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DIFFERENTIATED ANNUITIES IN A POOLING EQUILIBRIUM

by

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Differentiated Annuities in a Pooling Equilibrium

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Abstract

Regular annuities provide payment for the duration of an owner's lifetime. Period-Certain annuities provide additional payment after death to a beneficiary provided the insured dies within a certain period after annuitization. It has been argued that the bequest option offered by the latter is dominated by life insurance which provides non-random bequests. This is correct if competitive annuity and life insurance markets have full information about individual longevities. In contrast, this paper shows that when individual longevities are *private information*, a competitive pooling equilibrium which offers annuities at common prices to all individuals may have positive amounts of both types of annuities in addition to life insurance. In this equilibrium, individuals self-select the types of annuities that they purchase according to their longevity prospects. The break-even price of each type of annuity reflects the average longevity of its buyers. The broad conclusion that emerges from this paper is that *adverse-selection* due to asymmetric information is reflected not only in the *amounts* of insurance purchased but, importantly, also in the choice of *insurance products* suitable for different individual characteristics. This conclusion is supported by recent empirical work about the UK annuity market (Finkelstein and Poterba (2004)).

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1 Introduction

Regular annuities (sometimes called *'life-annuities'*) provide payouts, fixed or variable, for the duration of the owner's lifetime. No payments are made after the death of the annuitant. There are also *period-certain* annuities which provide additional payments after death to a beneficiary in the event that the insured individual dies within a specified period after annuitization[1]. Ten-year and Twenty-year certain periods are common (see Brown, Mitchell, Poterba and Warshawsky (2001)). Of course, expected benefits during life plus expected payments after death are adjusted to make the price of period-certain annuities commensurate with the price of regular annuities.

Period-certain annuities thus provide a bequest option not offered by regular annuities. It has been argued (e.g. Davidoff, Brown and Diamond (2005)) that a superior policy for risk-averse individuals who have a bequest motive is to purchase regular annuities and a *life insurance* policy. The latter provides a certain amount upon death, while the amount provided by period-certain annuities is random, depending on the time of death.

In a competitive market for annuities with full information about longevities, annuity prices will vary with annuitants' life expectancies. Such *'separating equilibrium'* in the annuity market, together with a competitive market for life insurance ensures that any combination of period-certain annuities and life insurance is indeed dominated by some combination of regular annuities and life-insurance.

The situation is different, however, when individual longevities are *private information* which cannot be revealed by individuals' choices and hence each type of annuities is sold at a common price available to all potential buyers. This is called a *'pooling equilibrium'*. In this case, the equilibrium price of each type of annuity is equal to the average longevity of the buyers of this type of annuity, weighted by the equilibrium amounts purchased. Consequently, these prices are higher than the average expected lifetime of the buyers, reflecting the *'adverse-selection'* caused by the larger amounts of annuities purchased by individuals with higher longevities[2].

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[1] TIAA-CREF, for example, calls these After-Tax-Retirement-Annuities (ATRA) with Death Benefits.

[2] It is assumed, that the amount of purchased annuities, presumably from different firms,

When regular annuities and period-certain annuities are available in the market, self-selection by individuals tends to segment annuity purchasers into different groups. Those with relatively short expected life span and a high probability of early death after annuitization will purchase period-certain annuities (and life insurance). Those with a high life expectancy and a low probability of early death will purchase regular annuities (and life-insurance) and those with intermediate longevity prospects will hold both types of annuities.

The theoretical implications of our modelling are supported by recent empirical findings reported in Finkelstein and Poterba (2002, 2004), who studied the UK annuity market. In a pioneering paper (2004), they test two hypotheses. One, "that higher-risk individuals self-select into insurance contracts that offer features that, at a given price, are most valuable to them". The second is that "the equilibrium pricing of insurance policies reflects variation in the risk pool across different policies". They find that the UK data supports both hypotheses.

Our modelling provides the theoretical underpinning for this observation: adverse selection in insurance markets may be largely revealed by self-selection of different insurance *instruments*, in addition to varying *amounts* of insurance purchased.

2 Modelling and First-Best

Consider individuals on the verge of retirement who face an uncertain lifetime. They derive utility from consumption and from leaving bequests after death. For simplicity, it is assumed that utilities are separable and independent of age. Denote the instantaneous utility from consumption by $u(a)$, where a is the flow of consumption, and $v(b)$ is the utility from bequests at the level of b. The functions $u(a)$ and $v(b)$ are assumed to be strictly concave, differentiable, and satisfy $u'(0) = v'(0) = \infty$ and $u'(\infty) = v'(\infty) = 0$. These assumptions ensure that individuals will choose strictly positive levels of both a and b.

Expected lifetime utility, U, is

$$U = u(a)\bar{z} + v(b) \tag{1}$$

cannot be monitored. This is a standard assumption. See, for example, Brugiavini (1993).

where $\bar{z}$ is expected lifetime. Individuals have different longevities represented by a parameter α, $\bar{z} = \bar{z}(\alpha)$. An individual with $\bar{z}(\alpha)$ is termed 'type α'. Assume that α varies continuously over the interval $[\underline{\alpha}, \bar{\alpha}]$, $\bar{\alpha} > \underline{\alpha}$. We take a higher α to indicate lower longevity: $\bar{z}'(\alpha) < 0$[3]. Let $G(\alpha)$ be the distribution function of α in the population.

Social welfare, $\bar{U}$, is the sum of individual expected utilities,

$$\bar{U} = \int_{\underline{\alpha}}^{\bar{\alpha}} [u(a(\alpha))\bar{z}(\alpha) + v(b(\alpha))]dG(\alpha) \tag{2}$$

where $(a(\alpha), b(\alpha))$ is consumption and bequests, respectively, of type α individuals.

Assume a zero rate of interest, so resources can be carried forward or backward in time at no cost. Hence, given total resources, W, the economy's resource constraint is

$$\int_{\underline{\alpha}}^{\bar{\alpha}} [a(\alpha)\bar{z}(\alpha) + b(\alpha)]dG(\alpha) = W \tag{3}$$

Maximization of (2) s.t. (3) yields a unique *First-Best* allocation, (a^*, b^*), independent of α, which equalizes the marginal utilities of consumption and bequests:

$$u'(a^*) = v'(b^*) \tag{4}$$

Conditions (3) and (4) jointly determine (a^*, b^*) and the corresponding optimum expected utility of type α individuals $U^*(\alpha) = u(a^*)\bar{z}(\alpha) + v(b^*)$. Note that while First-Best consumption and bequests are equalized across individuals with different longevities, U^* increases with longevity: $U^{*\prime}(\alpha) = u(a^*)\bar{z}'(\alpha) < 0$.

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[3]Let $F(z, \alpha)$ be probability that an individual survives to age z; $F(0, \alpha) = 1$, $\dfrac{\partial F(z,\alpha)}{\partial z} < 0$ and $F(T, \alpha) = 0$, where T is maximum lifetime. Average life expectancy is $\bar{z}(\alpha) = \int_0^T F(z, \alpha)dz$. It is assumed that $\bar{z}(\alpha)$ is well-defined when $T = \infty$. An increase in α is taken to reduce survival probabilities, $\dfrac{\partial F(z,\alpha)}{\partial \alpha} < 0$, for all z, hence $\bar{z}'(\alpha) < 0$.

Example: $F(z, \alpha) = \dfrac{e^{-\alpha z} - e^{-\alpha T}}{1 - e^{-\alpha T}}$, which becomes $F(z, \alpha) = e^{-\alpha z}$ when $T = \infty$.

3 Separating Equilibrium

Consumption is financed by annuities (for later reference these are called *'regular annuities'*) while bequests are provided by the purchase of life insurance. Each annuity pays a flow of one unit of consumption, contingent on the annuity holder's survival. Denote the price of annuities by p_a. A unit of life insurance pays upon death one unit of bequests and its price is denoted by p_b. Under full information about individual longevities, the competitive equilibrium price of an annuity varies with the purchaser's longevity, being equal (with a zero interest rate) to life expectancy, $p_a = p_a(\alpha) = \bar{z}(\alpha)$. Since each unit of life insurance pays 1 with certainty, its equilibrium price is unity: $p_b = 1$. This competitive *separating equilibrium* is always efficient, satisfying condition (4), and for a particular income distribution can support the First-Best allocation[4].

4 Pooling Equilibrium

Suppose that longevity is private information and hence annuities are sold at the same price, p_a, to all individuals.

Assume that all individuals have the same income, W, so their budget constraint is[5]:

$$p_a a + p_b b = W \tag{5}$$

Maximization of (1) s.t. (5) yields demand functions for annuities, $\hat{a}(p_a, p_b; \alpha)$, and for life insurance, $\hat{b}(p_a, p_b; \alpha)$[6]. Given our assumptions, $\frac{\partial \hat{a}}{\partial p_a} < 0$, $\frac{\partial \hat{a}}{\partial \alpha} < 0$, $\frac{\partial \hat{a}}{\partial p_b} \gtreqless 0$, $\frac{\partial \hat{b}}{\partial p_b} < 0$, $\frac{\partial \hat{b}}{\partial \alpha} > 0$, $\frac{\partial \hat{b}}{\partial p_a} \gtreqless 0$.

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[4]Individuals who maximize (1) s.t. budget constraint $\bar{z}(\alpha)a + b = W$ will select (a^*, b^*) iff $W(\alpha) = \gamma W + (1-\gamma)b^*$, where $\gamma = \gamma(\alpha) = \dfrac{\bar{z}(\alpha)}{\int_{\underline{\alpha}}^{\bar{\alpha}} \bar{z}(\alpha)dG(\alpha)} > 0$. Note that $W(\alpha)$ strictly decreases with α (increases with life expectancy).

[5]As noted above, allowing for different incomes is important for welfare analysis. The joint distribution of incomes and longevity is essential, for example, when considering tax/subsidy policies. Our focus is on the possibility of pooling equilibria with different types of annuities, given *any* income distribution. For simplicity, we assume equal incomes.

[6]The dependence on W is suppressed.

Profits from the sale of annuities, π_a, and from the sale of life insurance, π_b, are:

$$\pi_a(p_a, p_b) = \int_{\underline{\alpha}}^{\overline{\alpha}} (p_a - \bar{z}(\alpha))\hat{a}(p_a, p_b; \alpha)dG(\alpha) \tag{6}$$

and

$$\pi_b(p_a, p_b) = \int_{\underline{\alpha}}^{\overline{\alpha}} (p_b - 1)\hat{b}(p_a, p_b; \alpha)dG(\alpha) \tag{7}$$

Definition 1 *A pooling equilibrium is a pair of prices* $(\hat{p}_a, \hat{p}_b)$ *that satisfy* $\pi_a(\hat{p}_a, \hat{p}_b) = \pi_b(\hat{p}_a, \hat{p}_b) = 0$.

Clearly, $\hat{p}_b = 1$, because marginal costs of a life insurance policy are constant and equal to 1. In view of (6),

$$\hat{p}_a = \frac{\int_{\underline{\alpha}}^{\overline{\alpha}} \bar{z}(\alpha)\hat{a}(\hat{p}_a, 1; \alpha)dG(\alpha)}{\int_{\underline{\alpha}}^{\overline{\alpha}} \hat{a}(\hat{p}_a, 1; \alpha)dG(\alpha)}. \tag{8}$$

The equilibrium price of annuities is an average of marginal costs (equal to life expectancy), weighted by the equilibrium amounts of annuities.

It is seen from (8) that $\bar{z}(\bar{\alpha}) < \hat{p}_a < \bar{z}(\underline{\alpha})$. Furthermore, since $\hat{a}$ and $\bar{z}(\alpha)$ decrease with α, $\hat{p}_a > E(\bar{z}) = \int_{\underline{\alpha}}^{\overline{\alpha}} \bar{z}(\alpha)dG(\alpha)$. The equilibrium price of annuities is higher than the population's average expected lifetime, reflecting the *'adverse-selection'* present in a pooling equilibrium.

Regarding price dynamics out of equilibrium, we follow the standard assumption that the price of each good changes in opposite direction to the sign of profits from sales of this good.

The following assumption about the relation between the elasticity of demand for annuities and longevity will be shown to ensure uniqueness and stability of the pooling equilibrium. Let $\varepsilon_{ap_a}(p_a, p_b; \alpha) = \frac{p_a}{\hat{a}(p_a, p_b; \alpha)} \frac{\partial \hat{a}(p_a, p_b; \alpha)}{\partial p_a}$ be the price elasticity of the demand for annuities (at a given α).

Assumption 1. *For any* (p_a, p_b), ε_{ap_a} *non-decreases in* α.

We can now state:

Proposition 1. *Under Assumption 1, the pooling equilibrium,* $\hat{p}_a$, *satisfying (8), and* $\hat{p}_b = 1$ *is unique and stable.*

Proof. The solution $\hat{p}_a$ and $\hat{p}_b = 1$ satisfying (6) - (7) is unique and stable if the matrix

$$\begin{bmatrix} \frac{\partial \pi_a}{\partial a} & \frac{\partial \pi_a}{\partial p_b} \\ \frac{\partial \pi_b}{\partial p_a} & \frac{\partial \pi_b}{\partial p_b} \end{bmatrix}, \tag{9}$$

is *positive definite at* $(\hat{p}_a, 1)$. It can be shown that $\frac{\partial \pi_b}{\partial p_a} = 0$, $\frac{\partial \pi_b}{\partial p_b} = \hat{b}(\hat{p}_a, 1) > 0$, $\frac{\partial \pi_a}{\partial p_a} = \hat{a}(\hat{p}_a, 1) + \int_{\underline{\alpha}}^{\overline{\alpha}} (\hat{p}_a - \bar{z}(\alpha)) \frac{\partial \hat{a}(\hat{p}_a, 1; \alpha)}{\partial p_a} dG(\alpha)$ and $\frac{\partial \pi_a}{\partial p_b} = \int_{\underline{\alpha}}^{\overline{\alpha}} (\hat{p}_a - \bar{z}(\alpha)) \frac{\partial \hat{a}(\hat{p}_a, 1; \alpha)}{\partial p_b} dG(\alpha)$, where $\hat{a}(p_a, 1) = \int_{\underline{\alpha}}^{\overline{\alpha}} \hat{a}(\hat{p}_a, 1; \alpha) dG(\alpha)$ and $\hat{b}(\hat{p}_a, \mathbf{1}) = \int_{\underline{\alpha}}^{\overline{\alpha}} \hat{b}(\hat{p}_a, 1; \alpha) dG(\alpha)$ are aggregate demands for annuities and life insurance, respectively.

Rewrite

$$\int_{\underline{\alpha}}^{\overline{\alpha}} (\hat{p}_a - \bar{z}(\alpha)) \frac{\partial \hat{a}(\hat{p}_a, 1; \alpha)}{\partial p_a} dG(\alpha) =$$

$$= \frac{1}{\hat{p}_a} \int_{\underline{\alpha}}^{\overline{\alpha}} (\hat{p}_a - \bar{z}(\alpha)) \; \hat{a}(\hat{p}_a, 1; \alpha) \varepsilon_{p_a a}(\hat{p}_a, 1; \alpha) dG(\alpha) \tag{10}$$

By (6), $\hat{p}_a - \bar{z}(\alpha)$ changes sign once over $(\underline{\alpha}, \bar{\alpha})$, say at $\tilde{\alpha}$, $\underline{\alpha} < \tilde{\alpha} < \bar{\alpha}$, such that: $\hat{p}_a - \bar{z}(\alpha) \lesseqqgtr 0$ as $\alpha \lesseqqgtr \tilde{\alpha}$. It now follows from Assumption 1 and from (6) that

$$\int_{\underline{\alpha}}^{\overline{\alpha}} (\hat{p}_a - \bar{z}(\alpha)) \frac{\partial \hat{a}(\hat{p}_a, 1; \alpha)}{\partial p_a} dG(\alpha) \geq$$

$$\geq \frac{\varepsilon_{p_a a}(\hat{p}_a, 1; \tilde{\alpha})}{\hat{p}_a} \int_{\underline{\alpha}}^{\overline{\alpha}} (\hat{p}_a - \bar{z}(\alpha)) \hat{a}(\hat{p}_a, 1; \alpha) dG(\alpha) = 0 \tag{11}$$

It follows that $\frac{\partial \pi_a(\hat{p}_a, 1)}{\partial p_a} > 0$, which implies that (9) is positive-definite.

Figure 1 (drawn for $\frac{\partial \pi_a}{\partial p_b} < 0$) displays Proposition 1.

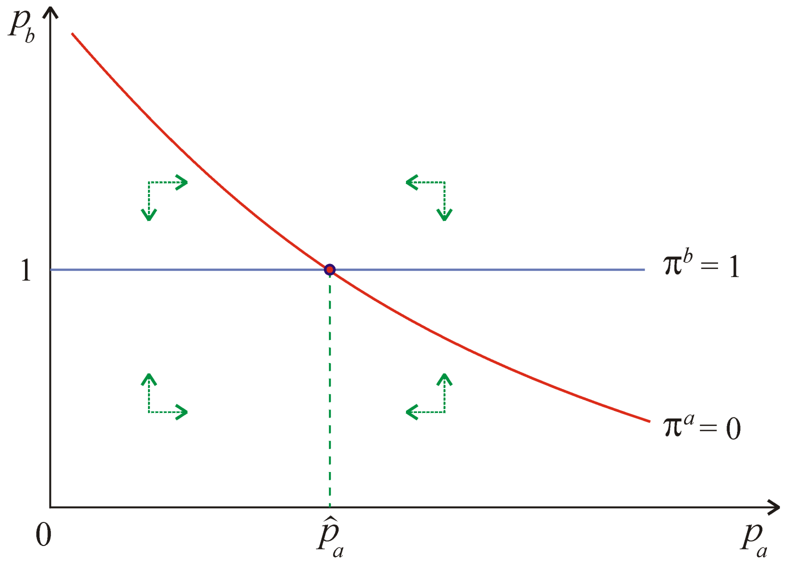

Figure 1

5 Period-Certain Annuities and Life Insurance

We have assumed that annuities provide payouts for the duration of the owner's lifetime and no payments are made after death of the annuitant. We called these *regular annuities*. There exist also *period-certain* annuities which provide additional payments to a designated beneficiary after death of the insured individual, provided death occurs within a specified period after annuitization[7]. Ten-year and Twenty-year certain periods are common and more annuitants choose them than regular annuities (see Brown, Mitchell, Poterba and Warshawsky (2001)). Of course, benefits during life plus expected payments after death are adjusted to make the price of period-certain annuities commensurate with the price of regular annuities.

(a) Inferiority of Period-Certain Annuities Under Full Information

Suppose that there are regular annuities and X-year-certain annuities (in short, X-annuities) who offer a unit flow of consumption while alive and an additional

[7]TIAA-CREF, for example, calls these After-Tax-Retirement Annuities (ATRA) with death benefits.

amount if the individual dies before age X. We continue to denote the amount of regular annuities by a and denote the amount of X-annuities by a_x. The additional payment that X-annuities offer if death occurs before age X is δa_x, where $\delta > 0$ is the payment per X-annuity.

Consider the First-Best allocation when both types of annuities are available. Social welfare is

$$\bar{U} = \int_{\underline{\alpha}}^{\bar{\alpha}} [u(a(\alpha) + a_x(\alpha))\bar{z}(\alpha) + v(b(\alpha) + \delta a_x(\alpha))p(\alpha) + v(b(\alpha))(1 - p(\alpha))]dG(\alpha) \tag{12}$$

and the resource constraint is

$$\int_{\underline{\alpha}}^{\bar{\alpha}} [(a(\alpha) + a_x(\alpha))\bar{z}(\alpha) + \delta a_x p(\alpha) + b(\alpha)]dG(\alpha) = W \tag{13}$$

where $p(\alpha)$ is the probability that a type α individual (with longevity $\bar{z}(\alpha)$) will die before age X[8]. Maximization of (12) s.t. (13) yields $a_x(\alpha) = 0$, $\underline{\alpha} < \alpha < \bar{\alpha}$. Thus, the First-Best has no X-annuities. This outcome also characterizes any competitive equilibrium under full information about individual longevities. *In a competitive separating equilibrium, the random bequest option offered by X-annuities is dominated by regular annuities and life insurance which jointly provide for non-random consumption and bequests.*

However, we shall now show that X-annuities may be held by individuals in a pooling equilibrium. Self-selection leads to a market equilibrium segmented by the two types of annuities: individuals with low longevities and high probability of early death purchase only X-annuities and life insurance, while individuals with high longevities and low probabilities of early death purchase only regular annuities and life insurance. In a range of intermediate longevities individuals hold both types of annuities.

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[8]Let $f(z, \alpha)$ be the probability of death at age z: $f(z, \alpha) = \dfrac{\partial}{\partial z}(1 - F(z, \alpha)) = -\dfrac{\partial F}{\partial z}(z, \alpha)$. Then $p(\alpha) = \int_0^X f(z, \alpha)dz$. The typical stipulations of X-annuities are that the holder of an X-annuity who dies at age z, $0 < z < x$, receives payment proportional to the *remaining period* until age X, $X - z$. Thus, expected payment is proportional to $\int_0^X (X - z)\ f(z, \alpha)dz$. In our formulation, therefore, δ should be interpreted as the *certainty-equivalence* of this amount.

(b) Pooling Equilibrium with Period Certain Annuities

Suppose first that only X-annuities and life insurance are available. Denote the price of X-annuities by p_a^x. The individual's budget constraint is

$$p_a^x \; a_x + b_x = W \tag{14}$$

where b_x is the amount of life insurance purchased jointly with X-annuities. The equilibrium price of life insurance is, as before, unity.

For any α, expected utility, U_x, is given by

$$U_x = u(a_x)\bar{z}(\alpha) + v(b_x + \delta a_x)p(\alpha) + v(b_x)(1 - p(\alpha)) \tag{15}$$

Maximization of (15) s.t. (14) yields (strictly) positive amounts $\hat{a}_x(p_a^x;\alpha)$ and $\hat{b}_x(p_a^x;\alpha)$[9]. It can be shown that $\frac{\partial \hat{a}_x}{\partial p_a^x} < 0$, $\frac{\partial \hat{a}_x}{\partial \alpha} < 0$, $\frac{\partial \hat{b}_x}{\partial \alpha} > 0$ and $\frac{\partial \hat{b}_x}{\partial p_a^x} \gtreqless 0$. Optimum expected utility, $\hat{U}_x$, may increase or decrease with α: $\frac{d\hat{U}_x}{d\alpha} = u(\hat{a}_x)\;\bar{z}'(\alpha) + [v(\hat{b}_x + \delta \hat{a}_x) - v(\hat{b}_x)]\;p'(\alpha)$. We shall assume that $p'(\alpha) > 0$, which is reasonable (though not necessary) since $\bar{z}'(\alpha) < 0$[10]. Hence, the sign of $\frac{d\hat{U}_x}{d\alpha}$ is indeterminate.

Total revenue from annuity sales is $p_a^x \hat{a}_x(p_a^x)$, where $\hat{a}_x(p_a^x) = \int_{\underline{\alpha}}^{\bar{\alpha}} \hat{a}_x(p_a^x;\alpha)dG(\alpha)$ is the aggregate demand for X-annuities, and expected payout is $\int_{\underline{\alpha}}^{\bar{\alpha}} (\bar{z}(\alpha) + \delta p(\alpha))\; \hat{a}_x(p_a^x;\alpha)dG(\alpha)$. The condition for zero expected profits is therefore

$$\hat{p}_a^x = \frac{\int_{\underline{\alpha}}^{\bar{\alpha}} (\bar{z}(\alpha) + \delta p(\alpha))\hat{a}_x(\hat{p}_a^x;\alpha)dG(\alpha)}{\int_{\underline{\alpha}}^{\bar{\alpha}} \hat{a}_x(\hat{p}_a^x;\alpha)} \tag{16}$$

where $\hat{p}_a^x$ is the equilibrium price of X-annuities. It is seen to be an average of longevities plus δ times the probability of early death, weighted by the equilibrium amounts of X-annuities. Assumption 1 regarded regular annuities. Similarly, it is assumed that the demand elasticity of X-annuities

[9]Henceforth, we suppress the price of life insurance, $\hat{p}_b = 1$ and the dependence on δ.

[10]For example, with $F(z,\alpha) = e^{-\alpha z}$, $f(z,\alpha) = \alpha e^{-\alpha z}$ and $p(\alpha) = \int_0^x f(z,\alpha)dz = 1 - e^{-\alpha x}$, which implies $p'(\alpha) > 0$.

increases with α. In addition to this assumption, a sufficient condition for the uniqueness and stability of a pooling equilibrium with X-annuities is the following:

Assumption 2. $\hat{p}_a^x - \bar{z}(\alpha) + \delta p(\alpha)$ *increases with* α.

This is not a vacuous assumption because $\bar{z}'(\alpha) < 0$ and $p'(\alpha) > 0$. It states that the first effect dominates the second.

Proposition 2. *Under Assumptions 1 and 2, the pooling equilibrium,* $\hat{p}_a^x$*, satisfying (16), and* $\hat{p}_b = 1$ *is unique and stable.*

Proof. Follows the same steps as the proof of Proposition 1[11].

6 Mixed Pooling Equilibrium

Now suppose that the market offers regular and X-annuities as well as life insurance. We shall show that, depending on the distribution $G(\alpha)$, *self-selection* of individuals in the pooling equilibrium may lead to following market segmentation: those with high longevities and low probabilities of early death to purchase only regular annuities, those with low longevities and high probabilities of early death to purchase only X-annuities, and individuals with intermediate longevities and probabilities of early death may hold both types. We call this a *'mixed pooling equilibrium'*.

Given p_a, p_a^x, $\bar{z}(\alpha)$ and $p(\alpha)$, the individual maximizes expected utility

$$U = u(a + a_x)\bar{z}(\alpha) + v(b + \delta a_x)p(\alpha) + v(b)(1 - p(\alpha)) \tag{17}$$

subject to the budget constraint

$$p_a a + p_a^x a_x + b = W. \tag{18}$$

The F.O.C. for an interior maximum are:

$$u'(\hat{a} + \hat{a}_x)\bar{z}(\alpha) - \lambda p_a = 0 \tag{19}$$

$$u'(\hat{a} + \hat{a}_x)\bar{z}(\alpha) + v'(\hat{b} + \delta\hat{a}_x)\delta p(\alpha) - \lambda p_a^x = 0 \tag{20}$$

[11] The specific condition is $\hat{a}_x(\hat{p}_a^x) + \int_{\underline{\alpha}}^{\overline{\alpha}} (\hat{p}_a^x - \bar{z}(\alpha) - \delta p(\alpha)) \dfrac{\partial \hat{a}_x}{\partial p_a^x}(p_a^x; \alpha) dG(\alpha) > 0$. Positive monotonicity of the price elasticity of $\hat{a}_x$ w.r.t. α is a sufficient condition.

$$v'(\hat{b}+\delta\hat{a}_x)p(\alpha)+v'(\hat{b})(1-p(\alpha))-\lambda=0 \tag{21}$$

where $\lambda>0$ is the Lagrangean associated with (18). Equations (18) - (21) jointly determine positive amounts $\hat{a}(p_a,p_a^x;\alpha)$, $\hat{a}_x(p_a,p_a^x;\alpha)$ and $\hat{b}(p_a,p_a^x;\alpha)$.

Note first that from (19) - (21) it follows that

$$p_a<p_a^x<p_a+\delta \tag{22}$$

is a necessary condition for an interior solution. When the L.H.S. inequality in (22) does not hold, then X-annuities, each paying a flow of 1 while alive *plus* δ with probability p after death, dominate regular annuities for all α. When the R.H.S. inequality in (22) does not hold, then regular annuities dominate X-annuities because the latter pay a flow of 1 while alive and δ after death with probability $p<1$.

Second, given our assumption that $u'(0)=v'(0)=\infty$, it follows that $\hat{b}>0$ and *either* $\hat{a}>0$ or $\hat{a}_x>0$ for all α. It is impossible to have $\hat{a}=\hat{a}_x=0$ at any α.

$\underline{\hat{a}>0,\ \hat{a}_x=0}$

Condition (20) becomes an inequality

$$u'(\hat{a})\bar{z}+v'(\hat{b})\delta p(\alpha)-\lambda p_a^x\leq 0 \tag{23}$$

while (19) and (21) (with $\hat{a}_x=0$) continue to hold. From these conditions it follows that in this case

$$p(\alpha)\leq\frac{p_a^x-p_a}{\delta} \tag{24}$$

Denote the R.H.S. of (24) by $p(\alpha_0)$. Since $p(\alpha)$ increases in α, it follows that individuals with $\underline{\alpha}\leq\alpha\leq\alpha_0$ purchase only regular annuities (and life insurance).

$\underline{\hat{a}=0,\ \hat{a}_x>0}$

Condition (19) becomes an inequality

$$u'(\hat{a}_x)\bar{z}(\alpha)-\lambda p_a\leq 0 \tag{25}$$

while (20) and (21) continue to hold (with $\hat{a}=0$).

Let

$$\varphi(\alpha) = \frac{1}{1 + \frac{v'(\hat{b} + \delta \hat{a}_x)}{v'(\hat{b})}\left(\frac{1 - p(\alpha_0)}{p(\alpha_0)}\right)} \tag{26}$$

It is seen that at $\alpha = \alpha_0$, $\varphi(\alpha_0) = p(\alpha_0)$. From (19) - (21) it can be further deduced that $p(\alpha) = \varphi(\alpha)$ at any interior solution ($\hat{a} > 0$, $\hat{a}_x > 0$). As α increases from α_0, $\hat{a}(\alpha)$ decreases while $\hat{a}_x(\alpha)$ increases (see Appendix). Let $\hat{a}(\alpha_1) = 0$ for some α_1, $\alpha_0 < \alpha_1 < \bar{\alpha}$. From (25) and (20) - (21) it can be seen that $p(\alpha) \geq \varphi(\alpha)$ whenever $\hat{a} = 0$ ($\hat{a}_x > 0$). It follows that if $\varphi(\alpha)$ non-increases with α for all $\alpha > \alpha_1$, then all individuals with $\alpha_1 < \alpha < \bar{\alpha}$ will hold only X-annuities (and life insurance). We shall now state a sufficient condition for this to hold.

Assumption 3. $v''(x)/v'(x)$ non-decreases with x.

(Exponential and power functions satisfy this assumption).

Proposition 3. *Under Assumption 3, all individuals with $\alpha_1 < \alpha < \bar{\alpha}$ hold only X-annuities.*

Proof. $\varphi(\alpha)$ non-increases in α iff $\frac{v'(\hat{b} + \delta \hat{a}_x)}{v'(\hat{b})}$ non-decreases in α. Using the budget constraint (18) with $\hat{a} = 0$,

$$\frac{\partial}{\partial \alpha}\left(\frac{v'(\hat{b} + \delta \hat{a}_x)}{v'(\hat{b})}\right) = \frac{v'(\hat{b} + \delta \hat{a}_x)}{v'(\hat{b})}\left[\left(\frac{v''(\hat{b})}{v'(\hat{b})} - \frac{v''(\hat{b} + \delta \hat{a}_x)}{v'(\hat{b} + \delta \hat{a}_x)}\right) p_a^x + \right.$$
$$\left. + \delta \frac{v''(\hat{b} + \delta \hat{a}_x)}{v'(\hat{b} + \delta \hat{a}_x)}\right] \frac{\partial \hat{a}_x}{\partial \alpha} \tag{27}$$

Since $\frac{\partial \hat{a}_x}{\partial \alpha} < 0$ (see Appendix), Assumption 3 is seen to ensure that (27) is strictly positive, implying that $\varphi(\alpha)$ decreases with α.

The pattern of optimum annuity holdings and life insurance is described schematically in Figure 2. For justification of this pattern in the three regions I - III, see Appendix.

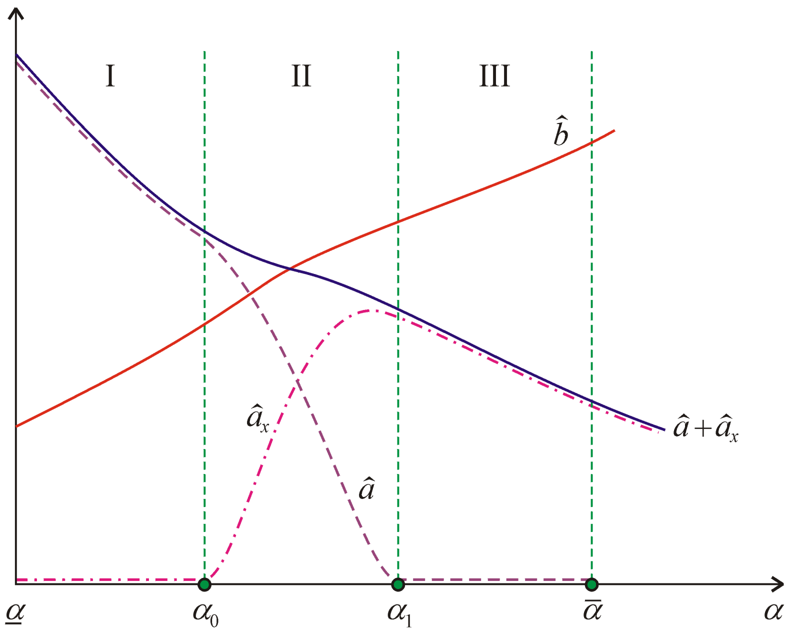

Figure 2
Optimum Annuity Holdings

Equilibrium prices satisfy a zero expected profits condition for each type of annuity, taking account of the self-selection discussed above: $\pi_a(\hat{p}_a, \hat{p}_a^x, 1) = \pi_a^x(\hat{p}_a, \hat{p}_a^x, 1) = \pi_b(\hat{p}_a, \hat{p}_a^x, 1) = 0$. These conditions can be written (suppressing $\hat{p}_b = 1$):

$$\hat{p}_a = \frac{\int_{\underline{\alpha}}^{\overline{\alpha}} \bar{z}(\alpha)\hat{a}(\hat{p}_a, \hat{p}_a^x; \alpha)dG(\alpha)}{\int_{\underline{\alpha}}^{\overline{\alpha}} \hat{a}(\hat{p}_a, \hat{p}_a^x; \alpha)dG(\alpha)} \tag{28}$$

and

$$\hat{p}_a^x = \frac{\int_{\underline{\alpha}}^{\overline{\alpha}} (\bar{z}(\alpha) + \delta p(\alpha))\hat{a}(\hat{p}_a, \hat{p}_a^x; \alpha)dG(\alpha)}{\int_{\underline{\alpha}}^{\overline{\alpha}} \hat{a}(\hat{p}_a, \hat{p}_a^x; \alpha)dG(\alpha)} \tag{29}$$

In Section 4 we stated conditions that ensure uniqueness and stability of the pooling equilibrium. Similar conditions can be formulated to ensure this applies to a mixed pooling equilibrium[12].

[12] These conditions ensure that the matrix of the partial derivatives of expected profits w.r.t. p_a, p_a^x and p_b is positive definite around $\hat{p}_a, \hat{p}_a^x$ and $\hat{p}_b = 1$.

7 Summary:

Recapitulating: in efficient full-information equilibria, the holdings of any period-certain annuities and life insurance is dominated by holdings of some combination of regular annuities and life insurance. However, when information about longevities is private, a competitive pooling equilibrium may support the coexistence of differentiated annuities and life insurance, with some individuals holding only one type of annuity and some holding both types of annuities.

Reassuringly, Finkelstein and Poterba (2004) find evidence of such self-selection in the UK annuity market. More specifically, our analysis suggests a hypothesis complementary to their observation of self-selection: those with high longevities hold regular annuities, while those with low longevities hold period-certain annuities, with mixed holdings for intermediate longevities.

Appendix

Deriving the dependence of the demands for annuities and life insurance on α. Maximizing (17) s.t. the budget constraint (18), yields solutions $\hat{a}$, $\hat{a}_x$ and $\hat{b}$. Given our assumption that $v'(0) = \infty$, $\hat{b} > 0$ for all α. Regarding annuities, we distinguish three regards: I. $\hat{a} \geq 0$, $\hat{a}_x = 0$; II. $\hat{a} \geq 0$, $\hat{a}_x \geq 0$ and III. $\hat{a} = 0$, $\hat{a}_x \geq 0$.

I. $\underline{\hat{a} \geq 0,\ \hat{a}_x = 0 \quad (\underline{\alpha} < \alpha < \alpha_0)}$

$$u'(\hat{a})\bar{z}(\alpha) - v'(\hat{b})p_a = 0 \tag{A.1}$$

$$W - p_a\hat{a} - \hat{b} = 0 \tag{A.2}$$

Differentiating totally:

$$\frac{\partial \hat{a}}{\partial \alpha} = -\frac{u'(\hat{a})\bar{z}'(\alpha)}{\Delta_1} < 0, \quad \frac{\partial \hat{b}}{\partial \alpha} = \frac{p_a u'(\hat{a})\bar{z}'(\alpha)}{\Delta_1} > 0 \tag{A.3}$$

$$\left(\frac{\partial \hat{a}}{\partial p_a} \gtreqless 0\right)$$

where

$$\Delta_1 = u''(\hat{a})\bar{z}(\alpha) + v''(\hat{b})p_a^2 < 0 \tag{A.4}$$

II. $\underline{\hat{a} \geq 0,\ \hat{a}_x \geq 0 \quad (\alpha_0 < \alpha < \alpha_1)}$

Equations (19) - (21) and the budget constraint hold:

$$u'(\hat{a} + \hat{a}_x)\bar{z}(\alpha) - \lambda p_a = 0 \tag{A.5}$$

$$u'(\hat{a} + \hat{a}_x)\bar{z}(\alpha) + v'(\hat{b} + \delta\hat{a}_x)\delta p(\alpha) - \lambda p_a^x = 0 \tag{A.6}$$

$$v'(\hat{b} + \delta\hat{a}_x)p(\alpha) + v'(\hat{b})(1 - p(\alpha)) - \lambda = 0 \tag{A.7}$$

$$W - p_a\hat{a} - p_a^x\hat{a}_x - \hat{b} = 0 \tag{A.8}$$

(A.5) - (A.8) are four equations in $\hat{a}$, $\hat{a}_x$, $\hat{b}$ and λ. The second-order conditions can be shown to hold:

$$\begin{aligned}\Delta_2 \quad = \quad & -\left(u''(\hat{a}+\hat{a}_x)\bar{z}(\alpha)\right)^2 - u''(\hat{a}+\hat{a}_x)\bar{z}(\alpha)[v''(\hat{b}+\delta\hat{a}_x)p(\alpha)(p_a^x - p_a - \delta)^2 + \\ & +v''(\hat{b}+\delta\hat{a}_x)p(\alpha)p_a(p_a^x-\delta) + v''(\hat{b})(1-p(\alpha))(p_a^x - p_a)^2 \\ & +v''(\hat{b})(1-p(\alpha))p_a p_a^x] - p_a^2 v''(\hat{b}+\delta\hat{a}_x)\delta^2 p(\alpha)v''(\hat{b})(1-p(\alpha)) \quad < 0\end{aligned} \tag{A.9}$$

provided $p_a^x - \delta > 0$.

The sign of $\dfrac{\partial\hat{a}}{\partial\alpha}$ and $\dfrac{\partial\hat{a}_x}{\partial\alpha}$ cannot be established for all α in this range without further restrictions. However, at $\alpha = \alpha_0$, differentiating (A.5) - (A.8) totally w.r.t. α, using, (24), $p(\alpha_0) = \dfrac{p_a^x - p_a}{\delta}$, we obtain after some manipulations:

$$\begin{aligned}\frac{\partial\hat{a}}{\partial\alpha} \;=\; & \frac{-1}{\Delta_2}[v''(\hat{b})(p_a^x - p_a)(p_a^x - p_a - \delta)u'(\hat{a})\bar{z}'(\alpha_0) + \\ & +(u''(\hat{a})\bar{z}'(\alpha) + v''(\hat{b})p_a^2)\; v'(\hat{b})\delta p'(\alpha_0)] < 0\end{aligned} \tag{A.10}$$

$$\frac{\partial\hat{a}_x}{\partial\alpha} = \frac{-1}{\Delta_2}[u''(\hat{a})\bar{z}(\alpha) + p_a^2 v''(\hat{b})]v'(\hat{b})\delta p'(\alpha_0) > 0 \tag{A.11}$$

and $\dfrac{\partial\hat{b}}{\partial\alpha} > 0$, where

$$\Delta_2 = (p_a^x - p_a)(p_a^x - p_a - \delta)(u''(\hat{a})\bar{z}'(\alpha) + p_a^2 v''(\hat{b}))v''(\hat{b}) < 0 \tag{A.12}$$

Furthermore,

$$\frac{\partial\hat{a}}{\partial\alpha} + \frac{\partial\hat{a}_x}{\partial\alpha} = \frac{-1}{\Delta_2}(p_a^x - p_a)(p_a^x - p_a - \delta)v''(\hat{b})u'(\hat{a})\bar{z}'(\alpha_0) < 0 \tag{A.13}$$

As α increases from $\alpha = \alpha_0$, $\hat{a}$ decreases, $\hat{a}_x$ increases and $\hat{a}+\hat{a}_x$ decreases, while $\hat{b}$ increases.

This justifies the general pattern displayed in Figure 2 at α_0. Individuals with $\alpha > \alpha_0$ hold positive amounts of *both types* of annuities and, while substituting regular with period-certain annuities, decrease the total amount of annuities as longevity decreases.

We cannot establish that the direction of these changes is monotone at all α, but we have proved the main point: generally, X-annuities are held in a pooling equilibrium.

III. $\underline{\hat{a} = 0,\ \hat{a}_x \geq 0 \quad (\alpha_1 < \alpha < \bar{\alpha})}$

$$u'(\hat{a}_x)\bar{z}(\alpha) + v'(\hat{b} + \delta\hat{a}_x)\delta p(\alpha) - \lambda p_a^x = 0 \tag{A.14}$$

$$v'(\hat{b} + \delta\hat{a}_x)p(\alpha) + v'(\hat{b})(1 - p(\alpha)) - \lambda = 0 \tag{A.15}$$

$$W - p_a^x\hat{a}_x - \hat{b} = 0 \tag{A.16}$$

The second-order condition is satisfied:

$$\Delta_3 = -u''(\hat{a}_x)\bar{z}(\alpha) - v''(\hat{b} + \delta\hat{a}_x)(p_a^x - \delta)^2 p(\alpha) - p_a^{x^2} v''(\hat{b} + \delta\hat{a}_x)(1 - p(\alpha)) > 0 \tag{A.17}$$

and

$$\frac{\partial\hat{a}_x}{\partial\alpha} = \frac{1}{\Delta_3}[u'(\hat{a}_x)\bar{z}'(\alpha) + (p_a^x v'(\hat{b} + \delta\hat{a}_x) - v'(\hat{b} + \delta\hat{a})(p_a^x - \delta))p'(\alpha)] \tag{A.18}$$

$$\frac{\partial\hat{b}}{\partial\alpha} = \frac{p_a^x}{\Delta_3}[-u'(\hat{a}_x)\bar{z}'(\alpha) + (p_a^x - \delta)v'(\hat{b} + \delta\hat{a}_x) + p_a^x v'(\hat{b} + \delta\hat{a}_x)] \tag{A.19}$$

It is seen from (A.18) and (A.19) that $\dfrac{\partial\hat{a}_x}{\partial\alpha} < 0$ and $\dfrac{\partial\hat{b}}{\partial\alpha} > 0$ provided $p_a^x - \delta > 0$.

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