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COMPUTING AN OPTIMAL CONTRACT IN SIMPLE TECHNOLOGIES

by

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Computing an Optimal Contract in Simple Technologies

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Abstract

We study an economic setting in which a principal motivates a team of strategic agents to exert costly effort toward the success of a joint *project*. The action taken by each agent is *hidden* and affects the (binary) outcome of the agent's individual *task* stochastically. A Boolean function, called *technology*, maps the individual tasks' outcomes into the outcome of the whole project. The principal induces a Nash equilibrium on the agents' actions through payments that are conditioned on the project's outcome (rather than the agents' actual actions) and the main challenge is that of determining the Nash equilibrium that maximizes the principal's net utility, referred to as the *optimal contract*.

Babaioff, Feldman and Nisan [1] suggest and study a basic *combinatorial agency* model for this setting. Here, we concentrate mainly on two extreme cases: the AND and OR technologies. Our analysis of the OR technology resolves an open question and disproves a conjecture raised in [1]. In particular, we show that while the AND case admits a polynomial-time algorithm, computing the optimal contract in the OR case is NP-hard. On the positive side, we devise an FPTAS for the OR case, which also sheds some light on optimal contract approximation of general technologies.

1 Introduction

We consider the setting in which a principal motivates a team of rational agents to exert costly effort towards the success of a joint project, where their actions are hidden from her. The outcome (usually, success or failure of the project) is stochastically determined by the set of actions taken by the agents and is visible to all. As agents' actions are invisible, their compensation depends on the outcome and the principal's challenge is to design contracts (conditional payments to the agents) as to maximize her net utility, given the payoff that she obtains from a successful outcome.

The problem of hidden-action in production teams has been extensively studied in the economics literature [6, 8, 11, 7, 12]. More recently, the problem has been examined from a computational

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perspective [4, 1, 2]. This line of research complements the field of Algorithmic Mechanism Design (AMD) [10, 9, 3] that received much attention in the last decade. While AMD studies the design of mechanisms in scenarios characterized by private information held by the individual agents, our focus is on the complementary problem, that of hidden-action taken by the individual agents. In [1], the authors concentrated on the case of homogeneous users, i.e., agents with identical capabilities. The current work extends the original work to the more complex (yet realistic) case, that of heterogeneous agents.

For example, consider an executive board that assigns stock options to the company's employees in attempt to motivate them to excel so that the value of the company increases. While the exact contribution of each individual may be difficult to measure, the stock's market price is visible to all, hence it serves as the groundwork in determining future payments to the staff. Given the significance of each employee (position, rank, etc.), how many stock options should he get?

For another example, consider the following scenario. A sender wishes to send a packet of information to a distant destination in a network in which intermediate routers are owned and operated by autonomous individuals or firms with diverse economic interests. The packet reaches the destination only if it successfully traverses through all hops in (at least) one network path to the target. Each intermediate router decides whether to exert "effort" (e.g., allocation of bandwidth, memory, storage or CPU's processing power) when attempting to forward the packet. If it does, it incurs some positive cost, but the packet traverses that hop with higher probability. While the final outcome of whether a packet reached its destination is clearly visible, it is rarely feasible to monitor the exact amount of effort exerted by each intermediate router. Therefore, the sender can motivate the intermediate routers by payments conditional on the final outcome alone. Given the payoff that the sender attains from the packet transmission, what is the optimal incentive structure she should impose? What is the complexity of computing optimal incentives in the above examples? This is the type of questions that motivate us in this work.

The model. We use the model presented in [1] (which is an extension of the model devised in [13]). In this model, a principal employs a set[1] N of agents in a joint *project* on her behalf. Each agent i takes an action $a_i \in \{0, 1\}$, which is known only to him, and succeeds or fails in his own *task* probabilistically and independently. The individual outcome of agent i is denoted by $x_i \in \{0, 1\}$. If the agent shirks ($a_i = 0$), he succeeds in his individual task ($x_i = 1$) with probability $0 < \gamma_i < 1$ and incurs no cost. If, however, he decides to exert effort ($a_i = 1$), he succeeds with probability $0 < \delta_i < 1$, where $\delta_i > \gamma_i$, but incurs some positive real *cost* $c > 0$.

A key component of the model is the way in which the individual outcomes determine the outcome of the whole project. We assume a monotone Boolean function $\varphi : \{0, 1\}^n \to \{0, 1\}$ which determines whether the project succeeds as a function of the individual outcomes of the

[1]Unless stated otherwise, we assume that $N = [n]$, where $[n]$ denotes the set $\{1, \ldots, n\}$.

n agents' tasks (and is not determined by any set of $n-1$ agents). Two fundamental examples of such Boolean functions are AND and OR. The AND function is the logical conjunction of x_i ($\varphi(x_1,\ldots,x_n) = \bigwedge_{i\in N} x_i$), representing the case in which the project succeeds only if *all* agents succeed in their tasks. In this case, we say that the agents *complement* each other. The OR function represents the other extreme, in which the project succeeds if *at least one* of the agents succeeds in his task. This function is the logical disjunction of x_i ($\varphi(x_1,\ldots,x_n) = \bigvee_{i\in N} x_i$), and we say that the agents *substitute* each other.

Given the action profile $a = (a_1,\ldots,a_n) \in \{0,1\}^n$ and a monotone Boolean function $\varphi : \{0,1\}^n \to \{0,1\}$, the *effectiveness* of the action profile a, denoted by $f(a)$, is the probability that the whole project succeeds under a and φ according to the distribution specified above. That is, the effectiveness $f(a)$ is defined as the probability that $\varphi(x_1,\ldots,x_n) = 1$, where $x_i \in \{0,1\}$ is determined probabilistically by a_i: if $a_i = 0$, then $x_i = 1$ with probability γ_i; if $a_i = 1$, then $x_i = 1$ with probability δ_i. The monotonicity of φ and the assumption that $\delta_i > \gamma_i$ for every $i \in N$ imply the monotonicity of the effectiveness function f, i.e., if we denote by $a_{-i} \in \{0,1\}^{n-1}$ the vector of actions taken by all agents excluding agent i (namely, $a_{-i} = (a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n)$), then the effectiveness function must satisfy $f(1,a_{-i}) > f(0,a_{-i})$ for every $i \in N$ and $a_{-i} \in \{0,1\}^{n-1}$.

The agents' success probabilities, the cost of exerting effort, and the monotone Boolean function that determines the final outcome determine the *technology* which is known to all. Formally, a technology $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi\rangle$ is a five-tuple, where N is a (finite) set of agents; γ_i (respectively, δ_i) is the probability that $x_i = 1$ when agent i shirks (resp., when agent i exerts effort), where $\delta_i > \gamma_i$; c is the cost incurred on an agent for exerting effort; and $\varphi : \{0,1\}^n \to \{0,1\}$ is the monotone Boolean function that maps the individual outcomes $x_1,\ldots,x_n$ to the outcome of the whole project. We sometimes abuse notation and refer to the Boolean function φ as the technology.

Since exerting effort entails some positive cost, an agent will not exert effort unless induced to do so by appropriately designed incentives. The principal can motivate the agents by offering them individual *payments*. However, due to the non-visibility of the agents' actions, the individual payments cannot be directly contingent on the actions of the agents, but rather only on the success of the whole project. The *conditional payment* to agent i is thus given by a real value $p_i \geq 0$ that is granted to agent i by the principal if the project succeeds (otherwise, the agent receives 0 payment[2]).

The expected *utility* of agent i under the profile of actions $a = (a_1,\ldots,a_n)$ and the conditional payment p_i is $p_i \cdot f(a)$ if $a_i = 0$; and $p_i \cdot f(a) - c$ if $a_i = 1$. Given a real *payoff* $v \geq 0$ that the principal obtains from a successful outcome of the project, the principal wishes to design the payments p_i as to maximize her own expected *utility* defined as $U_a(v) = f(a) \cdot \left(v - \sum_{i\in N} p_i\right)$, where the action

[2]We impose the *limited liability* constraint, implying that the principal can pay the agents but not fine them. Thus, all the payments must be non-negative.

profile a is assumed to be at Nash-equilibrium with respect to the payments p_i (i.e., no agent can improve his utility by a unilateral deviation). As multiple Nash equilibria may exist, we focus on the one that maximizes the utility of the principal. This is as if we let the principal choose the desired Nash equilibrium, and "suggest" it to the agents. The following observation is established in [1].

Observation. *The best conditional payments (from the principal's point of view) that induce the action profile $a \in \{0,1\}^n$ as a Nash equilibrium are $p_i = 0$ for agent i who shirks ($a_i = 0$), and $p_i = \frac{c}{\Delta_i(a_{-i})}$ for agent i who exerts effort ($a_i = 1$), where $\Delta_i(a_{-i}) = f(1, a_{-i}) - f(0, a_{-i})$. (Note that the monotonicity of the effectiveness function guarantees that $\Delta_i(a_{-i})$ is always positive.)*

The last observation implies that once the principal chooses the action profile $a \in \{0,1\}^n$, her (maximum) expected utility is determined to be $U_a(v) = f(a) \cdot (v - p(a))$, where $p(a)$ is the total *payment* (in case of a successful outcome of the project), given by $p(a) = \sum_{i|a_i=1} \frac{c}{\Delta_i(a_{-i})}$. Therefore the principal's goal is merely to choose a subset $S \subseteq N$ of agents that exert effort (the rest of the agents shirk) so that her expected utility is maximized. The agent subset S is referred to as a *contract* and we say that the principal *contracts with agent* i if $i \in S$. We sometimes abuse notation and denote $f(S)$, $p(S)$ and $U_S(v)$ instead of $f(a)$, $p(a)$ and $U_a(v)$, respectively, where $a_i = 1$ if $i \in S$ and $a_i = 0$ if $i \notin S$. Given the principal's payoff $v \geq 0$, the *optimal contract* is defined as $S_v^* = \text{argmax}_{S \subseteq N}\{U_S(v)\}$.

While finding the optimal set of payments that induces the contracted agents to exert effort is a straight-forward task (and can be efficiently computed), finding the optimal contract S_v^* for a given payoff $v \geq 0$ is the main challenge addressed in this paper. It is easy to see that for sufficiently low payoffs, no agent will ever be contracted while for sufficiently high payoffs, all agents will always be contracted. The problem becomes interesting for intermediate payoffs. Given a technology t, we refer to the collection of contracts that can be obtained as an optimal contract for some payoff as the *orbit* of t (ties between different contracts are broken according to a lexicographic order[3]).

Let $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi \rangle$ be an arbitrary technology. Once the contract $S \subseteq N$ is chosen, the expected utility of the principal $U_S(v) = f(S)(v - p(S))$ becomes a linear function of the payoff v. Therefore each contract S corresponds to some line in the 2-dimensional plane. It follows that computing the orbit of t is equivalent to identifying the top envelope of the lines collection $\{U_S(\cdot) \mid S \subseteq N\}$.

Our contribution. Multi-agent projects may exhibit delicate combinatorial structures of dependencies between the agents' actions. In the general case, these complex dependencies may be represented by a wide range of monotone Boolean functions. In the two extremes of this range reside two simple and natural functions, namely AND and OR, which represent the respective cases of pure complementarities and pure substitutabilities. These functions are shown in [1] to exhibit

[3] This implies that there are no two contracts with the same effectiveness in the orbit.

very different properties with respect to the optimal contract problem. Yet, the authors leave many questions open. Here, we provide a thorough analysis of the optimal contract problem in these two technologies, and in particular, resolve an open question and disprove a conjecture raised in [1]. In addition to the analysis of the AND and OR technologies, we obtain an interesting property exhibited by all technologies.

The substance of our analysis concerns the OR technology, which is left, to the most part, unresolved in [1]. In particular, it is left as an open question whether computing the optimal contract in any OR technology can be done in polynomial time. The first theorem proved in this paper addresses this question.

Theorem 1. *The problem of computing the optimal contract in OR technologies is NP-hard*[4].

Theorem 1 is addressed in Section 3. It is interesting to note that aside from establishing the computational hardness of the problem, our analysis implies the existence of OR technologies that admit exponential-size orbits. This disproves a conjecture raised in [1]. On the positive side, in Section 2 we prove the following theorem.

Theorem 2. *The problem of computing the optimal contract in OR technologies admits a fully polynomial-time approximation scheme (FPTAS).*

For the other extreme, the family of AND technologies, it is already established in [1] that the orbit size of any AND technology is at most $n + 1$. Here we show that the orbit can be efficiently computed, thus establishing the following theorem proved in Section 4.

Theorem 3. *There exists an efficient algorithm for the problem of computing the optimal contract in AND technologies.*

In addition to the analysis of the AND and OR technologies, we obtain a positive result regarding the general case. Consider an arbitrary technology t and let $\mathcal{S}$ be a collection of contracts. Given some real $\alpha > 1$, we say that $\mathcal{S}$ is an *α-approximation* of t's orbit if for every payoff v, there exists a contract $S \in \mathcal{S}$ such that $U_S(v) \geq \frac{U_{S_v^*}(v)}{1+\epsilon}$. The following theorem is proved in Section 2.

Theorem 4. *For every technology $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi \rangle$ and for any $\epsilon > 0$, the orbit of t admits a $(1+\epsilon)$-approximation of size polynomial in $\frac{1}{\epsilon}$ and $|t|$, where $|t|$ stands for the number of bits required for the binary representation of $\{\gamma_i\}_{i=1}^n$ and $\{\delta_i\}_{i=1}^n$.*

2 Approximated contracts

In this section we prove Theorem 2 by presenting an FPTAS for the optimal contract problem in OR technologies. Theorem 4, whose proof is simpler and relies on some of the arguments presented in the context of Theorem 2, is addressed at the end of the section. Consider some technology $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi \rangle$ and let $S \subseteq N$ be an arbitrary contract. We first observe that if φ is the AND function, then the effectiveness of S is given by $f(S) = \prod_{i \in S} \delta_i \prod_{i \in N - S} \gamma_i$. For the OR

[4] The problem remains NP-hard even for the special case in which $\delta_i = 1 - \gamma_i$ for every $i \in N$.

function, we have $f(S) = 1 - \prod_{i \in S}(1-\delta_i)\prod_{i \in N-S}(1-\gamma_i)$. Therefore if all agents shirk, then the effectiveness under an AND technology is $\prod_{i \in N} \gamma_i$. On the other hand, if all agents exert effort, then the effectiveness under an OR technology is $1-\prod_{i \in N}(1-\delta_i)$. Fix $\Delta = \min\left\{\prod_{i \in N}\gamma_i, \prod_{i \in N}(1-\delta_i)\right\}$. It is easy to verify that if t is an AND technology or an OR technology, then $f(S) \in [\Delta, 1-\Delta]$. The following lemma generalizes this property to the whole range of technologies.

Lemma 2.1. *The effectiveness $f(S)$ satisfies $f(S) \in [\Delta, 1-\Delta]$ regardless of the choice of the monotone Boolean function $\varphi : \{0,1\}^n \to \{0,1\}$.*

Proof. consider the underlying n-variables truth table of the Boolean function $\varphi(x_1, \ldots, x_n)$. Since φ is not a function of any $n-1$ variables, it cannot assign 0 to all rows of the table. Therefore, the minimum possible effectiveness is achieved when φ assigns 1 to exactly one row (otherwise, it can achieve a lower value by replacing a single 1 value with 0). By the monotonicity of φ, this single row must correspond to $x_1 = \cdots = x_n = 1$. (This is exactly the truth table of the AND function.) Clearly, the minimum possible effectiveness is achieved when all agents shirk. Combined together, the minimum possible effectiveness is simply $\mathbb{P}\left(x_1 = 1 \wedge \cdots \wedge x_n = 1 \mid a = (0, \ldots, 0)\right) = \prod_{i \in N}\gamma_i$. The proof that the maximum possible effectiveness is $\mathbb{P}\left(x_1 = 1 \vee \cdots \vee x_n = 1 \mid a = (1, \ldots, 1)\right) = 1 - \prod_{i \in N}(1-\delta_i)$ is analogous. ∎

Next, we establish the sub-modularity of OR technologies. We say that a function $h : 2^N \to \mathbb{R}$ is *strictly sub-modular* if $h(S) + h(T) \geq h(S \cup T) + h(S \cap T)$ for every $S, T \subseteq N$, where equality holds (if and) only if $S \subseteq T$ or $T \subseteq S$.

Lemma 2.2. *The effectiveness function of every OR technology is strictly sub-modular.*

Proof. Consider an arbitrary OR technology $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi\rangle$. We need to show that $f(S) + f(T) > f(S \cup T) + f(S \cap T)$ for every two contracts $S, T \subseteq N$ such that $S - T \neq \emptyset$ and $T - S \neq \emptyset$. By definition, we have

$$f(S) + f(T) \;=\; 2 - \prod_{i \in S}(1-\delta_i)\prod_{i \in N-S}(1-\gamma_i) - \prod_{i \in T}(1-\delta_i)\prod_{i \in N-T}(1-\gamma_i)$$

and

$$f(S \cup T) + f(S \cap T) \;=\; 2 - \prod_{i \in S \cup T}(1-\delta_i)\prod_{i \in N-(S \cup T)}(1-\gamma_i) - \prod_{i \in S \cap T}(1-\delta_i)\prod_{i \in N-(S \cap T)}(1-\gamma_i) \;.$$

Dividing both equations by $\prod_{i \in S \cap T}(1-\delta_i)\prod_{i \in N-(S \cup T)}(1-\gamma_i)$, we conclude that it is sufficient to prove that

$$\begin{aligned}&\prod_{i \in S-T}(1-\delta_i)\prod_{i \in T-S}(1-\gamma_i) \;+\; \prod_{i \in T-S}(1-\delta_i)\prod_{i \in S-T}(1-\gamma_i)\\ -\;&\prod_{i \in S-T}(1-\delta_i)\prod_{i \in T-S}(1-\delta_i) \;-\; \prod_{i \in S-T}(1-\gamma_i)\prod_{i \in T-S}(1-\gamma_i) \;<\; 0 \;.\end{aligned}$$

The last inequality holds if and only if

$$\prod_{i\in S-T}(1-\delta_i)\left(\prod_{i\in T-S}(1-\gamma_i)-\prod_{i\in T-S}(1-\delta_i)\right) + \prod_{i\in S-T}(1-\gamma_i)\left(\prod_{i\in T-S}(1-\delta_i)-\prod_{i\in T-S}(1-\gamma_i)\right) \ < \ 0 \ ,$$

which in turn, can be rewritten as

$$\left(\prod_{i\in T-S}(1-\gamma_i)-\prod_{i\in T-S}(1-\delta_i)\right)\left(\prod_{i\in S-T}(1-\delta_i)-\prod_{i\in S-T}(1-\gamma_i)\right) \ < \ 0 \ .$$

The assertion is now established as $\delta_i > \gamma_i$ for every $i \in N$. □

We employ Lemma 2.2 to characterize an important property of OR technologies. This property is a key ingredient in the analysis of our FPTAS algorithm.

Lemma 2.3. *Consider an arbitrary OR technology $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi\rangle$ and let $S \subseteq N$ be some contract such that $|S| = m \geq 2$. If S is optimal for the payoff $v > 0$, then $\frac{v}{p(S)} \geq 1 + \frac{1}{m}$.*

Proof. Let $S \subseteq N$ be an arbitrary contract of size $|S| = m \geq 2$. We first argue that $p(\{i\}) < p_i(S)$ (namely, that the total payment to the contract consisting of agent i alone is smaller than the payment to agent i under the contract S) for every agent $i \in S$. To justify this argument recall that $p(\{i\}) = \frac{c}{f(\{i\})-f(\emptyset)}$ and $p_i(S) = \frac{c}{f(S)-f(S-\{i\})}$, hence we have to show that $f(\{i\})+f(S-\{i\}) > f(S) + f(\emptyset)$, which is guaranteed by Lemma 2.2.

Fix $j^* = \text{argmin}_{i\in N} \frac{1-\delta_i}{1-\gamma_i}$. In the remainder, we prove that if $\frac{v}{p(S)} < 1 + \frac{1}{m}$, then $U_{\{j^*\}}(v) > U_S(v)$, in contradiction to the fact that S is optimal for v. We leave it to the reader to verify that $f(\{j^*\}) > f(\{i\})$ and $p(\{j^*\}) < p(\{i\})$ for every $i \neq j^*$ (can be established by a straightforward calculation). Since $p(S) = \sum_{i\in S} p_i(S)$, we conclude that $p(S) > m \cdot p(\{j^*\})$. On the other hand, Lemma 2.2 implies that $f(S) < \sum_{i\in S} f(\{i\})$, hence $f(S) < m \cdot f(\{j^*\})$.

Now, suppose that $\frac{v}{p(S)} < 1+\frac{1}{m}$. This implies $vm-p(S)(m+1) < 0$. As $m \geq 2$, we may multiply the inequality by $m-1$, obtaining $vm(m-1)-p(S)(m+1)(m-1) = vm(m-1)-p(S)(m^2-1) < 0$. Rearranging the last inequality, we conclude that $\frac{v-p(S)/m}{m} - (v - p(S)) > 0$. Since $p(S)/m > p(\{j^*\})$, we have $\frac{v-p(\{j^*\})}{m} - (v - p(S)) > 0$. As $f(S) > 0$, we may multiply the inequality by $f(S)$, obtaining $\frac{f(S)}{m}(v - p(\{j^*\})) - f(S)(v - p(S)) > 0$, and since $f(S)/m < f(j^*)$, we get $U_{\{j^*\}}(v) - U_S(v) = f(\{j^*\})(v - p(\{j^*\})) - f(S)(v - p(S)) > 0$. The lemma follows. □

We are now ready to present our FPTAS algorithm, referred to as Algorithm `Calibrate`. Consider the OR technology $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi\rangle$ input to Algorithm `Calibrate` and let $\epsilon > 0$ be the *performance parameter* of the FPTAS. Recall that Lemma 2.1 guarantees that $f(S) \in [\Delta, 1-\Delta]$ for every contract $S \subseteq N$, where $\Delta = \min\left\{\prod_{i\in N}\gamma_i, \prod_{i\in N}(1-\delta_i)\right\}$. Algorithm

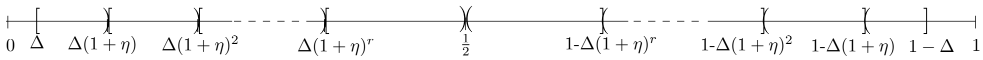

Figure 1: A scale of precision $1 + \eta$.

Calibrate generates a collection $\mathcal{C}$ of contracts in time $O\left(n^3 \log \frac{1}{\Delta} \cdot \max\left\{1, \frac{1}{\epsilon}\right\}\right)$. (Note that the binary representation of $\{\gamma_i\}_{i=1}^n$ and $\{\delta_i\}_{i=1}^n$ requires $\Omega\left(\log \frac{1}{\Delta}\right)$ bits.) We will soon prove that for every payoff v, there exists a contract $S \in \mathcal{C}$ such that $\frac{U_{S_v^*}(v)}{U_S(v)} \leq 1 + \epsilon$, where S_v^* is the optimal contract for v.

Let $\eta = \min\left\{\frac{1}{2n^2+1}, \frac{\epsilon}{2n(n+1+\epsilon n)}\right\}$ and let $r = \max\left\{k \in \mathbb{Z}_{\geq 0} \mid \Delta(1+\eta)^k < \frac{1}{2}\right\}$. Since $r < \log_{1+\eta}\left(\frac{1}{2\Delta}\right) = \log \frac{1}{2\Delta} \cdot \log_{1+\eta}(2)$, and since $\log_{1+\eta}(2) \leq \frac{1}{\eta}$, we conclude that $r < \frac{1}{\eta} \log \frac{1}{\Delta}$. We partition the interval $[\Delta, 1-\Delta]$ into the $2r+3$ smaller intervals $[\Delta, \Delta(1+\eta)), [\Delta(1+\eta), \Delta(1+\eta)^2), \ldots, [\Delta(1+\eta)^{r-1}, \Delta(1+\eta)^r), [\Delta(1+\eta)^r, \frac{1}{2}), [\frac{1}{2}, \frac{1}{2}], (\frac{1}{2}, 1-\Delta(1+\eta)^r], (1-\Delta(1+\eta)^r, 1-\Delta(1+\eta)^{r-1}], \ldots, (1-\Delta(1+\eta)^2, 1-\Delta(1+\eta)], (1-\Delta(1+\eta), 1-\Delta]$. The collection of these smaller intervals is called the *scale*. Refer to Figure 1 for an illustration of the scale. The *precision* of the scale is defined as $1+\eta$. We say that contract S is *calibrated* to interval $\mathcal{I}$ in the scale if $f(S) \in \mathcal{I}$.

Observation 2.4. *Let $S, S' \in N$ be some two contracts. The scale is designed to ensure that if S and S' are calibrated to the same interval, then $f(S) \leq (1+\eta) f(S')$ and $1 - f(S) \leq (1+\eta)(1-f(S'))$.*

Throughout the execution, Algorithm Calibrate maintains a collection $\mathcal{C}$ of contracts. The algorithm guarantees that no two contracts in $\mathcal{C}$ are calibrated to the same interval, thus $|\mathcal{C}| \leq 2r+3$ at any given moment. Algorithm Calibrate works in a dynamic programming fashion. On the m^{th} stage for $m = 1, \ldots, n$, the algorithm considers the m-agents OR technology t^m determined by $\{\gamma_i\}_{i=1}^m$ and $\{\delta_i\}_{i=1}^m$. (The cost c remains unchanged.) Given some contract $S \subseteq [m]$, we denote the effectiveness and payment of S under t^m by $f^m(S)$ and $p^m(S)$, respectively. The collection $\mathcal{C}$ at the end of the m^{th} stage is denoted by $\mathcal{C}^m$. Therefore at the end of the m^{th} stage, we have $S \subseteq [m]$ for every $S \in \mathcal{C}^m$. Moreover, for any two (different) contracts $S, S' \in \mathcal{C}^m$, if $f^m(S) \in \mathcal{I}$, where $\mathcal{I}$ is some interval in the scale, then $f^m(S') \notin \mathcal{I}$.

At the beginning of the $(m+1)^{\text{th}}$ stage the algorithm calibrates the contracts $\{S, S \cup \{m+1\} \mid S \in \mathcal{C}^m\}$ to a new scale according to their effectiveness f^{m+1} under t^{m+1}. Consequently, there may exist some interval in the new scale to which two (or more) contracts are calibrated (a conflict). Let $\mathcal{I}$ be such an interval and suppose that $S_1, \ldots, S_l$ are the contracts that were calibrated to $\mathcal{I}$, that is, $f^{m+1}(S_i) \in \mathcal{I}$ for every $1 \leq i \leq l$. Assume without loss of generality that S_l admits a minimum payment under t^{m+1}, i.e., $p^{m+1}(S_l) \leq p^{m+1}(S_i)$ for every $1 \leq i < l$. The algorithm then resolves the conflict by removing the contracts $S_1, \ldots, S_{l-1}$ from the new scale so that S_l remains the only contract calibrated to $\mathcal{I}$. In that case we say that the contracts $S_1, \ldots, S_{l-1}$ were *compensated* by the contract S_l. Thus the new collection $\mathcal{C}^{m+1}$ contains at most one contract for every interval and we may proceed with the next stage. At the end of the n^{th} stage Algorithm Calibrate returns the collection $\mathcal{C} = \mathcal{C}^n$.

We turn to the analysis of Algorithm `Calibrate`. The running time of the algorithm is determined by the number of stages (n) and by the size of the collection $\mathcal{C}$. The latter cannot exceed the number of intervals in the scale which is $O\left(\frac{1}{\eta}\log\frac{1}{\Delta}\right)$. In order to analyze the performance guarantee of the algorithm, we first define the following notion. Given two contracts $S, S' \subseteq N$ and a real $\alpha > 1$, we say that S is an *α-estimation* of S' under the technology t if (1) $f(S) \geq \frac{f(S')}{\alpha}$; (2) $1 - f(S) \geq \frac{1-f(S')}{\alpha}$; and (3) $p(S) \leq \alpha p(S')$. We say that a collection $\mathcal{S}$ of contracts is an *α-estimation* of the technology t if for every contract S' there exists a contract $S \in \mathcal{S}$ such that S is an α-estimation of S' under t. We are now ready to establish the main lemma of this section.

Lemma 2.5. *The collection $\mathcal{C}^m$ is a $(1+\eta)^m$-estimation of the technology t^m for every $1 \leq m \leq n$.*

Proof. The proof is by induction on m. To see that the assertion holds for $m = 1$, note that the technology t^1 admits exactly two contracts: $\emptyset$ and $\{1\}$. If these two contracts are calibrated to different intervals on the first stage, then $\mathcal{C}^1$ contains both of them and we are done. Otherwise, the two contracts are calibrated to the same interval ($\delta_1 - \gamma_1$ must be very small) and only the contract admitting smaller payment under t^1 (the empty contract in our case) survives. In that case the assertion holds due to Observation 2.4.

Assume that the assertion holds for $m-1$ and consider the m^{th} stage of the algorithm. Let $S^* \subseteq [m]$ be an arbitrary contract and fix $\bar{S}^* = S - \{m\}$. By the inductive hypothesis, there exists a contract $\bar{S} \in \mathcal{C}^{m-1}$ such that $\bar{S}$ is a $(1+\eta)^{m-1}$-estimation of $\bar{S}^*$ under the technology t^{m-1}. We define the contract $S \subseteq [m]$ as follows: if $m \in S^*$, then $S = \bar{S} \cup \{m\}$; if $m \notin S^*$, then $S = \bar{S}$. Given an arbitrary contract $R \subseteq [m]$, the effectiveness of R under the technology t^m can be expressed as

$$f^m(R) \;=\; 1 - \prod_{i \in R}(1-\delta_i) \prod_{i \in [m]-R}(1-\gamma_i) \;=\; 1 - (1-\zeta(R))(1 - f^{m-1}(R - \{m\})) \qquad (1)$$

$$= \zeta(R) + f^{m-1}(R - \{m\})(1 - \zeta(R)) \;, \qquad (2)$$

where $\zeta(R) = \begin{cases} \delta_m & \text{if } m \in R \\ \gamma_m & \text{if } m \notin R \end{cases}$.

By Plugging S and S^* into equation (1), and since $1 - f^{m-1}(\bar{S}) \geq \frac{1-f^{m-1}(\bar{S}^*)}{(1+\eta)^{m-1}}$, we conclude that $1 - f^m(S) \geq \frac{1-f^m(S^*)}{(1+\eta)^{m-1}}$. By plugging S and S^* into equation (2), and since $f^{m-1}(\bar{S}) \geq \frac{f^{m-1}(\bar{S}^*)}{(1+\eta)^{m-1}}$, we conclude that $f^m(S) \geq \frac{f^m(S^*)}{(1+\eta)^{m-1}}$. The contract S was considered during the m^{th} stage of the algorithm and calibrated to some interval $\mathcal{I}$ in the scale. Afterwards it was compensated by some contract $S' \subseteq [m]$ that was calibrated to $\mathcal{I}$ as well (if S remains in $\mathcal{C}^m$, then assume that $S' = S$) so that $S' \in \mathcal{C}^m$. By Observation 2.4, it follows that $1 - f^m(S') \geq \frac{1-f^m(S^*)}{(1+\eta)^m}$ and $f^m(S') \geq \frac{f^m(S^*)}{(1+\eta)^m}$ as required.

It remains to prove that $p^m(S') \leq p^m(S^*)$. Given an arbitrary contract $R \subseteq [m]$, the payment of R under the technology t^m can be expressed as

$$\begin{aligned} p^m(R) &= \sum_{i \in R} \frac{c}{f^m(R) - f^m(R - \{i\})} \\ &= \chi(m \in R) \cdot \frac{c}{f^m(R) - f^m(R - \{m\})} + \sum_{i \in R - \{m\}} \frac{c}{f^m(R) - f^m(R - \{i\})} , \end{aligned} \quad (3)$$

where $\chi(m \in R)$ is the characteristic function of the predicate $m \in R$ (if $m \notin R$, then the first term should be ignored). By equation (2), we have $f^m(R) - f^m(R - \{m\}) = (\delta_m - \gamma_m)(1 - f^{m-1}(R - \{m\}))$ and $f^m(R) - f^m(R - \{i\}) = \left(f^{m-1}(R - \{m\}) - f^{m-1}(R - \{i, m\})\right)(1 - \zeta(R))$ for every $i \in R - \{m\}$. Plugging it into equation (3), we get

$$p^m(R) = \chi(m \in R) \cdot \frac{c}{(\delta_m - \gamma_m)(1 - f^{m-1}(R - \{m\}))} + \frac{1}{1 - \zeta(R)} \cdot p^{m-1}(R - \{m\}) . \quad (4)$$

Note that by the definition of S, we have $\zeta(S) = \zeta(S^*)$, hence by plugging S and S^* into equation (4), and since $p^{m-1}(\bar{S}) \leq (1 + \eta)^{m-1} p^{m-1}(\bar{S}^*)$ and $1 - f^{m-1}(\bar{S}) \geq \frac{1 - f^{m-1}(\bar{S}^*)}{(1+\eta)^{m-1}}$, we conclude that $p^m(S) \leq (1 + \eta)^{m-1} p^m(S^*)$. The assertion follows as $p^m(S') \leq p^m(S)$. □

Consider an arbitrary payoff $v > 0$ and let $S_v^* \subseteq N$ be the optimal contract for v. Lemma 2.5 guarantees that the contracts collection $\mathcal{C}$ returned by Algorithm `Calibrate` contains a contract $S \subseteq N$ such that S is a $(1 + \eta)^n$-estimation of S_v^*. In particular, S satisfies $f(S) \geq \frac{f(S_v^*)}{(1+\eta)^n}$ and $p(S) \leq (1 + \eta)^n p(S_v^*)$. Therefore $\frac{U_{S_v^*}(v)}{U_S(v)} = \frac{f(S_v^*)(v - p(S_v^*))}{f(S)(v - p(S))} \leq (1 + \eta)^n \cdot \frac{v - p(S_v^*)}{v - p(S)}$. Since $\eta < \frac{1}{2n^2} \leq \frac{1}{2n}$, it follows that $(1+\eta)^n < 1 + 2n\eta < 1 + \frac{1}{n}$. Lemma 2.3 then implies that $p(S_v^*)(1+\eta)^n < v$, and therefore we can substitute $v - p(S)$ with $v - p(S_v^*)(1+\eta)^n$, concluding that $\frac{U_{S_v^*}(v)}{U_S(v)} \leq (1 + \eta)^n \cdot \frac{v - p(S_v^*)}{v - p(S_v^*)(1+\eta)^n}$.

Employing Lemma 2.3 once again, we derive $\frac{U_{S_v^*}(v)}{U_S(v)} \leq (1 + \eta)^n \cdot \frac{\left(1 + \frac{1}{n}\right) p(S_v^*) - p(S_v^*)}{\left(1 + \frac{1}{n}\right) p(S_v^*) - p(S_v^*)(1+\eta)^n} = (1 + \eta)^n \cdot \frac{1/n}{1 + (1/n) - (1+\eta)^n}$. As $(1+\eta)^n < 1 + 2n\eta$, and since $\eta < \frac{1}{2n^2}$, we can substitute $1 + (1/n) - (1 + \eta)^n$ with $1 + (1/n) - (1 + 2n\eta)$, concluding that $\frac{U_{S_v^*}(v)}{U_S(v)} < (1 + 2n\eta) \frac{1/n}{1 + (1/n) - (1 + 2n\eta)} = \frac{1 + 2n\eta}{1 - 2n^2\eta}$. The promised bound $\frac{U_{S_v^*}(v)}{U_S(v)} \leq 1 + \epsilon$ follows as $\eta \leq \frac{\epsilon}{2n(n+1+\epsilon n)}$, thus Theorem 2 is established.

Theorem 4 can now be established as well. Consider an arbitrary technology $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi \rangle$ and some $\epsilon > 0$. The contracts collection $\mathcal{C}$ is constructed in a single stage of Algorithm `Calibrate`: we first calibrate all contracts in 2^N into a scale of precision $1 + \epsilon$ and then remove from each interval all contracts excluding the one with minimum payment (under t). More formally, the collection $\mathcal{C}$ contains at most one contract S that is calibrated to the interval $\mathcal{I}$, in this case $p(S) \leq p(S')$ for every contract $S' \subseteq N$ such that S' is calibrated to $\mathcal{I}$. Following the line of arguments presented earlier in this section, we show that $|\mathcal{C}| = O\left(\frac{1}{\epsilon} \log \frac{1}{\Delta}\right)$. Moreover, if an arbitrary contract $S^* \subseteq N$ is not in $\mathcal{C}$, then it was compensated by some contract $S \in \mathcal{C}$ such that S and S^* are calibrated to the same interval. Therefore $f(S) \geq \frac{f(S^*)}{1+\epsilon}$ and since $p(S) \leq p(S^*)$, it follows that $\frac{U_{S^*}(v)}{U_S(v)} \leq 1 + \epsilon$ for every payoff $v \geq 0$. Unfortunately, in the case of arbitrary technologies (as opposed to OR technologies) we do not know how to construct the collection $\mathcal{C}$ efficiently.

i	0	1	2		m	m+1	m+2	m+3
$u^1, \ldots, u^n$	0	$u_i^j = 1$ *if* $i \in \Gamma_j$ and $u_i^j = 0$ otherwise				0	0	0
u^α	0	0				0	1	0
u^β	0	0				0	0	1
u^A	0	0				1	0	0
u^B	1	0				1	0	0
u^C	2	0				1	0	0

Figure 2: The $n+5$ vectors representing the $n+5$ agents of the technology. The first n agents correspond to the n variables of the 3-CNF formula ϕ, and the additional 5 agents are assigned with the vectors u^α, u^β, u^A, u^B and u^C.

3 NP-hardness of OR technologies

We present a polynomial-time Turing reduction from X3SAT (Problem LO4 in [5]) to the problem of computing an optimal contract for an OR technology. A 3-CNF formula ϕ in which all literals are positive is solvable under X3SAT if there exists a truth assignment for the variables of ϕ that assigns true to exactly one literal in every clause. Given a 3-CNF formula ϕ with m clauses and n variables in which all literals are positive, we construct an OR technology $t = \langle N, \{\gamma_j\}_{j=1}^{n+5}, \{\delta_j\}_{j=1}^{n+5}, c, \varphi \rangle$ such that (1) the agent set N contains $n+5$ agents ($N = [n+5]$); (2) the cost incurred on an agent for exerting effort is $c = 1$; and (3) $\gamma_j = 1 - \delta_j$ for every $j \in N$. The construction is designed to guarantee that by performing $O(n)$ queries, each reveals the optimal contract for some carefully chosen payoff, we can decide whether ϕ is solvable under X3SAT.

Let $\mathcal{W} = \{0,1,2,3\}^{m+2} \times \{0,1\}^2$. Each agent $j \in N$ is assigned with a vector $\mathbf{u}^j \in \mathcal{W}$. The first n agents correspond to the n variables of the 3-CNF formula ϕ. Assuming that variable j appears in clauses $\Gamma_j \subseteq \{1, \ldots, m\}$, the vector $\mathbf{u}^j = (u_0^j, \ldots, u_{m+3}^j)$ is defined so that $u_i^j = 1$ if $i \in \Gamma_j$; and $u_i^j = 0$ if $i \notin \Gamma_j$; for every $1 \leq j \leq n$ and $0 \leq i \leq m+3$.

Agents $n+j$ for $j = 1, \ldots, 5$ are provided for the sake of the analysis. To avoid cumbersome indexing, we denote $n+1$ and $n+2$ by α and β, respectively, and $n+3$, $n+4$ and $n+5$ by A, B and C, respectively. Agents α and β are assigned with the vectors $\mathbf{u}^\alpha = (0, \ldots, 0, 1, 0) \in \mathcal{W}$ and $\mathbf{u}^\beta = (0, \ldots, 0, 0, 1) \in \mathcal{W}$, respectively. Agents A, B and C are assigned with the vectors $\mathbf{u}^A = (0, \ldots, 0, 1, 0, 0) \in \mathcal{W}$, $\mathbf{u}^B = (1, 0, 0, \ldots, 0, 1, 0, 0) \in \mathcal{W}$ and $\mathbf{u}^C = (2, 0, 0, \ldots, 0, 1, 0, 0) \in \mathcal{W}$, respectively (see Figure 2). Observe that the first n agents affect coordinates $1, \ldots, m$; agents α and β affect coordinates $m+2$ and $m+3$; and agents A, B and C affect coordinates 0 and $m+1$.

We extend the assignment of vectors to sets of agents (a.k.a. contracts) in a natural way. Given a contract $S \subseteq N$, we define the vector $\mathbf{u}^S = \sum_{j \in S} \mathbf{u}^j$. As each clause in ϕ contains (at most) three variables, and by the definition of the vectors $\mathbf{u}^\alpha$, $\mathbf{u}^\beta$, $\mathbf{u}^A$, $\mathbf{u}^B$ and $\mathbf{u}^C$, it follows that $\mathbf{u}^S \in \mathcal{W}$ for every contract $S \subseteq N$. Observe that different contracts may be assigned with the same vector in

$\mathcal{W}$.

The reduction relies on the following fact: the formula ϕ is solvable under X3SAT if and only if there exists a contract S with vector $\mathbf{u}^S = (1, \ldots, 1) \in \mathcal{W}$. To justify this fact, note that a truth assignment that assigns true to exactly one variable in every clause is translated to a contract S with $\mathbf{u}^S = (u_0^S, 1, 1, \ldots, 1, u_{m+1}^S, u_{m+2}^S, u_{m+3}^S)$. Agents α, β and B can be added to S, thus setting $u_0^S = u_{m+1}^S = u_{m+2}^S = u_{m+3}^S = 1$, without affecting any other coordinate. We will show that if such a contract exists, then it is optimal for some payoff v^* which will be determined later on.

Consider the vector $\mathbf{x} = (x_0, \ldots, x_{m+3})$ in $\mathcal{W}$. Let $\sigma(\mathbf{x}) = \sum_{i=0}^{m+1} x_i 4^i$ and fix $\mu = 4^{5(m+2)}$. We define the *partial evaluation* of $\mathbf{x}$ to be $\tau_p(\mathbf{x}) = \left(1 + \frac{1}{\mu}\right)^{\sigma(\mathbf{x})}$. Note that since $x_i \leq 3$ for every $0 \leq i \leq m+1$, and since $\mu > \left(\sum_{i=0}^{m+1} 3 \cdot 4^i\right)^5$, it follows that $\mu > (\sigma(\mathbf{x}))^5$ for any $\mathbf{x} \in \mathcal{W}$. The partial evaluation of $\mathbf{x}$ can be rewritten as $\tau_p(\mathbf{x}) = \sum_{j=0}^{\sigma(\mathbf{x})} \binom{\sigma(\mathbf{x})}{j} \mu^{-j}$, thus $1 + \mu^{-1} \leq \tau_p(\mathbf{x}) \leq 1 + O(\mu^{-4/5})$ and in general,

$$\tau_p(\mathbf{x}) \;=\; \sum_{j=0}^{k-1} \binom{\sigma(\mathbf{x})}{j} \mu^{-j} + O\left(\mu^{-4k/5}\right) \tag{5}$$

for any $0 < k \leq \sigma(\mathbf{x})$. The *full evaluation* of $\mathbf{x}$ is defined to be $\tau(\mathbf{x}) = \tau_p(\mathbf{x}) \cdot \mu^{2x_{m+2}} \cdot \mu^{5x_{m+3}}$. Observe that $\tau(\mathbf{x}) = \tau_p(\mathbf{x})$ if $x_{m+2} = x_{m+3} = 0$. Moreover, $\tau(\mathbf{x}) \leq (1 + O(\mu^{-4/5}))\mu^7$ for every vector $\mathbf{x} \in \mathcal{W}$. Fix $\chi = 2\mu^7$ (so that $\chi > \tau(\mathbf{x})$ for every vector $\mathbf{x} \in \mathcal{W}$). Clearly, for any two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{W}$, the full evaluation of $\mathbf{x}$ is greater than the full evaluation of $\mathbf{y}$ if and only if $\mathbf{x}$ is lexicographically greater[5] than $\mathbf{y}$.

Proposition 3.1. *Let $\mathbf{x} = (x_0, \ldots, x_{m+3})$ and $\mathbf{y} = (y_0, \ldots, y_{m+3})$ be two vectors in $\mathcal{W}$ such that $\mathbf{x}$ is lexicographically greater than $\mathbf{y}$. The difference $\tau(\mathbf{x}) - \tau(\mathbf{y})$ satisfies (i) if $x_{m+2} \neq y_{m+2}$ or $x_{m+3} \neq y_{m+3}$, then $\tau(\mathbf{x}) - \tau(\mathbf{y}) = (1 + o(1))\mu^{2x_{m+2}+5x_{m+3}}$; and (ii) if $x_{m+2} = y_{m+2}$ and $x_{m+3} = y_{m+3}$, then $\mu^{2x_{m+2}+5x_{m+3}-1} < \tau(\mathbf{x}) - \tau(\mathbf{y}) \leq O(\mu^{2x_{m+2}+5x_{m+3}-(4/5)})$.*

Proof. The bound in (i) follows immediately from the definition of full evaluation as the partial evaluation is $1 + o(1)$. To establish (ii), note that since $\tau(\mathbf{x}) > \tau(\mathbf{y})$ although $x_{m+2} = y_{m+2}$ and $x_{m+3} = y_{m+3}$, we must have $\tau_p(\mathbf{x}) > \tau_p(\mathbf{y})$. By the definition of partial evaluation, it follows that $\frac{\tau_p(\mathbf{x})}{\tau_p(\mathbf{y})} = \left(1 + \mu^{-1}\right)^{\sigma(\mathbf{x}) - \sigma(\mathbf{y})}$, hence $1 + \mu^{-1} \leq \frac{\tau_p(\mathbf{x})}{\tau_p(\mathbf{y})} \leq 1 + O(\mu^{-4/5})$. Therefore

$$\mu^{-1} \;<\; \tau_p(\mathbf{y})(1 + \mu^{-1} - 1) \;\leq\; \tau_p(\mathbf{x}) - \tau_p(\mathbf{y}) \;\leq\; \tau_p(\mathbf{y})(1 + O(\mu^{-4/5}) - 1) \;\leq\; O(\mu^{-4/5})$$

The proof is completed as $\tau(\mathbf{x}) - \tau(\mathbf{y}) = \mu^{2x_{m+2}+5x_{m+3}}(\tau_p(\mathbf{x}) - \tau_p(\mathbf{y}))$. □

Let $\epsilon = \mu^{-\kappa}$, where κ is a sufficiently large constant that will be determined later on. We would have wanted to define the effectiveness factors of the OR technology by fixing $\gamma_j = 1 - \delta_j = \tau(\mathbf{u}^j) \cdot \epsilon$ for every $j \in N$. Unfortunately, the standard binary representation of $\tau(\mathbf{u}^j)$ may be

[5]The vector $\mathbf{x} = (x_0, \ldots, x_{m+3})$ is lexicographically greater than the vector $\mathbf{y} = (y_0, \ldots, y_{m+3})$ if there exists a coordinate $0 \leq j \leq m+3$ such that $x_i = y_i$ for every $i > j$ and $x_j > y_j$.

much larger than the binary representation of ϕ for some j, and in particular, exponential in m. We handle this obstacle by estimating the partial vector evaluations (see (5)). Given a vector $\mathbf{x} = (x_0, \ldots, x_{m+3}) \in \mathcal{W}$, let $\widetilde{\tau}_p(\mathbf{x}) = \sum_{j=0}^{\lceil 5(\kappa+7)/4 \rceil - 1} \binom{\sigma(\mathbf{x})}{j} \mu^{-j} = \tau_p(\mathbf{x}) - O(\mu^{-\kappa-7}) = \tau_p(\mathbf{x}) - O(\epsilon\mu^{-7})$ and $\widetilde{\tau}(\mathbf{x}) = \widetilde{\tau}_p(\mathbf{x}) \cdot \mu^{2x_{m+2}} \cdot \mu^{5x_{m+3}} = \tau(\mathbf{x}) - O(\epsilon)$. Note that the binary representation of $\widetilde{\tau}(\mathbf{x})$ is polynomial (linear actually) in m. The technology t is now determined by fixing

$$\gamma_j \;=\; 1 - \delta_j \;=\; \widetilde{\tau}(\mathbf{u}^j)\epsilon \;=\; \tau(\mathbf{u}^j)\epsilon - O(\epsilon^2) \tag{6}$$

for all $j \in N$.

Let $S \subseteq N$ be some contract and assume that $|S| = k > 0$. Let ν be the maximum among all constants hidden in the O notation of (6), that is, $\tau(\mathbf{u}^j)\epsilon - \gamma_j \leq \nu\epsilon^2$ for every $j \in N$. By the definition of OR technologies, we have

$$\begin{aligned} f(S) &= 1 - \prod_{j \in S}(1 - \delta_j) \prod_{j \in N-S}(1 - \gamma_j) \\ &= 1 - \prod_{j \in S} \epsilon\left(\tau(\mathbf{u}^j) - O(\epsilon)\right) \prod_{j \in N-S}\left(1 - \epsilon\left(\tau(\mathbf{u}^j) - O(\epsilon)\right)\right) \\ &= 1 - \epsilon^k \prod_{j \in S} \tau(\mathbf{u}^j) - \sum_{l=1}^{n+5}(-1)^l \epsilon^{k+l} \cdot O\left(\nu^l \chi \binom{n+5}{l}\right) \\ &= 1 - \tau(\mathbf{u}^S)\epsilon^k - \sum_{l=1}^{n+5}(-1)^l \epsilon^{k+l} \cdot O\left(\nu^l \chi \binom{n+5}{l}\right) . \end{aligned}$$

Taking $\epsilon < \left(\frac{1}{\nu\chi(n+5)}\right)^2$ guarantees that

$$f(S) \;=\; 1 - \tau(\mathbf{u}^S)\epsilon^k \pm O(\epsilon^{k+(1/2)}) . \tag{7}$$

Following a similar line of arguments, we conclude that $f(\emptyset) = O(\epsilon^{1/2})$. The next proposition can now be established.

Proposition 3.2. *Let $S, S' \subseteq N$ be some two contracts and let $k = |S|$, $k' = |S'|$. Then $f(S) < f(S')$ if and only if (i) $k < k'$; or (ii) $k = k'$ and $\tau(\mathbf{u}^S) > \tau(\mathbf{u}^{S'})$.*

Proof. The first claim follows immediately from (7) by taking $\epsilon \ll \chi^{-1}$. For the second claim, note that by (7), it is sufficient to prove that $\tau(\mathbf{u}^S) - \tau(\mathbf{u}^{S'}) = \omega(\epsilon^{1/2})$. This is guaranteed due to Proposition 3.1 by taking $\epsilon \ll \mu^{-2}$. □

A direct consequence of Proposition 3.2 is that $f(S) = f(S')$ if and only if $|S| = |S'|$ and

$\mathbf{u}^S = \mathbf{u}^{S'}$. The conditional payment to the agents in S, where $|S| = k$, can now be expressed as

$$\begin{aligned} p(S) &= \sum_{j \in S} \frac{1}{f(S) - f(S-j)} \\ &= \sum_{j \in S} \left[1 - \tau(\mathbf{u}^S)\epsilon^k \pm O(\epsilon^{k+(1/2)}) - 1 + \tau(\mathbf{u}^{S-j})\epsilon^{k-1} \pm O(\epsilon^{k-(1/2)}) \right]^{-1} \\ &= \sum_{j \in S} \left[\tau(\mathbf{u}^{S-j})\epsilon^{k-1} \pm O(\epsilon^{k-(1/2)}) \right]^{-1} \\ &= \sum_{j \in S} \left[\tau^{-1}(\mathbf{u}^{S-j})\epsilon^{1-k} \pm O(\epsilon^{(3/2)-k}) \right] \\ &= \sum_{j \in S} \tau^{-1}(\mathbf{u}^{S-j})\epsilon^{1-k} \pm O(\epsilon^{(5/4)-k}) , \end{aligned}$$

where $S - j$ denotes the contract $S - \{j\}$ and the last equation follows from taking $\epsilon < (n+5)^{-4}$. Define $\pi(S) = \sum_{j \in S} \tau^{-1}(\mathbf{u}^{S-j})$, so that

$$p(S) = \pi(S)\epsilon^{1-k} \pm O(\epsilon^{(5/4)-k}) . \tag{8}$$

Note that $\pi(S) < |S|$ for every contract $S \subseteq N$ since each term in the sum is smaller than 1.

Let $S \subseteq N$ be some contract and assume that $|S| = k > 0$. By plugging (7) and (8) into the definition of utility, we get

$$\begin{aligned} U_S(v) &= \left(1 - \tau(\mathbf{u}^S)\epsilon^k \pm O(\epsilon^{k+(1/2)})\right) \left(v - \pi(S)\epsilon^{1-k} \pm O(\epsilon^{(5/4)-k})\right) \\ &= v - \pi(S)\epsilon^{1-k} \pm O(\epsilon^{(5/4)-k}) - \tau(\mathbf{u}^S)v\epsilon^k + \pi(S)\tau(\mathbf{u}^S)\epsilon \pm O(\tau(\mathbf{u}^S)\epsilon^{5/4}) \\ &\pm O(v\epsilon^{k+(1/2)}) \pm O(\pi(S)\epsilon^{3/2}) \pm O(\epsilon^{7/4}) \\ &= v - \pi(S)\epsilon^{1-k} - \tau(\mathbf{u}^S)v\epsilon^k \pm O(\epsilon^{(5/4)-k}) \pm O(v\epsilon^{k+(1/2)}) , \end{aligned}$$

where the last equation is guaranteed by taking $\epsilon < ((n+5)\chi)^{-4/3}$. For the empty contract, we have $p(\emptyset) = 0$ and $U_\emptyset(v) = v \cdot O(\epsilon^{1/2})$.

Consider some two contracts $S, T \subseteq N$. Assuming that $f(S) \neq f(T)$, we refer to the payoff on which the lines $U_S(\cdot)$ and $U_T(\cdot)$ intersect as the *intersection payoff* of S and T, denoted $v[S,T]$, namely, $U_S(v[S,T]) = U_T(v[S,T])$. The next lemma correlates the intersection payoffs to the size of the contracts and to the vectors representing the contracts.

Lemma 3.3. *Let $S, S' \subseteq N$ be some two contracts such that $f(S) \neq f(S')$. Define $k = |S|$ and $k' = |S'|$. The intersection payoff $v[S,S']$ satisfies (i) if $0 < k = k'$, then*

$$v[S,S'] = \epsilon^{1-2k} \frac{\pi(S') - \pi(S) \pm O(\epsilon^{1/4})}{\tau(\mathbf{u}^S) - \tau(\mathbf{u}^{S'}) \pm O(\epsilon^{1/2})} ;$$

and (ii) if $k \neq k'$, $k, k' \geq 0$, then

$$\Omega(\epsilon^{(5/4)-k-k'}) \leq v[S,S'] \leq O(\epsilon^{(3/4)-k-k'}) .$$

(Observe that the case $0 = k = k'$ is irrelevant as there is only one empty contract.)

Proof. Assume without loss of generality that $k \leq k'$. Suppose first that $k > 0$. By comparing the utilities of S and S' on payoff $v[S, S']$, we get

$$\begin{aligned} & \pi(S)\epsilon^{1-k} + \tau(\mathbf{u}^S)v[S,S']\epsilon^k \pm O(\epsilon^{(5/4)-k}) \pm O(v[S,S']\epsilon^{k+(1/2)}) \\ = \ & \pi(S')\epsilon^{1-k'} + \tau(\mathbf{u}^{S'})v[S,S']\epsilon^{k'} \pm O(\epsilon^{(5/4)-k'}) \pm O(v[S,S']\epsilon^{k'+(1/2)}) \ , \end{aligned}$$

hence

$$v[S,S'] \ = \ \frac{\pi(S')\epsilon^{1-k'} - \pi(S)\epsilon^{1-k} \pm O(\epsilon^{(5/4)-k'})}{\tau(\mathbf{u}^S)\epsilon^k - \tau(\mathbf{u}^{S'})\epsilon^{k'} \pm O(\epsilon^{k+(1/2)})} \ .$$

By setting $k = k'$, (i) is established. Otherwise, if $0 < k < k'$, then, by taking $\epsilon < \min\{(n+5)^{-2}, \chi^{-2}\}$, we get

$$v[S,S'] \ = \ \frac{\pi(S')\epsilon^{1-k'} \pm O(\epsilon^{(5/4)-k'})}{\tau(\mathbf{u}^S)\epsilon^k \pm O(\epsilon^{k+(1/2)})} \ = \ \epsilon^{1-k'-k}\frac{\pi(S') \pm O(\epsilon^{1/4})}{\tau(\mathbf{u}^S) \pm O(\epsilon^{1/2})} \ . \tag{9}$$

It remains to consider the case $0 = k < k'$. Once again by comparing the utilities of S and S' on payoff $v[S, S']$, we have

$$v[S,S'] - \pi(S')\epsilon^{1-k'} - \tau(\mathbf{u}^{S'})v[S,S']\epsilon^{k'} \pm O(\epsilon^{(5/4)-k'}) \pm O(v[S,S']\epsilon^{k'+(1/2)}) \ = \ v[S,S'] \cdot O(\epsilon^{1/2}) \ ,$$

hence, by taking $\epsilon < \chi^{-2}$, we get

$$v[S,S'] \ = \ \frac{\pi(S')\epsilon^{1-k'} \pm O(\epsilon^{(5/4)-k'})}{1 - O(\epsilon^{1/2})} \ . \tag{10}$$

Taking $\epsilon < (\max\{(n+5), \chi\})^{-4}$ guarantees the bounds in (ii) due to (9) and (10). □

Let $\mathbf{x} = (x_0, \ldots, x_{m+3})$ be a vector in $\mathcal{W}$. We say that $\mathbf{x}$ is *protected* if $x_{m+2} = x_{m+3} = 1$. For every $0 \leq k \leq n+5$, let $\Psi_k(\mathbf{x}) = \{S \subseteq N \mid \mathbf{u}^S = \mathbf{x} \text{ and } |S| = k\}$. We argue that if $\mathbf{x}$ is a protected vector in $\mathcal{W}$, and if $\Psi_k(\mathbf{x}) \neq \emptyset$, then at least one contract in $\Psi_k(\mathbf{x})$ is in the top envelope of the lines collection $\{U_S(\cdot) \mid S \subseteq N\}$. We first establish some bounds related to $\pi(\cdot)$.

Proposition 3.4. *Let $S \subseteq N$ be a contract. If $\mathbf{u}^S$ is protected, then $\pi(S) = \Theta(\mu^{-2})$ and in particular, $\tau^{-1}(\mathbf{u}^{S-\beta}) \leq \pi(S) \leq (1 + O(\mu^{-3}))\tau^{-1}(\mathbf{u}^{S-\beta})$. If $\mathbf{u}^S$ is not protected, then $1 - o(1) \leq \pi(S) \leq |S|$.*

Proof. Suppose that $\mathbf{u}^S$ is protected. First observe that since $\alpha \in S - \beta$, it follows that $\tau(\mathbf{u}^{S-\beta}) = \Theta(\mu^2)$. Therefore if $\tau^{-1}(\mathbf{u}^{S-\beta}) \leq \pi(S) \leq (1 + O(\mu^{-3}))\tau^{-1}(\mathbf{u}^{S-\beta})$, then $\pi(S)$ is indeed $\Theta(\mu^{-2})$. Recall that $\pi(S) = \sum_{j \in S} \tau^{-1}(\mathbf{u}^{S-j}) = \sum_{j \in S - \{\alpha,\beta\}} \tau^{-1}(\mathbf{u}^{S-j}) + \tau^{-1}(\mathbf{u}^{S-\alpha}) + \tau^{-1}(\mathbf{u}^{S-\beta})$. For every $j \in S - \{\alpha, \beta\}$, we have $\frac{\tau^{-1}(\mathbf{u}^{S-j})}{\tau^{-1}(\mathbf{u}^{S-\beta})} = \frac{\tau(\mathbf{u}^j)}{\tau(\mathbf{u}^\beta)} = \frac{1+O(\mu^{-4/5})}{\mu^5}$, and $\frac{\tau^{-1}(\mathbf{u}^{S-\alpha})}{\tau^{-1}(\mathbf{u}^{S-\beta})} = \frac{\tau(\mathbf{u}^\alpha)}{\tau(\mathbf{u}^\beta)} = \frac{1}{\mu^3}$. Therefore $\frac{\pi(S)}{\tau^{-1}(\mathbf{u}^{S-\beta})} = \frac{(k-2)(1+O(\mu^{-4/5}))}{\mu^5} + \frac{1}{\mu^3} + 1$. Since $k - 2 \leq n + 3 \leq 3m + 3 \ll \mu$, we have $\pi(S) \leq (1 + \frac{O(1)}{\mu^3})\tau^{-1}(\mathbf{u}^{S-\beta})$.

Now suppose that $\mathbf{u}^S$ is not protected. We choose agent j' as follows. If $\alpha \in S$ or $\beta \in S$, then let j' be the (sole) agent in $S \cap \{\alpha, \beta\}$. (Recall that $\{\alpha, \beta\} \nsubseteq S$ as S is not protected.) Otherwise, let j' be any agent in S. Denote $\mathbf{u}^{S-j'} = (u_0, \ldots, u_{m+3})$. Since $\mathbf{u}^S$ is not protected, it follows that $u_{m+2} = u_{m+3} = 0$. Therefore $\tau(\mathbf{u}^{S-j'}) = \tau_p(\mathbf{u}^{S-j'}) = 1 + O(\mu^{-4/5})$, and $\pi(S) \geq \tau^{-1}(\mathbf{u}^{S-j'}) = 1 - o(1)$. □

Proposition 3.5. *Let $S, S' \subseteq N$ be two contracts such that $\mathbf{u}^S$ is protected and $\tau(\mathbf{u}^S) > \tau(\mathbf{u}^{S'})$. Then $\pi(S') - \pi(S) = \Omega(\mu^{-3})$.*

Proof. If $\mathbf{u}^{S'}$ is not protected, then Proposition 3.4 guarantees that $\pi(S') - \pi(S) = \Omega(1)$. Assume that $\mathbf{u}^{S'}$ is protected. Since coordinate $m+2$ is set in both $\mathbf{u}^S$ and $\mathbf{u}^{S'}$, we have $\frac{\tau(\mathbf{u}^{S-\beta})}{\tau(\mathbf{u}^{S'-\beta})} = \frac{\tau_p(\mathbf{u}^S)}{\tau_p(\mathbf{u}^{S'})} \geq 1 + \mu^{-1}$. By Proposition 3.4, we have $\pi(S') \geq \tau^{-1}(\mathbf{u}^{S'-\beta})$ and $\pi(S) \leq (1 + O(\mu^{-3}))\tau^{-1}(\mathbf{u}^{S-\beta})$. Therefore $\pi(S') - \pi(S) \geq \tau^{-1}(\mathbf{u}^{S-\beta})(1 + \mu^{-1} - 1 - O(\mu^{-3}))$. As $\tau^{-1}(\mathbf{u}^{S-\beta}) = \Theta(\mu^{-2})$, it follows that $\pi(S') - \pi(S) = \Omega(\mu^{-3})$. □

Consider the collection $\mathcal{F}$ of all continuous functions $g : \mathbb{R} \to \mathbb{R}$. Let H be a finite subset of $\mathcal{F}$ and let g be a function in $\mathcal{F}$. We say that g is *dominated* by the functions in H if for every $v \in \mathbb{R}$, there exists a function $g' \in H$ such that $g(v) \leq g'(v)$. Suppose that g and the functions in H are linear. Following some standard geometric arguments, one can show that if g is not dominated by any two functions in H, then g is not dominated by all functions in H. Given a contract $S \subseteq N$ and a subset of contracts $H \subseteq 2^N$, we say that S is *dominated* by the contracts in H if $U_S(\cdot)$ is dominated by the functions in $\{U_T(\cdot) \mid T \in H\}$.

We now turn to state the main lemma of this section, namely, that a contract assigned with a protected vector cannot be dominated by any two contracts assigned with different vectors.

Lemma 3.6. *Let $S \subseteq N$ be a contract such that $\mathbf{u}^S$ is protected and let $k = |S|$. Consider some two contracts $R, T \notin \Psi_k(\mathbf{u}^S)$. Then there exists a payoff v for which $U_S(v) > \max\{U_R(v), U_T(v)\}$.*

Proof. Assume without loss of generality that $f(R) \leq f(T)$. Proposition 3.2 implies that $f(S) \neq f(R)$ and $f(S) \neq f(T)$, hence it is sufficient to consider the case $f(R) < f(S) < f(T)$ (otherwise, S cannot be dominated by R and T). We prove that $v[R, S] < v[S, T]$. This establishes the lemma as it implies that $U_S(v) > \max\{U_R(v), U_T(v)\}$ for all $v[R, S] < v < v[S, T]$.

Let $k^R = |R|$ and $k^T = |T|$. We know, due to Proposition 3.2, that $k^R \leq k \leq k^T$. Lemma 3.3 is employed in order to analyze the following four cases. First if $k^R < k < k^T$, then $v[R, S] = O(\epsilon^{(3/4)-k^R-k})$ and $v[S, T] = \Omega(\epsilon^{(5/4)-k-k^T})$, thus $\frac{v[S,T]}{v[R,S]} = \Omega(\epsilon^{(1/2)-k^T+k^R}) \gg 1$, so the assertion holds. If $k^R < k = k^T$, then, by Proposition 3.2, we have $\tau(\mathbf{u}^S) > \tau(\mathbf{u}^T)$. By taking $\epsilon \ll \mu^{-12}$, Proposition 3.5 implies that $v[S, T] = \epsilon^{1-2k}\Omega(\mu^{-11})$. Hence, taking $\epsilon < \mu^{-22}$ guarantees that $v[S, T] = \Omega(\epsilon^{(3/2)-2k})$. As $v[R, S] = O(\epsilon^{(3/4)-k^R-k})$, we have $\frac{v[S,T]}{v[R,S]} = \Omega(\epsilon^{(3/4)-k+k^R}) \gg 1$, so the assertion holds. If $k^R = k < k^T$, then, by Proposition 3.2, we have $\tau(\mathbf{u}^R) > \tau(\mathbf{u}^S)$. By Proposition 3.4 and Proposition 3.1, it follows that $v[R, S] = O(\epsilon^{1-2k})$. As $v[S, T] = \Omega(\epsilon^{(5/4)-k-k^T})$, we have $\frac{v[S,T]}{v[R,S]} = \Omega(\epsilon^{(1/4)-k^T+k}) \gg 1$, so the assertion holds.

In what follows we assume that $k^R = k^T = k$ and $\tau(\mathbf{u}^R) > \tau(\mathbf{u}^S) > \tau(\mathbf{u}^T)$. We have to show that $\frac{\pi(S)-\pi(R)\pm O(\epsilon^{1/4})}{\tau(\mathbf{u}^R)-\tau(\mathbf{u}^S)\pm O(\epsilon^{1/2})} < \frac{\pi(T)-\pi(S)\pm O(\epsilon^{1/4})}{\tau(\mathbf{u}^S)-\tau(\mathbf{u}^T)\pm O(\epsilon^{1/2})}$. By taking $\epsilon < \chi^{-4}$, it is sufficient to prove that $(\pi(T) - \pi(S))(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^S)) - (\pi(S) - \pi(R))(\tau(\mathbf{u}^S) - \tau(\mathbf{u}^T)) > \epsilon^{1/8}$. Instead, we take $\epsilon \ll \mu^{-8}$ and establish the stronger bound

$$\pi(T)\left(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^S)\right) + \pi(R)\left(\tau(\mathbf{u}^S) - \tau(\mathbf{u}^T)\right) - \pi(S)\left(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^T)\right) \;=\; \Omega(\mu^{-1})\;. \tag{11}$$

Since $\mathbf{u}^S$ is protected, and since $\tau(\mathbf{u}^R) > \tau(\mathbf{u}^S)$, we conclude that $\mathbf{u}^R$ must be protected too. As for $\mathbf{u}^T$, we have to consider both cases (protected or not). If $\mathbf{u}^T$ is not protected, then we establish equation (11) by proving that $\pi(T)(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^S)) - \pi(S)\tau(\mathbf{u}^R) = \Omega(\mu^6)$. Proposition 3.4 and Proposition 3.1 guarantee that $\pi(T)(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^S)) = \Omega(\mu^6)$ and $\pi(S)\tau(\mathbf{u}^R) = O(\mu^5)$, thus the assertion holds. In the remainder of this proof we assume that $\mathbf{u}^R$, $\mathbf{u}^S$ and $\mathbf{u}^T$ are all protected.

We will soon show that

$$\frac{\tau_p(\mathbf{u}^R) - \tau_p(\mathbf{u}^S)}{\tau_p(\mathbf{u}^T)} + \frac{\tau_p(\mathbf{u}^S) - \tau_p(\mathbf{u}^T)}{\tau_p(\mathbf{u}^R)} - \frac{\tau_p(\mathbf{u}^R) - \tau_p(\mathbf{u}^T)}{\tau_p(\mathbf{u}^S)} \;=\; \Omega(\mu^{-3})\;, \tag{12}$$

thus, by the definition of full evaluation, it follows that

$$\begin{aligned}&\tau^{-1}(\mathbf{u}^{T-\beta})(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^S)) \;+\; \tau^{-1}(\mathbf{u}^{R-\beta})(\tau(\mathbf{u}^S) - \tau(\mathbf{u}^T))\\ &-\tau^{-1}(\mathbf{u}^{S-\beta})(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^T)) \;=\; \Omega(\mu^2)\;.\end{aligned}$$

As Proposition 3.1 guarantees that $\tau^{-1}(\mathbf{u}^{S-\beta})(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^T)) = o(\mu^5)$, we conclude that

$$\begin{aligned}&\tau^{-1}(\mathbf{u}^{T-\beta})(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^S)) \;+\; \tau^{-1}(\mathbf{u}^{R-\beta})(\tau(\mathbf{u}^S) - \tau(\mathbf{u}^T))\\ &-(1 + O(\mu^{-3}))\tau^{-1}(\mathbf{u}^{S-\beta})(\tau(\mathbf{u}^R) - \tau(\mathbf{u}^T)) \;=\; \Omega(\mu^2)\;.\end{aligned}$$

Equation (11) follows due to Proposition 3.4 and the assertion holds.

To establish Equation (12), let $a = \sigma(\mathbf{u}^R) - \sigma(\mathbf{u}^S)$ and $b = \sigma(\mathbf{u}^S) - \sigma(\mathbf{u}^T)$. Equation (12) can be rewritten as

$$(1+\mu^{-1})^{a+b} + (1+\mu^{-1})^{-a} + (1+\mu^{-1})^{-b} - (1+\mu^{-1})^{-a-b} - (1+\mu^{-1})^{a} - (1+\mu^{-1})^{b} \;=\; \Omega(\mu^{-3})\;,$$

which is equivalent to

$$\begin{aligned}&\sum_{j=0}^{a+b}\binom{a+b}{j}\mu^{-j} \;+\; \sum_{j=0}^{\infty}(-1)^j\binom{a+j-1}{j}\mu^{-j} \;+\; \sum_{j=0}^{\infty}(-1)^j\binom{b+j-1}{j}\mu^{-j}\\ -&\sum_{j=0}^{\infty}(-1)^j\binom{a+b+j-1}{j}\mu^{-j} \;-\; \sum_{j=0}^{a}\binom{a}{j}\mu^{-j} \;-\; \sum_{j=0}^{b}\binom{b}{j}\mu^{-j} \;=\; \Omega(\mu^{-3})\end{aligned} \tag{13}$$

due to the Taylor expansions

$$(1+z)^q = \sum_{j=0}^{q}\binom{q}{j}z^j \qquad \text{and} \qquad (1+z)^{-q} = \sum_{j=0}^{\infty}(-1)^j\binom{q+j-1}{j}z^j\;.$$

We leave it to the reader to verify that the jth terms of the six sums on the left hand side of equation (13) cancel each other for $j = 0, 1, 2$. For $j = 3$, the terms on the left hand side of equation (13) sums up to $\left(\binom{a+b}{3} - \binom{a+2}{3} - \binom{b+2}{3} + \binom{a+b+2}{3} - \binom{a}{3} - \binom{b}{3}\right)\mu^{-3} = (a^2b + ab^2)\mu^{-3} = \Omega(\mu^{-3})$.

It remains to show that the absolute value of the sums on the left hand side of equation (13) for $j = 4, 5, \ldots$ is $o(\mu^{-3})$. Instead we bound the larger expression

$$\begin{aligned} &\sum_{j=4}^{a+b}\binom{a+b}{j}\mu^{-j} + \sum_{j=4}^{\infty}\binom{a+j-1}{j}\mu^{-j} + \sum_{j=4}^{\infty}\binom{b+j-1}{j}\mu^{-j} \\ &+ \sum_{j=4}^{\infty}\binom{a+b+j-1}{j}\mu^{-j} + \sum_{j=4}^{a}\binom{a}{j}\mu^{-j} + \sum_{j=4}^{b}\binom{b}{j}\mu^{-j} \\ &\leq 6 \cdot \sum_{j=4}^{\infty}\binom{a+b+j-1}{j}\mu^{-j} \ . \end{aligned}$$

As $\binom{a+b+j}{j+1}/\binom{a+b+j-1}{j} \leq a + b$ for every positive j, we have

$$\sum_{j=4}^{\infty}\binom{a+b+j-1}{j}\mu^{-j} \ \leq \ \binom{a+b+3}{4}\mu^{-4}\sum_{j=0}^{\infty}\left(\frac{a+b}{\mu}\right)^j \ = \ O\left(\mu^{\frac{4}{5}-4}\right) \cdot O(1) \ = \ o(\mu^{-3}) \ ,$$

where the middle equality follows from $\mu = \Omega\left((a+b)^5\right)$. Therefore equation (13) is satisfied and the assertion holds. $\square$

The next corollary follows.

Corollary 3.7. *If* $\mathbf{x}$ *is a protected vector in* $\mathcal{W}$*, then for every* $0 \leq k \leq n + 5$*, either* $\Psi_k(\mathbf{x}) = \emptyset$ *or there exists a contract* $S \in \Psi_k(\mathbf{x})$ *and a payoff* v *such that* S *is optimal for* v*.*

Consider the vector $\mathbf{x} = (1, \ldots, 1) \in \mathcal{W}$. Recall that our goal is to decide whether there exists a contract S with $\mathbf{u}^S = \mathbf{x}$. Note that S is of size at least 4 as it must contain agents α, β, B and at least one more agent. For every $4 \leq k \leq n + 5$, Corollary 3.7 guarantees that if $\Psi_k(\mathbf{x})$ is not empty, then such a contract S is optimal for some payoff v_k^*. If we know the payoffs v_k^* for all $4 \leq k \leq n + 5$, then we can query all of them, thus deciding whether or not there exists a contract S with $\mathbf{u}^S = \mathbf{x}$.

Consider some $4 \leq k \leq n + 5$ and assume that $\Psi_k(\mathbf{x})$ is not empty. Recall that $\mathbf{u}^A = (0, \ldots, 0, 1, 0, 0)$, $\mathbf{u}^B = (1, 0, 0, \ldots, 0, 1, 0, 0)$ and $\mathbf{u}^C = (2, 0, 0, \ldots, 0, 1, 0, 0)$. Let $\mathbf{w} = (2, 1, 1, \ldots, 1) \in \mathcal{W}$ and let $\mathbf{y} = (0, 1, 1, \ldots, 1) \in \mathcal{W}$. Since $\mathbf{u}^A$, $\mathbf{u}^B$ and $\mathbf{u}^C$ determine the value of coordinates 0 and $m+1$ in $\mathcal{W}$ without affecting any other coordinate, and since $B \in S$ and $A, C \notin S$ for every contract S such that $\mathbf{u}^S = \mathbf{x}$, it follows that $\Psi_k(\mathbf{w}) \neq \emptyset$ and $\Psi_k(\mathbf{y}) \neq \emptyset$ (as $\Psi_k(\mathbf{x}) \neq \emptyset$ and agent B can be replaced by agent A or C in S). Let $\lambda_k^{w,x} = \max\{v[S, T] \mid S \in \Psi_k(\mathbf{w}) \text{ and } T \in \Psi_k(\mathbf{x})\}$ and let $\lambda_k^{x,y} = \min\{v[S, T] \mid S \in \Psi_k(\mathbf{x}) \text{ and } T \in \Psi_k(\mathbf{y})\}$ (see Figure 3). Note that $\lambda_k^{w,x}$ and

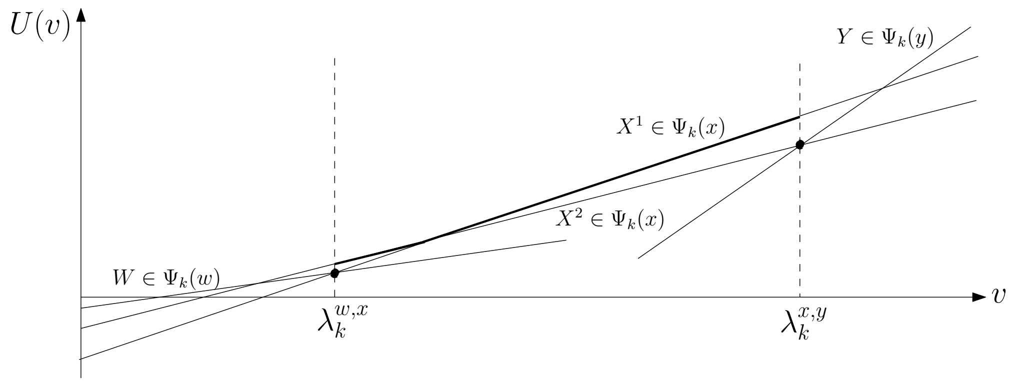

Figure 3: The contracts $X^1 \in \Psi_k(\mathbf{x})$ and $W \in \Psi_k(\mathbf{w})$ realizes $\lambda_k^{w,x}$; the contracts $X^2 \in \Psi_k(\mathbf{x})$ and $Y \in \Psi_k(\mathbf{y})$ realizes $\lambda_k^{x,y}$. For every payoff $\lambda_k^{w,x} \le v \le \lambda_k^{x,y}$, there exists a contract in $\Psi_k(\mathbf{x})$ which is optimal for v (bold lines).

$\lambda_k^{x,y}$ are well defined as $\Psi_k(\mathbf{w})$, $\Psi_k(\mathbf{x})$ and $\Psi_k(\mathbf{y})$ are not empty. Define $v_k^* = \frac{\epsilon^{1-2k}}{(1+\xi\mu^{-1})\mu^9}$, where $\xi = 2 \cdot \sum_{j=0}^{m+1} 4^j$. Observe that the binary representation of v_k^* is polynomial in m.

Lemma 3.8. *The payoff v_k^* satisfies $\lambda_k^{w,x} < v_k^* < \lambda_k^{x,y}$.*

Proof. Define $\mathbf{w}' = (2, 1, 1, \ldots, 1, 0) \in \mathcal{W}$, $\mathbf{x}' = (1, \ldots, 1, 0) \in \mathcal{W}$ and $\mathbf{y}' = (0, 1, 1, \ldots, 1, 0) \in \mathcal{W}$. By Lemma 3.3 and by Proposition 3.4, we have $\lambda_k^{w,x} \le \epsilon^{1-2k} \frac{(1+O(\mu^{-3}))\tau^{-1}(\mathbf{x}') - \tau^{-1}(\mathbf{w}') + O(\epsilon^{1/4})}{\tau(\mathbf{w}) - \tau(\mathbf{x}) - O(\epsilon^{1/2})}$ and $\lambda_k^{x,y} \ge \epsilon^{1-2k} \frac{\tau^{-1}(\mathbf{y}') - (1+O(\mu^{-3}))\tau^{-1}(\mathbf{x}') - O(\epsilon^{1/4})}{\tau(\mathbf{x}) - \tau(\mathbf{y}) + O(\epsilon^{1/2})}$. Propositions 3.1 and 3.5 imply that

$$\lambda_k^{w,x} \ \le \ \epsilon^{1-2k} \left(\frac{(1+O(\mu^{-3}))\tau^{-1}(\mathbf{x}') - \tau^{-1}(\mathbf{w}')}{\tau(\mathbf{w}) - \tau(\mathbf{x})} + o(\epsilon^{1/4}) \right)$$

and

$$\lambda_k^{x,y} \ \le \ \epsilon^{1-2k} \left(\frac{\tau^{-1}(\mathbf{y}') - (1+O(\mu^{-3}))\tau^{-1}(\mathbf{x}')}{\tau(\mathbf{x}) - \tau(\mathbf{y})} - o(\epsilon^{1/4}) \right) .$$

As $\frac{\tau^{-1}(\mathbf{y}')}{\tau^{-1}(\mathbf{x}')} = \frac{\tau^{-1}(\mathbf{x}')}{\tau^{-1}(\mathbf{w}')} = \frac{\tau(\mathbf{x})}{\tau(\mathbf{y})} = \frac{\tau(\mathbf{w})}{\tau(\mathbf{x})} = 1 + \mu^{-1}$, it follows that

$$\lambda_k^{w,x} \ \le \ \epsilon^{1-2k} \left(\frac{\tau^{-1}(\mathbf{w}')}{\tau(\mathbf{x})}(1 + O(\mu^{-2})) + o(\epsilon^{1/4}) \right)$$

and

$$\lambda_k^{x,y} \ \ge \ \epsilon^{1-2k} \left(\frac{\tau^{-1}(\mathbf{x}')}{\tau(\mathbf{y})}(1 - O(\mu^{-2})) - o(\epsilon^{1/4}) \right) .$$

By the definition of full evaluation, we have $\frac{\tau^{-1}(\mathbf{w}')}{\tau(\mathbf{x})} = (1+\mu^{-1})^{-(\xi+1)}\mu^{-9}$ and $\frac{\tau^{-1}(\mathbf{x}')}{\tau(\mathbf{y})} = (1 + \mu^{-1})^{-(\xi-1)}\mu^{-9}$, thus taking $\epsilon < \mu^{-44}$ guarantees that $\lambda_k^{w,x} \le \epsilon^{1-2k}(1 + \mu^{-1})^{-(\xi+1)}\mu^{-9}(1 + O(\mu^{-2}))$ and $\lambda_k^{x,y} \ge \epsilon^{1-2k}(1 + \mu^{-1})^{-(\xi-1)}\mu^{-9}(1 - O(\mu^{-2}))$. Since $\mu > \xi^5$, it follows that $(1 + \mu^{-1})^{\xi+1} = (1+\mu^{-1})(1+\xi\mu^{-1}+O(\mu^{-8/5})) \ge (1+\mu^{-1})(1+\xi\mu^{-1})$ and $(1+\mu^{-1})^{\xi-1} = 1+(\xi-1)\mu^{-1}+O(\mu^{-8/5}) \le$

$1 + \xi\mu^{-1} - \mu^{-1}/2$, hence

$$\frac{\lambda_k^{w,x}}{v_k^*} \leq \frac{(1+\xi\mu^{-1})(1+O(\mu^{-2}))}{(1+\mu^{-1})^{\xi+1}} \leq \frac{1+O(\mu^{-2})}{1+\mu^{-1}} < 1$$

and

$$\frac{\lambda_k^{x,y}}{v_k^*} \geq \frac{(1+\xi\mu^{-1})(1-O(\mu^{-2}))}{(1+\mu^{-1})^{\xi-1}} \geq \frac{1+\xi\mu^{-1}-O(\mu^{-2})}{1+\xi\mu^{-1}-\mu^{-1}/2} > 1 .$$

The assertion follows. □

The analysis is completed with the following lemma, which together with Lemma 3.8 derive Theorem 1.

Lemma 3.9. *The optimal contract for the payoff v is in $\Psi_k(\mathbf{x})$ for every $\lambda_k^{w,x} < v < \lambda_k^{x,y}$.*

Proof. Consider an arbitrary payoff $\lambda_k^{w,x} < \bar{v} < \lambda_k^{x,y}$ and suppose towards deriving contradiction that there exists a contract $T \notin \Psi_k(\mathbf{x})$ such that T is optimal for $\bar{v}$. Recall that Proposition 3.2 implies that $f(S^w) < f(S^x) < f(S^y)$ for every three contracts $S^w \in \Psi_k(\mathbf{w})$, $S^x \in \Psi_k(\mathbf{x})$ and $S^y \in \Psi_k(\mathbf{y})$. Therefore by the definition of $\lambda_k^{w,x}$ and $\lambda_k^{x,y}$, it follows that $T \notin \Psi_k(\mathbf{w})$ and $T \notin \Psi_k(\mathbf{y})$. Let $R^w \in \Psi_k(\mathbf{w})$ and $R^x \in \Psi_k(\mathbf{x})$ be the contracts that realize $\lambda_k^{w,x}$ and let $S^x \in \Psi_k(\mathbf{x})$ and $S^y \in \Psi_k(\mathbf{y})$ be the contracts that realize $\lambda_k^{x,y}$, i.e., $v[R^w, R^x] = \lambda_k^{w,x}$ and $v[S^x, S^y] = \lambda_k^{x,y}$.

We argue that T must satisfy $f(R^w) \leq f(T) \leq f(S^y)$. This can be justified as follows. If $f(T) < f(R^w)$, then since $U_T(\bar{v}) > U_{R^w}(\bar{v})$, we have $U_T(v) > U_{R^w}(v)$ for every $v < \bar{v}$. As $U_{R^x}(v) > U_{R^w}(v)$ for every $v > \lambda_k^{w,x}$, and since $\bar{v} > \lambda_k^{w,x}$, it follows that R^w is dominated by T and R^x, in contradiction to Lemma 3.6. The case where $f(T) > f(S^y)$ is analogous. Proposition 3.2 implies that $|T| = k$ and $\tau(\mathbf{y}) < \tau(\mathbf{u}^T) < \tau(\mathbf{w})$ as otherwise, we get $f(T) < f(R^w)$ or $f(T) > f(S^y)$. But this implies that $\mathbf{u}^T = \mathbf{x}$, in contradiction to the assumption, as $\mathbf{x}$ is the only vector in $\mathcal{W}$ which is lexicographically smaller than $\mathbf{w}$ and greater than $\mathbf{y}$. The assertion follows. □

4 AND technologies

It is shown in [1] that the orbit of every homogeneous AND technology is of size 2, and that the orbit size of every heterogeneous AND technology cannot exceed $n+1$. In what follows, we prove a stronger result, stating that for every k, the contract $S_k \subseteq N$ consisting of the k agents admitting the highest $\frac{\delta_i}{\gamma_i}$ ratios dominates any other k-size contract in the sense that S_k provides both a higher effectiveness and a lower payment. This result implies a trivial polynomial-time algorithm for computing the optimal contract in AND technologies.

Let $t = \langle N, \{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n, c, \varphi \rangle$ be some AND technology and assume without loss of generality that $\frac{\delta_i}{\gamma_i} \geq \frac{\delta_{i+1}}{\gamma_{i+1}}$ for every $1 \leq i < n$. Denote $S_k = \{1, \ldots, k\}$ for every $0 \leq k \leq n$. In order to prove Theorem 3, we show that $f(S_k) \geq f(S)$ and $p(S_k) \leq p(S)$ for any contract $S \subseteq N$ such that $|S| = k$.

Consider an arbitrary bijection $b : S_k - S \to S - S_k$. By the definition of S_k, it follows that $\frac{\delta_j}{\gamma_j} \geq \frac{\delta_{b(j)}}{\gamma_{b(j)}}$ for every $j \in S_k - S$. We first argue that $f(S_k) \geq f(S)$. Recall that $f(S_k) = \prod_{i \in S_k} \delta_i \prod_{i \in N - S_k} \gamma_i$ and $f(S) = \prod_{i \in S} \delta_i \prod_{i \in N-S} \gamma_i$. Therefore we have to show that

$$\prod_{i \in S_k - S} \delta_i \prod_{i \in S_k \cap S} \delta_i \prod_{i \in S - S_k} \gamma_i \prod_{i \in N - (S_k \cup S)} \gamma_i \geq \prod_{i \in S - S_k} \delta_i \prod_{i \in S \cap S_k} \delta_i \prod_{i \in S_k - S} \gamma_i \prod_{i \in N - (S \cup S_k)} \gamma_i$$

The last inequality is equivalent to

$$\frac{\prod_{i \in S_k - S} \delta_i}{\prod_{i \in S_k - S} \gamma_i} \geq \frac{\prod_{i \in S - S_k} \delta_i}{\prod_{i \in S - S_k} \gamma_i},$$

which can be rewritten as $\prod_{i \in S_k - S} \frac{\delta_i}{\gamma_i} \geq \prod_{i \in S_k - S} \frac{\delta_{b(i)}}{\gamma_{b(i)}}$. The argument follows.

We now turn to argue that $p(S_k) \leq p(S)$. Recall that for every contract $R \subseteq N$, we have $p(R) = \sum_{i \in R} p_i(R)$, where $p_i(R) = \frac{c}{f(R) - f(R - \{i\})} = \frac{c}{f(R)\left(1 - \frac{\gamma_i}{\delta_i}\right)}$. Thus for every $i \in S_k \cap S$, we have $p_i(S_k) \leq p_i(S) \Leftrightarrow \frac{c}{f(S_k)\left(1 - \frac{\gamma_i}{\delta_i}\right)} \leq \frac{c}{f(S)\left(1 - \frac{\gamma_i}{\delta_i}\right)}$, which holds as $f(S_k) \geq f(S)$ (the previous argument). Similarly, for every $i \in S_k - S$, we have $p_i(S_k) \leq p_{b(i)}(S) \Leftrightarrow \frac{c}{f(S_k)\left(1 - \frac{\gamma_i}{\delta_i}\right)} \leq \frac{c}{f(S)\left(1 - \frac{\gamma_{b(i)}}{\delta_{b(i)}}\right)}$, which holds as $f(S_k) \geq f(S)$ and $\frac{\delta_i}{\gamma_i} \geq \frac{\delta_{b(i)}}{\gamma_{b(i)}}$. The argument is established since $U_{S_k}(v) = f(S_k)(v - p(S_k))$ and $U_S(v) = f(S)(v - p(S))$.

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