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TRADE WITH HETROGENEOUS BELIEFS

by

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Trade with Heterogeneous Beliefs[1]

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Abstract:

The paper analyzes an economy with asymmetric information in which agents trade in contingent assets. The new feature in the model is that each agent may have any prior belief on the states of nature and thus the posterior belief of an agent maybe any probability distribution that is consistent with his private information. We study two solution concepts: Equilibrium, which assumes rationality and market clearing, and common knowledge equilibrium (CKE) which makes the stronger assumption that rationality, market clearing, and the parameters which define the economy are common knowledge. The two main results characterize the set of equilibrium prices and the set of CKE prices in terms of parameters which specify for each state s and event E the amount of money in the hands of agents who know the event E at the state s. The characterizations that are obtained apply to a broad class of preferences which include all preferences that can be represented by the expectation of a state dependent monotone utility function. One implication of these results is a characterization of the information that is revealed in a CKE.

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1 Introduction

The paper analyzes an economy with asymmetric information in which agents trade in contingent assets. Different non-trade theorems (for example, Milgrom and Stokey 1982) establish that if risk averse agents have a common prior on the set of states then there will be no trade. In particular, there is no trade in a rational expectations equilibrium (REE) [3]. The main results in this paper characterize the set of equilibrium and CKE prices when agents may have different beliefs on the state of nature. The characterizations that are obtained apply to a broad class of preferences over uncertain outcomes and highlight the way in which the knowledge that agents have restricts the set of possible trades. We turn now to a more detailed description of the set-up and the results.

Let $S = \{1, .., n\}$ be the set of states of nature. For $s \in S$, A_s is an asset that pays \$1 in the state s and zero at any other state. The assets are commitments to make contingent payments which are issued by some agents and bought by others. An agent who buys (sells) one unit of the asset A_s gets(pays) \$1 if the state is s and zero otherwise. We study two solution concepts. The first concept, which we simply call equilibrium, assumes rationality and market clearing. The second concept, common knowledge equilibrium (CKE), makes the stronger assumption that the rationality of the agents (R), market clearing (MC), and the parameters that define the economy ($\mathcal{E}$) are all common knowledge among the agents. Let $\overline{P}^s$ and P^s denote, respectively, the set of equilibrium prices and the set of CKE prices at the state $s \in S$. More specifically, let I denote the set of agents and let $\psi^i(s)$ denote the private information of agent i at the state s. The set $\overline{P}^s$ is the set of equilibrium prices that are generated by all profiles of (subjective) probabilities $\gamma = \{\gamma^i\}_{i \in I}$ such that $\gamma^i \in \Delta(\psi^i(s))$. ($\gamma^i$ is the posterior probability of agent i at the state s.). Similarly, the set P^s is the set of equilibrium prices that are generated by all profiles of beliefs that are consistent with common knowledge of R, MC, and $\mathcal{E}$.

Our two main results, theorem 1 and theorem 2, characterize the set of equilibrium prices and the set of CKE prices respectively. These theorems have two notable features:

(a) The sets $\overline{P}^s$ and P^s are characterized in terms of the parameters

[3] The formal statement and proof of this proposition can be found in the appendix (proposition 1).

$m(I_{s'}^E)$, $s' \in S$, $E \subseteq S$, where $m(I_{s'}^E)$ specifies the aggregate amount of money in the hands of agents who know the event E at the state s'. In particular,we obtain that the set $\overline{P}^s$ is the core of a cooperative game where the set of players is the set of states of nature and the value of a coalition E, $E \subseteq S$, is $m(I_s^E)$. Thus, the way in which the information that agents have restricts the set of equilibrium trades can be captured by the concept of the core.

(b) The characterizations in theorems 1 and 2 apply to a broad class of preferences over uncertain outcomes. This class which we denote by $\mathcal{M}$ (for monotonicity) includes all preferences that can be represented by an expectation of a monotone state dependent utility from money[4]. Thus, theorem 1 (theorem 2) implies that for every profile of preferences in $\mathcal{M}$ the set of equilibrium prices (CKE prices) is the same set. In particular, the set of equilibrium prices (CKE prices) does not depend on whether agents are risk averse or on their degree of risk aversion[5].

An interesting implication of theorem 2 is a characterization of the information that is revealed in a CKE at a given state s. Specifically, we characterize the minimal set of states that is common knowledge in every CKE at a given state s.

There is some previous work which examines the implications of rationality and market clearing in economies with asymmetric information in which agents have heterogeneous beliefs. MacAllister (1990) and Dutta and Morris (1997) propose a solution concept, Belief Equilibrium, which is stronger than CKE as it assumes that in addition to common knowledge of rationality and market clearing there is also common knowledge of the beliefs of the players. Desgranges (1999) was the first to propose the concept of CKE. Desgranges studies exchange economies and the main focus in his work is on determining conditions under which CKE implies the REE outcome. Ben-Porath and Heifetz (2006) propose the concept of $CKRMC$ (for common knowledge of

[4]A bundle of assets z defines an outcome $x(z) \in R^n$ that specifies the amount of money that z generates at each state $s \in S$. To say that the preference of an agent i can be represented by the expectation of a state dependent utility from money $u(\cdot, s)$ means that i evaluates a bundle z by the expectation of $u(x(z)_s, s)$ w.r.t his subjective probability distribution γ^i.

[5]A clarifying comment maybe in place: Obviously, for a specific profile of beliefs the set of equilibrium prices (CKE prices) depends on the preferences of the agents. What theorem 1 says is that if we pick a profile of preferences for the agents, $\succeq \equiv (\succeq^i)_{i \in I}$, and then look at the set of equilibrium prices that is generated when we go over all profiles of subjective beliefs then this set of prices is the same set for every profile $\succeq$ in $\mathcal{M}$. Theorem 2 establishes a similar result for the set of CKE prices.

rationality and market clearing). $CKRMC$ differs from CKE in that it assumes a situation where the agents have a common prior on the set of states of nature but have different subjective beliefs on the price function (the function that associates a vector of prices with each state of nature.) Theorem 1 in BH establishes that under general conditions, which apply to asset economies, the set of $CKRMC$ prices at a given state s equals the set of CKE prices at s. (Roughly speaking, the result establishes that a common prior on the state of nature with subjective beliefs on the price function is equivalent to subjective beliefs on the state of nature.) I focus here on CKE (rather than $CKRMC$) because it is easier to apply. A different approach to the possibility of trade with heterogeneous beliefs has been taken by Morris (1994). Morris studies an exchange economy with asymmetric information and explores conditions on the beliefs of the players which will yield non-trade results in a set-up where the trade contracts can be conditioned only on payoff relevant states but not on the private signals that players observe. In our model a state of nature defines both the payoff of the players and their knowledge. For most of the paper we assume that any state contingent transaction can be executed. In section 7 we show how our main results, theorem 1 and theorem 2, can be extended to the case where there are some parameters that define the state of nature (such as the private signals of the agents) on which it is impossible to contract. In any case our focus is on a characterization of equilibrium and CKE prices and the characterization of the information that is revealed in a CKE. A situation where there is no trade in equilibrium (CKE) will be reflected in our model by the fact that the only equilibrium (CKE) price vector at a state s is the vector where the price of the asset A_s is 1 and the price of every other asset $A_{s'}$, $s' \neq s$, is zero.

The paper is organized as follows: In section 2 we present the model and the main results (theorem 1 and theorem 2). The proof of theorem 1 is presented in section 3 (the proof of theorem 2 from theorem 1 is simple and is given in section 2.) In section 4 we present a characterization of the information that is revealed in a CKE. In section 5 we discuss the relationship between CKE, REE, and $CKRMC$. Section 6 provides an epistemic foundation for CKE. In section 7 we present extensions of theorem 1 and theorem 2 to asset economies that are incomplete. Section 8 concludes. All the proofs that are not presented in the body of the paper can be found in the appendix.

2 The model and the results.

In this section I define an economy with asymmetric information in which agents trade in contingent assets. I then present the main results, theorem 1 and theorem 2, and demonstrate them by means of a simple example.

The economy is defined as follows:

The set of agents is $I \equiv [0,1]$. The set of states of nature is $S \equiv \{1, .., n\}$. For $s \in S$ we let A_s denote the asset which pays \$1 in the state s and zero otherwise. The asset A_{n+1} is money, i.e., A_{n+1} pays \$1 in every state. For $E \subseteq S$ define $A_E \equiv \sum_{s \in E} A_s$. A_E is the composite asset which pays \$1 in the event E and zero otherwise. The assets are commitments to make contingent payments which are issued by some agents and bought by others. An agent who buys (sells) one unit of the asset $A_s, s \in S$, gets (pays) \$1 if the state is s and zero otherwise. A bundle of assets is a vector $z = (z_1, ..., z_{n+1}) \in R^{n+1}$ where $z_k, k = 1, .., n+1$, is the number of units of the asset A_k in the bundle. For $s \in S$ $z_s < 0$ means that $|z_s|$ units of the asset A_s have been sold. A bundle $z = (z_1, ..., z_{n+1})$ defines an outcome $x \in R^n, x = x(z)$, as follows:

For $s \in S$ $x_s \equiv z_s + z_{n+1}$.

x_s is the number of \$ that an agent who holds the bundle z will have if the true state is s.

We assume that each agent is restricted to the choice of bundles that generate outcomes in $X \equiv R^n_+$. (The reason for this restriction will become clear later on). Thus, each agent $i \in I$ is characterized by:

(1) $m^i -$ an initial amount of money.

(2) $\psi^i -$ an information partition of S. For $s \in S$ $\psi^i(s)$ is the event that i knows at the state s.

(3) $\succsim^i \equiv \left\{ \succeq^i_\gamma \right\}_{\gamma \in \Delta(S)}$ where $\succsim^i_\gamma$ is the preference relation of agent i on X w.r.t. the subjective probability distribution γ.

We make only two assumptions on $\succsim^i_\gamma$: For $x, y \in X$,

(1) Monotonicity, (M): If $x \geqq y$ then $x \succsim^i_\gamma y$ and if for some $s \in S$ s.t. $\gamma(s) > 0$ $x_s > y_s$ then $x \succ^i_\gamma y$.

(2) Null events don't count (N): If $x_s \neq y_s \Rightarrow \gamma(s) = 0$ then $x \sim^i_\gamma y$.

We denote by $\mathcal{M}$ the class of preferences that satisfies (M) and (N).

Remarks:

(1) The class $\mathcal{M}$ includes all the preferences that can be represented by an expectation of a monotone state-dependent utility function.

(2) The class $\mathcal{M}$ includes incomplete preferences. We say that a bundle z is an optimal choice for an agent i w.r.t $\succsim^i_\gamma$ and a choice (budget) set B if there is no bundle $z' \in B$ such that $x(z') \succ^i_\gamma x(z)$.

(3) We assume that the function $m(i) = m^i$ is integrable and we normalize the aggregate amount of money to 1. Thus, $\int_{i \in I} m^i = 1$.

We let $p = (p_s)_{s \in S}$ denote a vector of prices of the assets.

Since $A_S \equiv \sum_{s \in S} A_s$ is equivalent to money non-arbitrage implies $\sum_{s \in S} p_s = 1$. (Non-arbitrage is implied by the definition of an equilibrium which will be given later on.)

It is convenient to think of the economy as operating in two periods. In period 1 nature selects a state $\widehat{s}$. Each agent i gets his private signal $\psi^i(\widehat{s})$ and then as a function of the vector of prices p and his posterior subjective belief γ^i on $\psi^i(\widehat{s})$[6] agent i chooses a bundle of assets z^i in his budget set $B(p, m^i)$ ($B(p, m^i)$ is defined in the next paragraph.) In period 2 the state $\widehat{s}$ becomes common knowledge and the transactions which the assets define are implemented.

The budget set $B(p, m)$ is defined as follows:

A vector $z \in R^n$ belongs to $B(p, m)$ iff:

(1) Income constraint (IC): $\sum_{s \in S} p_s \cdot z_s + z_{n+1} = m$.

(2) No borrowing (NB): $z_{n+1} \geqq 0$.

(3) Complete Coverage (CC): $\forall s \in S \ -z_s \leqq z_{n+1}$.

The constraints (IC) and (NB) are standard. The constraint (CC) requires that an agent will be able to pay back at every state. In particular, if an agent sold $|z_s|$ units of A_s then (CC) requires that the amount of money that is available for him at the state s, $\$z_{n+1}$, is sufficient to cover the payment that he has to make which is $\$-z_s$. To further motivate (CC) and get a

[6]We assume that the prior probability of each agent i assigns a positive probability to each state.

better sense of the model we present lemma 1. Lemma 1 states that the purchase of an asset A_E, $E \subseteq S$, is equivalent to the sale of the complementary asset $A_{S \setminus E}$ in that both transactions generate the same outcome. Furthermore, the purchase of A_E satisfies (NB) iff the sale of $A_{S \setminus E}$ satisfies (CC). Put differently, If (CC) is relaxed then the agent is in effect in a situation where he can borrow money because any outcome that can be generated by borrowing money to buy some asset $A_E, E \subseteq S$, can also be generated by selling short the complementary asset $A_{S \setminus E}$.

Lemma 1:

Let p be a price vector s.t. $\sum_{s \in S} p_s = 1$ and let $m \geqq 0$ be an initial endowment of money. Let $\overline{z}^y_E$ be the bundle where the agent buys y units of the asset E and let $z^y_{S \setminus E}$ be the bundle where the agent sells y units of the asset $S \setminus E$, that is

$$(\overline{z}^y_E)_k \equiv \begin{cases} y & k \in E \\ 0 & k \in S \setminus E \\ m - y \cdot (\sum_{s \in E} p_s) & k = n+1 \end{cases}$$

$$(z^y_{S \setminus E})_k \equiv \begin{cases} 0 & k \in E \\ -y & k \in S \setminus E \\ m + y \cdot (\sum_{s \in S \setminus E} p_s) & k = n+1 \end{cases}$$

then $x(\overline{z}^y_E) = x(z^y_{S \setminus E})$ and $\overline{z}^y_E$ satisfies the constraint (NB) iff z^y_E satisfies (CC).

Definition: A belief and demand realization (BDR) is a profile $(\gamma^i, z^i)_{i \in I}$ which specifies a belief $\gamma^i \in \Delta(S)$ and a bundle $z^i \in R^{n+1}$ for each agent $i \in I$.

We are now ready to give a definition of an equilibrium in the economy.

Definition: Let $(\gamma^i, z^i)_{i \in I}$ be a BDR and let $p = (p_s)_{s \in S}$ be a price vector. We say that $((\gamma^i, z^i)_{i \in I}, p)$ is an equilibrium at the state $\widehat{s}$ if

(1) Rationality: $\gamma^i \in \Delta(\psi^i(\widehat{s}))$ and z^i is optimal w.r.t $\succsim^i_{\gamma^i}$in the budget set $B(p, m^i)$.

(2) Market Clearing: $\int_{i \in I} z^i = (0, ..., 0, 1)$.

Definition: A vector of prices p is an equilibrium at a state $\widehat{s}$ if there exists a BDR $(\gamma^i, z^i)_{i \in I}$ such that $((\gamma^i, z^i)_{i \in I}, p)$ is an equilibrium at $\widehat{s}$.

To state our main result we need two additional definitions. Let $\widehat{s} \in S$ and let $E \subseteq S$ be an event. Define $I_{\widehat{s}}^{E} \equiv \left\{ i \,\middle|\, i \in I \ \psi^{i}(\widehat{s}) \subseteq E \right\}$. $I_{\widehat{s}}^{E}$ is the set of agents who know the event E at the state $\widehat{s}$. Let $J \subseteq I$ be a measurable set of agents. Define $m(J)= \int_{i\in J} m^{i}$. $m(J)$ is the aggregate amount of money in the hands of agents in J. In particular, $m(I_{\widehat{s}}^{E})$ is the aggregate amount of money in the hands of agents who know E at the state $\widehat{s}$[7].

Theorem 1:

The price vector $p = (p_s)_{s\in S}$ is an equilibrium price at a state $\widehat{s}$ iff:

1. $\sum_{s\in S} p_s = 1$.
2. For every $E \subseteq S$ $m(I_{\widehat{s}}^{E}) \leq \sum_{s\in E} p_s$.

Remarks:

1. The set of price vectors that satisfy conditions 1 and 2 is independent of the profile of preferences $\{\succsim^{i}\}_{i\in I}$ of the agents. Thus, theorem 1 implies in particular that for any profile of preferences in $\mathcal{M}$ the set of equilibrium prices is the same set.

2. For $E \subseteq S$ $\sum_{s\in E} p_s$ is the price of the asset A_E. Since A_S is equivalent to money

condition 1 is just a non-arbitrage constraint. Condition 2 states that for any event $E \subseteq S$ the price of the asset A_E is greater or equal to the aggregate amount of money in the hands of agents who know E at the state $\widehat{s}$. Thus, the set of equilibrium prices, $\overline{P}^{\widehat{s}}$ is the core of a cooperative game where the set of players is S and the value of a coalition E, $E \subseteq S$, is $m(I_{\widehat{s}}^{E})$. In particular, theorem 1 establishes that the set of equilibrium prices at a state $\widehat{s}$ is determined by the parameters $m(I_{\widehat{s}}^{E})$, $E \subseteq S$.

3. To get some immediate sense for the result we consider two extreme cases:

(a) Every agent knows the true state $\widehat{s}$. In this case $m(I_{\widehat{s}}^{\{\widehat{s}\}}) = 1$ and therefore conditions 1 and 2 imply that $p_{\widehat{s}} = 1$ and for $s \neq \widehat{s}$ $p_s = 0$ (which is of course what we would expect.)

[7] We assume that for every $E \subseteq S$ and $\widehat{s} \in S$ $I_{\widehat{s}}^{E}$ is measurable.

(b) No one knows anything, i.e., $\psi^i(\widehat{s}) = S$ for every $i \in I$. In this case $m(I^E_{\widehat{s}}) = 0$ for every $E \subsetneq S$ and therefore theorem 1 implies that a price vector p is an equilibrium price iff $\sum\limits_{s \in S} p_s = 1$.

We now present a simple example which on one hand demonstrates theorem 1 and on the other hand motivates the definition of common knowledge equilibrium (CKE)[8]

<u>Example 1</u>

$S = \{1, 2\}$. $I = I_1 \cup I_2$ where $I_1 = [0, \delta]$ and $I_2 = (\delta, 1]$. Every agent i in I_1 knows the true state $(\psi^i(s) = s)$ while every agent j in I_2 does not know anything $(\psi^j(s) = S$.) All the agents have an initial endowment of \$1 and all of them evaluate an outcome by its expectation. That is, for every $i \in I$ $\gamma \in \Delta(S)$ and $x, y \in R^2$ $x \succsim^i_\gamma y$ iff $\gamma(1) \cdot x_1 + \gamma(2) \cdot x_2 \geq \gamma(1) \cdot y_1 + \gamma(2) \cdot y_2$[9]. Let $\overline{P}^s$ denote the set of equilibrium prices at the state s. We will now apply theorem 1 to solve $\overline{P}^1$. Table 1 summarizes the different constraints that are imposed by condition 2

Event E	$m(I^E_1) \leq \sum\limits_{s \in E} p_s$
$\{1\}$	$\delta \leq p_1$
$\{2\}$	$0 \leq p_2$
$\{1, 2\}$	$1 \leq p_1 + p_2$

<u>Table 1</u>

Adding the constraint $p_1 + p_2 = 1$ which is implied by condition 1 we obtain that

$\overline{P}^1 = \{p = (p_1, p_2) \,|\, \delta \leq p_1,\ p_1 + p_2 = 1\}$

We observe that the set $\overline{P}^1$ depends on δ, the fraction of agents who know the true state, in an intuitive way, as δ increases $\overline{P}^1$ shrinks. In a similar way we obtain that $\overline{P}^2$, the set of equilibrium prices at the state 2, is given by

$\overline{P}^2 = \{p = (p_1, p_2) \,|\, \delta \leq p_2,\ p_1 + p_2 = 1\}$.

We now use example 1 to motivate the introduction of CKE. Consider a price $p \in \overline{P}^1$ such that $0 < p_2 < \delta$ $(1 > p_1 > 1 - \delta)$. The price p does not

[8] Examples of a similar nature in the context of exchange economies are studied in Desgranges and Guesnerie (2002), Desgranges(1999), and Ben-Porath and Heifetz(2006). I present the example here because it nicely demonstrates theorem 1.

[9] Since an agent in I_1 knows the true state she assigns it a probability 1.

belong to the set $\overline{P}^2$. This means that at the state 2 p is not consistent with the assumption of rationality and market clearing. Thus, an agent in I_2 who knows the parameters that define the economy, knows that every agent made a rational choice, and knows that the markets clear at p should conclude that the true state must be 1, but if all agents reach this conclusion then p cannot be a clearing price (the clearing price is $(1, 0)$). Thus, p is not consistent with common knowledge (CK) of the parameters that define the economy ($\mathcal{E}$), the rationality of the agents (R), and market clearing (MC). We now present the concept of common knowledge equilibrium, CKE, and then use the result in theorem 1 to characterize the set of CKE prices[10].

Definition:

1. Let $\widehat{S} \subseteq S$ be a set of states, let $((\gamma^{i,s}, z^{i,s})_{i\in I})_{s\in\widehat{S}}$ be a tuple of BDRs,

a BDR for

each state $s \in \widehat{S}$ and let $p = (p_s)_{s\in S}$ be a price vector. We say that the tuple $e_{\widehat{S}} \equiv (((\gamma^{i,s}, z^{i,s})_{i\in I})_{s\in\widehat{S}}, p)$ is a CKE w.r.t the set $\widehat{S}$ if for every $s \in \widehat{S}$ $((\gamma^{i,s}, z^{i,s})_{i\in I}), p)$ is an equilibrium at the state s and $\gamma^{i,s} \in \Delta(\psi^i(s) \cap \widehat{S})$.

We say that e is a CKE at a state $\widehat{s}$ if there exists a set $\widehat{S} \subseteq S$ s.t. $\widehat{s} \in \widehat{S}$ and e is a CKE w.r.t $\widehat{S}$.

2. We say that $p \in R^n$ is a CKE price w.r.t a set $\widehat{S} \subseteq S$ if there is a tuple of BDRs $((\gamma^{i,s}, z^{i,s})_{i\in I})_{s\in\widehat{S}}$ such that $e_{\widehat{S}} \equiv (((\gamma^{i,s}, z^{i,s})_{i\in I})_{s\in\widehat{S}}, p)$ is a CKE w.r.t $\widehat{S}$.

The price p is a CKE equilibrium price at the state $\widehat{s}$ if there exist a set $\widehat{S}$ s.t. $\widehat{s} \in \widehat{S}$ and p is a CKE price w.r.t. $\widehat{S}$[11].

The idea that underlies the definition of CKE is that if p can be supported at every state $s \in \widehat{S}$ by some profile of beliefs $\{\gamma^{i,s}\}_{i\in I}$ with support in $\widehat{S}$ (and such that $\gamma^{i,s}$ respects the private information of agent i at the state s) then p is consistent with CK of $\mathcal{E}$, R, and MC at any state $s \in \widehat{S}$ because each belief $\gamma^{i,s}$ assigns a positive probability to some set of states $\widehat{S}^{i,s} \subseteq \widehat{S}$ in which p is an equilibrium price. (Furthermore, in each state $s' \in \widehat{S}^{i,s}$ p is supported by beliefs $\{\gamma^{j,s'}\}_{j\in I}$ such that $\gamma^{j,s'}$ assigns a positive probability

[10] The concept of CKE was first defined by Desgranges (1999). It is also studied in Ben-Porath and Heifetz (2006).

[11] Equivalently: p is a CKE price at the state $\widehat{s}$ if there exists a CKE at $\widehat{s}$ in which the price is p.

only to some set of states $\widehat{S}^{j,s'} \subseteq \widehat{S}$ in which p is an equilibrium price, and so forth.) In section 6 we present a formal framework in which the argument that p is consistent with CK of $\mathcal{E}, R$, and MC at a state $\widehat{s}$ iff p is a CKE at $\widehat{s}$ is made precise. Finally, we observe that if p is a CKE price w.r.t the set S^1 and w.r.t the set S^2 then it is a CKE price w.r.t $S^1 \cup S^2$ as well. Define $S(p) \equiv \{s \,|p \text{ is a } CKE \text{ price at } s\}$. It is easy to see that $S(p)$ is the maximal set w.r.t which p is a CKE.

Define:
For $\widehat{s} \in S$ $\;P^{\widehat{s}} \equiv \{p\,|p \text{ is a } CKE \text{ price at } \widehat{s}\}$
For $\overline{S} \subseteq S$ $\;P_{\overline{S}} \equiv \left\{p\,\middle|p \text{ is a } CKE \text{ price w.r.t } \overline{S}\right\}$
For $\overline{S} \subseteq S$ and $s \in \overline{S}$
$P^s_{\overline{S}} \equiv \left\{p\,\middle|\text{there exists an equilibrium } ((\gamma^{i,s}, z^{i,s})_{i\in I}, p) \text{ at } s \text{ s.t. } \gamma^{i,s} \in \Delta(\psi^i(s) \cap \overline{S})\right\}$
The definition of CKE implies that:
(1.1) $P^{\widehat{s}} = \bigcup_{\overline{S},\widehat{s}\in\overline{S}} P_{\overline{S}} = \bigcup_{\overline{S},\widehat{s}\in\overline{S}} (\bigcap_{s\in\overline{S}} P^s_{\overline{S}})$

Consider example 1. We will compute P^1. From the left equation of (1.1) we have
(1.2) $P^1 = \bigcup_{\overline{S},1\in\overline{S}} P_{\overline{S}} = P_{\{1\}} \cup P_{\{1,2\}}$.

$P_{\{1\}}$ is the set of equilibrium prices when the state is 1 and every agent assigns probability 1 to the state 1. Clearly, $P_{\{1\}} = \{(1,0)\}$.

From the right equation in (1.1) we have
(1.3) $P_{\{1,2\}} = P^1_{\{1,2\}} \cap P^2_{\{1,2\}}$.
Now,
(1.4) $P^1_{\{1,2\}} = \overline{P}^1 = \{p\,|p_1 + p_2 = 1 \text{ and } \delta \le p_1\}$ and similarly
(1.5) $P^2_{\{1,2\}} = \overline{P}^2 = \{p\,|p_1 + p_2 = 1 \text{ and } \delta \le p_2\}$
(we remind that $\overline{P}^1$ and $\overline{P}^2$ are, respectively, the set of equilibrium prices at the state 1 and the set of equilibrium prices at the state 2.)

Putting (1.3), (1.4) and (1.5) together we obtain that
If $\delta \le \frac{1}{2}$ $P_{\{1,2\}} = \{p\,|p_1 + p_2 = 1 \;\delta \le p_1 \le 1-\delta\}$ and if $\delta > \frac{1}{2}$ then $P_{\{1,2\}} = \varnothing$.

Now from (1.2) we get
If $\delta \le \frac{1}{2}$ then $P^1 = \{(1,0)\} \cup \{p\,|p_1 + p_2 = 1 \;\delta \le p_1 \le 1-\delta\}$
If $\delta > \frac{1}{2}$ then $P^1 = \{(1,0)\}$.
Similarly,
If $\delta \le \frac{1}{2}$ then $P^2 = \{(0,1)\} \cup \{p\,|p_1 + p_2 = 1 \;\delta \le p_1 \le 1-\delta\}$
If $\delta > \frac{1}{2}$ then $P^2 = \{(0,1)\}$.

In particular, when $\delta > \frac{1}{2}$ the unique CKE price at a state $s, s = 1, 2$, is the fully revealing price, p^s, where the price of the asset A_s is 1 and the price of the other asset is zero. Thus, when $\delta > \frac{1}{2}$ CK of $\mathcal{E}, R$, and MC select the fully revealing rational expectations equilibrium even when players may have heterogeneous beliefs.

We now show that the result in theorem 1 provides a characterization for the set of CKE prices in a given state $\widehat{s}$ in terms of the parameters $m(I_s^E), s \in S, E \subseteq S$. Specifically, for a set $\overline{S} \subseteq S$ and a state $s \in \overline{S}$ we now demonstrate that the set $P_{\overline{S}}^s$ can be characterized in terms of the set parameters $m(I_s^E)$, $E \subseteq S$.

To see this we remind that $p \in P_{\overline{S}}^s$ iff there is a profile of beliefs $(\gamma^{i,s})_{i \in I}$ and a profile of bundles $(z^{i,s})_{i \in I}$ such that $((\gamma^{i,s}, z^{i,s})_{i \in I}, p)$ is an equilibrium at s and

(1.6) $\gamma^{i,s} \in \Delta(\psi^i(s) \cap \overline{S})$.

Define now an economy $\mathcal{E}_{\overline{S}}$ that is obtained from the original economy $\mathcal{E}$ by restricting the set of states to $\overline{S}$ and defining the information partition of an agent i, $\overline{\psi}^i$, as follows: For $s' \in \overline{S}$ $\overline{\psi}^i(s') \equiv \psi^i(s') \cap \overline{S}$.

Let $p \in R^n$ and let $\overline{S} \subseteq S$. Define $p_{\overline{S}} \equiv (p_s)_{s \in \overline{S}}$. For $\overline{S} \subseteq S$ and $s \in \overline{S}$ define

$P(\mathcal{E}_{\overline{S}}, s) \equiv \left\{ p \middle| p \in R^n, p_{\overline{S}} \text{ is an equilibrium price in } \mathcal{E}_{\overline{S}} \text{ at } s \text{ and } p_{S \setminus \overline{S}} = 0. \right\}.$

Lemma 2: $p \in P_{\overline{S}}^s$ iff $p \in P(\mathcal{E}_{\overline{S}}, s)$.

The point of lemma 2 is that the sets $P(\mathcal{E}_{\overline{S}}, s)$ can be characterized in terms of the parameters $m(I_s^E), E \subseteq S$, of the original economy $\mathcal{E}$. Specifically, let $E \subseteq \overline{S}$. It is easy to see that the set of agents that know the event E in the economy $\mathcal{E}_{\overline{S}}$ at the state $s \in \overline{S}$ is $I_s^{E \cup (S \setminus \overline{S})}$ (which is the set of agents who know the event $E \cup (S \setminus \overline{S})$ in the original economy $\mathcal{E}$). Applying now theorem 1 to the economy $\mathcal{E}_{\overline{S}}$ we obtain that $p \in P(\mathcal{E}_{\overline{S}}, s)$ iff

(1) $\sum_{s' \in S} p_{s'} = 1$.

(2) For every $E \subseteq \overline{S}$ $m(I_s^{E \cup (S \setminus \overline{S})}) \leq \sum_{s' \in E} p_{s'}$.

Putting (1.1) and lemma 2 together we obtain theorem 2 which characterizes the set of prices $P^{\widehat{s}}$ that are CKE at a state $\widehat{s}$ in terms of the parameters $m(I_s^E), s \in S, E \subseteq S$.

Theorem 2: $P^{\widehat{s}} = \bigcup_{\overline{S}, \widehat{s} \in \overline{S}} \left(\bigcap_{s \in \overline{S}} P(\mathcal{E}_{\overline{S}}, s) \right).$

We note that just like in the case of theorem 1 the characterization of $P^{\widehat{s}}$ in theorem 2 applies to every profile of preferences in the class $\mathcal{M}$.

In section 4 we show that the result in theorem 2 can be used to characterize the information that is revealed in a CKE at a given state $\widehat{s}$.

3 The proof of theorem 1.

We show, first, that if p is an equilibrium price at a state $\widehat{s}$ then p satisfies conditions (1) and (2). Start with condition (1). Since the asset $A_S \equiv \sum_{s \in S} A_s$ is equivalent to money the argument that the price of A_S, $\sum_{s \in S} p_s$, must equal 1 is an argument that points to the possibility of arbitrage if this equation is not satisfied. Specifically, assume by contradiction that $\sum_{s \in S} p_s > 1$. For every number $x > 0$ an agent i can obtain an outcome which gives $\$x \cdot (\sum_{s \in S} p_s - 1)$ in every state s by selling x units of A_S. (The constraint (CC) in the definition of the budget set is satisfied because the sale commits agent i to a payment of $\$x$ in each state while the amount of money in his hands is $\$1 + x \cdot (\sum_{s \in S} p_s)$.) It follows that each agent can obtain an unbounded amount of money in every state. Since the aggregate amount of money in the economy is 1 p cannot be a clearing price. Similarly, assume by contradiction that $\sum_{s \in S} p_s < 1$. An optimal bundle z^* must satisfy $z^*_{n+1} = 0$. Otherwise, the agent could obtain an outcome that pays better in every state by using his $\$z^*_{n+1}$ to buy the asset A_S. Now, if every agent does not want to hold money then there is an excess supply of money and therefore p is not an equilibrium price.

We turn now to condition (2).

Lemma 3: Let $p \in R^n$ be a price vector such that $\sum_{s \in S} p_s = 1$ and let $z \in R^{n+1}$ be a bundle for agent i in the budget set $B(p, m^i)$. There exists a bundle $\overline{z} \in B(p, m^i)$ such that:

(1) $\overline{z}_k \geq 0$ for every $k = 1,..,n+1$.

(2) $x(\overline{z}) = x(z)$.

In words, every outcome that can be generated by a bundle of assets in $B(p, m^i)$ can also be generated by a bundle in which agent i does not sell any asset.

The proof of lemma 3 is similar to the proof of lemma 1 and is given in the appendix. We now show how it implies condition (2). Let p be an equilibrium price in $\widehat{s}$ and assume by contradiction that there exists an event E such that $m(I^E_{\widehat{s}}) > \sum_{s \in E} p_s$. Let $((\gamma^i, z^i)_{i \in I}, p)$ be an equilibrium at $\widehat{s}$. Lemma 3 implies that for any $i \in I^E_{\widehat{s}}$ there exists a bundle $\overline{z}^i \in B(p, m^i)$ such that $\overline{z}^i_k \geq 0$ for every $k = 1,..,n+1$ and $x(\overline{z}^i) = x(z^i)$. Since z^i is optimal for agent i in $B(p, m^i)$ so is $\overline{z}^i$. We claim that $\overline{z}^i_k = 0$ for $k \notin E$. To see that we observe that for $s \notin E$ agent i will not buy the asset A_s because he assigns the state s probability zero. Also, $\overline{z}^i_{n+1} > 0$ is impossible because agent i assigns probability 1 to the event E and therefore could obtain a better outcome by spending the $\$\overline{z}^i_{n+1}$ on buying the asset A_E. For $s \in E$ define $\overline{m}^i_s \equiv p_s \cdot \overline{z}^i_s$ and $\overline{m}_s \equiv \int_{i \in I^E_{\widehat{s}}} \overline{m}^i_s$. Thus, $\overline{m}^i_s$ is the number of \$ that agent i spends on buying the asset A_s and $\overline{m}_s$ is the aggregate amount of money that agents in $I^E_{\widehat{s}}$ spend on A_s. Since $\overline{z}^i_k = 0$ for every $i \in I^E_{\widehat{s}}$ and $k \notin E$ we have $\sum_{s \in E} \overline{m}_s = m(I^E_{\widehat{s}})$. It follows that $\sum_{s \in E} \overline{m}_s > \sum_{s \in E} p_s$ and therefore there exists a state $\widetilde{s}$ such that $\overline{m}_{\widetilde{s}} > p_{\widetilde{s}}$. Thus, we obtain that

$$\int_{i \in I^E_{\widehat{s}}} x(\overline{z}^i)_{\widetilde{s}} = \frac{\overline{m}_{\widetilde{s}}}{p_{\widetilde{s}}} > 1.$$

That is, the aggregate amount of money that agents in $I^E_{\widehat{s}}$ obtain in the state $\widetilde{s}$ is larger than 1. Since the aggregate amount of money in the economy is 1 p cannot be an equilibrium price. Thus, we have obtained a contradiction to the assumption that there exists an event E that does not satisfy condition (2).

We have shown that if p is an equilibrium price at a state $\widehat{s}$ then conditions (1) and (2) must be satisfied. We now show that if p satisfies conditions (1) and (2) w.r.t. the state $\widehat{s}$ then p is an equilibrium price at $\widehat{s}$. Lemma 4 below plays a central role in this part.

Lemma 4: Let $p = (p_1, ..., p_n)$ be a price vector that satisfies conditions (1) and (2) w.r.t. the state $\widehat{s}$. There exists a partition of I, $\widehat{I}_1, ..., \widehat{I}_n$, such that for $s = 1, ..., n$:

(a) If $i \in \widehat{I}_s$ then $s \in \psi^i(\widehat{s})$.

(b) $m(\widehat{I}_s) = p_s$.

We first prove the theorem from the lemma. Define a profile of subjective beliefs as follows: For $i \in \widehat{I}_s$ $\gamma^i(s) = 1$ (and $\gamma^i(s') = 0$ for $s' \neq s$). Define $\widetilde{S} \equiv \{s \,|p_s > 0\}$. We can ignore the agents in $\bigcup\limits_{s \in S \setminus \widetilde{S}} I_s$ because $m(\bigcup\limits_{s \in S \setminus \widetilde{S}} I_s) = 0$ and thus their behavior does not influence the equilibrium price. Let $\widetilde{s}$ be some arbitrary state in $\widetilde{S}$. For every $s \in \widetilde{S} \setminus \{\widetilde{s}\}$ an agent i that belongs to the set I_s chooses a bundle z^i in which he spends all his money on the purchase of the asset A_s. Lemma 3 implies that the bundle z^i is an optimal choice for agent i w.r.t the belief γ^i. Since $m(\widehat{I}_s) = p_s$ the aggregate demand for each asset $A_s, s \in \widetilde{S} \setminus \{\widetilde{s}\}$, is 1. Consider now the agents in $\widehat{I}_{\widetilde{s}}$. Define $\overline{S} \equiv \widetilde{S} \setminus \{\widetilde{s}\}$. Each agent $i \in I_{\widetilde{s}}$ chooses a bundle z^i where he uses all his income as a cover for the sale of the asset $A_{\overline{S}}$. Let y^i denote the number of units of $A_{\overline{S}}$ that i sells ($y^i = |z^i_s|$ for $s \in \overline{S}$). Since i uses all his income as a cover for the sale of $A_{\overline{S}}$ y^i is defined by the equation $m^i + y^i \cdot (\sum\limits_{s \in \overline{S}} p_s) = y^i$ (The RHS is the payment that i will have to make at a state $s \in \overline{S}$ while the LHS is the amount of money that he holds.) It follows that

(3.2) $y^i = \frac{m^i}{1 - \sum\limits_{s \in \overline{S}} p_s} = \frac{m^i}{p_{\widetilde{s}}}$

(where the right equality follows from $\sum\limits_{s \in \widetilde{S}} p_s = 1$.)

It is easy to see that since $\gamma^i(\widetilde{s}) = 1$ the outcome that z^i generates is equivalent to the outcome that is generated by a bundle where agent i spends all his income on the purchase of the asset $A_{\widetilde{s}}$ (In both cases agent i gets $\$\frac{m^i}{p_{\widetilde{s}}}$ in the state $\widetilde{s}$.) It follows from lemma 3 that z^i is an optimal choice for agent i w.r.t γ^i. Also, since $m(\widehat{I}_{\widetilde{s}}) = p_{\widetilde{s}}$, (3.2) implies that the aggregate supply of $A_{\overline{S}}$ by the agents in $I_{\widetilde{s}}$ is 1. Thus, we have shown that for every $i \in I$ z^i is optimal w.r.t γ^i in $B(p, m^i)$ and that the markets for all the assets clear. It follows that $((\gamma^i, z^i)_{i \in I}, p)$ is an equilibrium at $\widehat{s}$.

Proof of lemma 4:

The proof of lemma 4 is based on a continuous version of the famous marriage lemma (Hall 1935)[12] which is due to Hart and Kohlberg (1974)[13].

Lemma 5 (Hart and Kohlberg):

Let $(\Omega, \mathcal{B}, \mu)$ be a non-atomic measure space and let $\{F_i\}_{i=1}^n \subset \Omega$ and $\{\alpha_i\}_{i=1}^n \in R_+$ such that for all $L \subseteq \{1, .., n\}$:

(1) $\mu(\cup_{i \in L} F_i) \geq \sum\limits_{i \in L} \alpha_i$

(2) $\mu(\cup_{i=1}^n F_i) = \sum\limits_{i=1}^n \alpha_i$.

Then there exist disjoint sets $\{T_i\}_{i=1}^n$ such that $T_i \subseteq F_i$ and $\mu(T_i) = \alpha_i$.

Lemma 4 is now proved as follows. Let $(\Omega, \mathcal{B}, \mu)$ be the measure space where $\Omega = I = [0,1]$, $\mathcal{B}$ is the set of Borel sets and for $J \in \mathcal{B}$ μ is defined by $\mu(J) \equiv \int\limits_{i \in J} m^i$. (That is, $\mu(J)$ is the aggregate amount of money in the hands of agents in J.) For $s \in S$ define $J_s \equiv \left\{ i \,\middle|\, s \in \psi^i(\widehat{s}) \right\}$. Condition (2) states that for every $E \subseteq S$ $\sum\limits_{s \in E} p_s \geqq \mu(I_{\widehat{s}}^E)$. Since $\sum\limits_{s \in S} p_s = 1$ and $\mu(I) = 1$ we obtain that $\sum\limits_{s \in E^c} p_s \leqq \mu((I_{\widehat{s}}^E)^C)$ where $(I_{\widehat{s}}^E)^C$ is the complement of $I_{\widehat{s}}^E$. Now since $(I_{\widehat{s}}^E)^C = \bigcup\limits_{s \in E^C} J_s$ we obtain that for every $E \subseteq S$ $\sum\limits_{s \in E^C} p_s \leqq \mu(\bigcup\limits_{s \in E^C} J_s)$. This, of course, means that for every $E \subseteq S$ $\sum\limits_{s \in E} p_s \leqq \mu(\bigcup\limits_{s \in E} J_s)$.In addition, we have $\sum\limits_{s \in S} p_s = \mu(\bigcup\limits_{s \in S} J_s) = 1$. Applying lemma 5 by setting $\alpha_s \equiv p_s$ and $F_s \equiv J_s, s \in S$, we obtain that there exist disjoint sets $\widehat{I}_s, s \in S$, such that $\widehat{I}_s \subseteq J_s$ and $m(\widehat{I}_s) = \mu(\widehat{I}_s) = p_s$. The proof of lemma 4 is now complete[14].

[12] Let B be a set of boys and let G be a set of girls. Assume that $|B| = |G| = n$. For $b \in B$ let $I(b)$ denote the set of girls that b knows. For $B' \subseteq B$ define $I(B') \equiv \bigcup\limits_{b \in B'} I(b)$. A match $m, m : B \to G$, is a one-to-one function s.t. for every $b \in B$ $m(b) \in I(B)$. Clearly, a neccessary condition for the existence of a match is that

$(*)$ for every $B' \subseteq B$ $|m(B')| \geq |B'|$.

Hall's lemma establishes that $(*)$ is also a sufficient condition.

[13] I thank Sergiu Hart for pointing out that lemma 5 can be used to prove lemma 4.

[14] Since $\sum\limits_{s \in S} \mu(\widehat{I}_s) = 1$ we can assume w.l.o.g. that the set $\widehat{I}_s, s \in S$, form a partition.

4 Information Revelation

An interesting implication of theorem 2 is a characterization of the information that is revealed in a CKE at a given state $\widehat{s}$. Let $\mathcal{S}(\widehat{s})$ be the minimal set of states that is common knowledge in every CKE at the state $\widehat{s}$. Thus, if the economy is in a CKE in the state $\widehat{s}$ then the set $\mathcal{S}(\widehat{s})$ is common knowledge. (The point is that one can say that $\mathcal{S}(\widehat{s})$ is common knowledge without knowing which particular CKE has materialized.)

Define $C(\widehat{s}) \equiv \left\{s \left| P^s \cap P^{\widehat{s}} \neq \emptyset\right.\right\}$. We claim that $\mathcal{S}(\widehat{s}) = C(\widehat{s})$. A precise formulation and proof of this claim requires the framework which we define in section 6 (theorem 4), however, the proof is fairly simple and it is worthwhile to present an outline of the argument here. Let s' be a state such that $s' \notin C(\widehat{s})$ and let e be some CKE in $\widehat{s}$ where the price is $\widehat{p}$. We have $\widehat{p} \in P^{\widehat{s}}$ and therefore $\widehat{p} \notin P^{s'}$. Now since it is assumed that the model is common knowledge the set $P^{s'}$is common knowledge and therefore it is common knowledge at e that state is not s'. It follows that $s' \notin \mathcal{S}(\widehat{s})$. Thus, we have shown that $\mathcal{S}(\widehat{s}) \subseteq C(\widehat{s})$. We now show that $C(\widehat{s}) \subseteq \mathcal{S}(\widehat{s})$. Let s' be a state such that $s' \in C(\widehat{s})$ and let $\widehat{p}$ be a price such that $\widehat{p} \in P^{s'} \cap P^{\widehat{s}}$. It follows that both s' and $\widehat{s}$ are members of the set $S(\widehat{p})$. (We remind that $S(\widehat{p})$ is the set of states in which p is a CKE price.) Let e be a CKE w.r.t. the set $S(\widehat{p})$. A simple argument[15] establishes that there exist agents i and j and a state $\widetilde{s}$ such that $\widetilde{s} \in \psi^i(\widehat{s}) \cap S(\widehat{p})$ and $s' \in \psi^j(\widetilde{s}) \cap S(\widehat{p})$. Thus, in the CKE e player i cannot exclude the state $\widetilde{s}$ at the state $\widehat{s}$ and at the state $\widetilde{s}$ player j cannot exclude the state s'. It follows that at the state $\widehat{s}$ player i cannot exclude the possibility that player j does not exclude the state s'. Therefore, the set of states that is common knowledge at $\widehat{s}$ contains s' and thus $s' \in \mathcal{S}(\widehat{s})$.

Consider example 1. We have shown that if $\delta > \frac{1}{2}$ then the only CKE price at a state s is p^s (we remind that p^s is the price vector where the price of the asset A_s is 1 and the price of every other asset is zero.) In particular, we have $P^1 = \{(1,0)\}$ and $P^2 = \{(0,1)\}$. Thus, when $\delta > \frac{1}{2}$ then $P^1 \cap P^2 = \emptyset$ and our characterization implies (as we would expect) that $\mathcal{S}(s) = \{s\}$ for $s = 1, 2$. This means that the event $\{s\}$ is common knowledge in every CKE at s. If $\delta \leq \frac{1}{2}$ then

$P^1 \cap P^2 = \{p \,|p_1 + p_2 = 1, \delta \leq p_1 \leq 1 - \delta\ \} \neq \emptyset$ and hence $\mathcal{S}(s) = S$ for $s = 1, 2$. That is, the minimal set that is common knowledge in every CKE

[15] See lemma 4.1 in the proof of theorem 4 (section 6).

is S^{16}.

We now apply our characterization to an example in which each agent has partial information on the true state.

Example 2:
$S \equiv \{1,2,3\}$. $I \equiv I_1 \cup I_2$ where $I_1 = [0,\alpha]$ and $I_2 = (\alpha, 1]$.

Define
$\Psi^1 \equiv \{\{1\},\{2,3\}\}$
$\Psi^2 \equiv \{\{1,2\},\{3\}\}$
The information partition of an agent i in the set I_k is Ψ^k. All the agents in the economy have an initial endowment of \$1.

We will compute the sets $P^s, s = 1,2,3$ and then apply the characterization, $\mathcal{S}(s) = C(s)$, to compute the sets $\mathcal{S}(s), s = 1,2,3$.

We remind that the definition of CKE implies that

(A.3.1) $P^s = \bigcup_{S' \subseteq S, \text{ s.t. } s \in S'} P_{S'}$

(where $P_{S'}$ is the set of prices that are CKE w.r.t S').

The following lemma simplifies the computation of the sets P^s.

Lemma 3.1: $P_S = \emptyset$.

Proof: The definition of CKE w.r.t. the set S implies that $P_S = \bigcap_{s \in S} \overline{P}^s$ (we remind that $\overline{P}^s$ is the set of equilibrium prices at the state s.). We will show that $\overline{P}^1 \cap \overline{P}^3 = \emptyset$. To compute the sets $\overline{P}^1$and $\overline{P}^3$ we apply theorem 1. Consider first, $\overline{P}^1$. We have: $m(I_1^{\{1\}}) = \alpha$, $m(I_1^{\{1,2\}}) = 1$, and $m(I_1^{\{1,3\}}) = \alpha$ (the last equation is redundant). Theorem 1 implies that $\overline{P}^1$ is the set of price vectors that satisfy the following conditions: $\sum_{s=1}^{3} p_s = 1$, $\alpha \leq p_1$, and $1 \leq p_1 + p_2$. These conditions imply that if $p \in \overline{P}^1$ then $p_3 = 0$.

A symmetric calculation yields that $\overline{P}^3$ is the set of price vectors that satisfy: $\sum_{s=1}^{3} p_s = 1$, $1 - \alpha \leq p_3$, and $1 \leq p_2 + p_3$.

We obtain that if $p \in \overline{P}^1$ then $p_1 > 0$ while if $p \in \overline{P}^3$ then $p_1 = 0$. It follows that $\overline{P}^1 \cap \overline{P}^3 = \emptyset$.

[16] There is of course a CKE at the state s where s is common knowledge this is a CKE where the price of A_s is 1 and each agent assigns the state s probability 1.

The equation (A.3.1) and lemma 3.1 imply that
$P^1 = P_{\{1\}} \cup P_{\{1,2\}}$.
$P^2 = P_{\{2\}} \cup P_{\{1,2\}} \cup P_{\{2,3\}}$.
$P^3 = P_{\{3\}} \cup P_{\{2,3\}}$.
We have
$P_{\{1\}} = \{(1,0,0)\}\,,\ P_{\{2\}} = \{(0,1,0)\}\,,\ P_{\{3\}} = \{(0,0,1)\}\,.$

The calculation of the sets $P_{\{1,2\}}$ and $P_{\{2,3\}}$is similar to the calculation that was done in the solution of example 1[17]. We obtain that:

$$P_{\{1,2\}} = \left\{ \begin{array}{ll} \emptyset & \text{if } \alpha > \frac{1}{2} \\ \{(p_1, p_2, 0)\,|p_1 + p_2 = 1,\ \alpha \le p_1 \le 1 - \alpha\} & \text{if } \alpha \le \frac{1}{2} \end{array} \right\}$$

$$P_{\{2,3\}} = \left\{ \begin{array}{ll} \emptyset & \text{if } \alpha < \frac{1}{2} \\ \{(0, p_2, p_3)\,|p_2 + p_3 = 1,\ 1 - \alpha \le p_3 \le \alpha\} & \text{if } \alpha \ge \frac{1}{2} \end{array} \right\}$$

It is now simple to compute the sets $\mathcal{S}(s), s = 1,2,3$ using the characterization, $\mathcal{S}(s) = C(s)$. We obtain that:

$$\mathcal{S}(1) = \begin{cases} \{1\} & \alpha > \frac{1}{2} \\ \{1,2\} & \alpha \le \frac{1}{2} \end{cases}$$

$$\mathcal{S}(2) = \begin{cases} \{2,3\} & \alpha > \frac{1}{2} \\ \{1,2,3\} & \alpha = \frac{1}{2} \\ \{1,2\} & \alpha < \frac{1}{2} \end{cases}$$

$$\mathcal{S}(3) = \begin{cases} \{2,3\} & \alpha \ge \frac{1}{2} \\ \{3\} & \alpha < \frac{1}{2} \end{cases}$$

Thus, in the case where $\alpha > \frac{1}{2}$ we obtain that at the state 1 the fact that the state is 1 is common knowledge in every CKE. In both the state 2 and the state 3 the minimal set of states that is common knowledge in every CKE is $\{2,3\}$. One implication of this result is that when $\alpha > \frac{1}{2}$ agents in I_2 learn the true state in every state and in every CKE (This follows because an agent in I_2 can distinguish between the state 2 and the state 3.) Similarly, when $\alpha < \frac{1}{2}$ agents in I_1 learn the true state in every state and every CKE. Consider now the case where $\alpha = \frac{1}{2}$. At the state 2 there are three CKE prices: $p^1 = (\frac{1}{2}, \frac{1}{2}, 0)$, $p^2 = (0, \frac{1}{2}, \frac{1}{2})$, and $p^3 = (0,1,0)$. If the CKE price is p^1 then the minimal set of states that is CK is $\{1,2\}$, if the price is p^2 it is the set $\{2,3\}$, and if the price is p^3 then the state 2 is CK. Thus, in every CKE the minimal set of states that is CK is strictly contained in S. However, $\mathcal{S}(2) = S$. (Indeed, it is easy to see directly that the minimal set of

[17] For example, to solve $P_{\{1,2\}}$ we solve for the set of CKE w.r.t the whole set of states in the economy $\mathcal{E}_{\{1,2\}}$which is the restriction of the original economy to the set $\{1,2\}$. The economy $\mathcal{E}_{\{1,2\}}$ is equivalent to the economy in example 1.

states that is CK in <u>every</u> CKE at the state 2 is S.)

5 CKE, *Rational Expectations Equilibrium*, and $CKRMC$

In this section we discuss the relationship between CKE, *rational expectations equilibrium* (REE) and the concept of $CKRMC$ (for common knowledge of rationality and market clearing) that is proposed in Ben-Porath and Heifetz (2006). Given the prominence of the concept of REE for the analysis of economies with asymmetric information the interest in discussing its relationship with CKE is obvious. There are two reasons for the discussion of $CKRMC$. First, $CKRMC$ provides a link between REE and CKE. Second, $CKRMC$ is an alternative solution concept for economies where agents have heterogeneous beliefs. Theorem 1 in BH(2006) establishes that under general conditions, which apply to asset economies, the set of CKE prices at a given state s equals the set of $CKRMC$ prices at s. This equivalence provides an additional perspective on theorem 2.

The concept of REE assumes a situation where all the agents in the economy have a common prior on the states of nature and know the price function. (The price function associates with each state of nature a vector of prices for the different assets.) Let p^s denote the price vector where the price of the asset A_s is 1 and the price of every other asset is zero. Proposition (1) establishes that if agents are risk averse then there is no trade. Furthermore, if the information of all the agents put together pins down the true state then the only REE is the fully revealing equilibrium where the price at state s is p^s. (Proposition 1 is formulated and proved in the appendix. It is similar to other non-trade and full-revelation results that appear in the literature.) This result stands in sharp contrast to the predictions of CKE where trade is possible in a broad class of economies. In particular, for the simple economy discussed in example 1 REE implies full revelation for every positive δ[18] while trade is consistent with CKE whenever $\delta < \frac{1}{2}$.

The concept of $CKRMC$ introduces heterogeneous beliefs by assuming that different agents may have different beliefs on the price function. More

[18] We remind that δ is the fraction of agents who know the true state.

specifically, a set of price functions F is $CKRMC$ if every price function f in F can be supported by a profile of beliefs $\{\mu_f^i\}_{i\in I}$ where μ_f^i is the subjective belief of agent i on F. We now make this informal definition precise. $CKRMC$ assumes, (like REE), that all the agents have a common prior α on S. The prior belief of an agent i on the space $S\times F$ is a product measure $\alpha\times\mu^i$ where μ^i is the subjective belief[19] of agent i on F. Let $L^i \equiv \{\psi^i(s)\,|s \in S\}$ be the set of private signals for player i. Given a signal $l_i \in L_i$ and a price vector $p \in R^n$ agent i computes a posterior on $S\times F$ which in particular gives a posterior on S. A demand function for player i, $d^i : L^i \times R^n \rightarrow R^{n+1}$, is a function that associates with a private signal and a price a bundle $z \in R^{n+1}$. We can now define concept of $CKRMC$.

Definition:A set of functions F[20] is $CKRMC$ if for every $f \in F$ there exits a profile of demand functions $\{d_f^i\}_{i\in I}$ and a profile of beliefs $\{\mu_f^i\}_{i\in I}$ such that for every $s \in S$: (a) The demand of each each agent $i \in I$ at s, $d_f^i(\psi^i(s), f(s))$, is optimal w.r.t. the posterior of agent i. (b) All the markets clear, that is, $\int\limits_{i\in I} d_f^i(\psi^i(s), f(s)) = (0, .., 0, 1)$.

We say that a price function f is $CKRMC$ if there exist a set of functions F such that F is $CKRMC$ and $f \in F$.

We say that a price vector $p \in R^n$ is a $CKRMC$ price at a state s if there exist a $CKRMC$ function f such that $f(s) = p$.

An argument that is identical to the proof of theorem 1 in BH(2006) implies:

Proposition (2): Let $\mathcal{E}$ be an asset economy. The price p is $CKRMC$ at a state s iff p is a CKE price at s.

To summarize, the main difference between the concept of $CKRMC$ and CKE is that $CKRMC$ assumes (like REE) that each agent has a complete theory about what price might materialize in each state. This theory is represented by a belief on price functions. Given a private signal and a price agent i updates his probability on S. By contrast CKE does not assume

[19] For reasons that are explained in BH(2006) a beleif μ^i of a player i on F is defined to be a lexicogaphic sequence of probabilities (on F). However, for the purposes of the description here it would be more useful to think of μ^i as just one subjective probability on F.

[20] For the purposes of the description here it would be useful to think of F as a finite set.

that an agent has a belief on price functions, rather, given a price p he forms a subjective belief on the states of nature which is consistent with common knowledge of rationality and market clearing. Proposition (2) establishes that in asset economies these two concepts are equivalent. I have focused on CKE in this paper because it is simpler to define and characterize.

6 An Epistemic Foundation for CKE

In this section we define an epistemic model where each state ω contains a complete description of the system. That is, the state ω specifies the state of nature, the vector of prices, and the knowledge, beliefs, and demand of each agent. In particular, a state ω specifies whether the markets clear and whether the choice of each agent i is rational. In this model the event that there is common knowledge of rationality (R) and market clearing (MC) is well defined and thus the argument that p is consistent with common knowledge of R and MC at a state of nature $\widehat{s}$ iff p is a CKE at $\widehat{s}$ can be made precise[21]. (Theorem 3 below). Next, we use the epistemic model to provide a precise formulation and proof to the claim that the minimal set of states that is common knowledge in every CKE at a given state $\widehat{s}$ is $C(\widehat{s})$[22]. (Theorem 4).

A knowledge and belief model, M, for the economy $\mathcal{E}$ is a tuple $M = \left\langle \Omega, \mathbf{s}, \mathbf{p}, (\boldsymbol{\chi}^{\mathbf{i}}, \boldsymbol{\mu}^{\mathbf{i}}, \mathbf{z}^{\mathbf{i}})_{i \in I} \right\rangle$[23] where Ω is a set of comprehensive states (henceforth, states, as opposed to states of nature which are elements of S) and $\mathbf{s}, \mathbf{p}, \boldsymbol{\chi}^{\mathbf{i}}, \boldsymbol{\mu}^{\mathbf{i}}, \mathbf{z}^{\mathbf{i}}$, are functions that associate with each state ω, respectively, a state of nature $\mathbf{s}(\omega)$, a price vector $\mathbf{p}(\omega)$, an event $\boldsymbol{\chi}^{\mathbf{i}}(\omega) \subseteq \Omega$ which is the event that agent i knows at the state ω, a belief for agent i $\boldsymbol{\mu}^{\mathbf{i}}(\omega)$ which is a probability distribution on Ω, and a bundle $\mathbf{z}^{\mathbf{i}}(\omega)$ for agent i. The knowledge and belief functions satisfy the following properties:

[21] The fact that the parameters which define the economy are common knowledge is implicit.

There is no problem to define this common knowledge explicitly but it would make the model more cumbersome without providing any important clarification.

[22] We remind that $C(\widehat{s}) \equiv \left\{ s \,\middle|\, P^s \cap P^{\widehat{s}} \neq \varnothing \right\}$ where P_s is the set of CKE prices at the state s.

[23] The parameters that are written in a bold font are functions.

1. $\omega \in \boldsymbol{\chi}^{\mathbf{i}}(\omega)$.
2. $\Pi^i \equiv \left\{\boldsymbol{\chi}^{\mathbf{i}}(\omega) \,|\omega \in \Omega\right\}$ is a partition of Ω.
3. $\boldsymbol{\chi}^{\mathbf{i}}(\widehat{\omega}) \subseteq \left\{\omega \left|\mathbf{s}(\omega) \in \psi^i(\mathbf{s}(\widehat{\omega})), \mathbf{p}(\omega) = \mathbf{p}(\widehat{\omega}), \mathbf{z}^{\mathbf{i}}(\omega) = \mathbf{z}^{\mathbf{i}}(\widehat{\omega}), \boldsymbol{\mu}^{\mathbf{i}}(\omega) = \boldsymbol{\mu}^{\mathbf{i}}(\widehat{\omega})\right.\right\}$.
4. $\boldsymbol{\mu}^{\mathbf{i}}(\widehat{\omega})\left[\boldsymbol{\chi}^{\mathbf{i}}(\widehat{\omega})\right] = 1$.

Properties 1 and 2 are standard assumptions on a knowledge operator. Property 3 requires consistency with the assumption that each agent i knows his private signal, the price vector, his demand , and his beliefs. Property 4 requires that the belief of each agent i at a state $\widehat{\omega}$ be consistent with his knowledge in that state.

A state $\widehat{\omega} \in \Omega$ satisfies rationality (R) if for every $i \in I$ the bundle $\mathbf{z}^{\mathbf{i}}(\widehat{\omega})$ is optimal w.r.t. the price $\mathbf{p}(\widehat{\omega})$ and the belief $\boldsymbol{\mu}^{\mathbf{i}}(\widehat{\omega})$[24].

A state $\widehat{\omega} \in \Omega$ satisfies market Clearing (MC) if $\int\limits_{i \in I} \mathbf{z}^{\mathbf{i}}(\widehat{\omega}) = (0,..,0,1)$.

We let R and MC denote, respectively, the set of states that satisfy rationality and the set of states that satisfy market clearing.

We turn now to the definition of common knowledge (CK).

Let $E \subseteq \Omega$ be an event. Define

$\boldsymbol{\sigma}(E) = \bigcup\limits_{i \in I, \omega \in E} \boldsymbol{\chi}^{\mathbf{i}}(\omega)$

We note that since for every $\omega \in \Omega$ $\omega \in \boldsymbol{\chi}^{\mathbf{i}}(\{\omega\})$ (property 1) we have

$E \subseteq \boldsymbol{\sigma}(E)$.

Let $\widehat{\omega} \in \Omega$. Define the operators $\boldsymbol{\sigma}^k$, $k = 1, 2, ...$, inductively as follows:

$\boldsymbol{\sigma}^1(\widehat{\omega}) = \boldsymbol{\sigma}(\{\widehat{\omega}\})$

$\boldsymbol{\sigma}^k(\widehat{\omega}) = \boldsymbol{\sigma}(\boldsymbol{\sigma}^{k-1}(\widehat{\omega}))$

$\boldsymbol{\sigma}^\infty(\widehat{\omega}) = \bigcup\limits_{k=1}^{\infty} \boldsymbol{\sigma}^k(\widehat{\omega})$.

$\boldsymbol{\sigma}^1(\widehat{\omega})$ is the minimal event that every agent i knows in the state $\widehat{\omega}$, $\boldsymbol{\sigma}^2(\widehat{\omega})$ is the minimal event E such that at $\widehat{\omega}$ every agent knows that every other agent knows E. A simple induction establishes that $\boldsymbol{\sigma}^k(\widehat{\omega})$ is the minimal event E with the property that at the state $\widehat{\omega}$ every proposition of the following type is true: Player i_1 knows that player i_2 knows... that player i_k knows that the event E occurred. It follows that if an event E is CK at $\widehat{\omega}$ then $\boldsymbol{\sigma}^\infty(\widehat{\omega}) \subseteq E$. On the other hand it is easy to see (again, a simple induction) that for every k $\boldsymbol{\sigma}^k(\boldsymbol{\sigma}^\infty(\widehat{\omega})) \subseteq \boldsymbol{\sigma}^\infty(\widehat{\omega})$ and therefore $\boldsymbol{\sigma}^\infty(\widehat{\omega})$ is CK at $\widehat{\omega}$. Thus, $\boldsymbol{\sigma}^\infty(\widehat{\omega})$ is the minimal event that is CK at $\widehat{\omega}$. Let $CK(\widehat{\omega})$

[24] Optimality w.r.t. $\boldsymbol{\mu}^{\mathbf{i}}(\widehat{\omega})$ is equivalent, of course, to optimality w.r.t. the marginal of $\boldsymbol{\mu}^{\mathbf{i}}(\widehat{\omega})$ on S.

denote $\boldsymbol{\sigma}^{\infty}(\widehat{\omega})$. We summarize the observations in the last paragraph with the following definition:

Definition: An event $E \subseteq \Omega$ is CK at a state $\widehat{\omega}$ if $CK(\widehat{\omega}) \subseteq E$.[25]

We are now ready to present theorem 3 which establishes the equivalence of CKE with CK of R and MC.

Theorem 3: Let $\mathcal{E}$ be an asset economy. The price p is a CKE at a state $\widehat{s} \in S$ iff there exists a model M for the economy $\mathcal{E}$ and a state $\widehat{\omega} \in \Omega(M)$ s.t. $\mathbf{s}(\widehat{\omega}) = \widehat{s}$, $\mathbf{p}(\widehat{\omega}) = \widehat{p}$, and the event $R \cap MC$ is CK at $\widehat{\omega}$.

Our next result characterizes the minimal set of states of nature that is common knowledge in every CKE at a given state $\widehat{s}$. Let $\mathcal{S}(\widehat{s})$ denote this set. To define $\mathcal{S}(\widehat{s})$ formally we introduce the following notation. Given an event $E \subseteq \Omega$ define $\mathbf{s}(E) = \{\mathbf{s}(\omega) \,|\omega \in E\}$. We now define $\mathcal{S}(\widehat{s})$ as follows:

$$\mathcal{S}(\widehat{s}) \equiv \left\{ s \;\middle|\; \begin{array}{c} \text{There exists a model } M \text{ and a state } \widehat{\omega} \in \Omega(M) \text{ s.t.} \\ \mathbf{s}(\widehat{\omega}) = \widehat{s}, CK(\widehat{\omega}) \subseteq R \cap MC, \text{ and } s \in \mathbf{s}(CK(\widehat{\omega})) \end{array} \right\}$$

Theorem 4: $\mathcal{S}(\widehat{s}) = C(\widehat{s})$.

7 An Extension

In this section we extend theorems 1 and 2 to the case where the asset economy is incomplete, that is, we relax the assumption that for every $s \in S$ the asset A_s exists. The motivation for this extension is that it may be impossible to verify some features that define the state of nature and in such a case it will be impossible to enforce contracts which condition on such features. For example, it may be impossible to condition on the features of the state of nature that determine the private information of some agents (see Morris 1994 for such a model). Formally, let $\Phi \subseteq 2^S$ be a collection of events. We think of Φ as the set of events on which the agents can contract. Define $\mathcal{E}_\Phi$ to be the economy where the set of assets is $\mathcal{A}_\Phi \equiv \{A_E \,|E \in \Phi\}$. We say that Φ is regular if there exists a partition of S, $\mathcal{F} = \{F_1, .., F_K\}$, such that Φ is the set of unions of elements in $\mathcal{F}$. In such a case we will say that Φ is

[25] The pararaph which precedes the definition of CK shows the equivalence of our definition to a definition of common knowledge in terms of a knowledge predicat. This equivalence is well known.

generated by $\mathcal{F}$ and write $\Phi = \Phi(\mathcal{F})$. It is easy to see that a collection of events Φ can be generated by a partition iff the following two conditions are satisfied: (1) For every $E, E' \in \Phi$ $E \cap E' \in \Phi$ or $E \cap E' = \emptyset$. (2) If $E \in \Phi$ then $E^C \in \Phi$. Thus, assuming regularity means that if agents can contract on the events E and E' they can contract on the events $E \cap E'$ and E^c as well. To characterize the set of equilibrium outcomes in the economy $\mathcal{E}_{\Phi(\mathcal{F})}$ we can restrict attention to the set of basic assets $\mathcal{A}_{\mathcal{F}} = \{A_F \,|F \in \mathcal{F}\}$ because, clearly, in equilibrium the price p_E of an asset $A_E, E \in \Phi$, is just the sum of the prices of the basic assets that compose A_E, that is, $p_E = \sum\limits_{F \subseteq E, F \in \mathcal{F}} p_F$.

Let $p = (p_F)_{F \in \mathcal{F}}$ be a vector of prices of the assets in $\mathcal{F}$. The definitions of an equilibrium and CKE at a state $s \in S$ in the economy $\mathcal{E}_{\Phi(\mathcal{F})}$ are immediate and obvious modifications of the definitions that were given in section 2 for the complete asset economy. We now present theorems 5 and 6 which extend theorems 1 and 2 respectively to the case of incomplete asset economies.

Theorem 5:

Let $\mathcal{F} = \{F_1, .., F_K\}$ be a partition of S. The price vector $p = (p_F)_{F \in \mathcal{F}}$ is an equilibrium price at a state $\widehat{s}$ in the economy $\mathcal{E}_{\Phi(\mathcal{F})}$ iff:

(1) $\sum\limits_{F \in \mathcal{F}} p_F = 1$

(2) For every $E \subseteq \Phi(\mathcal{F})$ $m(I_{\widehat{s}}^E) \leq \sum\limits_{F \subseteq E, F \in \mathcal{F}} p_F$.

The proof of theorem 5 is similar to the proof of theorem 1 and therefore is omitted.

For $\widehat{s} \in S$ we let $P_{\mathcal{F}}^{\widehat{s}}$ denote the set of CKE prices at $\widehat{s}$ in the economy $\mathcal{E}_{\Phi(\mathcal{F})}$.

For $\overline{S} \subseteq S$ and $s \in \overline{S}$ define

$P_{\overline{S},\mathcal{F}}^{s} \equiv \left\{p \,\middle|\, \text{there is an equilibrium } ((\gamma^{i,s}, z^{i,s})_{i \in I}, p) \text{ at } s \text{ in } \mathcal{E}_{\Phi(\mathcal{F})} \text{ s.t. } \gamma^{i,s} \in \Delta(\psi^i(s) \cap \overline{S})\right\}$

The definition of CKE implies that

$$(7.1)\quad P_{\mathcal{F}}^{\widehat{s}} = \bigcup_{\overline{S}, \widehat{s} \in \overline{S}} \left(\bigcap_{s \in \overline{S}} P_{\overline{S},\mathcal{F}}^{s} \right)$$

We now show that the sets $P_{\overline{S},\mathcal{F}}^{s}$ can be characterized in terms of the parameters $m(I_s^E), E \subseteq S, s \in S$. To prove this it will be useful to define an economy $\mathcal{E}_{\overline{S},\mathcal{F}}$ in the following way: The set of states is $\overline{S}$. The set of basic assets is $\mathcal{F}_{\overline{S}} \equiv \left\{F \,\middle|\, \exists F' \in \mathcal{F} \text{ s.t. } F = F' \cap \overline{S} \right\}$. The information partition of an agent i, $\overline{\psi}_i$, is defined as follows: For $s' \in \overline{S}$ $\overline{\psi}^i(s') \equiv \psi^i(s') \cap \overline{S}$.

Let $p = (p_F)_{F\in\mathcal{F}}$ be a vector of prices in the economy $\mathcal{E}_{\Phi(\mathcal{F})}$. Define a vector of prices $p^{\overline{S}}$ in the economy $\mathcal{E}_{\overline{S},\mathcal{F}}$ as follows: Let $F \in \mathcal{F}_{\overline{S}}$ and let $F' \in \mathcal{F}$ be the set such that $F = F' \cap \overline{S}$, define $p_F^{\overline{S}} \equiv p_{F'}$.

Now let $s \in \overline{S}$ and let $P(\mathcal{F}, \overline{S}, s)$ denote the set of price vectors $p = (p_F)_{F\in\mathcal{F}}$ that satisfy the following two conditions: (1) The price vector $p^{\overline{S}}$ is an equilibrium price vector in the economy $\mathcal{E}_{\overline{S},\mathcal{F}}$ at the state s. (2) If $F \cap \overline{S} = \emptyset$ then $p_F = 0$.

An argument which is identical to the proof of lemma 2 establishes that the sets $P^s_{\overline{S},\mathcal{F}}$ and $P(\mathcal{F}, \overline{S}, s)$ are equal. Putting this equality with 7.1 we obtain:

Theorem 6: $P^{\widehat{s}}_{\mathcal{F}} = \bigcup_{\overline{S},\widehat{s}\in\overline{S}} \left(\bigcap_{s\in\overline{S}} P(\mathcal{F}, \overline{S}, s) \right).$

Theorem 6 is useful because the sets $P(\mathcal{F}, \overline{S}, s)$, $\overline{S} \subseteq S, s \in \overline{S}$, can be characterized in terms of the parameters $m(I_s^E), E \subseteq S, s \in S$. To see this we observe that the set of agents who know an event an event $E \subseteq \overline{S}$ in the economy $\mathcal{E}_{\overline{S},\mathcal{F}}$ at the state $s \in E$ is $I_s^{E\cup(S\setminus\overline{S})}$. Applying now theorem 5 to the economy $\mathcal{E}_{\overline{S},\mathcal{F}}$ we obtain that $p \in P(\mathcal{F}, \overline{S}, s)$ iff

(1) $\sum_{F\in\mathcal{F}} p_F = 1$

(2) For every $E \subseteq \Phi(\mathcal{F}_{\overline{S}})$ $m(I_s^{E\cup(S\setminus\overline{S})}) \leq \sum_{F\subseteq E, F\in\mathcal{F}_{\overline{S}}} p_F^{\overline{S}}$.

8 Conclusion

The paper analyzes an economy with asymmetric information in which agents trade in contingent assets. The new feature in the model is that different agents may have different priors on the state of nature. We have proposed two solution concepts: Equilibrium, which assumes rationality and market clearing, and common knowledge equilibrium (CKE) which makes the stronger assumption that rationality, market clearing, and the parameters which define the economy are common knowledge. The two main results, theorem 1 and theorem 2, characterize the set of equilibrium prices, $\overline{P}^s$, and the set of CKE prices, P^s, at a given state s. Theorem 1 and theorem 2 have two notable features:

(1) The sets $\overline{P}^s$ and P^s are characterized in terms of the parameters $m(I_{s'}^E), s' \in S\ E \subseteq S$, where $m(I_{s'}^E)$ specifies the amount of money in the hands of agents who know the event E at the state s'.

(2) The characterizations in theorems 1 and 2 apply to a broad class of preferences over uncertain outcomes. In particular, theorem 1 (theorem 2) implies that for every profile of preferences that satisfy a basic monotonicity requirement the set of equilibrium prices (CKE prices) is the same set.

Theorem 2 implies a characterization of the information that is revealed in a CKE at a given state s. Specifically, we characterize the minimal event that is common knowledge in every CKE at s.

Finally, we have discussed the relationship between CKE, REE, and $CKRMC$, provided an epistemic foundation for CKE and extended theorem 1 and theorem 2 to the case where the asset economy is incomplete.

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Appendix

Section 2:

Proof of lemma 1:

It is easy to see that the definition of the outcome function $x(\cdot)$ and the equation $\sum_{s\in S} p_s = 1$ imply that $x(\overline{z}^y_E)_s = x(z^y_E)_s = \begin{cases} m + y\cdot(\sum_{s\in S\setminus E} p_s) & s\in E \\ m - y\cdot(\sum_{s\in E} p_s) & s\in S\setminus E \end{cases}$

Also, $\overline{z}^y_E$ satisfies the constraint (NB) iff $m \geq y\cdot(\sum_{s\in E} p_s) \Leftrightarrow y \leq \frac{m}{\sum_{s\in E} p_s} \Leftrightarrow$ $y \leq \frac{m}{1-\sum_{s\in S\setminus E} p_s} \Leftrightarrow y \leq m + y\cdot(\sum_{s\in S\setminus E} p_s)$. Now, we observe that $z^y_{S\setminus E}$ satisfies the constraint (CC) iff $y \leq m + y\cdot(\sum_{s\in S\setminus E} p_s)$. (The LHS is the payment that the agent will have to make at a state $s\in S\setminus E$ and the RHS is the amount of money that he has.)

Proof of lemma 2:

Let $p \in P^s_{\overline{S}}$ and let $e = ((\gamma^{i,s}, z^{i,s})_{i\in I}, p)$ be an equilibrium at s such that $\gamma^{i,s} \in \Delta(\psi^i(s)\cap\overline{S})$. First, we show that $p_{s'} = 0$ for every $s' \in S\setminus\overline{S}$. Assume by contradiction that there exists a state $s' \in S\setminus\overline{S}$ such that $p_{s'} > 0$. In the equilibrium e every agent i believes that he can make a profit (for example, by selling the asset $A_{s'}$.) It follows that every agent i is trading (i.e., selling or buying some assets.) It is easy to see that if an agent i is selling some assets he will sell some quantity of the asset $A_{s'}$ as well[26]. On the other hand there is no agent who is willing to buy $A_{s'}$ because everyone assigns the state s' probability zero. It follows that there is a positive supply of the asset $A_{s'}$ but zero demand and thus the market for $A_{s'}$ does not clear. We have thus obtained a contradiction to the assumption that there exists a state s' such

[26] If i is selling then $z^{i,n+1} > 0$. If $z^{i,s'} = 0$ i could move to a better bundle by selling $z^{i,n+1}$ units of $A_{s'}$.

that $p_{s'} > 0$. We now show that $p_{\overline{S}}$ is an equilibrium in the economy $\mathcal{E}_{\overline{S}}$. For every $i \in I$ define the bundle $\overline{z^{i,s}}$ as follows:

$$\overline{z_k^{i,s}} \equiv \begin{cases} z_k^{i,s} & k \in \overline{S} \\ z_k^{i,s} & k = n+1 \\ 0 & k \in S \backslash \overline{S} \end{cases}$$

It is easy to see that since $p_{s'} = 0$ for every $s' \in S\backslash\overline{S}$ and since $\gamma^{i,s}(S\backslash\overline{S}) = 0$ for every $i \in I$ the bundle $\overline{z^{i,s}}$ belongs to the budget set $B(p, m^i)$ and is optimal in this set w.r.t. the belief $\gamma^{i,s}$. Now, since $\gamma^{i,s} \in \Delta(\psi^i(s) \cap \overline{S})$, $\gamma^{i,s}$ is consistent with the knowledge of the agent i in state s in the economy $\mathcal{E}_{\overline{S}}$. Also, the fact that $(z^{i,s})_{i\in I}$ satisfies market clearing implies that the markets clear in $(\overline{z^{i,s}})_{i\in I}$ as well. It follows that the tuple $((\gamma^{i,s}, \overline{z^{i,s}})_{i\in I}, p)$ is an equilibrium in the economy $\mathcal{E}_{\overline{S}}$.

We turn now to the second direction. Let p be a price vector such that $p_{s'} = 0$ for every $s' \in S\backslash\overline{S}$ and such that $p_{\overline{S}}$ is an equilibrium price vector in the economy $\mathcal{E}_{\overline{S}}$ at the state s. Let $((\gamma^{i,s}, \overline{z^{i,s}})_{i\in I}, p_{\overline{S}})$ be an equilibrium in $\mathcal{E}_{\overline{S}}$ at the state s. Obviously, we have $\gamma^{i,s} \in \Delta(\psi^i(s) \cap \overline{S})$. For every $i \in I$ define

$$z_k^{i,s} \equiv \begin{cases} \overline{z_k^{i,s}} & k \in \overline{S} \\ \overline{z_k^{i,s}} & k = n+1 \\ 0 & k \in S \backslash \overline{S} \end{cases}$$

It is easy to see that the profile $(z^{i,s})_{i\in I}$ satisfies market clearing in the economy $\mathcal{E}$. Also, since $p_{s'} = 0$ for every $s' \in S\backslash\overline{S}$ no agent will benefit from selling the asset $A_{s'}$ and since $\gamma^{i,s}(S\backslash\overline{S}) = 0$ for every $i \in I$ no agent believes that he will benefit from buying an asset $A_{s'}$, $s' \in S\backslash\overline{S}$. It follows that $((\gamma^{i,s}, z^{i,s})_{i\in I}, p)$ is an equilibrium in the economy $\mathcal{E}$.

Section 3:

Proof of lemma 3:

Let $z \in B(p, m^i)$.

Define $\overline{S} \equiv \{s \,|s \in S \; z_s < 0\}$. $\overline{S}$ is the set of assets that are sold in the bundle z. Define $y \equiv \min_{s\in\overline{S}} |z_s|$. We will now construct a bundle $\overline{z}$ such that $\overline{z} \in B(p, m^i), x(\overline{z}) = x(z)$, and such that the set of assets that are sold in $\overline{z}$ is strictly contained in $\overline{S}$. the bundle $\overline{z}$ is obtained from z by reducing the sale of each asset $A_s, s \in \overline{S}$, by y units and then using $\$y \cdot (\sum_{s \in S \setminus \overline{S}} p_s)$ to buy y units of the asset $A_{S \setminus \overline{S}}$. Since $\sum_{s \in S} p_s = 1$ we obtain that the bundle $\overline{z}$ is defined as follows:

$$\overline{z}_k \equiv \begin{cases} z_k + y & k \in S \\ z_{n+1} - y & k = n+1 \end{cases}$$

It is easy to see that $x(\overline{z}) = x(z)$. To see that $\overline{z}$ does indeed belong to the budget set $B(p, m^i)$ we observe that since the sales of the asset $A_{\overline{S}}$ in the bundle $\overline{z}$ is lower by y units in comparison to the sale in the bundle z the commitment of agent i to pay back is lower by $\$y$. On the other hand the reduction in the sales of $A_{\overline{S}}$ decreases the amount of money in the hands of the agent by $\$y \cdot (\sum_{s \in \overline{S}} p_s)$. Putting this together we see that the reduction of y units in the sale of the asset $A_{\overline{S}}$ releases $\$(y - y \cdot (\sum_{s \in \overline{S}} p_s)) = \$y \cdot (\sum_{s \in S \setminus \overline{S}} p_s)$ that can be used for the purchase of the asset $A_{S \setminus \overline{S}}$. It follows that $\overline{z}$ belongs to $B(p, m^i)$. Also, the set of assets that are sold in $\overline{z}$ is strictly contained in $\overline{S}$. If $\overline{z} \geqq 0$ we are done. Otherwise, we repeat the procedure and obtain, after at most $|\overline{S}|$ steps, a bundle $\widetilde{z} \in B(p, m^i)$ such that $\widetilde{z} \geqq 0$ and $x(\widetilde{z}) = x(z)$.

Section 5

Consider an asset economy where all the agent have a common prior $\alpha \in \Delta(S)$. We assume w.l.o.g. that $\alpha(s) > 0$ for $s \in S$ and that every agent $i \in I$ has a positive amount of money. The concept of REE assumes that in equilibrium all the agents know the true price function, that is, they know which price will materialize in each state. When an agent i knows that the true price function is f then at the state s he knows the event $\psi^i(s) \cap f^{-1}(f(s))$ (where $f^{-1}(f(s)) \equiv \{s' \,|f(s') = f(s)\}$.) Let ψ^i_f denote the information partition that represents this knowledge, that is, $\psi^i_f(s) \equiv \psi^i(s) \cap f^{-1}(f(s))$. We can now define a REE.

Definition: Let f be a price function and let $\{d^i\}_{i \in I}$ be a profile of demand functions. The pair $(f, \{d^i\}_{i \in I})$ is a REE if

(1) For every $s \in S$ $d^i(\psi^i(s), f(s))$ is an optimal choice for agent i in the budget set $B(f(s), m^i)$ w.r.t to the posterior probability $\alpha\left(\cdot \,\middle|\psi^i_f(s)\right)$.

(2) All the markets clear, that is, $\int_{i \in I} d^i(\psi^i(s), f(s)) = (0, ..., 0, 1)$.

To present the non-trade and full-revelation properties of REE we need two additional definitions.

We say that agent i is risk averse if there exits a strictly monotone and strictly concave utility function $u^i : R \to R$ such that the preference of agent i on the space of outcomes X is defined by the expectation of u^i. More formally, let γ be the posterior probability of agent i on S then for $x, y \in X$

$$x \succsim_i^\gamma y \Leftrightarrow \sum_{s \in S} \gamma(s) \cdot u^i(x_s) \geq \sum_{s \in S} \gamma(s) \cdot u^i(y_s).$$

Let $\mathcal{E}$ be an asset economy. We say that $\mathcal{E}$ satisfies AIR (the aggregate information is revealing) if for every $s, s' \in S, s \neq s'$, the set $\left\{ i \,\middle|\, s' \notin \psi^i(s) \right\}$ has a positive measure.

Proposition (1): Let $\mathcal{E}$ be an economy with risk averse agents and let $(f, \{d^i\}_{i \in I})$ be a REE in $\mathcal{E}$. Then:

(a) For every $s \in S$ either $f(s) = p^s$ or there exits a set of agents of measure 1, I', such that for every $i \in I'$ $d^i(\psi^i(s), f(s))$ is the initial bundle of agent i.

(b) If $\mathcal{E}$ satisfies AIR then f is the fully revealing price function, that is, $f(s) = p^s$ for every $s \in S$.

Remark: If the price function is fully-revealing then the price of each asset equals its true value and thus agents are indifferent between trading and not trading.

Proof : Start with part (a). Let $\overline{s} \in S$. If $f(\overline{s}) = p^{\overline{s}}$ then there is nothing to prove. So assume that $f(\overline{s}) = p$ where $p \neq p^{\overline{s}}$. Define $S(p) \equiv \{s \,|\, f(s) = p\}$. Clearly, $|S(p)| \geq 2$ (otherwise $S(p) = \{\overline{s}\}$ which would imply that $\overline{s}$ is revealed and then p must equal $p^{\overline{s}}$.) Next, for every $s \in S(p)$ $p_s > 0$ (This follows, because $\alpha(s) > 0$ and hence at the state s the posterior of each agent assigns a positive probability to s. If $p_s = 0$ there would be an unbounded demand for A_s.)

Let $p \in R^n$ be a vector of prices, let $z \in R^{n+1}$ be a bundle of assets, and let $s \in S$. We let $\pi(z, p, s)$ denote the profit form the bundle z at the state s when the price is p, that is, $\pi(z, p, s) = z_s - \sum_{s' \in S} p_{s'} \cdot z_{s'}$. Clearly, for every agent i and every state $\widetilde{s} \in S(p)$

(A.5.1) $E\left[\pi(d^i(\psi^i(\widetilde{s}), p), p, s) \,\middle|\, \psi^i(\widetilde{s}) \cap S(p)\right] \geq 0$[27]

because otherwise it follows from the risk aversion of agent i that in the event $\psi^i(\widetilde{s}) \cap S(p)$ the bundle $d^i(\psi^i(\widetilde{s}), p)$ is inferior to the initial bundle

[27] We remind that the demand function of agent i is measurable w.r.t. ψ_f^i and therefore in every state $s \in \psi^i(\widetilde{s}) \cap S(p)$ agent i demands the same bundle, $d^i(\psi^i(\widetilde{s}), p)$.

(i.e., agent i would be better of not trading.) Now (A.5.1) implies that the expected profit of each agent i conditional on the event $S(p)$ is non-negative as well. That is, we have

(A.5.2) $E\left[\pi(d^i(\psi^i(s),p),p,s)\,|S(p)\right] \geq 0$

On the other hand the aggregate profit in every state must equal zero, that is, for every $s \in S(p)$ we have

(A.5.3) $\int\limits_{i \in I} \pi(d^i(\psi^i(s),p),p,s) = 0$

(A.5.3) and (A.5.2) imply that there exists a set of agents of measure 1, I', such that for every $i \in I'$ (A.5.2) is satisfied as an equality. This in turn implies that for every $i \in I'$ and $\widetilde{s} \in S(p)$ (A.5.1) is satisfied as an equality. It follows that for every $i \in I'$ and $\widetilde{s} \in S(p)$ the bundle $d^i(\psi^i(\widetilde{s}),p)$ is the initial bundle of agent i because any bundle that has an expected profit of zero and which is different than the initial bundle exposes agent i to some risk without giving him any benefit in terms of expected profit.

We have completed the proof of part (a). The proof of part (b) given the proof of part (a) is quick. Assume by contradiction that there exists a state, $\overline{s}$, such that $f(\overline{s}) = p \neq p^{\overline{s}}$. We have seen in part (a) that $|S(p)| \geq 2$. Let s and s' be two states in $S(p)$. Define $I_{s,s'} \equiv \left\{i \,\middle|\, s' \notin \psi^i(s)\right\}$. AIR implies that the set $I_{s,s'}$ has a positive measure. As we have seen in part (a) $p_{s'} > 0$. It follows that at the state s each agent in $I_{s,s'}$ knows that he can make a positive profit by selling the asset $A_{s'}$. Thus, there is set of agents of a positive measure who know at the state s that they can make a positive profit. This contradicts the conclusion in part (a). We obtain that $f(s) = p^s$ for every $s \in S$.

<u>Section 6:</u>

<u>Proof of theorem 3:</u>

One direction is simple; Assume that there exists a model M and a state $\widehat{\omega} \in \Omega$ s.t. $\mathbf{s}(\widehat{\omega}) = \widehat{s}$, $\mathbf{p}(\widehat{\omega}) = \widehat{p}$, and there is CK of R and MC at $\widehat{\omega}$. We have to show that $\widehat{p}$ is a CKE price in $\widehat{s}$. We now show that $\widehat{p}$ is a CKE price w.r.t $\mathbf{s}(CK(\widehat{\omega}))$. Since $\widehat{s} \in \mathbf{s}(CK(\widehat{\omega}))$ we will obtain that $\widehat{p}$ is a CKE price in $\widehat{s}$. Let $s \in \mathbf{s}(CK(\widehat{\omega}))$ and let ω be an element in $CK(\widehat{\omega})$ such that $\mathbf{s}(\omega) = s$. It is easy to see that property 3 in the definition of a model M implies that $\mathbf{p}(\omega) = \widehat{p}$. Define $\boldsymbol{\gamma}^{\mathbf{i}}(\omega)$ to be the marginal of $\boldsymbol{\mu}^{\mathbf{i}}(\omega)$ on S. Properties 3 and 4, the fact that $\omega \in R \cap MC$, and the fact that $\mathbf{s}(\boldsymbol{\sigma}(\omega)) \subseteq \mathbf{s}(CK(\widehat{\omega}))$ imply that $((\boldsymbol{\gamma}^{\mathbf{i}}(\omega), \mathbf{z}^{\mathbf{i}}(\omega))_{i \in I}, \widehat{p})$ is an equilibrium in the state s and that for every

$i \in I$ $\boldsymbol{\gamma}^{\mathbf{i}}(\omega)$ assigns probability 1 to the set $\mathbf{s}(CK(\widehat{\omega}))$. It follows that $\widehat{p}$ is a CKE price w.r.t. $\mathbf{s}(CK(\widehat{\omega}))$.

Consider now the second direction. Let $\widehat{p}$ be a CKE price at a state of nature $\widehat{s}$. We will construct a model M for the economy $\mathcal{E}$ with a state $\widehat{\omega} \in \Omega$ such that $\mathbf{s}(\widehat{\omega}) = \widehat{s}$ and $\mathbf{p}(\widehat{\omega}) = \widehat{p}$ and such that R and MC are satisfied in every $\omega \in \Omega$. Such a construction clearly implies that there is CK of R and MC in $\widehat{\omega}$. The definition of a CKE implies that for every $s \in S(\widehat{p})$ there exists an equilibrium $((\gamma^{i,s}, z^{i,s})_{i\in I}, \widehat{p})$ where the support $\gamma^{i,s}$ is contained in $\psi^i(s) \cap S(\widehat{p})$. Before going into the formal definition of the model M we explain the idea that underlies the construction. To have markets clear in a state s we want each agent j to have the demand $z^{j,s}$. To have a particular player i make a demand of $z^{i,s}$ we construct a set of comprehensive states where the belief of player i on the states of nature is $\gamma^{i,s}$. Specifically, with each state of nature $\overline{s} \in \psi^i(s) \cap S(\widehat{p})$ we associate a comprehensive state $(\overline{s}, s, i)$ which should be interpreted as follows: The state of nature is $\overline{s}$ and every agent j except agent i has a demand $z^{j,\overline{s}}$ and a belief with a marginal on S that equals $\gamma^{j,\overline{s}}$. Player i has a belief that assigns to a state of nature $\widetilde{s} \in \psi^i(s) \cap S(\widehat{p})$ the probability $\gamma^{i,s}(\widetilde{s})$. The demand of agent i is $z^{i,s}$. We note that because there is a continuum of agents and since each agent j that is different from i has the demand $z^{j,\overline{s}}$ all the markets clear. The states $(\overline{s}, s, i)$, $\overline{s} \in \psi^i(s) \cap S(\widehat{p})$, should be thought of as possible contingencies in the mind of player i. In each such state player i assigns probability $\gamma^{i,s}(\widetilde{s})$ to the state $(\widetilde{s}, s, i)$ which is the event that the true state is $\widetilde{s}$ and that the agents different from i make demands that clear the market. With this in mind we turn now to the formal definition of a model M in which R and MC are satisfied in each state and where there is a state $\widehat{\omega}$ that specifies the state of nature $\widehat{s}$ and the price $\widehat{p}$. First, define

$\overline{\Omega} \equiv \left\{ (\overline{s}, s, i) \,\middle|\, s \in S(\widehat{p}), i \in I, \overline{s} \in \psi^i(s) \cap S(\widehat{p}) \right\}$.

The set of states Ω is defined as follows:

$\Omega \equiv \overline{\Omega} \cup (S \backslash S(\widehat{p}))$.

The states $S \backslash S(\widehat{p})$ are just used to complete the model and have no important role in the construction. In a state $s \in S \backslash S(\widehat{p})$ there is full revelation, so every agent knows that the state s has materialized. In particular, the vector of prices is the vector p^s in which the price of the asset A_s is 1 and the prices of the other assets are zero. Formally, for a state $s \in S \backslash S(\widehat{p})$ we define the functions $\mathbf{s}, \mathbf{p}, \boldsymbol{\chi}^{\mathbf{j}}, \boldsymbol{\mu}^{\mathbf{j}}, \mathbf{z}^{\mathbf{j}}$ as follows: $\mathbf{s}(s) = s, \mathbf{p}(s) = p^s, \boldsymbol{\chi}^{\mathbf{j}}(s) = \{s\}, \boldsymbol{\mu}^{\mathbf{j}}(s)$ assigns the state s probability 1, and $\mathbf{z}^{\mathbf{i}}(s) = (0, .., 0, m^i)$, that is, each agent keeps his initial bundle. (We remind that m^i is the initial amount of money of agent i.) Clearly, keeping the initial bundle is optimal w.r.t the price p^s and the belief $\boldsymbol{\mu}^{\mathbf{i}}(s)$. We turn now to the definition of the functions $\mathbf{s}, \mathbf{p}, \boldsymbol{\chi}^{\mathbf{j}}, \boldsymbol{\mu}^{\mathbf{j}}, \mathbf{z}^{\mathbf{j}}$ on the set $\overline{\Omega}$. For $\omega = (\overline{s}, s, i) \in \overline{\Omega}$ define:

1. $\mathbf{s}(\omega) = \overline{s}$.
2. $\mathbf{p}(\omega) = \widehat{p}$.
3. $\mathbf{z}^{\mathbf{j}}(\omega) = \begin{cases} z^{i,s} & j = i \\ z^{j,\overline{s}} & j \neq i \end{cases}$
4. For the agent i
   $\boldsymbol{\mu}^{\mathbf{i}}(\omega)\,[\omega'] = \begin{cases} \gamma^{i,s}(\widetilde{s}) & \text{If } \omega' = (\widetilde{s}, s, i) \text{ s.t. } \widetilde{s} \in S(\widehat{p}) \cap \psi^i(s) \\ 0 & \text{otherwise} \end{cases}$
   For an agent $j \neq i$
   $\boldsymbol{\mu}^{\mathbf{j}}(\omega)\,[\omega'] = \begin{cases} \gamma^{j,\overline{s}}(\widetilde{s}) & \text{If } \omega' = (\widetilde{s}, \overline{s}, j) \text{ s.t. } \widetilde{s} \in S(\widehat{p}) \cap \psi^j(\overline{s}) \\ 0 & \text{otherwise} \end{cases}$
5. For every agent $j \in I$
   $\boldsymbol{\chi}^j(\omega) = \left\{\omega' \,\middle|\, \mathbf{s}(\omega') \in S(\widehat{p}) \cap \psi^j(\overline{s}), \mathbf{p}(\omega') = \widehat{p}, \boldsymbol{\mu}^{\mathbf{j}}(\omega') = \boldsymbol{\mu}^{\mathbf{j}}(\omega), \mathbf{z}^{\mathbf{j}}(\omega') = \mathbf{z}^{\mathbf{j}}(\omega)\right\}$

It is easy to see that properties (1)-(4) in the definition of a knowledge and belief model are satisfied and that in each state ω the demand of each agent is optimal w.r.t his belief on S and the markets clear. It follows that M is a model for the economy $\mathcal{E}$ such that R and MC are satisfied in each $\omega \in \Omega$. Since for any state $\widehat{\omega} \in \overline{\Omega}$ such that $\widehat{\omega} = (\widehat{s}, \widehat{s}, i)$ we have $\mathbf{s}(\widehat{\omega}) = \widehat{s}$ and $\mathbf{p}(\widehat{\omega}) = \widehat{p}$ the proof of theorem 3 is complete.

Proof of theorem 4:

First, we show that $\mathcal{S}(\widehat{s}) \subseteq C(\widehat{s})$. Let s' be a state of nature such that $s' \notin C(\widehat{s})$. We will show that $s' \notin \mathcal{S}(\widehat{s})$. Let M be a model and $\widehat{\omega} \in \Omega$ a state such that $\mathbf{s}(\widehat{\omega}) = \widehat{s}$ and $CK(\widehat{\omega}) \subseteq R \cap MC$. Let $\widehat{p} = \mathbf{p}(\widehat{\omega})$. The first part of the proof of theorem 3 implies that the price $\widehat{p}$ is a CKE price w.r.t. the set $\mathbf{s}(CK(\widehat{\omega}))$. Since $s' \notin C(\widehat{s})$ $P^{s'} \cap P^{\widehat{s}} = \varnothing$ and therefore $\widehat{p}$ is not a CKE price at the state s'. It follows that $s' \notin \mathbf{s}(CK(\widehat{\omega}))$ which implies our claim.

We now show that $C(\widehat{s}) \subseteq \mathcal{S}(\widehat{s})$. Let $s' \in C(\widehat{s})$ we need to show that there exists a model M and a state $\widehat{\omega} \in \Omega(M)$ such that $\mathbf{s}(\widehat{\omega}) = \widehat{s}$, $CK(\widehat{\omega}) \subseteq R \cap MC$, and $s' \in \mathbf{s}(CK(\widehat{\omega}))$. Let $\widehat{p}$ be a price such that $\widehat{p} \in P^{\widehat{s}} \cap P^{s'}$. We have $\widehat{s}, s' \in S(\widehat{p})$[28]. We will show that $s' \in \mathbf{s}(CK(\widehat{\omega}))$ for the model M and the state $\widehat{\omega} \in \Omega(M)$ that are defined in the second part of the proof of theorem 3. For $s \in S(\widehat{p})$ define

[28]We remind that $S(\widehat{p})$ is the set of states w.r.t which p is a CKE price.

$\psi(s,\widehat{p}) \equiv \bigcup_{i\in I} \psi^i(s) \cap S(\widehat{p})$

Lemma 4.1: $\psi(\widehat{s},\widehat{p}) \cap \psi(s',\widehat{p}) \neq \emptyset$.

Proof: Assume by contradiction that $\psi(\widehat{s},\widehat{p}) \cap \psi(s',\widehat{p}) = \emptyset$ and let $e = (((\gamma^{i,s}, z^{i,s})_{i\in I})_{s\in S(\widehat{p})}, \widehat{p})$ be a CKE w.r.t. $S(\widehat{p})$. The assumption that $\psi(\widehat{s},\widehat{p}) \cap \psi(s',\widehat{p}) = \emptyset$ implies that for $s \in \psi(s',\widehat{p})$ and $i \in I$ $s \notin \psi^i(\widehat{s}) \cap S(\widehat{p})$. It follows that for $s \in \psi(s',\widehat{p})$ and every $i \in I$ $\gamma^{i,\widehat{s}}(s) = 0$ and hence $\widehat{p}_s = 0$. (If $\widehat{p}_s > 0$ then every agent who sells some asset at $\widehat{s}$ will want to sell A_s as well. However, this is impossible in equilibrium because no agent wants to buy A_s at a positive price.) Thus, $\widehat{p}_s = 0$ for every $s \in \psi(s',\widehat{p})$. However, this means that at the state s' every agent knows that he can make an unbounded profit by buying the asset $A_{\psi(s',\widehat{p})}$. This, of course, contradicts the assumption that e is a CKE .

Lemma 4.1 implies that there exists agents i and j and a state $\widetilde{s} \in S$ such that $\widetilde{s}$ is an element in both $\psi^i(\widehat{s}) \cap S(\widehat{p})$ and in $\psi^j(s') \cap S(\widehat{p})$. Consider now the model M that is defined in the second part of the proof of theorem 3. Define three states: $\widehat{\omega} = (\widehat{s},\widetilde{s},i), \widetilde{\omega} = (\widetilde{s},\widetilde{s},i)$, and $\omega' = (s',\widetilde{s},j)$. The states $\widehat{\omega}, \widetilde{\omega}$ and ω' satisfy the following properties: $\mathbf{s}(\widehat{\omega}) = \widehat{s}, \mathbf{s}(\omega') = s', \widetilde{\omega} \in \boldsymbol{\chi}^{\mathbf{i}}(\widehat{\omega})$, and $\omega' \in \boldsymbol{\chi}^{\mathbf{j}}(\widetilde{\omega})$. This means that in the state $\widehat{\omega}$ agent i does not exclude the possibility that the state is $\widetilde{\omega}$ which is a state in which player j does not exclude the possibility that the state is ω' (where the state of nature is s'.) It follows that $\omega' \in CK(\widehat{\omega})$ which implies that $s' \in \mathbf{s}(CK(\widehat{\omega}))$ and hence $s' \in \mathcal{S}(\widehat{s})$.