{s_{-1}}\}_{s_{-1} \in S_{-1}}$ be an optimal strategy for the column player that assures her an expected payoff smaller than the value for every strategy of the row player. Let $\overline{x}^* = \{x^*_{s_1}\}_{s_1 \in S_1}$ be an optimal strategy for the row player that assures him an expected payoff greater than the value for every strategy of the column player. So we have

$$(\#) \sum_{s_{-1} \in S_{-1}} \overline{y}^*_{s_{-1}} g(s_1, s_{-1}) = H_{s_1}(\overline{y}) \leq a, \text{ for every } s_1 \in S_1.$$

On the other hand, for $s_1 \in S_1$ s.t. $x^*_{s_1} > 0$ we of course have equality in (#), and so we get $\overline{y}^*$ as a best reply vector for that s_1. According to Proposition 1, we have a correlated equilibrium π of $2G$ s.t. $(\pi \mid s_1^*) = \overline{y}^*$. So we get a as a rational expectation, $a \in C(2G)$. □

We have proved that $a, b \in C(2G)$, and they are also the boundaries of $C(2G)$ from below and above respectively. From Part a of Theorem 2 (convexity of $C(2G)$ for semi-elementary games) we deduce that $C(2G) = [a, b]$, and thus we have proved Part b. □

Examples

In this section, using examples from [3], we demonstrate the use of Theorem 2.

	L	R
T	$6,6$	$2,7$
B	$7,2$	$0,0$

Figure 1

The game depicted in Figure 1 is a two-person elementary game (chicken). To see this, take a correlated equilibrium that assigns an equal probability of $\frac{1}{3}$ to the profile $(T, L), (T, R)$, and (B, L). Using Theorem 2, we see that the right-hand point of the RE interval is the maximal payoff for Player 1, that is, 7, and the left-hand point is the *maxmin* payoff, that is, 2. The RE's set is therefore $[2, 7]$.

	L	C	R
T	$0,0$	$4,5$	$5,4$
M	$5,4$	$0,0$	$4,5$
B	$4,5$	$5,4$	$0,0$

Figure 2

The game depicted in Figure 2, due to Lloyd S. Shapley (see [5]), is a two-person elementary game.[3] Again using Theorem 2, we see that the right-

[3]To see this, take a correlated equilibrium that assigns an equal probability of $\frac{1}{6}$ to

hand point of the interval is the maximal payoff for Player 1, that is, 5, and the left-hand point is the *maxmin* payoff for player 1, that is, 3. The RE set is therefore $[3, 5]$.

Failure of Theorem 2 without Semi-elementarity

The question naturally arises whether we can go one step further and abandon the demand for semi-elementarity, i.e., whether the conclusion of Theorem 2 holds for full games that are not semi-elementary games. As the following example demonstrate, the answer to this question is, unfortunately, no.

$G =$

	L	R
T	$1, -1$	$-1, 1$
M	$-1, 1$	$1, -1$
B	$-4, 0$	$2, 0$

Figure 3

$G' =$

	L	R
T	$1, -1$	$-1, 1$
B	$1, -1$	$-1, 1$

Figure 4

The game G' is a two-person zero-sum game with a unique correlated equilibrium, which is also a Nash equilibrium, that assigns an equal probability of $\frac{1}{4}$ to every profile of strategies in G'. Every correlated equilibrium of the game G that assigns positive probability to one of the first two strategies has to satisfy the same constraint in the game G'. As a result, every

every profile of strategies with a non-zero payoff.

correlated equilibrium of the game G assigns equal probability to the profiles $(T, L), (T, R), (M, L)$, and (M, R).

Therefore we can deduce that the set of correlated equilibria of G is the following:

	L	R
T	$\frac{(1-\alpha)}{4}$	$\frac{(1-\alpha)}{4}$
M	$\frac{(1-\alpha)}{4}$	$\frac{(1-\alpha)}{4}$
B	$\alpha\beta$	$\alpha(1-\beta)$

$$\text{for} \quad 0 \leq \alpha \leq 1, 0 \leq \beta \leq \tfrac{1}{4}.$$

Proposition 2. The set of conditional correlated equilibrium payoffs for Player 1 in the game $2G$ is the same as in the game G.

Proof. It is clear that the set of payoffs for the two copies of the third strategy is the same in G and in $2G$. This follows from the fact that for every distribution vector $v \in \Delta(S_{-1})$ where the third strategy is a best reply to it, there exists a correlated equilibrium π of G s.t. $(\pi \mid s_1^3) = v$.

Now, let $\mu = \{\mu_{ij}\}_{1 \leq i \leq 6, 1 \leq j \leq 2}$ be a correlated equilibrium of $2G$ that assigns a positive probability to one of the copies of the first two strategies. Let $a = \sum_{i,j=1}^{2} \mu_{ij}$, $a > 0$. We can define a correlated equilibrium λ of $2G'$ using μ as follows:

$$\lambda_{ij} = \frac{1}{a}\mu_{ij} \text{ for } 1 \leq i \leq 4,\ 1 \leq j \leq 2.$$

The fact that λ is a correlated equilibrium follows from:

a. μ is a correlated equilibrium of $2G$.

b. Given that Player 1 plays the third strategy, the payoff for Player 2 is 0.

□

Now every two-person zero-sum game has a unique RE (see Theorem A in [3]), which is also the value; in this case, it is 0. Thus

$$C(G) = C(2G) = \{0\} \cup [\frac{1}{2}, 2],$$

and convexity fails.

Myerson [4] describes a way to reduce every strategic n-person game to an elementary game. This process is obtained by looking at the stationary distribution of a Markov chain deriving from the dual problem to the one that defines the correlated equilibria of the game. If we apply the reduction process to the game depicted in Figure 3, the only RE we get for Player 1 in the reduced elementary game is 0. Thus, the reduction process can eliminate rational expectations from the original game.

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