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# DETERMINISTIC APPROXIMATION OF BEST-RESPONSE DYNAMICS FOR THE MATCHING PENNIES GAME

by

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# Deterministic Approximation of Best-Response Dynamics for the Matching Pennies Game*

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**Abstract**

We consider stochastic dynamics for the Matching Pennies game, which behave, in expectation, like the best-response dynamics (i.e., the continuous fictitious play). Since the corresponding vector field is *not* continuous, we cannot apply the deterministic approximation results of Benaïm and Weibull [2003]. Nevertheless, we prove such results for our dynamics by developing the notion of a "leading coordinate."

## 1 Introduction

Benaïm and Weibull [2003] provide deterministic approximations for stochastic processes that arise from evolutionary games. They establish a precise connection between the stochastic process and the system of ordinary differential equations derived from a suitable averaging of the transition probabilities of the Markov chain.

However, in Benaïm and Weibull [2003], as well as in other approximations results,[1] continuity of the corresponding vector field is crucial, and so these results cannot be used to analyze the long-run behavior of stochastic

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[1]See, e.g., Ljung [1977], and, for processes with constant step size, Benaïm [1998], Benaïm and Hirsch [1999], and Fort and Pagès [1999].

processes with non-continuous transitions. Since many models in evolutionary games use best- (or better-) response dynamics, which are *not* continuous, we need to develop different tools in order to analyze such dynamics.

We will show how a deterministic approximation can be obtained for such dynamics by applying the approximation to only one of the coordinates. Specifically, we use the fact that at each point where the vector field is discontinuous, it is nevertheless Lipschitz in one coordinate, and it is that coordinate that "leads" the process through the discontinuity.

We start with the simplest game, the Matching Pennies game, and we consider stochastic best-response dynamics for this game. In Gorodeisky [2005] we analyzed a specific dynamic model, and estimated the transition times of the Markov chain to prove convergence of the stochastic process to the unique mixed Nash equilibrium. Here, however, we consider a more general class of best-response dynamics, and develop deterministic approximations for these processes. The deterministic approximation is the continuous fictitious play, which we describe in Section 2. Using the approximation, we show that any stochastic dynamic that behaves, on average, like the deterministic best-response dynamic converges (like the best-response dynamic) to the unique equilibrium.

In Section 2 we present the dynamics and the Main Theorem. In Section 3 we give an example of the use of the Main Theorem. In Section 4 we prove the Main Theorem and the deterministic approximation, and describe the idea of the proof of the deterministic approximation.

## 2 The Model and the Main Result

In this section we describe the dynamics we use and present the main result. We start with the deterministic dynamic, which is used to approximate the stochastic dynamics. The deterministic dynamic is the best-response dynamic for the Matching Pennies game, which, after an appropriate rescaling of time — which does not change the orbits — is equivalent to the continuous time fictitious play.

Thus, the dynamic is of the form $\xi(t) = (\xi_1(t), \xi_2(t))$, where $\xi_i(t)$ is a mixed strategy of player $i$, and satisfies

$$\dot{\xi}_1 \in BR_1(\xi_2) - \xi_1, \qquad \dot{\xi}_2 \in BR_2(\xi_1) - \xi_2, \tag{2.1}$$

where $BR_i(p)$ is the set of best responses of player $i$ against the strategy $p$ of the other player $(-i)$.

For convenience, we shift and normalize the dynamic. Let $\Omega = [-1,1]^2$, and let[2] $F : \Omega \to I\!R^2$ be defined as $F = (F_1, F_2)$, where

$$F_1(x,y) = \begin{cases} \frac{1}{2}(1-x) & \text{if } y > 0, \\ -\frac{1}{2}(1+x) & \text{if } y < 0, \\ 0 & \text{if } y = 0. \end{cases} \quad F_2(x,y) = \begin{cases} \frac{1}{2}(1-y) & \text{if } x < 0, \\ -\frac{1}{2}(1+y) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.2}$$

Given the deterministic process $\dot{\xi} = F(\xi)$, we use it to approximate stochastic processes that behave "like" $\xi$. A discrete stochastic process $X = \{X_n\}$ behaves like the deterministic process when its step size is small, and its expected difference per step size is given by $F$, the change of the deterministic process. Thus, $X$ can be written as

$$X_{n+1} - X_n = \delta(F(X_n) + U_{n+1}), \tag{2.3}$$

where $U_{n+1}$ is a random variable with zero expectation ("noise"), and $\delta$ is the step size ($\delta$ is assumed to be small and $U_{n+1}$ is assumed to be bounded).

The processes we consider here are in fact slightly more general than (2.3), in that the expected difference is close to $\delta F(X_n)$ (rather than equal to it). Therefore, let $X = \{X_n\}$ be a Markov process on a state space $\Omega' \subset \Omega = [-1,1]^2$ that satisfies

$$\begin{gathered} X_{n+1} - X_n = \delta\left(F(X_n) + Y_n + U_{n+1}\right), \text{ where} \\ E[U_{n+1} \,|\, X_n] = 0. \end{gathered} \tag{2.4}$$

We will refer to $\delta$ in (2.4) as the *step size* of $\{X_n\}$.

Our main result is that if the stochastic process is close to the determin-

[2] $F_i$ is the normalized version of $BR_i(x_{-i}) - x_i$. $F$ and $\Omega$ will be fixed from now on.

istic process (i.e., $\delta$ and $\{Y_n\}$ are small), then any invariant distribution of it[3] is close to the invariant distribution of the deterministic process.

**Main Theorem.** *For each $\delta > 0$, let $X^\delta = \{X_n^\delta\}_n$ be a discrete-time Markov process on $\Omega^\delta \subset \Omega$ satisfying* (2.4) *with step size $\delta$. Let $\mu^\delta$ be an invariant distribution of $X^\delta$. If $\sup_n \left\|Y_n^\delta\right\| \to 0$ as $\delta \to 0$ and $\sup_{n,\delta} \left\|U_n^\delta\right\| < \infty$, then $\mu^\delta$ converge weakly as $\delta \to 0$ to $1_{(0,0)}$, the Dirac measure on the point* $(0,0)$.

*Remark* 2.1. In the Main Theorem, we assume that for every $\delta > 0$ there exists a Markov process $X^\delta$. The result holds also for a sequence of Markov processes $\{X^k\}_{k\in\mathbb{N}}$, that satisfy (2.4), each with step size $\delta_k$ such that $\delta_k \to 0$, when $\sup_n \left\|Y_n^k\right\| \to 0$ as $k \to \infty$, and $\sup_{n,k} \left\|U_n^k\right\| < \infty$.

*Remark* 2.2. The assumption that $\sup_{n,\delta} \left\|U_n^\delta\right\| < \infty$ is not necessary for the proof of the Main Theorem, and it is sufficient to assume that $\sqrt{\delta}\sup_n \left\|U_n^\delta\right\| \to 0$ as $\delta \to 0$.

# 3 An Evolutionary Model with Finite Populations

As an example of the use of the main result, we describe a natural stochastic evolutionary process that behaves in a similar manner to the deterministic best-response dynamic, and we therefore can use the main result to analyze them. This example is based on the *basic model* of Hart [2002].

The process is an evolutionary game played by two finite populations. Each individual in each population plays a pure strategy (an action), i.e., $T$ or $B$ for the individual from population 1 and $L$ or $R$ for the individual from population 2. The actions of the individuals change each period, either because of selection or because of mutation. The changes that occur in the populations are in accordance with the payoffs of the Matching Pennies game.

[3] $\mu$ is an invariant distribution of $X^\delta$ if for every continuous and bounded function $f : \mathbb{R}^2 \to \mathbb{R}$ and every $n$ we have

$$\int_{\Omega^\delta} E[f(X_n^\delta) \mid X_0^\delta = x] d\mu(x) = \int_{\Omega^\delta} f(x) d\mu(x).$$

Let $N \in I\!N$ be the size of the populations, and let $r \geq 0$ be the probability for mutation (the mutation rate). The process is a stationary Markov chain $\{X_n = (X_{n,1}, X_{n,2})\}_n$ on the state space[4] $\Omega^N = \{(i/N, j/N) : 0 \leq i, j \leq N\}$, where in a state $(i/N, j/N)$, there are $i$ individuals in population 1 that play $T$ (and $N - i$ that play $B$), and $j$ individuals in population 2 that play $L$ (and $N - j$ that play $R$).

Each period one randomly chosen individual of each population may change his action. With probability $r$ there is a mutation, and the action is changed randomly, and with probability $1 - r$ there is selection, and the action is changed to the best-response strategy against the other population.[5]

This process has the following transition probabilities:

- Only one individual in each population may change his action,

$$|X_{n+1,i} - X_{n,i}| \in \left\{0, \pm\frac{1}{N}\right\}.$$

- $X_{n,1}$ increases if the chosen individual from population 1 plays $B$ (and there is a proportion of $1 - X_{n,1}$ of such individuals), and changes his action to $T$ either because of mutation (with probability $r/2$), or because of selection (with probability of $1 - r$), provided that $T$ is the unique best reply.

  Therefore,

$$Pr[X_{n+1,1} - X_{n,1} = \frac{1}{N}] = \begin{cases} (1-X_{n,1})\left(1-\frac{r}{2}\right) & \text{if } BR^1(X_{n,2}) = \{T\} \\ (1-X_{n,1})\frac{r}{2} & \text{otherwise,} \end{cases}$$

  and, similarly

$$Pr[X_{n+1,1} - X_{n,1} = -\frac{1}{N}] = \begin{cases} X_{n,1}\left(1-\frac{r}{2}\right) & \text{if } BR^1(X_{n,2}) = \{B\} \\ X_{n,1}\frac{r}{2} & \text{otherwise.} \end{cases}$$

[4]For simplicity we describe the process as a process on the space of mixed strategies and do not normalize it to be a process on $\Omega = [-1, 1]^2$.

[5]When the best response is not unique, we assume that there is no change.

The transition probabilities for $X_{n,2}$ are similar.

Let

$$Y_{n+1} = E[N(X_{n+1} - X_n) - F(X_n) \,|\, X_n], \text{ and}$$
$$U_{n+1} = N(X_{n+1} - X_n) - (F(X_n) + Y_{n+1}),$$

then $E[U_{n+1} \,|\, X_n] = 0$, and $\{X_n\}$ satisfies (2.4) with step size $\delta = 1/N$. As $\|Y_n\| \leq r/2$ and $\|U_n\| \leq 2 + r$, we obtain from the Main Theorem

**Proposition 3.1.** *Let $\{r^k\}_k$ be a sequence of mutation rates such that $r^k \geq 0$ and $r^k \to 0$ as $k \to \infty$, and let $\{N^k\}$ be a sequence of population sizes such that $N^k \to \infty$ as $k \to \infty$. For each $k$, let $X^k$ be the evolutionary process described above with population size $N^k$ and mutation rate $r^k$, and let $\mu^k$ be an invariant distribution of $X^k$. Then, $\mu^k$ converges weakly to the Dirac measure on the equilibrium.*

**Remarks:**

1. If $r^k = 0$ for all $k$, we obtain Theorem 3.1 of Gorodeisky [2005].

2. If $r^k > 0$, then $X^k$ is an ergodic Markov chain with unique invariant distribution that describes the long-run behavior of the process independently of the initial state. Therefore, for $k$ large enough, the evolutionary process will be most of the time near the equilibrium.

# 4 Proofs

In this section we provide the deterministic approximation of the stochastic process (Theorem 4.13), and use it to prove the main result. The idea of the proof is given in Section 4.1. The detailed proof is given in Section 4.4, following some preliminary results on the deterministic dynamic in Section 4.2 and on the stochastic dynamics in Sections 4.3.

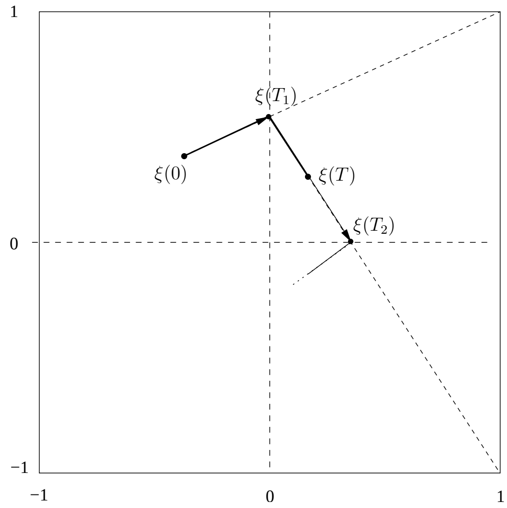


Figure 1: The deterministic dynamics

## 4.1 The Idea of the Proof

As in Benaïm and Weibull [2003], we will use the deterministic approximation to show that the probability that the stochastic process deviates from the deterministic process, in a given time interval, is small.

Let $T > 0$, and let $\xi(t)$ be a trajectory of the deterministic dynamic (see Figure 1 and Section 4.2). As in Benaïm and Weibull [2003], for every time $t$ that $F$ is Lipschitz in a neighborhood of $\xi(t)$, both processes — the deterministic one and the stochastic one — move in a similar direction near $\xi(t)$, and with high probability are close to one another.

Take now a time $t = T_1$ where $F$ is *not* Lipschitz at $\xi(T_1)$ (w.l.o.g, $\xi_1 = 0$, and player 2 is indifferent between his two strategies; again, see Figure 1). There the two processes may move in different directions. Nevertheless, this movement in different directions is controlled by the fact that the first coordinate of $F$ is Lipschitz and so it “leads” both processes away from the non-Lipschitz region (namely, the line $x_1 = 0$).

Thus, the first coordinates of the two processes are close to one another (Proposition 4.8). Using this result, we next show that, except for a small time interval around $T_1$, both processes are on the same side of the line $x_1 = 0$ — i.e., in the region where $F$ is Lipschitz — and therefore must in fact be close to one another in *both* coordinates (Proposition 4.9).

In summary, the way we handle the critical lines where $F$ is not Lipschitz is by using the "L-coordinate" — that one coordinate that *is Lipschitz* there — and showing that it *leads* the stochastic process fast enough so that it remains close to the deterministic path.

## 4.2 The Deterministic Best-Response Dynamic

The Matching Pennies game is a zero-sum game, and therefore, the fictitious play converges to the equilibrium by Robinson [1951] (a proof for the continuous model can be seen in Berger [2001]; for the geometric structure of the dynamic see Metrick and Polak [1994]). Moreover, (2.1) has a unique solution.

For our result, we need more than just the convergence of the dynamic. We need to analyze the behavior of the dynamics in points where it is discontinuous, i.e., points where the best response changes. We therefore show a few more properties of the dynamic.

$F$ (defined in (2.2)) is Lipschitz in each open quadrant (or "best-response region") of $\Omega$ (e.g., $\Omega \cap \{x_1 > 0, x_2 > 0\}$), and $F_i$ is Lipschitz in each open half space (e.g., $\Omega \cap \{x_i > 0\}$). Let $\lambda$ be the maximum of all these Lipschitz constants. $F$ is not continuous at each point at the boundary between two best-response regions (i.e., at all $(x, y) \neq (0, 0) \in \Omega$ such that $xy = 0$), where the best response changes. We call such points *discontinuity points*.

W.l.o.g. let $x = (x_1, x_2) \in \Omega$ be such that $x_1 < 0$ and $x_2 \geq 0$, and let $\rho > 0$ be such that $\|x\|_1 > \rho$. Let $\xi(t) = (\xi_1(t), \xi_2(t))$ be a best-response path starting at $x$; i.e., $\xi(t)$ satisfies $\dot{\xi} = F(\xi)$ and $\xi(0) = x$. A *discontinuity time* of $\xi$ is $t > 0$ such that $\xi(t)$ is a discontinuity point of $F$. Let $T_1 > 0$ be the minimal discontinuity time of $\xi$ (so $\xi_1(T_1) = 0$; see Figure 1). Solving $\dot{\xi} = F(\xi)$ yields for all $0 \leq t \leq T_1$,

$$\begin{aligned} \xi_1(t) &= 1 - (1 - x_1)e^{-t/2}, \\ \xi_2(t) &= 1 - (1 - x_2)e^{-t/2}, \text{ and} \\ T_1 &= 2\ln(1 - x_1). \end{aligned} \tag{4.1}$$

Define $T_n$ as the $n$-th discontinuity time of $\xi$.

**Lemma 4.1.** *Let $\xi(t)$ be a best-response path starting at $x$ with $\|x\|_1 \geq \rho > 0$. Then*

1. *$\|\xi(t)\|_1$ is strictly decreasing in $t$.*
2. *Let $i(n) = 1$ for $n$ odd and $i(n) = 2$ for $n$ even. Then*

$$\xi_{i(n)}(T_n) = \pm \frac{\|\xi(0)\|_1}{1 - x_1 + (n-1)\|\xi(0)\|_1}, \text{ and } \xi_{-i(n)}(T_n) = 0.$$

3. *$T_{n+1} - T_n \geq \rho/(1 + n + \rho)$ for all $n$.*
4. *$T_{n+1} \geq 2\rho \ln(n/2)$ for all $n$.*

*Proof.* 1. Follows from (4.1) and the symmetry of (2.2).

2. From (4.1), we get $\xi_2(T_1) = \|\xi(0)\|_1 / (1 - x_1)$, and by induction we get the equality for all $n$.

3. As in (4.1), it is easy to check that

$$\begin{aligned} T_{n+1} - T_n = 2\ln\left(1 + \left|\xi_{i(n)}\right|\right) &= 2\ln\left(1 + \frac{\|\xi(0)\|_1}{1 - x_1 + (n-1)\|\xi(0)\|_1}\right) \\ &\geq 2\ln\left(1 + \frac{\rho}{2(1+n)}\right) > \frac{2\rho}{2(1+n)+\rho}. \end{aligned}$$

4. From (3) we have

$$T_{n+1} = \sum_{i=0}^{n} T_{i+1} - T_i \geq \sum_{i=0}^{n} \frac{2\rho}{2(1+i)+\rho} \geq 2\rho \ln\left(\frac{n}{2}\right). \qquad \square$$

**Definition 4.2.** Let $x \neq (0,0)$ and $T > 0$, and let $\xi(t)$ be a best-response path starting at $x$. Define $n(x,T)$ as the number of discontinuity times of $\xi(t)$ for $0 \leq t \leq T$, and define $\Delta(x,T) = \min_{0 \leq t \leq T} \|\xi(t)\|_1$.

With these notations and Lemma 4.1 we have

**Lemma 4.3.** *Let $\rho > 0$ and $T > 0$ and let $C = C(\rho, T) = 2e^{T/(2\rho)} + 1$; then for every $x \in \Omega$ such that $\|x\|_1 \geq \rho$, we have $n(x, T) \leq C$, and $\Delta(x, T) \geq \rho/(2C)$.*

*Proof.* Let $n = C - 1$; then $T_{n+1} \geq 2\rho \ln(n/2) = T$, and therefore $n(x, T) \leq n + 1$. The second part is true as

$$\Delta(x, T) = \|\xi(T)\|_1 \geq \|\xi(T_{n+1})\|_1 = \frac{\|\xi(0)\|_1}{1 - x_1 + n \|\xi(0)\|_1} \geq \frac{\rho}{2 + 2n}. \qquad \square$$

We now show that the time the dynamic spends in a neighborhood of a discontinuity point is small. Let $\xi(t)$ be a best-response path, and let $t_d > 0$ be a discontinuity time. W.l.o.g. assume that $\xi_1(t_d) = 0$ and $\xi_2(t_d) > 0$. Let $T > t_d$, and assume that $\xi_2(t) > 0$ for all $0 \leq t \leq T$. Thus, we have $\xi_1(t) < 0$ for all $0 \leq t < t_d$, and $\xi_1(t) > 0$ for all $t_d < t \leq T$.

Let $r > 0$, and define

$$\tau^-(r) = \begin{cases} \min\{\tau : \xi_1(t_d - \tau) < -r\} & \text{if } \xi_1(0) < -r, \\ t_d & \text{otherwise,} \end{cases}$$

and

$$\tau^+(r) = \begin{cases} \min\{\tau : \xi_1(t_d + \tau) > r\} & \text{if } \xi_1(T) > r, \\ T - t_d & \text{otherwise.} \end{cases}$$

Let $\tau(r) = \max\{\tau^-(r), \tau^+(r)\}$; then $\xi_1(t) < -r$ for all $0 \leq t < t_d - \tau(r)$, and $\xi_1(t) > r$ for all $t_d + \tau(r) < t \leq T$; i.e., the time the dynamic spends in the $r$-neighborhood of the discontinuity point is less than $2\tau(r)$.

**Lemma 4.4.** *For all $0 < r < 1/2$ we have $\tau(r) \leq 4r$.*

*Proof.* For all $t_d \leq t < t_d + \tau^+(r)$ we have $\xi_1(t) \leq r < 1/2$ and therefore $\dot{\xi}_1 = F_1(\xi(t)) > 1/4$. Since $\xi_1(t_d + \tau^+(r)) - \xi_1(t_d) \leq r$, we have $\tau^+(r) \leq 4r$. The same holds for $\tau^-(r)$. $\square$

## 4.3 The Stochastic Dynamics

Let $X = \{X_n\}_n$ be a Markov process on $\Omega' \subset \Omega$ that satisfies (2.4) with step size $\delta > 0$. We now define some notations and show some properties of $X$.

1. Let $\bar{X}(t)$, for $t \geq 0$, be the continuous-time step process generated by $X$, i.e., $\bar{X}(t) = X_n$ for $n\delta \leq t < (n+1)\delta$.

2. Let $\hat{X}(t)$, for $t \geq 0$, be the interpolated continuous-time process defined by the piecewise affine interpolation of $X$, i.e.,

$$\hat{X}(t) = X_n + \frac{(t - n\delta)}{\delta}\left(X_{n+1} - X_n\right), \tag{4.2}$$

   for $n\delta \leq t < (n+1)\delta$.

3. Let $Y(t) = Y_n$ and $U(t) = U_{n+1}$ for $n\delta \leq t < (n+1)\delta$, and let[6] $\Psi(t) = \max_{0 \leq s \leq t} \left\| \int_0^s U(\tau) d\tau \right\|$.

4. Let

$$\begin{aligned} y &= \sup_n \|Y_n\|, \\ \Delta x &= \sup_n \|X_{n+1} - X_n\|, \text{ and} \\ u &= \sup_n \|U_n\|_2^2. \end{aligned} \tag{4.3}$$

**Lemma 4.5.** $\left\|\bar{X}(t) - \hat{X}(t)\right\| \leq \Delta x$ *for all* $t \geq 0$.

**Lemma 4.6.** *For every* $t \geq 0$ *we have*

$$\hat{X}(t) - \hat{X}(0) = \int_0^t \left(F(\bar{X}(s)) + Y(s) + U(s)\right) ds.$$

*Proof.* For any $n$ we have

$$X_{n+1} - X_n = \int_{n\delta}^{(n+1)\delta} \left(F(\bar{X}(s)) + Y(s) + U(s)\right) ds.$$

[6] We use $\|\cdot\|$ for $\|\cdot\|_\infty$.

Let $n$ be such that $n\delta \le t < (n+1)\delta$. Then

$$\int_0^t \big(F(\bar{X}(s)) + Y(s) + U(s)\big)\, ds = \sum_{k=0}^{n-1} (X_{k+1} - X_k)$$
$$+ \int_{n\delta}^t \frac{1}{\delta} (X_{n+1} - X_n)\, ds = \hat{X}(t) - \hat{X}(0). \qquad \square$$

As in the proof of Lemma 1 in Benaïm and Weibull [2003], we have

**Lemma 4.7.** *For every $\varepsilon > 0$ and $T > 0$ we have*

$$Pr[\Psi(T) \ge \varepsilon] \le 4 \exp\left(-\frac{\varepsilon^2}{2\delta T u}\right).$$

## 4.4 The Deterministic Approximation

$F$ is not continuous at all the discontinuity points, and it is not clear that deterministic trajectories passing through such points can be used to approximate the stochastic process. We will show here that, outside the rest point of the deterministic process $(0,0)$, the approximation can be made. To show this, we start with simple trajectories that pass through a single discontinuity point. By the symmetry of the system, we look, w.l.o.g. at trajectories passing through a discontinuity point $(0, \xi_2)$ with $\xi_2 > 0$.

Let $\xi(t)$ satisfy $\dot{\xi} = F(\xi)$, and let $T > 0$ be such that $\xi_2(t) > 0$ for all $0 \le t \le T$. Let $0 < t_d < T$ be a discontinuity time. Let $\{X_n\}$ be a Markov process on $\Omega' \subset \Omega$ that satisfies 2.4 with step size $\delta$. Let $\rho > 0$ be such that $\xi_2(t) > \rho$ for all $0 \le t \le T$, and assume that $X_0 \in \Omega'$ satisfies $\|\xi(0) - X_0\| \le d$ for some $d \le \rho$.

Let $\mathbf{T} = \inf\{t \ge 0 \,|\, \hat{X}_2(t) = 0\}$, and define $\mathbf{t} = \min\{t, \mathbf{T}\}$ for every $t > 0$. Define $D(t) = \max_{0 \le s \le t} \left\|\xi(s) - \hat{X}(s)\right\|$, $\mathbf{D}(t) = \max_{0 \le s \le t} \left\|\xi(\mathbf{s}) - \hat{X}(\mathbf{s})\right\|$, and $\mathbf{D}_1(t) = \max_{0 \le s \le t} \left|\xi_1(\mathbf{s}) - \hat{X}_1(\mathbf{s})\right|$.

Using the notations defined above and in Sections 4.2 and 4.3, define for every $\varepsilon > 0$

$$p(\varepsilon) = 4\exp\left(-\frac{\varepsilon^2}{2\delta T u}\right),$$
$$H(\varepsilon) = e^{\lambda T}\left(\varepsilon + d + T(y + \lambda\Delta x)\right), \text{ and}$$
$$G(\varepsilon) = e^{\lambda T}\left(\varepsilon + d + T(y + \lambda\Delta x) + 16(H(\varepsilon) + \Delta x)\right).$$

**Proposition 4.8.** *For every $\varepsilon >$ we have $Pr[\boldsymbol{D}_1(T) \geq H(\varepsilon)] \leq p(\varepsilon)$.*

*Proof.* Let $0 \leq t \leq T$. Since $\xi(t) - \xi(0) = \int_0^t F(\xi(s))ds$, we obtain, by Lemma 4.6,

$$\left|\hat{X}_1(t) - \xi_1(t)\right| \leq \left|\hat{X}_1(0) - \xi_1(0)\right| + \int_0^t |Y_1(s)|\, ds + \left\|\int_0^t U(s)ds\right\|$$
$$+ \int_0^t \left|F_1(\bar{X}(s)) - F_1(\hat{X}(s))\right| ds + \int_0^t \left|F_1(\hat{X}(s)) - F_1(\xi(s))\right| ds.$$

For every $0 \leq s \leq \mathbf{t}$, we have $\xi_2(s), \hat{X}_2(s), \bar{X}_2(s) > 0$. Therefore, by Lemma 4.5, we have

$$\left|F_1(\bar{X}(s)) - F_1(\hat{X}(s))\right| \leq \lambda\left|\hat{X}_1(s) - \bar{X}_1(s)\right| \leq \lambda\Delta x, \text{ and}$$
$$\left|F_1(\hat{X}(s)) - F_1(\xi(s))\right| \leq \lambda\left|\hat{X}_1(s) - \xi_1(s)\right|.$$

Therefore,

$$\left|\hat{X}_1(\mathbf{t}) - \xi_1(\mathbf{t})\right| \leq d + Ty + \lambda T\Delta x + \Psi(T) + \lambda\int_0^{\mathbf{t}} \left|\hat{X}_1(s) - \xi_1(s)\right| ds.$$

By Grönwall's inequality we obtain

$$\mathbf{D}_1(T) \leq \left[d + Ty + \lambda T\Delta x + \Psi(T)\right] e^{\lambda T} = H(\varepsilon) + \Psi(T)e^{\lambda T} - \varepsilon e^{\lambda T},$$

and therefore

$$Pr[\mathbf{D}_1(T) \geq H(\varepsilon)] \leq Pr[\Psi(T) \geq \varepsilon].$$

By Lemma 4.7, we have

$$Pr[\Psi(T) \geq \varepsilon] \leq 4\exp\left(-\frac{\varepsilon^2}{2\delta T u}\right) = p(\varepsilon). \qquad \square$$

**Proposition 4.9.** *If* $H(\varepsilon) + \Delta x < 1/2$ *then* $Pr[\boldsymbol{D}(T) \geq G(\varepsilon)] \leq 2p(\varepsilon)$.

*Proof.* Let $0 \leq t \leq T$, and let $r = H(\varepsilon) + \Delta x$. Define $t_d^- = t_d - \tau^-(r)$ and $t_d^+ = t_d + \tau^+(r)$. W.l.o.g. assume that $t \geq t_d^+$. Let $\mathbf{A}$ be the event $\mathbf{D}_1(T) < H(\varepsilon)$. Given $\mathbf{A}$, for all $0 \leq s < \mathbf{t}_d^-$, we have $\xi_1(s) < -r$, $\hat{X}_1(s) < \xi_1(s) + H < -\Delta x$, and (by Lemma 4.5) $\bar{X}_1(s) < \hat{X}_1(s) + \Delta x < 0$. Similarly, for all $\mathbf{t}_d^+ < s < \mathbf{t}$, we have $\xi_1(s), \hat{X}_1(s), \bar{X}_1(s) > 0$. Since $\xi_2(s), \hat{X}_2(s), \bar{X}_2(s) > 0$ for all $0 < s < \mathbf{t}$, for all $0 \leq s < \mathbf{t}_d^-$ and $\mathbf{t}_d^+ < s < \mathbf{t}$, we have

$$\left\| F(\bar{X}(s)) - F(\hat{X}(s)) \right\| \leq \lambda \Delta x, \text{ and}$$
$$\left\| F(\hat{X}(s)) - F(\xi(s)) \right\| \leq \lambda \left\| \hat{X}(s) - \xi(s) \right\|.$$

As in the proof of Proposition 4.8, by dividing the integrals into the segments $[0, \mathbf{t}_d^-]$, $[\mathbf{t}_d^-, \mathbf{t}_d^+]$ and $[\mathbf{t}_d^+, \mathbf{t}]$, we obtain (given $\mathbf{A}$)

$$\begin{aligned} \left\| \hat{X}(\mathbf{t}) - \xi(\mathbf{t}) \right\| &\leq \left\| \hat{X}(0) - \xi(0) \right\| + Ty + \lambda T \Delta x + \Psi(T) \\ &+ \lambda \int_0^{\mathbf{t}} \left\| \hat{X}(s) - \xi(s) \right\| ds + \int_{\mathbf{t}_d^-}^{\mathbf{t}_d^+} \left\| F(\bar{X}(s)) - F(\xi(s)) \right\| ds \\ &\leq d + Ty + \lambda T \Delta x + 4 \|F\| \tau(r) + \Psi(T) + \lambda \int_0^{\mathbf{t}} \left\| \hat{X}(s) - \xi(s) \right\| ds, \end{aligned}$$

and, by Lemma 4.4 and Grönwall's inequality, we obtain

$$\mathbf{D}(T) \leq G(\varepsilon) + \Psi(T) e^{\lambda T} - \varepsilon e^{\lambda T}.$$

Therefore,

$$Pr[\mathbf{D}(T) \geq G(\varepsilon) \,|\, \mathbf{A}] \leq Pr[\Psi(T) \geq \varepsilon \,|\, \mathbf{A}].$$

Let $\mathbf{A}^c$ be the complement of $\mathbf{A}$; then, by Proposition 4.8, we obtain

$$Pr[\mathbf{D}(T) \geq G(\varepsilon)] = Pr[\mathbf{D}(T) \geq G(\varepsilon) \,|\, \mathbf{A}]Pr[\mathbf{A}]$$
$$+Pr[\mathbf{D}(T) \geq G(\varepsilon) \,|\, \mathbf{A}^c]Pr[\mathbf{A}^c] \leq Pr[\Psi(T) \geq \varepsilon \,|\, \mathbf{A}]Pr[\mathbf{A}] + Pr[\mathbf{A}^c]$$
$$\leq Pr[\Psi(T) \geq \varepsilon] + p(\varepsilon) \leq 2p(\varepsilon).$$ □

**Corollary 4.10.** *If $G(\varepsilon) \leq \rho \leq 1$, then $Pr[D(T) \geq G(\varepsilon)] \leq 2p(\varepsilon)$.*

*Proof.* Notice first that $G(\varepsilon) \leq 1$ implies that $H(\varepsilon) + \Delta x < 1/2$. We have $\xi_2(t) > \rho$ for all $0 \leq t \leq T$; therefore $\left\|\hat{X}(\mathbf{t}) - \xi(\mathbf{t})\right\| < G(\varepsilon) \leq \rho$ implies that $\hat{X}_2(\mathbf{t}) > 0$ and, therefore, $\mathbf{t} = \min\{t, \mathbf{T}\} = t$, and $\left\|\hat{X}(t) - \xi(t)\right\| \leq G(\varepsilon)$. Therefore, $Pr[D(T) \geq G(\varepsilon)] \leq Pr[\mathbf{D}(T) \geq G(\varepsilon)]$. □

Let $\alpha = 17e^T(1 + T)$; then we have[7] $G(\varepsilon) \leq \alpha(\varepsilon + d + y + \Delta x)$. Let $\beta = \alpha(\varepsilon + y + \Delta x)$ and define $g_\varepsilon : I\!R \to I\!R$ as $g_\varepsilon(x) = \alpha x + \beta$; then $G(\varepsilon) \leq g_\varepsilon(d)$.

Therefore, Corollary 4.10 can be stated as follows.

**Proposition 4.11.** *Let $\xi(0) \in \Omega$ and $T > 0$ such that $\xi_2(t) > \rho$ for all $0 \leq t \leq T$. Then, for all $d$ such that $g_\varepsilon(d) < \rho$, we have*

$$Pr\left[D(T) \geq g_\varepsilon(d) \,\middle|\, \left\|\hat{X}(0) - \xi(0)\right\| \leq d\right] \leq 2p(\varepsilon).$$

We now extend Proposition 4.11 to a general best-response path that passes through discontinuity points more than once.

**Proposition 4.12.** *Let $\xi(0) \neq (0, 0)$, $T > 0$, and $\varepsilon > 0$. Let $n \geq n(\xi(0), T)$ and let $\rho \leq \Delta(\xi(0), T)/2$. Then, for any Markov process $X$, $d$, and $\varepsilon'$ such that $g^{n+2}_{\varepsilon'}(d) \leq \min\{\varepsilon, \rho\}$, we have*

$$Pr\left[D(T) \geq \varepsilon \,\middle|\, \left\|\hat{X}(0) - \xi(0)\right\| \leq d\right] \leq 2(n + 2)p(\varepsilon').$$

*Proof.* Let $0 < T_1 < \ldots < T_l < T$ be the discontinuity times of $\xi$; then $l \leq n$. It is easy to prove that there are times $L_i$ for $i = 0, \ldots, l + 2$, such

[7] For $F$ defined in (2.2) we have $\lambda = 1/2$.

that $T_0 = 0 < L_1 < T_1 < L_2 < \ldots < T_l < L_{l+1} < L_{l+2} = T$, and for all $i = 0, \ldots, l+1$, and for all $L_i \le t \le L_{i+1}$, we have $|\xi_1(t)| \ge \rho$ or $|\xi_2(t)| \ge \rho$.

Let $d_i = g^i_{\varepsilon'}(d)$ for $i = 1, \ldots, n+2$; then $d_i \le d_{i+1} \le d_{n+2} < \rho$ for all $i$. Therefore, for all $i = 0, \ldots, l+1$, we have, by Proposition 4.11,

$$Pr[D(L_{i+1}) \ge d_{i+1} \,|\, D(L_i) < d_i] \ \le 2p(\varepsilon').$$

Therefore, we have

$$\begin{aligned} Pr[D(L_{i+1}) \ge d_{i+1}] \le Pr[D(L_{i+1}) \ge d_{i+1} \,|\, D(L_i) \le d_i] \\ + Pr[D(L_i) \ge d_i] \le 2p(\varepsilon') + Pr[D(L_i) \ge d_i], \end{aligned}$$

and by induction we obtain

$$Pr[D(T) \ge \varepsilon] \le Pr[D(L_{n+2}) \ge d_{n+2}] \le 2(n+2)p(\varepsilon'),$$

whenever $\left\| \hat{X}(0) - \xi(0) \right\| \le d$. □

By the properties of the deterministic process, we can use Proposition 4.12 on all trajectories outside a neighborhood of $(0,0)$, and obtain the deterministic approximation of the stochastic process.

**Theorem 4.13.** *For all $\rho > 0$, $\varepsilon > 0$, and $T > 0$, there exist constants $\gamma$ and $C$, such that for all Markov processes $X$ on $\Omega' \subset \Omega$ that satisfy* (2.4) *with step size $\delta$ and with $y, \Delta x < \gamma$, we have*

$$Pr\left[ \max_{0 \le t \le T} \left\| \hat{X}(t) - \xi(t) \right\| \ge \varepsilon \,\middle|\, X_0 = \xi(0) = x \right] \le C \exp\left( -\frac{\gamma^2}{2\delta T u} \right),$$

*where $x \in \Omega'$ satisfies $\|x\|_1 > \rho$, $\xi(t)$ satisfies $\dot{\xi} = F(\xi)$, $\hat{X}$ is given by* (4.2), *and $y$, $u$, and $\Delta x$ are given by* (4.3).

*Proof.* Let $n_0 = 2e^{T/(2\rho)} + 1$ and $\rho_0 = \rho/2(2 + 4e^{t/(2\rho)})$; then by Lemma 4.3 we have $n(\xi(0), T) \le n_0$ and $\Delta(\xi(0), T)/2 \ge \rho_0$ for all $\xi(0)$ such that $\|\xi(0)\|_1 > \rho$.

Since

$$g_{\varepsilon'}^{n_0+2}(0) = \beta \sum_{i=0}^{n_0+1} \alpha^i \leq \beta(n_0+2)\alpha^{n_0+1} = (n_0+2)\alpha^{n_0+2}\left(\varepsilon' + y + \Delta x\right),$$

there exists $\gamma = \gamma(\rho, \varepsilon, T)$ such that $g_{\varepsilon'}^{n_0+2}(0) \leq \min\{\varepsilon, \rho_0\}$, whenever $\varepsilon' = \gamma$ and $y, \Delta x < \gamma$. Therefore, the proof follows from Proposition 4.12, with $\gamma$ and $C = 2(n_0 + 2)$. □

We now use Theorem 4.13 to show convergence to the equilibrium.

*Proof of the Main Theorem.* Let $\rho > 0$, let $O_\rho$ be a $\rho$-neighborhood of $(0, 0)$, and let $H_\rho = \Omega \setminus O_\rho$. By Theorem 4.13, for any $T > 0$ and $\varepsilon > 0$, we have $\lim_{\delta\to 0} Pr[D(T) \geq \varepsilon \mid \xi(0) = X^\delta(0)] = 0$ uniformly for $\xi(0) \in H_\rho$. Let $\mu$ be a limit point of $\{\mu^\delta\}$, when $\delta \to 0$, relative to the topology of weak convergence. By Corollary 3.2 in Benaïm [1998], $\mu$ is an invariant measure of the deterministic dynamic on $H_\rho$, i.e., $\mu(A) = \mu(\{\xi(T) : \xi(0) \in A\})$ for every measurable $A \subset H_\rho$ and $T > 0$. Therefore, $\mu(H_\rho) = 0$, and as $\rho$ is arbitrary, we have $\mu(0, 0) = 1$. Since the space of probability measures on $\Omega$ is compact, we have $\mu^\delta \xrightarrow{w} 1_{(0,0)}$ as $\delta \to 0$. □

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