

An Equivalence Principle for Perfectly Competitive Economies*

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Four axioms are placed on a correspondence from smooth, non-atomic economies to their allocations. We show that the axioms *categorically determine* the (coincident) competitive-core-value correspondence. Thus any solution is equivalent to the above three if, and only if, it satisfies the axioms. In this sense our result is tantamount to an “equivalence principle.” At the same time, our result implies that the three solutions themselves are determined by the axioms and so serves as an axiomatic characterization of the well-known competitive (or core, or value) correspondence. *Journal of Economic Literature* Classification Numbers: C71, D41, D51, D63. © 1997 Academic Press

INTRODUCTION

It has been much remarked that different solutions become equivalent in the setting of private goods economies in which there is “perfect competition,” i.e., no single individual can affect the overall outcome. The conjecture that core and competitive allocations coincide was made as far back as 1881 by Edgeworth [15]. This insight has been confirmed in a series of papers [1, 2, 9–11, 13, 14, 16, 20–22, 28] over the last four decades. Another line of inquiry originated with the recent introduction of a value

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for games by Shapley [25, 26]. It was found that—with a smoothness assumption on preferences—value allocations also coincided with the above two solutions [4, 8, 14, 27].¹

The equivalence phenomenon is striking in view of the fact that these solutions are posited on entirely different grounds. Let us briefly recount them. We will confine ourselves throughout to pure exchange economies, and will work with a set of these called $\tilde{\mathcal{M}}$. To represent perfect competition, the underlying space of traders in $\tilde{\mathcal{M}}$ is a non-atomic continuum (T, \mathcal{C}, μ) . (See Aumann's model in [2].) T is the set of traders, \mathcal{C} the sigma-algebra of coalitions and μ a finite, non-atomic, non-negative population measure on (T, \mathcal{C}) . An *ordinal economy* consists of a pair of functions (a, \succsim) which specify the characteristics of the traders. If n is the number of commodities in the economy, then the function a maps² T to \mathbb{R}_+^n and \succsim maps T to preference relations on \mathbb{R}_+^n . Denote $a(t), \succsim(t)$ by a^t, \succsim^t . The j th component a_j^t of a^t is the initial endowment of commodity j held by trader t ; and $\alpha \succsim^t \beta$ means that t likes the "consumption bundle" α no less than β . The domain $\tilde{\mathcal{M}}$ is made up of pairs (a, \succsim) which are subject to certain constraints. This will be made precise in Section 2. But, roughly, (T, \mathcal{C}, μ) is held fixed in $\tilde{\mathcal{M}}$ while the characteristics a, \succsim are allowed to vary. In particular there can be an arbitrary finite number of commodities. The initial endowment a is assumed to be bounded. The essential constraint on \succsim is that each \succsim^t be smooth, convex and monotonic.

Fix an ordinal economy $m = (a, \succsim)$ in $\tilde{\mathcal{M}}$. For any coalition S in \mathcal{C} , an S -allocation is an integrable map $x: S \rightarrow \mathbb{R}_+^n$ such that $\int_S x \, d\mu = \int_S a \, d\mu$. It describes a redistribution of commodities among the members of S . (T -allocations will often be called just "allocations.") Also denote by \succ^t (and, \sim^t) the strict preference (and, indifference) induced by \succsim^t , i.e., $\alpha \succ^t \beta$ if $\alpha \succsim^t \beta$ and it is not the case that $\beta \succsim^t \alpha$ (and, $\alpha \sim^t \beta$ if $\alpha \succsim^t \beta$ and $\beta \succsim^t \alpha$).

Competitive allocations in m are based on the idea of a state of "equilibrium" that is arrived at via the "law of supply and demand." There are prices $p \in \mathbb{R}_+^n$ for all commodities which cannot be influenced by the actions of any single trader. (The presence of a continuum of traders makes this hypothesis viable.) Any trader t can freely buy and sell each commodity j at the fixed price p_j , provided only that he balances his "budget," i.e., t can obtain any bundle α for which $p \cdot \alpha \leq p \cdot a^t$ (where \cdot denotes dot product). The market supply and demand is then got by aggregating the individualistically optimized choices over all $t \in T$. If they happen to be exactly matched, i.e. all commodity markets clear, we are at an equilibrium,

¹ The relation between value and competitive allocations has been further explored in [11, 12, 23].

² As is usual in the continuum setting, we always identify any two maps that coincide on their domain, except for a subset of μ -measure zero.

and the resulting allocation is called competitive. In symbols, $x: T \rightarrow \mathbb{R}_+^n$ is *competitive* with prices $p \in \mathbb{R}_+^n$ if x^t is \succsim^t -optimal in $\{\alpha \in \mathbb{R}_+^n : p \cdot \alpha \leq p \cdot a^t\}$ for almost all t , and $\int_T x^t d\mu = \int_T a^t d\mu$. On the other hand, the models behind the core and the value are considerably different. They both depend on the full game underlying the economy. An allocation is defined to belong to the core if no coalition can, on its own, improve upon it. More precisely, take an allocation y . If there does not exist any non-null $S \in \mathcal{C}$ and an S -allocation z such that $z^t \succ^t y^t$ for all t in S , then y is in the *core* of m . Thus the core represents a region of coalitional stability in the space of allocations. In contrast, a value allocation may be viewed as an arbitrated outcome of the game. For a thoroughgoing discussion we refer the reader to [4]. Here let us simply recall how it is determined. First assign utilities³ $u: T \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ to represent the preferences \succsim , i.e., denoting $u(t, x) \equiv u^t(x)$, $u^t(x) > u^t(y)$ if and only if $x \succ^t y$. Then u generates the market game $w_{a,u}: \mathcal{C} \rightarrow \mathbb{R}$ by the rule

$$w_{(a,u)}(S) = \sup \left\{ \int_S u^t(x^t) d\mu : x \text{ is an } S\text{-allocation in } (a, u) \right\}.$$

The number $w_{(a,u)}(S)$ can be interpreted (see [4, 8]) as the maximum welfare (under the representation u of \succsim) that S can guarantee to itself, no matter what the other players do. Let $Aw_{(a,u)}: \mathcal{C} \rightarrow \mathbb{R}$ be the Shapley value of the game $w_{(a,u)}$. (For a description of A , see [8].) In the context at hand, the value assigns to each trader t , the “contribution” made by him to social welfare, i.e., “the amount of welfare added to society by his presence” (see [4]). Also, for an allocation x in (a, u) let $u_x: \mathcal{C} \rightarrow \mathbb{R}$ be the payoff generated by x , i.e., $u_x(S) = \int_S u^t(x^t) d\mu$. Then x in $m = (a, \succsim)$ is called a *value-allocation* if we can find a utility-representation u for \succsim such that $u_x = Aw_{(a,u)}$. In other words it is an allocation at which, for some choice of individual utilities, each trader gets precisely his contribution to the welfare of society.

The three solutions—competitive, core, and value allocations—are, in general, quite different from each other. It is a remarkable fact that they become equivalent on $\tilde{\mathcal{M}}$. In this paper we make an attempt to understand the equivalence phenomenon.⁴ The question is: what are the crucial properties that are common across these solutions and on which—at bottom—the equivalence depends? To set the stage for the analysis, we take a map φ on $\tilde{\mathcal{M}}$ which maps each economy m to a nonempty subset $\varphi(m)$ of allocations in m . We restrict ourselves to the case where the elements of $\varphi(m)$ are both individually rational and Pareto-optimal in m . Then we look for an

³ The u^t are subject to several constraints (see [4, 8] and also Section 2).

⁴ The transferable utility case was presented in [14]. Here we extend that analysis to non-transferable utility.

irreducible list of axioms on φ that will uniquely categorize φ as the coincident competitive-core-value correspondence. Four axioms are presented which accomplish the job. They are named the “anonymity”, “equity”, “consistency” and “restricted continuity” axioms.

This may be viewed as a “meta-equivalence” theorem. For instance the fact that the value, core and competitive allocations coincide would follow from our result once it is verified that each of the maps

(the value correspondence): $m \mapsto$ set of value allocations of m

(the core correspondence): $m \mapsto$ set of core allocations of m

(the competitive correspondence): $m \mapsto$ set of competitive allocations of m

obeys our axioms. In general any solution on $\tilde{\mathcal{M}}$ coincides with the above correspondence if, and only if, it obeys the axioms.

From another point of view our result also sheds new light on the nature of the competitive equilibrium itself. The direct definition is “local.” It is based, as we saw, on a model of individualistic optimization of trade in the presence of fixed prices. And it refers only to the characteristics a, \succsim of a single economy. But our axioms, since they categorize it, also serve as a definition of the competitive correspondence on $\tilde{\mathcal{M}}$. Here a space of economies is needed, and the axioms express “global” properties, in which two or more economies are compared to each other. Worthy of note is the fact that the spirit of our axioms is quite different from that of the competitive equilibrium definition. Nowhere is the existence of prices put into the axioms. Nor is it postulated that traders behave like individual optimizers. Nor indeed is any notion of coalitional stability invoked. Instead our approach is akin to that taken in the study of social choice. The solution is viewed in its entirety on the whole domain $\tilde{\mathcal{M}}$ and we introduce certain constraints on its global behaviour via the axioms.

To describe the axioms, a little more notation is needed. For $m = (a, \succsim)$, let $A(m)$ be the set of all bounded allocations x in m that are individually rational and Pareto-optimal i.e., $x^t \succsim^t a^t$ for almost all $t \in T$, and there does not exist any allocation y in m such that $y^t \succ^t x^t$ for almost all $t \in T$. Consider a map

$$\tilde{\varphi}: \tilde{\mathcal{M}} \rightarrow \text{sets of allocations.}$$

Such a map is called *admissible* on $\tilde{\mathcal{M}}$ if $\emptyset \neq \tilde{\varphi}(m) \subset A(m)$ and

$$\left. \begin{array}{l} (a, \succsim) = m \in \tilde{\mathcal{M}} \\ x \in \tilde{\varphi}(m), y \in A(m) \\ x^t \sim^t y^t \quad \text{for a.a.t.} \end{array} \right\} \Rightarrow y \in \tilde{\varphi}(m).$$

Our first axiom says that no traits of a trader count for anything in the economy other than his endowments and preferences: he is fully characterized by these two items. To make this formal, let Θ be a μ -measure-preserving automorphism of (T, \mathcal{C}, μ) and take $m = (a, \succcurlyeq)$. Define the economy Θm by relabelling the traders according to Θ , i.e., $\Theta m = (\Theta a, \Theta \succcurlyeq)$, where $(\Theta a)^t = a^{\Theta(t)}$ and $(\Theta \succcurlyeq)^t = \succcurlyeq^{\Theta(t)}$. Then $\tilde{\varphi}(\Theta m)$ consists precisely of the (relabelled) allocations Θx for x in $\tilde{\varphi}(m)$. Summing up:

Axiom I (Anonymity). Suppose $m \in \tilde{\mathcal{M}}$ and Θ is a μ -measure-preserving automorphism of (T, \mathcal{C}) . Then $\tilde{\varphi}(\Theta m) = \Theta \tilde{\varphi}(m)$.

Our next axiom asserts the existence of an allocation in the solution, in which traders with identical characteristics are treated at par.

Axiom II (Equity). Suppose $(a, \succcurlyeq) \in \tilde{\mathcal{M}}$. Then there exists $x \in \tilde{\varphi}(a, \succcurlyeq)$ such that

$$\left. \begin{array}{l} a^t = a^{t'} \\ \succcurlyeq^t = \succcurlyeq^{t'} \end{array} \right\} \Rightarrow x^t \sim^t x^{t'}$$

for almost all t and t' in T .

(It will suffice, in fact, to require Equity to hold only in the context of finite-type economies. See Axiom B in Section 2.)

The consistency axiom has to do with enlarging or reducing the set of commodities in an economy. Suppose $m = (a, \succcurlyeq)$, $m' = (a', \succcurlyeq')$ and $m_* = (a_*, \succcurlyeq_*)$ are three economies in $\tilde{\mathcal{M}}$ with k, l and $k + l$ commodities respectively.

Consider any $(x, y) \in \mathbb{R}_+^{k+l}$, where $x \in \mathbb{R}_+^k$ and $y \in \mathbb{R}_+^l$. Then \succcurlyeq_*^t induces a preference $\succcurlyeq_*^{t,x}$ on \mathbb{R}_+^l [$\succcurlyeq_*^{t,y}$ on \mathbb{R}_+^k] by the rule $y' \succcurlyeq_*^{t,x} y'' \Leftrightarrow (x, y') \succcurlyeq_*^t (x, y'')$ [$x' \succcurlyeq_*^{t,y} x'' \Leftrightarrow (x', y) \succcurlyeq_*^t (x'', y)$]. Write $m_* = m \otimes m'$ if, for almost all $t \in T$,

- (a) $a_*^t = (a^t, a'^t)$
- (b) for any $x \in \mathbb{R}_+^k$ and $y \in \mathbb{R}_+^l$, $\succcurlyeq_*^{t,x} = \succcurlyeq^{t'}$ and $\succcurlyeq_*^{t,y} = \succcurlyeq^t$.

Axiom III (Consistency). Suppose $m_* = (a_*, \succcurlyeq_*)$, $m = (a, \succcurlyeq)$, $m' = (a', \succcurlyeq')$ are in $\tilde{\mathcal{M}}$ and $m_* = m \otimes m'$. Then

$$(i) \left. \begin{array}{l} x \in \tilde{\varphi}(m) \\ y \in \tilde{\varphi}(m') \\ (x, y) \text{ is Pareto-optimal in } m_* \end{array} \right\} \Rightarrow (x, y) \in \tilde{\varphi}(m_*)$$

$$(ii) \quad \left. \begin{array}{l} (x, y) \in \tilde{\varphi}(m_*) \\ y = a' \end{array} \right\} \Rightarrow x \in \tilde{\varphi}(m).$$

The economies m and m' constitute, in some sense, a factorization of m_* . One might wonder if it should be possible to arrive at a solution of m_* by solving each of the “sub-economies” separately and then putting together their solutions (i.e., if “partial equilibrium” analysis fits into the general). But the problem is that m and m' are not exactly “non-interacting” parts of m_* . While, by (b), \succcurlyeq_* is consistent with \succcurlyeq and \succcurlyeq' , “cross-effects” may appear, so that when we compose solutions of m and m' , the resultant allocation in m_* need not be Pareto-optimal there. Since we insist that $\tilde{\varphi}$ be Pareto-optimal we cannot, at the same time, require $\tilde{\varphi}$ to compose across the constituent sub-economies without qualification. At the very least we would have to put in the clause: “if this does not violate Pareto-optimality.” This is precisely the content⁵ of part (i). Part (ii) may be described as follows. If there is no trade in the last l commodities at a solution (x, y) of m_* , then $\tilde{\varphi}$ will induce the allocation x in m . This is because, being untraded, the l commodities will not appear in the economy, so that—in effect—we will be “left observing” just x in m . But $\tilde{\varphi}$ acts directly on m also. Thus (ii) says that the action of $\tilde{\varphi}$ induced on a sub-economy m should be consistent with the direct action of $\tilde{\varphi}$ on m . (The commodities that are not traded are like “irrelevant commodities” and the sub-economy is “independent” of them.)

Finally we turn to restricted continuity. Two versions can be given, called axioms IV and IV' below. The latter is particularly easy to state. Indeed, using $\| \quad \|$ to denote the bounded variation norm (see [8]), we have:

Axiom IV'. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any two representations $(a, u), (\bar{a}, \bar{u})$ of economies in $\tilde{\mathcal{M}}$:

$$\left. \begin{array}{l} \|w_{(a, u)} - w_{(\bar{a}, \bar{u})}\| < \delta \\ x \in \tilde{\varphi}(a, u), u_x = w_{(a, u)}(T) \\ y \in \tilde{\varphi}(\bar{a}, \bar{u}), \bar{u}_y = w_{(\bar{a}, \bar{u})}(T) \end{array} \right\} \Rightarrow \|u_x - \bar{u}_y\| < \varepsilon.$$

(The reader who wishes to quickly see one version of our theorem is invited, at this point, to skip ahead and read it with IV' in place of IV. The theorem is stated in the introduction in italics just after the discussion of axiom IV.)

⁵ Part (i) of the consistency axiom is inspired by the “conditional additivity” axiom in [5].

Axiom IV' is much stronger than IV. To the extent that it depicts a property of "coincident solutions" on the domain $\tilde{\mathcal{M}}$, its strength is a virtue. But, for the meta-equivalence aspect of our result, it is preferable to cast the axioms in as weak a form as possible, and to bring the maximum number of solutions within their ambit. Therefore we will focus on Axiom IV, even though some "build up" is required before it can be stated. As we shall see, Axiom IV has a more intuitive flavour than IV', and—being weaker—applies more widely than IV'. For instance it can be verified that the "NTU-Harsanyi value" (see [19]; also [17, 18]) satisfies IV but not IV'; and, working with IV, we find that the Harsanyi value fails to be equivalent to the other classical solutions not because it violates continuity but on account of the fact that it is not consistent in our sense.

The rough idea in Axiom IV is that $\tilde{\varphi}$ must be continuous over economies for which a "natural" notion of "nearness" can be defined. Of course the words within quotes must be made precise.

The conditions we impose on our economies are "classical" (as in [8]), and in particular they require that utilities eventually grow slower than any linear function (see (2.4)' in Section 2). Thus "transferable-utility-like" (TU-like) economies are excluded in $\tilde{\mathcal{M}}$. Nevertheless we will motivate our discussion of continuity by first recalling TU-like economies, since a very simple concept of distance exists for them.

Consider $m = (a, \succsim)$. Suppose there exists a utility representation $u: T \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ for \succsim such that setting $w_{a,u}(S) = \sup\{\int_T u^t(x^t) d\mu: \int_T x^t d\mu \leq \int_T a^t d\mu\}$ for $S \in \mathcal{C}$, the two sets $V_{a,u}(S) = \{S \xrightarrow{f} \mathbb{R}: f \text{ is measurable and } u^t(a^t) \leq f(t) \leq u^t(x^t) \text{ for almost all } t \in S, \text{ where } x \text{ is an } S\text{-allocation}\}$ and $V_{(a,u)}^+(S) = \{S \xrightarrow{f} \mathbb{R}: f \text{ is measurable and } \int_S f d\mu \leq w_{a,u}(S), f(t) \geq u^t(a^t) \text{ for almost all } t \in S\}$ are equal for all $S \in \mathcal{C}$, i.e., the individually rational "payoffs" of any coalition S (given by the first set) correspond to redistributions of the number $w_{a,u}(S)$. When this happens we say that the representation (a, u) of (a, \succsim) is TU-like.

It is worth emphasizing that there is no transfer of utilities that is in fact conceived in the above picture. Our basic focus is only on $V_{(a,u)}$. If it turns out (and this will happen for very rare (a, u)) that $V_{(a,u)} = V_{(a,u)}^+$ then (a, u) is called TU-like.

Given (a, u) and (\tilde{a}, \tilde{u}) , both of which are TU-like, there is a natural notion of distance between them, namely $\|w_{(a,u)} - w_{(\tilde{a},\tilde{u})}\|$, where $\|\cdot\|$ denotes the bounded variation norm (see [8]; and also Section 2). This distance looks at (a, u) and (\tilde{a}, \tilde{u}) in payoff (or utility) space, forgetting about the detailed microdata of endowments and preferences (including even the numbers of underlying commodities in the two economies).

There are no (a, u) in our domain that are exactly TU-like. But, if we do not look at them "too closely," then some of the (a, u) appear to be TU-like, permitting us to talk about how close they are to each other. To make

this idea precise, let $S \in \mathcal{C}$ be a coalition and $f: S \rightarrow \mathbb{R}$ an integrable (payoff) function. We want to describe the idea of “observing” f not at each point $t \in S$ (i.e., not each $f(t)$), but at a coarser, more aggregated “macro-level.” Suppose this level is $\delta > 0$. (As δ becomes smaller the observation will be finer). Let $\mathcal{P} = \{S_1, \dots, S_q\}$ be a partition of S with $\mu(S_i) \geq \delta\mu(S)$ for $1 \leq i \leq q$. (Such partitions will be called δ -partitions of S .) Then the observation of f through \mathcal{P} , denoted $f_{|\mathcal{P}}$, is defined by

$$f_{|\mathcal{P}} = \left(\int_{S_1} f \, d\mu, \dots, \int_{S_p} f \, d\mu \right).$$

Thus, if we look at $V_{(a,u)}(S)$, $V_{(a,u)}^+(S)$ through \mathcal{P} , the set observed is

$$V_{(a,u)}(S|\mathcal{P}) = \{f_{|\mathcal{P}} : f \in V_{(a,u)}(S)\}$$

$$V_{(a,u)}^+(S|\mathcal{P}) = \{f_{|\mathcal{P}} : f \in V_{(a,u)}^+(S)\}.$$

An economy (\bar{a}, \bar{u}) is said to be a δ -approximation of the economy (a, u) if:

(1) for all $S \in \mathcal{C}$ and all δ -partitions of S ,

$$V_{(a,u)}(S|\mathcal{P}) = V_{(a,u)}^+(S|\mathcal{P}) \quad \text{and} \quad V_{(\bar{a}, \bar{u})}(S|\mathcal{P}) = V_{(\bar{a}, \bar{u})}^+(S|\mathcal{P}).$$

(2) $\|w_{(a,u)} - w_{(\bar{a}, \bar{u})}\| < \delta$.

Condition (1) says that both (a, u) and (\bar{a}, \bar{u}) are TU-like at the level δ (the smaller δ , is the more TU-like they are); and (2) says that viewing these nearly TU-like representations as TU, the bounded variation distance between them is less than δ . Define

$$d((a, u), (\bar{a}, \bar{u})) = \inf\{\delta : (\bar{a}, \bar{u}) \text{ is a } \delta\text{-approximation of } (a, u)\}.$$

Then (\bar{a}, \bar{u}) is a δ -approximation of (a, u) if $d((a, u), (\bar{a}, \bar{u})) < \delta$.

For an allocation z and utilities u , recall the set function $u_z: \mathcal{C} \rightarrow \mathbb{R}$, given by $u_z(S) = \int_S u^i(z^i) \, d\mu$.

Axiom IV (Restricted Continuity). For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any two representations (a, u) , (\bar{a}, \bar{u}) of economies in $\tilde{\mathcal{M}}$:

$$\left. \begin{array}{l} d((a, u), (\bar{a}, \bar{u})) < \delta \\ x \in \tilde{\varphi}(a, u), u_x(T) = w_{a,u}(T) \\ y \in \tilde{\varphi}(\bar{a}, \bar{u}), \bar{u}_y(T) = w_{\bar{a}, \bar{u}}(T) \end{array} \right\} \Rightarrow \|u_x - \bar{u}_y\| < \varepsilon.$$

Note that this axiom is stated in payoff space. It abstracts from the microdata of the economies, and of their solutions, and looks at the images of both in payoff space, with the aid of utility representations. Then it says that if two economies (represented in payoff space) are nearly TU-like and close, so are any two “TU-compatible” points in their solutions (correspondingly represented in payoff space). In other words, only those points are considered which are compatible with the TU view, i.e. $\{u^t(x^t): t \in T\}$ is Pareto optimal in $V_{(a,u)}^+$, equivalently $u_x(T) = w_{(a,u)}(T)$ (and similarly for y).

Should the payoffs in $\varphi(a, u)$ depend only on $V_{(a,u)}$ (and not on the finer microdata given by a and u)? Once we agree to this, i.e., to factor the payoffs $\{u_x: x \in \varphi(a, u)\}$ through $V_{(a,u)}$, the continuity assumption is seen to be relatively mild. It restricts attention to pairs (a, u) , (\bar{a}, \bar{u}) , which form a very “thin” set in $\tilde{\mathcal{M}} \times \tilde{\mathcal{M}}$, and for which a measure of closeness is quite naturally defined.

Consider $\{u_x: x \in \varphi(a, u)\}$. We are not concerned here with the allocations x in $\varphi(a, u)$, but rather with the *payoffs* that arise from these allocations. But the moment we confine our attention to payoffs, $V_{(a,u)}$ provides “sufficient statistics” for (a, u) . The entire story of payoffs in the economy (a, u) is told for S by the set $V_{(a,u)}(S)$. The details contained in a and u , including number of underlying commodities, is irrelevant. This is so for all coalitions S . So, *in payoff space*, the economy (a, u) “equals” $V_{(a,u)}$. It is then only natural to conclude that the associated *payoffs* $\{u_x: x \in \varphi(a, u)\}$ of $\varphi(a, u)$ are a function of $V_{(a,u)}$ alone. Notice that in Axiom IV we do not use the full force of this argument, for we restrict to economies which are nearly TU-like, and to payoffs which are TU-compatible.

Our result is:

THEOREM A. *There is one, and only one, admissible $\tilde{\varphi}$ on $\tilde{\mathcal{M}}$ which satisfies Axioms I, II, III, IV and it is the competitive correspondence: $m \mapsto$ set of competitive allocations in m .*

In fact we actually prove a stronger result than this. To make the distinction it will be necessary to turn to *cardinal* economies. Our approach so far has been a strictly ordinal one. Even though utilities $u: T \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ did often show up in the analysis, they were brought in only as a convenient device to refer to properties of the ordinal preferences (which would have been painful to state otherwise). In short there was no role played by u other than to represent \succcurlyeq . It is true that we could have defined φ directly on pairs (a, u) —rather than on (a, \succcurlyeq) —but then, implicit in our analysis, was the assumption

$$\text{if } u \text{ and } w \text{ both represent } \succcurlyeq, \text{ then } \varphi(a, u) = \varphi(a, w). \quad (*)$$

One might ask what happens if utilities are taken into account. We will adopt the cardinal point of view, in which what matters are the intra-personal comparisons of utility differences $(u^t(x) - u^t(y))/(u^t(x') - u^t(y'))$ for x, x', y, y' in \mathbb{R}_+^n . Thus two utilities u and \tilde{u} must be deemed to be the same if (for almost all t) $u^t(\cdot) = q^t + \lambda^t \tilde{u}^t(\cdot)$ for some choice of measurable functions $q: T \rightarrow \mathbb{R}$ and $\lambda: T \rightarrow \mathbb{R}_{++}$. We say that u and \tilde{u} are "cardinally equivalent" in this case. If (a, u) and (a, \tilde{u}) are two economies with the same endowments, and cardinally equivalent utilities, we write $(a, u) \sim (a, \tilde{u})$.

Let \mathcal{M} denote a set of *cardinal* economies (a, u) , and consider $\varphi: \mathcal{M} \rightarrow$ sets of allocations. The fact that φ is a cardinal solution is summed up by

$$\left. \begin{array}{l} (a, u), (a, \tilde{u}) \in \mathcal{M}; \\ (a, u) \sim (a, \tilde{u}) \end{array} \right\} \Rightarrow \varphi(a, u) = \varphi(a, \tilde{u}).$$

The condition (**) is much weaker than (*). It leaves open the possibility that $\varphi(a, u)$ and $\varphi(a, u')$ may differ even when u and u' represent the same ordinal preference.

To any ordinal economy $m = (a, \succsim)$ we can attach a subset $\mathcal{L}(m) = \{(a, u) \in \mathcal{M} : u \text{ represents } \succsim\}$ of \mathcal{M} . There is no *a-priori* reason to expect φ to be constant across the different cardinal economies in $\mathcal{L}(m)$. However, we will impose *watered-down* versions of the previous four axioms (see Axioms A, B, C, D of Section 2) on φ and show that even then the competitive correspondence is categorically determined. (See Theorem B in Section 4) Thus the ordinal character of φ given by (*)—i.e., the invariance of φ on $\mathcal{L}(m)$ —is *deduced* starting with the purely cardinal viewpoint of (**).

This is stronger than the ordinal result on $\tilde{\varphi}$ that we cited earlier. For Axiom I (II, III, IV) implies Axiom A (B, C, D) in general, and the converse is only true in the presence of (*). Ex-post, of course, both sets of axioms yield the competitive correspondence on \mathcal{M} . However, if we step outside of \mathcal{M} , the ordinal axioms are far more stringent than the cardinal. For instance, if T consists of two traders, the Nash bargaining solution satisfies A, B, C, D but not I, II, III, IV.

Returning to the ordinal economies in $\tilde{\mathcal{M}}$ we know that even stronger properties hold for the competitive correspondence than those depicted in I, II, III, IV, and we could equally well have started with them. But, keeping the meta-equivalence aspect of our result in mind, we have attempted to put the axioms in as weak a form as possible and to show that they still categorize the competitive correspondence. It was this that motivated our choice of the cardinal context over the ordinal one.

It must be stressed that our analysis holds up in the context of \mathcal{M} alone. Dropping either the non-atomicity of the trader-set or the smoothness of utilities would make our theorem break down. But then the equivalence of

the competitive, core, and value allocations also breaks outside \mathcal{M} , so the restricted domain is just what was to be expected.

2. NON-ATOMIC SMOOTH CARDINAL ECONOMIES

Let \mathbb{R}^l be the Euclidean space of dimension l , and \mathbb{R}^l_+ its non-negative orthant. For any x in \mathbb{R}^l , x_j is the j th component of x . For x and y in \mathbb{R}^l , define $x > y$ ($x = y$, $x \geq y$) if $x_j > y_j$ ($x_j = y_j$, $x_j \geq y_j$) for all $1 \leq j \leq l$; and $x \geq y$ if $x \geq y$, but not $x = y$. The symbol 0 represents the zero vector in \mathbb{R}^l (the relevant dimension l will always be clear from the context). The l_1 -norm on \mathbb{R}^l is denoted $\| \cdot \|$, i.e. $\|x\| = \sum_{j=1}^l |x_j|$.

Fix a measure space (T, \mathcal{C}, μ) , where (T, \mathcal{C}) is isomorphic to the closed unit interval $[0, 1]$ with its Borel sets. T is the set of agents, \mathcal{C} the σ -algebra of coalitions, and μ the population measure. We assume that μ is finite, σ -additive, non-negative and non-atomic. W.l.o.g. $\mu(T) = 1$.

A cardinal economy⁶ with n commodities is a pair (a, u) , where $a: T \rightarrow \mathbb{R}^n_+$ and $u: T \times \mathbb{R}^n_+ \rightarrow \mathbb{R}$. Here $a(t) \equiv a^t$ is the initial endowment of agent t and $u(t, \cdot) \equiv u^t(\cdot)$ the utility function of agent t .

To describe the conditions on (a, u) , first some notation. Let f be a real-valued function on \mathbb{R}^l_+ . Then $f_k(y) \equiv \partial f / \partial x_k(y)$ if $y_k > 0$, provided the partial derivative exists; and $f_k(y) \equiv \lim_{h \rightarrow 0^+} [f(y + he_k) - f(y)]/h$ if $y_k = 0$, where e_k denotes the k th unit direction in \mathbb{R}^l , provided the limit exists. (We will often denote the gradient $(f_1(y), \dots, f_l(y))$ by $\nabla f(y)$.) Finally, given $S \in \mathcal{C}$, “almost all t in S ” (abbreviated a.a.t in S) means all t in S except for a μ -null subset of S ; and a.a.t means a.a.t in T .

When a function g from T (or $T \times \mathbb{R}^q_+$) to \mathbb{R}^l_+ is called *measurable*, it is with respect to the σ -fields \mathcal{C} (or the product σ -field $\mathcal{C} \times \mathcal{B}$) and \mathcal{B} , where \mathcal{B} represents the Borel sets of the relevant Euclidean space. Given a measurable $g: T \rightarrow \mathbb{R}^l_+$, we say that g is *uniformly positive* (or *bounded*) if there exist $a, b \in \mathbb{R}^l_{++}$ such that $g(t) \geq b$ (or $g(t) \leq b$) for a.a.t in T . Also for $g: T \times \mathbb{R}^q_+ \rightarrow \mathbb{R}^l_+$, g is (i) *uniformly continuous* at $x \in \mathbb{R}^q_+$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that: $\|x - y\| < \delta \Rightarrow \|g(t, x) - g(t, y)\| < \varepsilon$ for a.a.t in T ; (ii) *uniformly bounded* (or *uniformly positive*) if there exists $b \in \mathbb{R}^l_{++}$ such that $g(t, x) \leq b$ (or $g(t, x) \geq b$) for all $x \in \mathbb{R}^q_+$ and a.a.t in T . A weaker version of (ii) will also be used: g is *uniformly bounded* (or *uniformly positive*) *on compact sets* if, for every compact set K , there exists $b \in \mathbb{R}^l_{++}$ such that $g(t, x) \leq b$ (or $g(t, x) \geq b$) for all $x \in K$ and a.a.t in T .

The set \mathcal{M} of *cardinal economies* consists of all measurable pairs (a, u) which satisfy the conditions ((2.1) till (2.7)) below:

⁶ Recall that, in the continuum setting, any two maps that differ only on a set of measure zero are identified.

(2.1) a is bounded and $\int_T a \, d\mu > 0$.

(2.2) u^t is continuous and concave, and $u^t(0) = 0$, for a.a.t. in T .

(2.3) u is uniformly continuous at 0.

(2.4) u is uniformly bounded.

(2.5) for a.a.t, $u_j^t(y)$ exists and is positive whenever $y_j > 0$; $u_j^t(y) = \infty$ whenever $y_j = 0$; and $u_j^t(x) \rightarrow u_j^t(y)$ whenever $x \rightarrow y$.

(2.6) The derivative functions u_j are uniformly positive on compact sets.

(2.7) The function $t \mapsto u^t(a^t)$ is uniformly positive.

For some of our results we can weaken (2.4) to

(2.4)' $u^t = o(\|x\|)$ as $\|x\| \rightarrow \infty$, uniformly in t ; i.e., for every $\varepsilon > 0$, there exists $\eta > 0$ such that $|u^t(x)| < \varepsilon \|x\|$ whenever $\|x\| > \eta$, for a.a.t in T

With this weaker restriction, we get a larger class of economies \mathcal{M}' , which includes \mathcal{M} , and for which many of our results continue to hold.

Also the boundedness of the initial allocation a (in assumption 2.1) can be dispensed with, provided we strengthen the anonymity axiom and modify some of the conditions on u (See Remark 9).

For $m = (a, u)$ in \mathcal{M} and $S \in C$, an S -allocation is an integrable $x: S \rightarrow \mathbb{R}_+^n$ with $\int_T x \, d\mu = \int_T a \, d\mu$. A T -allocation x is defined to be:

Individually rational (I.R.) if $u^t(x^t) \geq u^t(a^t)$ for a.a.t in T ;

Pareto-optimal (P.O.) if there is no T -allocation y with $u^t(y^t) > u^t(x^t)$ for a.a.t in T .

For $m = (a, u)$ in \mathcal{M} , we denote by $A(m)$ the set of all bounded T -allocations that are I.R. and P.O.

Given $S \in C$, also define

$V_m(S) = \{f \in L_1(S): \text{there exists an } S\text{-allocation } x \text{ in } m \text{ such that } u^t(a^t) \leq f(t) \leq u^t(x^t) \text{ for a.a.t in } S\}$

$w_m(S) = \max\{\int_S u^t(x^t) \, d\mu: x \text{ is an } S\text{-allocation in } m\}$.

$V_m^+(S) = \{f \in L_1(S): f(t) \geq u^t(a^t) \text{ for a.a.t in } S \text{ and } \int_S f \, d\mu \leq w_m(S)\}$

Let $P = \{S_1, \dots, S_q\}$ be a measurable partition of $S \in C$. For any integrable $f: S \rightarrow \mathbb{R}$, let $f|P = (\int_{S_1} f \, d\mu, \dots, \int_{S_q} f \, d\mu)$. Define

$$V_m(S|P) = \{f|P: f \in V_m(S)\}$$

$$V_m^+(S|P) = \{f|P: f \in V_m^+(S)\}.$$

A δ -partition of S is a measurable partition $P = \{S_1, \dots, S_q\}$ of S with $\mu(S_i) > \delta\mu(S)$ for $1 \leq i \leq q$. We denote by $\mathcal{M}(\delta)$ the set of all economies m in \mathcal{M} such that, for every $S \in C$ and every δ -partition P of S ,

$$V_m^+(S|P) = V_m(S|P).$$

In short, $\mathcal{M}(\delta)$ is the set of economies that are *T.U. -like at the level δ* , as discussed in the introduction.

3. COMPETITIVE, CORE, AND VALUE ALLOCATIONS

Fix $m = (a, u) \in \mathcal{M}$ and a T -allocation x of m . Then x is called a *competitive allocation* if there exists a $p \in \mathbb{R}_+^n$ such that

$$u^t(x^t) = \max\{u^t(y) : y \in \mathbb{R}_+^n, p \cdot y \leq p \cdot a^t\}$$

for a.a.t.

Also x is a *core allocation* of m if there does not exist any S -allocation y with $\mu(S) > 0$ and $u^t(y^t) > u^t(x^t)$ for a.a.t in S .

To define value allocations, we first turn to cardinal equivalence. Two economies $m = (a, u)$ and $\bar{m} = (\bar{a}, \bar{u})$ in \mathcal{M} are called *cardinally equivalent* if $a = \bar{a}$, and there exists a measurable $\lambda: T \rightarrow \mathbb{R}_{++}$ such that $u^t = \lambda(t) \bar{u}^t$ for a.a.t in T . In this case we write $m \sim \bar{m}$. It is easily checked that \sim is an equivalence relation on \mathcal{M} .

Also recall from [8] that, for $m \in \mathcal{M}$, the Shapley value Aw_m of w_m is a countably additive function on \mathcal{C} . A T -allocation x is called a *value allocation* of m if there exists an economy $\bar{m} = (\bar{a}, \bar{u})$ in \mathcal{M} such that $m \sim \bar{m}$ and

$$\int_S \bar{u}^t(x^t) d\mu(t) = (Aw_{\bar{m}})(S)$$

for all $S \in C$.

Denote the set of competitive, core and value allocations of m by $\tau(m)$, $\sigma(m)$ and $\psi(m)$ respectively.

4. STATEMENT OF THEOREM B

The axioms will refer to a map

$$\varphi: \mathcal{M} \rightarrow \text{sets of allocations.}$$

Such a map will be called *admissible* on \mathcal{M} if, $\emptyset \neq \varphi(m) \subset A(m)$ and

$$m \sim \bar{m} \Rightarrow \varphi(m) = \varphi(\bar{m}),$$

and

$$\left. \begin{array}{l} m = (a, u) \\ x \in \varphi(m), y \in A(m) \\ u^t(x^t) = u^t(y^t) \quad \text{for a.a.t} \end{array} \right\} \Rightarrow y \in \varphi(m).$$

The first requirement says that φ selects from individually rational and Pareto-optimal allocations; the second says that φ is a cardinal solution (see (**)) of Introduction); the third says that T -allocations which are payoff-equivalent are treated at par by φ , and is in keeping with our point-of-view of factoring the solutions through payoff space. (This last requirement will reappear more forcibly when we discuss the continuity axiom below).

For any (a, u) and a μ -measure-preserving automorphism Θ of (T, \mathcal{C}, μ) define $(\Theta a, \Theta u)$ by $(\Theta a)^t = a^{\Theta(t)}$, $(\Theta u)^t = u^{\Theta(t)}$.

Axiom A (Anonymity). If $(a, u) \in \mathcal{M}$ and Θ is a μ -measure-preserving automorphism of (T, \mathcal{C}) , then $\varphi(\Theta a, \Theta u) = \Theta \varphi(a, u)$.

Given a partition of T into sets T_1, \dots, T_k in \mathcal{C} , we say that (a, u) has types T_1, \dots, T_k if $a^t = a^{t'}$ and $u^t = u^{t'}$ for all $t, t' \in T_i$ and all $1 \leq i \leq k$.

Axiom B (Equity). If $(a, u) \in \mathcal{M}$ has types T_1, \dots, T_k then there exists an $x \in \varphi(a, u)$ such that $u^t(x^t) = u^{t'}(x^{t'})$ for all $t, t' \in T_i$ and all $1 \leq i \leq k$.

Let $(a, u), (\hat{a}, \hat{u}), (\tilde{a}, \tilde{u})$ be economies in \mathcal{M} with $l, k, l+k$ commodities respectively. Write $(\tilde{a}, \tilde{u}) = (a, u) \times (\hat{a}, \hat{u})$ if, for almost all $t \in T$,

$$\begin{aligned} \tilde{a}^t &= (a^t, \hat{a}^t) \\ \tilde{u}^t(x, y) &= u^t(x) + \hat{u}^t(y). \end{aligned}$$

Axiom C (Consistency). Suppose $(a, u), (\hat{a}, \hat{u}), (\tilde{a}, \tilde{u}) \in \mathcal{M}$, and $(\tilde{a}, \tilde{u}) = (a, u) \times (\hat{a}, \hat{u})$. Then

$$\begin{array}{l} \text{(i)} \\ \text{(ii)} \end{array} \left. \begin{array}{l} x \in \varphi(a, u) \\ y \in \varphi(\hat{a}, \hat{u}) \\ (x, y) \text{ is Pareto optimal in } (\tilde{a}, \tilde{u}) \end{array} \right\} \Rightarrow (x, y) \in \varphi(\tilde{a}, \tilde{u})$$

$$\left. \begin{array}{l} (x, y) \in \varphi(\tilde{a}, \tilde{u}) \\ y = \hat{a} \end{array} \right\} \Rightarrow x \in \varphi(a, u).$$

Note that this is weaker than Axiom III, since $(\tilde{a}, \tilde{u}) = (a, u) \times (\hat{a}, \hat{u})$ implies $(\tilde{a}, \tilde{u}) = (a, u) \otimes (\hat{a}, \hat{u})$ but not vice versa. For example, any two-commodity economy decomposes in the obvious way into two one-commodity subeconomies in the sense of “ \otimes ”, but rarely in the sense of “ \times ”.

As was pointed out in the introduction, there are two axioms of continuity, either of which suffice for the theorem. The first, which we call Axiom D', is stated exactly as Axiom IV' of the ordinal case. But, for reasons already discussed, we will focus on the other version. First recall from [8] that a map $v: \mathcal{C} \rightarrow \mathbb{R}$ is called a set-function if $v(\emptyset) = 0$; and is called monotonic if $v(S) \leq v(T)$ whenever $S \subset T$. The bounded variation norm of a set-function v is $\|v\| = \inf\{v_1(T) + v_2(T)\}$, where the infimum is taken over all monotonic set-functions v_1 and v_2 such that $v = v_1 - v_2$. It follows from [8] that $\|w_m\|$ is finite for $m \in \mathcal{M}$.

Let $m, \bar{m} \in \mathcal{M}$ and let $\delta > 0$. We say that \bar{m} is a δ -approximation to m if $m \in \mathcal{M}(\delta)$, $\bar{m} \in \mathcal{M}(\delta)$ and $\|w_m - w_{\bar{m}}\| < \delta$.

Define

$$d(m, \bar{m}) = \inf\{\delta > 0 : m \in \mathcal{M}(\delta), \bar{m} \in \mathcal{M}(\delta), \|w_m - w_{\bar{m}}\| \leq \delta\}.$$

Then \bar{m} is a δ -approximation to m if $d(m, \bar{m}) < \delta$.

For a T -allocation x in $m = (a, u) \in \mathcal{M}$, recall that $u_x: \mathcal{C} \rightarrow \mathbb{R}$, where $u_x(S) = \int_S u'(x^t) d\mu$, is the payoff induced by x in m .

Axiom D (Restricted Continuity). For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left. \begin{aligned} m &= (a, u) \in \mathcal{M} \\ \bar{m} &= (\bar{a}, \bar{u}) \in \mathcal{M} \\ d(m, \bar{m}) &< \delta \end{aligned} \right\} \Rightarrow \|u_x - \bar{u}_y\| < \varepsilon$$

for every $x \in \varphi(a, u)$ and $y \in \varphi(\bar{a}, \bar{u})$, such that $u_x(T) = w_{(a, u)}(T)$ and $\bar{u}_y(T) = w_{(\bar{a}, \bar{u})}(T)$.

THEOREM B. *There is one, and only one, admissible map φ on \mathcal{M} which satisfies Axioms A, B, C, D and it is given by*

$$\varphi(m) = \tau(m) \equiv \text{the set of competitive allocations in } m.$$

Moreover, the competitive, core and value allocations coincide on \mathcal{M} , i.e., $\tau(m) = \sigma(m) = \psi(m)$ for all m in \mathcal{M} .

(The theorem also holds with axiom D' in place of D).

5. PROOF OF THEOREM B

For a straight proof of the theorem, the reader could read \mathcal{M} in place of \mathcal{M}' throughout what follows. But part of the equivalence is valid on the larger domain \mathcal{M}' (see Remark 5 in Section 6). In order to bring this out, we have stated various lemmas and remarks for \mathcal{M}' , even though this occasionally entails longer proofs.

For $1 \leq k \leq n$ denote by e_k the k -th unit vector in \mathbb{R}^n , and recall that u'_k is the directional derivative of u^t in the direction e_k , (see beginning of Section 2).

LEMMA 1. *Let (a, u) be in \mathcal{M}' . Then, for any Pareto optimal allocation $x: T \rightarrow \mathbb{R}_+^n$ in (a, u) ,*

$$\mu\{t \in T: x'_i = 0 \text{ and } x^t \neq 0\} = 0.$$

Proof. Let x be a Pareto-optimal allocation in (a, u) . According to the well-known "welfare theorem" (see, e.g., Theorem 4 in Section 4.3 of [20]) there exists $p \in \mathbb{R}_+^n$, $p \neq 0$, such that x is a competitive allocation of the economy (x, u) with prices p . Moreover, by condition (2.5) on the utilities, it is clear that $p > 0$. Consider any t with $x'_j > 0$ and $x'_k = 0$ for some j and k . Since x^t maximizes u^t on $\{y \in \mathbb{R}_+^n: p \cdot y \leq p \cdot x^t\}$, we must have $p_k u'_j(x^t) \geq p_j u'_k(x^t)$, which is impossible since (by (2.5)) $u'_j(x^t) < \infty$ and $u'_k(x^t) = \infty$. This proves the lemma. Q.E.D.

LEMMA 2. *Let (a, u) be in \mathcal{M}' . Then there exists $K > 0$ such that, for any coalition S and any S -allocation x which achieves $\max\{\int_S u^t(y^t) d\mu: y^t \text{ is an } S\text{-allocation}\}$, $\|x^t\| \leq K$, for a.a.t in S .*

Proof. Recall that for (a, u) in \mathcal{M}' ,

- (1) a is bounded.
- (2) u'_i are uniformly positive on compact sets.
- (3) $u^t(x) = o(\|x\|)$ uniformly in t (i.e., for every $\varepsilon > 0$, there is a $\eta > 0$ such that, for a.a.t. in T , $|u^t(x)| \leq \varepsilon \|x\|$ whenever $\|x\| \geq \eta$).

W.l.o.g. we assume (given (1)) that $\|a^t\| \leq 1$ for almost all t in T . By (2), there exists $M^- > 0$ such that, for almost all $t \in T$ and $1 \leq j \leq n$, $u'_j(y) > M^-$ whenever $\|y\| \leq 2$. Also, by (3), it follows that there exists $K > 1$ such that, for almost all t in T , $u^t(x) < M^- \|x\|$ if $\|x\| \geq K$. Let S be a coalition and let x be an S -allocation such that $\mu\{t \in S: \|x^t\| \geq K\} > 0$. Put $S^+ = \{t \in S: \|x^t\| \geq K\}$, $S^- = \{t \in S: \|x^t\| \leq 1\}$. As $\|a^t\| \leq 1$ for almost all t in S and $\mu(S) \geq \mu(S^+) > 0$, we have $\mu(S^-) > 0$. Consider the

non-atomic vector measure (ν_1, ν_2) on $(S, \mathcal{C}_{|S})$ where $(\nu_1(R), \nu_2(R)) = (\int_R x^t d\mu, \int_R u^t(x^t) d\mu)$ for $R \subset S, R \in \mathcal{C}$. For every $0 < \varepsilon < 1, 0 \leq \varepsilon\mu(S^-) < 1$, and therefore by Lyapunov's theorem there exists a measurable subset $S^+(\varepsilon)$ of S^+ , with

$$(\nu_1(S^+(\varepsilon)), \nu_2(S^+(\varepsilon))) = (1 - \varepsilon\mu(S^-))(v_1(S^+), v_2(S^+)),$$

i.e.,

$$\int_{S^+(\varepsilon)} x^t d\mu = (1 - \varepsilon\mu(S^-)) \int_{S^+} x^t d\mu$$

and

$$\int_{S^+(\varepsilon)} u^t(x^t) d\mu = (1 - \varepsilon\mu(S^-)) \int_{S^+} u^t(x^t) d\mu.$$

Consider the S -allocation $x(\varepsilon)$ that is given by

$$x^t(\varepsilon) = \begin{cases} x^t & \text{for } t \in S^+(\varepsilon) \\ 0 & \text{for } t \in S^+ \setminus S^+(\varepsilon) \\ x^t + \varepsilon x(S^+) & \text{for } t \in S^- \\ x^t & \text{for } t \in S \setminus (S^+ \cup S^-). \end{cases}$$

For an integrable function y from S into a Euclidean space, and measurable $R \subset S$ we denote the integral $\int_R y d\mu$ by $y(R)$. Note that $(x(\varepsilon))(S^+) + (x(\varepsilon))(S^-) = x(S^+) + x(S^-)$, so that $x(\varepsilon)$ is indeed an S -allocation.

For sufficiently small $\varepsilon > 0, \|x^t(\varepsilon)\| \leq 2$ for any t in S^- . Therefore,

$$\int_{S^-} (u^t(x^t(\varepsilon)) - u^t(x^t)) d\mu \geq M^- \varepsilon \|x(S^+)\| \mu(S^-),$$

and

$$\begin{aligned} & \int_{S^+} (u^t(x^t(\varepsilon)) - u^t(x^t)) d\mu \\ &= -\varepsilon\mu(S^-) \int_{S^+} u^t(x^t) d\mu > -M^- \varepsilon\mu(S^-) \int_{S^+} \|x^t\| d\mu \\ &= -M^- \varepsilon\mu(S^-) \|x(S^+)\|. \end{aligned}$$

But then

$$\max \left\{ \int_S u^t(y^t) : y \text{ is an } S\text{-allocation} \right\} \geq \int_S u^t(x^t(\varepsilon)) d\mu > \int_S u^t(x^t) d\mu,$$

a contradiction.

LEMMA 3. Let $m = (a, \tilde{u})$ be an economy in \mathcal{M}' and let $x \in A(m)$. Then, there exists measurable $\lambda: T \rightarrow \mathbb{R}_{++}$ (which is unique almost everywhere up to multiplication by a scalar) and a vector $p > 0$ such that

- (i) λ is bounded and uniformly positive;
- (ii) $\lambda(t) \nabla \tilde{u}^t(x^t) = p$ for a.a.t in T ;
- (iii) if we set $u^t = \lambda(t) \tilde{u}^t$, then

$$\int_T u^t(x^t) d\mu = w_{(a, u)}(T);$$

- (iv) $(a, u) \in \mathcal{M}'$, and $(a, u) \in \mathcal{M}$ if $(a, \tilde{u}) \in \mathcal{M}$.

Proof. Note that $u^t(0) = 0$ and $u^t(a^t) > 0$ for a.a.t by (2.2) and (2.7). Thus the I.R. property of x implies that $x^t \neq 0$ for a.a.t. Then, by Lemma 1, $x^t > 0$ for a.a.t. From the Pareto-optimality of x , we conclude that there exists a measurable λ and a vector p such that (ii) holds. By (2.5), clearly $p > 0$.

From (ii) and the concavity of the u^t (see (2.2)), we get

$$u^t(\alpha) - u^t(x^t) \leq p \cdot (\alpha - x^t)$$

for all $\alpha \in \mathbb{R}_+^n$ and a.a.t. Now, for any allocation y , $\int_T y^t d\mu = \int a^t d\mu = \int x^t d\mu$ and therefore, by integrating the inequality $u^t(y^t) - u^t(x^t) \leq p \cdot (y^t - x^t)$, it follows that $\int_T u^t(y^t) d\mu \leq \int_T u^t(x^t) d\mu$, i.e. (iii) holds.

Next we verify (i). Since, by definition, $A(m)$ contains only bounded allocations, there exists $K > 0$ such that $\|x^t\| < K$ for a.a.t; and, therefore, by (2.6) there exists $\alpha > 0$ such that $\tilde{u}_i^t(x^t) \geq \alpha$ for every $1 \leq i \leq n$ and a.a.t. Also, by (2.7), (2.3) and the fact that $\tilde{u}^t(0) = 0$ for a.a.t, there exists $\beta > 0$ such that $\|x^t\| > \beta$ for a.a.t. So, for almost all t and t' in T ,

$$\lambda^t / \lambda^{t'} = \tilde{u}_i^t(x^t) / \tilde{u}_i^{t'}(x^{t'}) \leq \tilde{u}_i^t(x^t) / \alpha$$

and thus

$$\beta(\lambda^t / \lambda^{t'}) \leq \sum_{i=1}^n x_i^{t'}(\lambda^t / \lambda^{t'}) \leq \sum_{i=1}^n x_i^{t'} \tilde{u}_i^t(x^t) / \alpha \leq \tilde{u}^t(x^t) / \alpha,$$

where the last inequality comes from the concavity of \tilde{u}^t . By conditions (2.2), (2.3) and the bound $\|x^t\| \leq K$ of Lemma 2, we get that $\tilde{u}^t(x^t)$ is

bounded; and therefore λ'/λ' is bounded for almost all t and t' in T , proving (i).

Finally (iv) is obvious from (i).

Q.E.D.

Let \mathcal{H} be a finite subfield of \mathcal{C} . Denote by $\mathcal{M}(\mathcal{H})$ the set of all economies (a, u) in \mathcal{M} such that a is measurable w.r.t. \mathcal{H} , and u is measurable w.r.t. the product σ -field $\mathcal{H} \times \mathcal{B}$. Note that every such \mathcal{H} is identified with a partition of T into finitely many measurable sets T_1, \dots, T_k (the atoms of \mathcal{H}) which represent the types of agents in the economy. A uniform field is a field \mathcal{H} with all of its atoms, T_1, \dots, T_k having equal μ -measure, i.e., $\mu(T_i) = k^{-1}$ for all $1 \leq i \leq k$. Consider any $m \in \mathcal{M}(\mathcal{H})$ where T_1, \dots, T_p are the atoms of \mathcal{H} . Let v denote the vector measure (v_1, \dots, v_p) where $v_i(S) = \mu(T_i \cap S)$. The range \mathbb{R}_m of v is compact and convex, and the function $f_m: \mathbb{R}_m \rightarrow \mathbb{R}$ defined on \mathbb{R}_m by

$$f_m(v(S)) = w_m(S)$$

is a ‘‘market function’’ (i.e. continuous, concave, and homogeneous of degree one).

If \mathcal{H} is a uniform field, we distinguish a subset $\mathcal{M}^*(\mathcal{H})$ of $\mathcal{M}(\mathcal{H})$, which consists of all those m in $\mathcal{M}(\mathcal{H})$ for which \mathbb{R}_m, f_m enjoy the following properties:

(i) each $v_i(T_i) = p^{-1}$ (thus $\mathbb{R}_m = [0, p^{-1}]^p$).

(ii) $f_m: [0, p^{-1}]^p \rightarrow \mathbb{R}$ is linear on a conical neighbourhood of the diagonal of $[0, p^{-1}]^p$, i.e., there exist $\beta > 0$ and $\alpha_1, \dots, \alpha_p$ in \mathbb{R} such that $f_m(x) = \sum \alpha_i x_i$ if $x \in \{y \in [0, p^{-1}]^p: |y_i - y_j| < \beta \sum y_j\}$.

For $S \in \mathcal{C}$, define

$$V_m(S, \mathcal{H}) = \{f \in V_m(S): f \text{ is measurable w.r.t. } \mathcal{H}\}$$

$$V_m^+(S, \mathcal{H}) = \{f \in V_m^+(S): f \text{ is measurable w.r.t. } \mathcal{H}\}.$$

Also define, for any real number $q > 1$, the inessential economy $e(q) = ({}^q a, {}^q u)$ by

$${}^q a^t = q$$

and

$${}^q u^t(x) = \begin{cases} 2x^{1/2} & \text{if } 0 \leq x \leq 1 \\ x + 1 & \text{if } 1 < x \leq q^2 \\ q^2 + 1 - (x - q^2 + 1)^{-1} & \text{if } q^2 < x \end{cases}$$

for all $t \in T$. Note that ${}^q u^t(x) \leq q^2 + 2$ for all $t \in T$ and $x \in \mathbb{R}_+$.

Finally define $\mathcal{M}'(\delta)$, $\mathcal{M}'(\mathcal{H})$ etc. exactly as $\mathcal{M}(\delta)$, $\mathcal{M}(\mathcal{H})$ etc. with \mathcal{M}' in place of \mathcal{M} .

LEMMA 4. *Let $m \in \mathcal{M}'$ and $\delta > 0$. There exists $\bar{q} > 0$ such that $m \times e(q) \in \mathcal{M}'(\delta)$ for all $q > \bar{q}$.*

Proof. Let $m = (a, u)$ and let $K > 0$ be given by Lemma 2. Put $\bar{B} = \text{ess sup}\{u^t(Ke) : t \in T\}$ where $e = (1, \dots, 1)$. (The ess sup is finite by the uniform continuity of u at 0 and the concavity of u .) Let $\bar{q} = 1 + \max\{\bar{B} + 1, \delta^{-1}(\bar{B} + 1), (1 + \delta)^{-1}\}$. Denote $(a, u) \times e(q)$ by (\bar{a}, \bar{u}) . Note that $\nabla^q u^t(qa^t) = 1$ for all t , so by Proposition 36.4 in [8],

- (i) ${}^q a$ achieves $w_{e(q)}(R)$ for any $R \in \mathcal{C}$; from which it follows that
- (ii) z achieves $w_{(a, u)}(R) \Rightarrow (z, {}^q a)$ achieves $w_{(\bar{a}, \bar{u})}(R)$.

Let S be a coalition in \mathcal{C} with $\mu(S) > 0$ and let P be a δ -partition of S , $P = \{T_1, \dots, T_k\}$.

Let $(b_1\mu(T_1), \dots, b_k\mu(T_k)) \in V_{(\bar{a}, \bar{u})}^+(S|P)$, so that,

- (iii) $\sum_{i=1}^k \mu(T_i) b_i \leq w_{(\bar{a}, \bar{u})}(S)$ and
- (iv) $\mu(T_i) b_i \geq \int_{T_i} (u^t(a^t) + q + 1) d\mu(t)$.

Let y be an allocation that achieves $w_{(a, u)}(S)$. Put $x^t = (y^t, \beta^t)$ where $\beta^t = b_i - u^t(y^t) - 1$ for $t \in T_i$. Since $u^t(a^t) > u^t(0) = 0$, (iv) implies that $b_i > q + 1$. Moreover, by Lemma 2 and the construction of \bar{q} , $q > 1 + u^t(y^t)$. Therefore, $\beta^t > 1$ for a.a.t.

On the other hand, by (ii) and the construction of \bar{q} ,

$$w_{(\bar{a}, \bar{u})}(S) = \int_S u^t(y^t) d\mu(t) + \int_S (q + 1) d\mu(t) \leq (1 + \delta) q\mu(S);$$

so, by (iii), it follows that

$$\delta b_i \mu(S) \leq (1 + \delta) q\mu(S),$$

and therefore, again by the construction of \bar{q} , $\beta^t \leq b_i \leq (1 + \delta^{-1}) q < q^2$.

Since $1 < \beta^t \leq q^2$, we see that β^t is in the "linear part" of ${}^q u$, i.e. ${}^q u^t(\beta^t) = \beta^t + 1$ and hence $\bar{u}(x^t) = u^t(y^t) + \beta^t + 1 = b_i$ for a.a.t in T_i and $1 \leq i \leq k$. Moreover $\int_S \beta^t d\mu(t) = \sum_{i=1}^k b_i \mu(T_i) - \int_S u^t(y^t) d\mu(t) - \mu(S) \leq w_{\bar{a}, \bar{u}}(S) - w_{(a, u)}(S) - \mu(S) = (q + 1)\mu(S) - \mu(S) = q\mu(S) = \int_S {}^q a^t d\mu$.

This shows that $V_{(a, u) \times e(q)}^+(S|\mathcal{P}) \subset V_{(\bar{a}, \bar{u}) \times e(q)}(S|\mathcal{P})$, for $q > \bar{q}$, $S \in \mathcal{C}$, and every δ -partition \mathcal{P} of S . The converse inclusion is obvious from the definitions. Q.E.D.

COROLLARY 1. *Let $m, \tilde{m} \in \mathcal{M}'$ and $\|w_m - w_{\tilde{m}}\| < \delta$. There exists $\tilde{q} > 0$ such that $d(m \times e(q), \tilde{m} \times e(q)) < \delta$ for all $q > \tilde{q}$.*

Proof. Construct \bar{q}_m and $\bar{q}_{\bar{m}}$ in accordance with Lemma 4. Noting the obvious equality $w_{m \times e(q)} - w_{\bar{m} \times e(q)} = w_m - w_{\bar{m}}$, the corollary follows. Q.E.D.

COROLLARY 2. *For any $m \in \mathcal{M}'(\mathcal{H})$, there exists $\hat{q} > 0$ such that*

$$V_{m \times e(q)}(T, \mathcal{H}) = V_{\bar{m} \times e(q)}^+(T, \mathcal{H})$$

for all $q > \hat{q}$.

Proof. Let T_1, \dots, T_k be the atoms of \mathcal{H} and let $\delta = \min\{\mu(T_i) : 1 \leq i \leq k\}$. Repeat the proof of Lemma 4, with \mathcal{H} (and, T) in place of \mathcal{P} (and, S). Q.E.D.

Remark 1. If $m \in \mathcal{M}'$ (or $m \in \mathcal{M}$) then clearly $m \times e(q) \in \mathcal{M}'$ (or $m \times e(q) \in \mathcal{M}$), so that Lemma 4 and its corollaries hold on both \mathcal{M} and \mathcal{M}' .

LEMMA 5. *Let φ be an admissible map on \mathcal{M} satisfying Axioms A, C, D. Then for any uniform field \mathcal{H} , any $m \in \mathcal{M}^*(\mathcal{H})$, and $x \in \varphi(m)$ with $u_x(T) = w_m(T)$, we have $u_x = \Lambda w_m$ (i.e., $x \in \psi(m)$).*

Proof. Let $x \in \varphi(m)$, $m \in \mathcal{M}^*(\mathcal{H})$ and $u_x(T) = w_{(a,u)}(T)$.

Let $\mu_i: \mathcal{C} \rightarrow R$ be defined by $\mu_i(S) = p\mu(S \cap T_i)$, where T_1, \dots, T_p are the atoms of \mathcal{H} . Note that the μ_i are mutually singular probability measures, and that there exists $\beta > 0$ and $\alpha_1, \dots, \alpha_p$ in \mathbb{R} such that $w_m(S) = \sum \alpha_i \mu_i(S)$ for any S for which $|\mu_i(S) - \mu_j(S)| < \beta \sum_{l=1}^p \mu_l(S)$ for all i, j . Put $\bar{v} = \sum \alpha_i \mu_i$. It is well-known (see [8]) that $\Lambda w_m = \bar{v}$. Therefore we have to prove that $v = \bar{v}$, where $v: \mathcal{C} \rightarrow R$ denotes u_x , i.e., $v(S) = \int_S u^t(x^t) d\mu$.

Suppose $v \neq \bar{v}$. By (3.13) in [24], there is a universal positive constant K such that, for every automorphism θ of the measurable space (T, \mathcal{C}) and every positive integer k , there are real numbers a_i^k with $|a_i^k| = 1$, $0 \leq i \leq k$ and $\|\sum_0^k a_i \theta^i(v - \bar{v})\| \geq 2K\sqrt{k} \|v - \bar{v}\|$. By the triangle inequality it follows that there is a subset $B(k)$ of $\{0, 1, \dots, k\}$ with $\|\sum_{i \in B(k)} \theta^i(v - \bar{v})\| \geq K\sqrt{k} \|v - \bar{v}\|$. Choose $\varepsilon > 0$ such that $K\|v - \bar{v}\| > \varepsilon$. By Lemma 5.5 of [14], there is a μ -measure preserving automorphism θ such that

$$(i) \quad \|\sum_{i \in B(k)} \theta^i(w_m - \bar{v})\|/\sqrt{k} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

For every i , let $\theta^i \mathcal{H}$ denote the field that is generated by $\theta^{-i}T_1, \dots, \theta^{-i}T_p$ and let $\mathcal{H}_k = \vee_{i \in B(k)} \theta^i \mathcal{H}$ (where \vee denotes ‘‘join’’). Observe that $\theta^i m \in \mathcal{M}^*(\theta^i \mathcal{H})$ and $m^k = (1/\sqrt{k}) \times_{i \in B(k)} \theta^i m \in \mathcal{M}(\mathcal{H}_k)$ (where, for any economy $m = (a, u)$ in \mathcal{M} and α in \mathbb{R}_{++} , the economy $(a, \alpha u)$ is denoted αm).

Let $\bar{m} = (\bar{a}, \bar{u})$ be a one-commodity economy in $\mathcal{M}(\mathcal{H})$ with $\bar{u}^t(\bar{a}^t) = (d\bar{v}/d\mu)(t)$ and $(d\bar{u}/dx)(t)|_{x=\bar{a}^t} = 1$. Then $\int_S \bar{u}^t(\bar{a}^t) d\mu(t) = \bar{v}(S)$ for every $S \in \mathcal{C}$ and $w_{\bar{m}} = \bar{v}$. Denote $\bar{m}^k = (1/\sqrt{k}) \times_{i \in B(k)} \theta^i \bar{m} \in \mathcal{M}^*(\mathcal{H}_k)$.

Note that $w_{m^k} = (1/\sqrt{k}) \sum_{i \in B(k)} \theta^i w_m$ and $w_{\bar{m}^k} = (1/\sqrt{k}) \sum_{i \in B(k)} \theta^i \bar{v}$, and therefore $\|w_{m^k} - w_{\bar{m}^k}\| = (1/\sqrt{k}) \|\sum_{i \in B(k)} \theta^i (w_m - \bar{v})\| = o(1)$, as $k \rightarrow \infty$ by (i). Thus, for any $\varepsilon > 0$, there exists \bar{k} such that: $k > \bar{k} \Rightarrow \|w_{m^k} - w_{\bar{m}^k}\| < \delta(\varepsilon)$ (where $\delta(\varepsilon)$ is chosen for ε in accordance with Axiom D). By Corollary 1 of Lemma 4, $d(m^k \times e(q), \bar{m}^k \times e(q)) < \delta(\varepsilon)$, for all sufficiently large q . Let $\bar{m} = (\bar{a}, \bar{u})$. The economy $\bar{m}^k \times e(q)$ has $|B(k)| + 1$ commodities and the allocation $\times_{i \in B(k)} \theta^i \bar{a} \times^q a$ is P.O. in $\bar{m}^k \times e(q)$. Moreover, any I.R. and P.O. allocation in $\bar{m}^k \times e(q)$ is payoff equivalent to the allocation $\times_{i \in B(k)} \theta^i \bar{a} \times^q a$. Thus, if we denote $\bar{m}^k \times e(q) \equiv (\bar{a}^{k,q}, \bar{u}^{k,q})$, then $\bar{u}_y^{k,q} = (1/\sqrt{k}) \sum_{i \in B(k)} \theta^i \bar{v} + w_{e(q)}$ for all $y \in \varphi(\bar{m}^k \times e(q))$.

By the anonymity Axiom A, $\theta^i x \in \varphi(\theta^i m)$. Thus, by the consistency Axiom C (part (i)) and the Pareto-optimality of $\times_{i \in B(k)} \theta^i x \times^q a$, we have $\times_{i \in B(k)} \theta^i x \times^q a \in \varphi(m^k \times e(q))$. Then Axiom D (restricted continuity) implies

$$\begin{aligned} \varepsilon &> \left\| \left((1/\sqrt{k}) \sum_{i \in B(k)} \theta^i v + w_{e(q)} - \left((1/\sqrt{k}) \sum_{i \in B(k)} \theta^i \bar{v} + w_{e(q)} \right) \right) \right\| \\ &= \left\| (1/\sqrt{k}) \sum_{i \in B(k)} \theta^i (v - \bar{v}) \right\| \geq K \|v - \bar{v}\| > \varepsilon, \end{aligned}$$

a contradiction.

Q.E.D.

Remark 2. It is evident from the proof and our earlier remarks that Lemma 5 holds if we replace \mathcal{M} by \mathcal{M}' .

LEMMA 6. For any $m \in \mathcal{M}$,

$$\tau(m) = \sigma(m) = \psi(m) \subset A(m).$$

*Proof.*⁷ By [2], $\tau(m) = \sigma(m)$, so it suffices to prove that $\psi(m) = \tau(m) \subset A(m)$.

Let $x \in \tau(m)$ be a competitive allocation with prices $p > 0$. (By (2.5), it is clear that no component of p is zero). Then, as is well-known, x is I.R. and P.O. We claim that it is also uniformly bounded. Indeed, by (2.1), there exists a positive number A such that $\|a^t\| \leq A$ for a.a.t. Let $\mathcal{A} = \{y \in \mathbb{R}_+^n : p \cdot y \leq p \cdot Ae\}$ where $e = e_1 + \dots + e_n$ is the unit vector in \mathbb{R}^n . Then $p \cdot x^t \leq p \cdot a^t \leq p \cdot Ae$ for a.a.t, i.e. $x^t \in \mathcal{A}$ for a.a.t. Since \mathcal{A} is compact, x is uniformly bounded. We conclude that $x \in A(m)$.

⁷ This is analogous to the proofs of Propositions 5.1 and 5.2 in [5], but our Lemma does not directly follow from these propositions because (on account of condition (2.5) on the utilities) our domain \mathcal{M} is disjoint from that of [5].

It follows from Lemma 3 that there exists a measurable, bounded and uniformly positive $\lambda: T \rightarrow \mathbb{R}_{++}$ such that, if $m = (a, u)$ and $\bar{u}^t = \lambda(t) u^t$, then $\nabla \bar{u}^t(x^t) = p$ for a.a.t. From the concavity of \bar{u}^t and the above equality, we get

$$\bar{u}^t(\alpha) - \bar{u}^t(x^t) \leq p \cdot (\alpha - x^t)$$

for all $\alpha \in \mathbb{R}_+^n$ and a.a.t. But since the u^t are strictly monotonic (see (2.5)) and agents maximize utility at a competitive equilibrium, we have $p \cdot x^t = p \cdot a^t$ for a.a.t. Thus the inequality holds for a.a.t with $p \cdot (\alpha - a^t)$ in place of $p \cdot (\alpha - x^t)$. This shows that $\langle x, p \rangle$ is a T.U.C.E. (transferable utility competitive equilibrium—see [8]) of the economy (a, \bar{u}) . As an immediate consequence of Proposition 32.3 of [8] (see the remark following the statement of the proposition)

$$\begin{aligned} A_{(a, \bar{u})}(S) &= \int_S (\bar{u}^t(x^t) - p \cdot (x^t - a^t)) d\mu \\ &= \int_S \bar{u}^t(x^t) d\mu \end{aligned}$$

for all $S \in C$. (The second equality again uses the fact that $p \cdot x^t = p \cdot a^t$ for a.a.t). But $(a, \bar{u}) \sim (a, u)$ by construction, and $(a, \bar{u}) \in \mathcal{M}$ by Lemma 3. We conclude that $x \in \psi(m)$.

To show the converse, let $x \in \psi(m)$ be a value allocation of $m = (a, u)$. Then there exists an $\bar{m} = (a, \bar{u})$ in \mathcal{M} , with $\bar{m} \sim m$, such that

$$A_{(a, \bar{u})}(S) = \int_S \bar{u}^t(x^t) d\mu$$

for all $S \in C$. Again by Proposition 32.3 of [8] and its accompanying remark, there exists a $p \in \mathbb{R}_+^n$ such that $\langle x, p \rangle$ is a T.U.C.E. of (a, \bar{u}) , i.e.,

$$\max_{y \in \mathbb{R}_+^n} [\bar{u}^t(y) - p \cdot (y - a^t)] = \bar{u}^t(x^t)$$

for a.a.t; and, moreover,

$$\int_S [\bar{u}^t(x^t) - p \cdot (x^t - a^t)] = A_{(a, \bar{u})}(S)$$

for all $S \in C$. Consider the last three equalities. The first and the third imply that $p \cdot x^t - p \cdot a^t = 0$ for a.a.t. The second then implies that

$$\bar{u}^t(x^t) = \max\{\bar{u}^t(y) : y \in \mathbb{R}_+^n, p \cdot y = p \cdot a^t\}.$$

Thus $x \in \tau(a, \bar{u})$. But $\tau(a, \bar{u}) = \tau(a, u)$ since $(a, u) \sim (a, \bar{u})$, so $x \in \tau(a, u)$ as well. Q.E.D.

LEMMA 7. For any $(a, u) \in \mathcal{M}$, there exists $\bar{q} \in \mathbb{R}_+$ such that if $q > \bar{q}$ then every competitive allocation $(x, b): T \rightarrow \mathbb{R}_+^I \times \mathbb{R}_+$ of $(a, u) \times e(q)$ has the property $1 < b^t < q^2$ for a.a.t in T . Moreover all competitive allocations give rise to the same payoffs.

Proof. By (2.4), there exists A in \mathbb{R}_+ with $A \geq u^t(\alpha)$ for every t in T and α in \mathbb{R}_+^I . Then, since (x, b) is individually rational, we have (for $q > A + 1$)

$$\begin{aligned} A + {}^q u^t(b^t) &\geq u^t(x^t) + {}^q u^t(b^t) \\ &\geq u^t(a^t) + {}^q u^t(q) \\ &\geq q + 1; \end{aligned}$$

and therefore ${}^q u^t(b^t) \geq q + 1 - A > 2$, implying that $b^t > 1$.

By (2.1) there exists B in \mathbb{R}_+ with $\sum_j a_j^t \leq B$ for a.a.t. in T . By (2.6), there exists C in \mathbb{R}_{++} such that $u_j^t(\alpha) > C$ for every $\alpha \in \mathbb{R}_+^I$ with $\sum_j \alpha_j \leq B$. Let $\bar{q} > 0$ be sufficiently large to ensure that $\bar{q} > A + 1$ and $\bar{q}^2 > \bar{q} + B/C$. Suppose $q > \bar{q}$. By the monotonicity of utilities ((2.5)) it is clear that any competitive price vector for (x, b) is strictly positive in each component. (Recall (x, b) is a competitive allocation of $(a, u) \times e(q)$). W.l.o.g. let $(p, 1)$ in $\mathbb{R}_{++}^I \times \mathbb{R}_{++}$ be such a price vector. Then, for a.a.t in T , we must have $p_j u_j^t(x^t) = ({}^q u^t)'(b^t)$ (where $'$ denotes derivative). Since $\bar{q} > A + 1$, we have $b_j^t > 1$ as shown earlier. This implies that $({}^q u^t)'(b^t) \leq 1$ for a.a.t. As $\sum_j a_j^t \leq A$ for a.a.t, it follows that there exists a trader set of positive measure with $\sum_j x_j^t \leq B$ and $u_j^t(x^t) \geq C$ for t in this set. We conclude that $p_j \leq 1/C$. Therefore, for a.a.t in T , $b^t \leq q + a^t \cdot p \leq q + B/C < q^2$.

To finish the proof of the lemma, it remains to be shown that all competitive allocations (x, b) of $(a, u) \times e(q)$, possessing the property: $1 < b^t < q^2$ for a.a.t in T , give rise to the same payoffs.

Consider the economy (even though it does not belong in \mathcal{M}) defined exactly as $(a, u) \times e(q)$, except that ${}^q u^t$ is replaced by $\bar{u}(z) = z + 1$ for all $t \in T$. It is immediate that (x, b) is also a competitive allocation of this economy with the same prices $(p, 1)$, because the gradients of agents' utilities at (x, b) remain the same and they all equal $(p, 1)$.

Then consider the economy obtained from the above economy by subtracting 1 from each agent's utility. Then (x, b) with prices $(p, 1)$ continues to be a CE of this last economy. By Proposition 32.1 in [8], (x, b) is a transferable utility competitive equilibrium (TUCE) of the economy (a, u) ; and by Theorem J in [8], all TUCE of (a, u) give rise to the same payoffs.

Q.E.D.

LEMMA 8. Let φ be an admissible map on \mathcal{M} satisfying Axioms A, B, C, D. Then $\varphi(m) \subset \psi(m)$ for any m in \mathcal{M} .

Proof. Let $m \in \mathcal{M}$ and let $x \in \varphi(m)$. By resealing utilities if necessary and applying Lemma 3, we can assume w.l.o.g. that $u_x(T) = w_m(T)$. Suppose $u_x \neq \Lambda w_m$. Then there exists $1 > \varepsilon > 0$ such that $\|u_x - \Lambda w_m\| \geq 2\varepsilon$. By Proposition 5.1 and its proof in [12], for any given $\varepsilon > \delta > 0$, there exists a uniform field \mathcal{H}^* and $m^* = (a^*, u^*) \in \mathcal{M}^*(\mathcal{H}^*)$ with $\|w_m - w_{m^*}\| < \delta$. By Corollaries 1 and 2 of Lemma 4, there exists \bar{q} such that for $q > \bar{q}$, $d(m \times e(q), m^* \times e(q)) < \delta$ and $V_{m^* \times e(q)}^+(T, \mathcal{H}) = V_{m^* \times e(q)}(T, \mathcal{H})$.

Now $A(e(q)) = \{^q a\}$, so $\varphi(e(q)) = \{^q a\}$. Moreover it is clear that $(x, ^q a)$ is Pareto-optimal in $m \times e(q)$ (indeed the gradient of each agent's utility at $(x^t, ^q a^t)$ is $(p, 1)$ for some p). Therefore, by (i) of axiom C, $(x, ^q a) \in \varphi(m \times e(q))$.

By Axiom B, there exists $y \in \varphi(m^* \times e(q))$ with $u_y \in V_{m^* \times e(q)}(T, \mathcal{H}^*)$. As $V_{m^* \times e(q)}^+(T, \mathcal{H}^*) = V_{m^* \times e(q)}(T, \mathcal{H}^*)$, it follows from the I.R. and P.O. properties of y , that $u_y^{*q}(T) = w_{m^* \times e(q)}(T)$, where we denote $m^* \times e(q) = (a^{*q}, u^{*q})$. By Lemma 5, $u_y^{*q} = \Lambda w_{m^* \times e(q)} = \Lambda w_{m^*} + \Lambda w_{e(q)}$. (The last equality comes from the linearity of Λ and the obvious fact that $w_{m^* \times e(q)} = w_{m^*} + w_{e(q)}$.) Denoting $m \times e(q) = (a^q, u^q)$ we observe that $u_{(x, ^q a)}^q = u_x + \Lambda w_{e(q)}$ (since $\Lambda w_{e(q)}(S) = \int_S ^q a \, d\mu$ for all $S \in \mathcal{C}$). Thus

$$\begin{aligned} \|u_{(x, ^q a)}^q - u_y^{*q}\| &= \|u_x - \Lambda w_{m^*}\| \\ &\geq \|\Lambda w_m - u_x\| - \|\Lambda w_{m^*} - \Lambda w_m\| \\ &\geq 2\varepsilon - \delta > \varepsilon, \end{aligned}$$

contradicting Axiom D.

LEMMA 9. *Let φ be an admissible map on \mathcal{M} satisfying Axioms A, B, C, D. Then $\psi(m) \subset \varphi(m)$ for any m in \mathcal{M} .*

Proof. Let $x \in \psi(m)$. Then $x \in A(m)$ by Lemma 6. By rescaling utilities if necessary and applying Lemma 3, we can assume w.l.o.g. that $u_x(T) = w_m(T)$ where $m = (a, u)$, and that $u_x = \Lambda w_m$. Therefore, for any positive $q > 1$, $(x, ^q a) \in \psi(m \times e(q))$. By assumption, $\varphi(m \times e(q)) \neq \emptyset$ and thus there exists $(y, b) \in \varphi(m \times e(q))$. By lemmas 8 and 6, $(y, b) \in \psi(m \times e(q)) = \tau(m \times e(q))$. Then (by Lemma 7) $u^t(x^t) + ^q a^t = u^t(y^t) + b^t$ for sufficiently large q ; and therefore (by the admissibility of φ) $(x, ^q a) \in \varphi(m \times e(q))$. The consistency axiom (ii) now implies that $x \in \varphi(m)$. Q.E.D.

LEMMA 10. *ψ is admissible and satisfies Axioms A, B, C, D on \mathcal{M} .*

Proof. It is well known that ψ satisfies A. By Lemma 6, ψ coincides with the competitive correspondence τ . It is obvious that τ satisfies Axiom B. We shall next show that τ satisfies Axiom C and is admissible.

Let $m = (a, u)$, $\tilde{m} = (\tilde{a}, \tilde{u})$, $x \in \tau(m)$, $\tilde{x} \in \tau(\tilde{m})$ and suppose (x, \tilde{x}) is Pareto-optimal in $m \times \tilde{m}$. Let p, \tilde{p} be CE prices for x, \tilde{x} . By Lemma 1, and the well-known IR property of competitive allocations, $x^t \geq 0$ and $\tilde{x}^t \geq 0$ for a.a.t in T . By Lemma 3 and the inclusion $\tau(m) \subset A(m)$ (see Lemma 6), there exist measurable $\lambda: T \rightarrow \mathbb{R}_{++}$, $\tilde{\lambda}: T \rightarrow \mathbb{R}_{++}$ such that $\nabla u^t(x^t) = \lambda^t p$ and $\nabla \tilde{u}^t(\tilde{x}^t) = \tilde{\lambda}^t \tilde{p}$ for a.a.t in T . But, again by Lemma 3, Pareto optimality of (x, \tilde{x}) in $m \times \tilde{m}$ implies that $\lambda^t/\tilde{\lambda}^t = K$ for some constant K and a.a.t. But then (x, \tilde{x}) is competitive with prices (Kp, \tilde{p}) i.e. $(x, \tilde{x}) \in \tau(m \times \tilde{m})$, verifying (i) of Axiom C. Let $(x, \tilde{x}) \in \tau(m \times \tilde{m})$ and $\tilde{x} = \tilde{a}$, and suppose (p, \tilde{p}) is a C.E. price for (x, \tilde{x}) . Then since $(x^t, \tilde{x}^t) \in \{(y^t, \tilde{x}^t): p \cdot y^t \leq p \cdot a^t, y^t \geq 0\} \subset \{(y^t, z^t): (p, \tilde{p}) \cdot (y^t, z^t) \leq (p, \tilde{p}) \cdot (a^t, \tilde{a}^t), y^t \geq 0, z^t \geq 0\}$, it follows that x is competitive in m with prices p , verifying (ii) of Axiom C.

It is well-known that $\emptyset \neq \tau(m) \subset A(m)$. (See, e.g., [3]). Suppose x is competitive in $m = (a, u)$ with prices p , and that $y \in A(a, u)$ with $u^t(x^t) = u^t(y^t)$ for a.a.t. From the strict monotonicity of u^t it follows that $p \cdot y^t = p \cdot x^t$ for a.a.t. Hence y^t is an "optimal bundle" in t 's "budget set" at prices p for a.a.t. Since y is a T -allocation by assumption, it follows that y is also competitive in (a, u) with prices p . This establishes that τ is admissible.

Finally let $0 < \varepsilon < 1$; $m = (a, u)$ and $\tilde{m} = (\tilde{a}, \tilde{u})$ be in $\mathcal{M}(\varepsilon)$; $x \in \psi(m)$; $\tilde{x} \in \psi(\tilde{m})$; $u_x(T) = w_m(T)$; $u_{\tilde{x}}(T) = w_{\tilde{m}}(T)$. There exists measurable $\lambda: T \rightarrow \mathbb{R}_{++}$ such that $\int_T \lambda^t u^t(x^t) d\mu \geq \int_T \lambda^t u^t(y^t) d\mu$ and also $\int_T u^t(x^t) d\mu \geq \int_T \lambda^t u^t(y^t) d\mu$, for every allocation y in m . By Lemma 3, $\lambda^t = K$ for a.a.t in T , and some constant K . Therefore $Ku_x = \lambda w_{(a, Ku)} = K\lambda w_{(a, u)}$, implying $u_x = \lambda w_{(a, u)}$. Similarly $u_{\tilde{x}} = \lambda w_{(\tilde{a}, \tilde{u})}$. As λ has norm 1 (see, e.g., [8]), $\|u_x - u_{\tilde{x}}\| \leq \|w_{(a, u)} - w_{(\tilde{a}, \tilde{u})}\| < \varepsilon$, proving that ψ satisfies Axiom D. Q.E.D.

Lemmas 8, 9 and 10 immediately yield the Theorem.

6. PROOF OF THEOREM A

Recall that the set $\tilde{\mathcal{M}}$ of ordinal economies is given by $\tilde{\mathcal{M}} = \{(a, \succ): \exists$ a representation u of \succ such that (a, u) satisfies conditions (2.1) till (2.7) $\}$. Let

$$\tilde{\varphi}: \tilde{\mathcal{M}} \rightarrow \text{sets of allocations}$$

be an admissible map on $\tilde{\mathcal{M}}$ which satisfies Axioms I, II, III, IV. Define

$$\varphi: \mathcal{M} \rightarrow \text{sets of allocations}$$

by $\varphi(a, u) = \tilde{\varphi}(a, \succ)$, where \succ is the preference underlying u . It is clear that the admissibility of $\tilde{\varphi}$ on $\tilde{\mathcal{M}}$ implies the admissibility of φ on \mathcal{M} . Also

from the fact that $\tilde{\varphi}$ satisfies Axioms I, II, III, IV it follows that φ satisfies Axioms A, B, C, D (respectively). But then, by Theorem B,

$$\varphi(a, u) = \tau(a, u) \equiv \text{the set of competitive allocations of } (a, u)$$

for $(a, u) \in \mathcal{M}$. From our definition of φ , we obtain $\tilde{\varphi}(a, \succcurlyeq) = \tau(a, u) \equiv$ the set of competitive allocations of (a, u) for any $(a, u) \in \mathcal{M}$, where u represents \succcurlyeq . Thus (as $\tau(a, u) = \tau(a, u')$ whenever u and u' represent the same preference), φ is the competitive correspondence. Finally, from the admissibility of τ and the fact that τ satisfies A, B, C, D (see Lemmas 6 and 10) we deduce that $\tilde{\varphi}$ is admissible and satisfies Axioms I, II, III, IV.

7. CONCLUDING REMARKS

Remark 3. It is clear from our proof that if an admissible map φ on \mathcal{M} (or \mathcal{M}') satisfies Axioms A, B, C(i), D, then $\varphi(m) \subset \psi(m)$ for all $m \in \mathcal{M}$ (or \mathcal{M}'). Thus part of the (axiomatic) equivalence principle retains its validity with a weaker consistency axiom, and also on a larger domain.

Remark 4. Consider the following variants of the domain \mathcal{M} :

- (a) All $m \in \mathcal{M}$ that are of finite type;
- (b) All (a, u) which satisfy the same conditions as before, except that we weaken (2.5) to: “ $u'_j(y)$ exists and is positive whenever $y_j > 0$; and $u'_j(x) \rightarrow u'_j(y)$ whenever $x \rightarrow y$ and $y_j > 0$,” and add

$$\{\alpha \in \mathbb{R}^l_+ : u'(\alpha) \geq u'(a')\} \subset \mathbb{R}^l_{++}$$

for a.a.t in T .

- (c) All (a, u) in (b) (or in (a)), with u' of differentiability class C^k on \mathbb{R}^l_+ , for a.a.t in T (with $k \geq 1$).

The domain of (a) is a subset of \mathcal{M} , of (b) is a superset of \mathcal{M} , of (c) is neither. But our theorem remains intact (with minor and obvious variations in the proof) on each of these domains. (Indeed, on the domain of (a), let $A(m)$ consist of allocations that are I R., P.O. and give identical payoffs to all agents of the same type. Then clearly Axiom B is redundant, and our theorem holds without it on this domain.)

Remark 5. We conjecture that our axioms are “tight,” i.e., removing any one of them would make the theorem break down. This is motivated by the following considerations.

- (i) *Continuity.* The map $m \mapsto A(m)$ is admissible and satisfies Axioms A, B, C but not D.

(ii) *Consistency*. We do not know what would happen if one were to drop just part (i) (or part (ii)) of Axiom C. But we cannot drop C in its entirety. Indeed, let H be the set of market games in pNA (see [8]). Consider a uniformly continuous (in the BV norm) map η from H to NA which satisfies Anonymity, Individual Rationality and Efficiency but violates Additivity. Take the domain of economies of Remark 5(a). Define a T -allocation x to be an “ η -allocation” of the economy m if there exists $\bar{m} = (\bar{a}, \bar{u})$ with $\int_S \bar{u}'(x^t) d\mu(t) = (\eta w_{\bar{m}})(S)$ for all $S \in \mathcal{C}$. Then the map $m \mapsto$ set of η -allocations of m , will be admissible and satisfy Axioms A, B, D but not C.

By way of another example, take the same domain and consider the “NTU-Harsanyi value allocations” defined in [19] (see also [17, 18]). This map satisfies Axioms A, B, D (the last requires some checking) but not C.

(iii) *Anonymity and Equity*. The two axioms are intimately related. In the theory of TU-values (or indeed any other single-point solution, where $\varphi(m)$ is assumed to have cardinality one), anonymity implies equity, and is the stronger axiom. It has been the tradition to favour anonymity. This is particularly so in the standard axiomatic derivations of the TU value (on finite games, or various classes of non-atomic games such as (see [8]) pNA , $bv'NA$, etc.), where anonymity is postulated, even though the weaker equity axiom would do equally well.

The only two instances we know, where anonymity can not *obviously* be replaced with equity, occurs in [24] and (for closely related reasons) in [14] (even if, in [14], φ is assumed to be a single-point solution). Our conjecture is that anonymity is indispensable in these two instances.

In our current NTU scenario, the set $\varphi(m)$ cannot be taken to be a singleton. Thus anonymity and equity become a priori independent axioms. We do not know if dropping either one of them would still leave our theorem intact. What is quite clear is that both cannot be dropped. Indeed, on the domain of Remark 5(a), consider the (non-symmetric) NTU-value $\tilde{\psi}$, defined exactly as ψ , but with \tilde{A} in place of A , where \tilde{A} is as in Example 6.13 of [14]. Then $\tilde{\psi}$ is admissible and satisfies Axioms C and D, but neither A nor B.

Remark 6. Our theorem shows that competitive, core and value allocations are equivalent and satisfy Axioms A, B, C, D; and, indeed, any other solution coincides with them if, and only if, it satisfies these axioms.

A weaker result could be stated which would entail a seemingly more coherent (and more aesthetic?) proof. Restrict attention to value allocations ψ as the concrete “reference case.” A corollary of our theorem is:

there is one, and only one, φ on \mathcal{M} which satisfies Axioms A, B, C and D, and it is ψ . The proof of this corollary can be made more “intrinsic.” One would simply reword Lemmas 6, 7 and their proofs so as to suppress any allusion to “extraneous” objects such as τ or σ (competitive or core allocations).

Our current proof not only yields a stronger theorem but has other advantages. Lemma 6 highlights the transparency of the basic argument in [4] for value equivalence. And Lemma 7 may be of some independent interest as a contribution to the literature on the uniqueness of competitive equilibria.

Remark 7. Our earlier analysis of TU economies in [14] invites immediate comparison with this paper. Not surprisingly, the TU result is sharper. Pareto-optimality is deduced from the axioms in [14] but made an axiom here. Moreover, in [14], it is not assumed that $\varphi(m)$ is nonempty, except for one-commodity economies. The existence, indeed uniqueness, of solutions in $\varphi(m)$ is inferred in [14]. Here we have had to directly postulate the nonemptiness of $\varphi(m)$. (Given our NTU scenario, uniqueness of course breaks down and $\varphi(m)$ is normally multiple-valued.) But, as is evident from the proof, it would be sufficient for us to assume somewhat less: for any $m \in \mathcal{M}$, $\varphi(m \times e(q))$ is nonempty for sufficiently large q . (Notice, by the way, that we also show in Lemma 7 that $\varphi(m \times e(q))$ is a singleton in this circumstance.) Finally, only the analogue of part (i) of the consistency axiom is needed in [14].

Remark 8. We have assumed that utilities are concave (in the cardinal case) and that preferences are convex (in the ordinal case). These assumptions were needed also in [4] to prove the equivalence of value and competitive allocations. On the other hand (see [2]) core and competitive allocations coincide without such convexity assumptions. It would be interesting to investigate how far the meta-equivalence phenomenon (including an axiomatic approach to it) can go without convexity.

We have also not explored the effect of keeping the number of commodities bounded or fixed in an axiomatic framework. (For the fixed case, our consistency axiom does not apply and would need to be reformulated.) Such an exploration would require a careful look at the set of TU characteristic functions that are generated as endowments and utilities vary on a fixed commodity space.

Remark 9. The boundedness of the initial allocation a (see 2.1) is an assumption which facilitates the presentation of our results. It could be dropped at the cost of a stronger anonymity axiom and also of some technical complications both in the statement of the theorem and its proof.

A similar situation arises in the study of both core equivalence—see [11] and of value equivalence—see [4], especially section 10.

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