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## ASYMMETRIC AUCTIONS: ANALYTIC SOLUTIONS TO THE GENERAL UNIFORM CASE

**by**

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## מרכז לחקר הרציונליות

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# Asymmetric Auctions: Analytic Solutions to the General Uniform Case.

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**Abstract**

While auction research, including asymmetric auctions, has grown significantly in recent years, there is still little analytical solutions of first-price auctions outside the symmetric case. Even in the uniform case, Griesmer et al. (1967) and Plum (1992) find solutions only to the case where the lower bounds of the two distributions are the same. We present the general analytical solutions to asymmetric auctions in the uniform case for two bidders, both with and without a minimum bid. We show that our solution is consistent with the previously known solutions of auctions with uniform distributions. Several interesting examples are presented including a class where the two bid functions are linear. We hope this result improves our understanding of auctions and provides a useful tool for future research in auctions.

## 1 Introduction

While auction research, including asymmetric auctions, has grown significantly in recent years, there is still little analytical solutions of first-price auctions outside the symmetric case. Surprisingly, the main existing result goes back to Griesmer et al. (1967) who study the following two distributions $V_1 \sim U[0,1], V_2 \sim U[0,\beta]$ and find solutions

$$v_1(b) = \frac{2b\beta^2}{\beta^2 - b^2(1-\beta^2)}$$
$$v_2(b) = \frac{2b\beta^2}{\beta^2 + b^2(1-\beta^2)}$$

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This result was later used by Lebrun (1998, 1999), Maskin & Riley (2000), and Cantillon (2002).

Plum (1992) extends this analytical result to cover the power distribution $F_1(x) = x^\mu$ and $F_2(x) = \left(\frac{x}{\beta}\right)^\mu$. Note that these again, have the same lower bound for the support of the two distributions.[1]

In this paper, we present the general analytical solutions to asymmetric auctions in the uniform case for two buyers (on an interval), both with and without a minimum bid. We show that our solution is consistent with the previously known solutions of auctions with uniform distributions. As we explain later, our solution also covers the general case of uniform distributions with atoms at the lower end of the interval. Several interesting examples are presented including a class where both bid functions are linear. We hope this result improves our understanding of auctions and provides a useful tool for future research in auctions.

## 2 The Model

We consider two general uniform distributions (on intervals): $U[\underline{v}_1, \overline{v}_1]$ for buyer 1 and the other $U[\underline{v}_2, \overline{v}_2]$ for buyer 2 (where $-\infty < \underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2 < \infty$, as clearly a uniform distribution is of finite support.) Without loss of generality, assume that $\underline{v}_1 \leq \underline{v}_2$. We allow for the possibility of a minimum bid $m$ which is assumed to be finite, to ensure that bids are bounded from below. The fact that the bids are bounded from below imply that no buyer wins by bidding less than $\underline{v}_1$(the argument for that is similar to one made by Kaplan & Wettstein, 2000).[2] In particular, in equilibrium, there is no bid $b$ lower than $\underline{v}_1$. Consequently, we shall assume from now on and without loss of generality, that $m \geq \underline{v}_1$.

Notice when $m \geq \min\{\overline{v}_1, \overline{v}_2\}$, we have the trivial equilibrium of at most one buyer placing a bid at $m$. In addition, if $\underline{v}_2 \geq 2\overline{v}_1 - \underline{v}_1$, then any Nash equilibrium must have buyer 2 always bidding $\overline{v}_1$ (and hence always wins the

[1] For presentation purposes, we have normalized here the first bidder's distribution to be on [0,1]. The key is that for both Griesmer et al. (1967) and Plum (1992) the lower end of the support of the distributions is the same while the asymmetry is derived from different higher end points of the support.

[2] The argument is along the following lines and by contradiction. Assume that there is a minimum bid $m$ and that bidding below $\underline{v}_1$ has strictly positive probability of winning. From this, bidders must have strictly positive profits for all values including $\underline{v}_1$. Take $b^*$ as the minimum possible equilibrium bid. The bidder bidding $b^*$ must have a no chance of winning since if not a slight increase in bid will yield a discrete jump in probability of winning. Since he has no chance of winning bidding $b^*$, it follows that the bidder has zero expected profits, providing a contradiction.

object at price $\overline{v}_1$). After eliminating these cases, the following conditions hold for all remaining possibilities are at the intersection of the following conditions:

$$\begin{aligned}
(i)\ \underline{v}_1 &< \overline{v}_1 \\
(ii)\ \underline{v}_1 &\leq \underline{v}_2 < \overline{v}_2 \\
(iii)\ \underline{v}_2 &< 2\overline{v}_1 - \underline{v}_1 \\
(iv)\ m &< \min\{\overline{v}_1, \overline{v}_2\}
\end{aligned}$$

In this region, we now look for strictly monotone, differentiable equilibrium bid functions $b_1(v)$ and $b_2(v)$. Denote the inverses of these bid functions as $v_1(b)$ with support $[\underline{b}_1, \overline{b}_1]$ and $v_2(b)$ with support $[\underline{b}_2, \overline{b}_2]$. Assume that (in equilibrium) a buyer with zero probability of winning bids his value (this includes any value below $m$).[3] In equilibrium, denote by $(\underline{b}, \overline{b}]$ the region where if a buyer submits a bid, he has a strictly positive probablity of winning. It must be the case that $\overline{b}_1 = \overline{b}_2 \equiv \overline{b}$ (otherwise, one buyer can lower the bid without changing the probability of winning) and that $\underline{b}_1 \leq \max\{\underline{b}_2, m\} \equiv \underline{b}$. (Since any bid $b$ is such that $b \geq \underline{v}_1$ and no one bids above one's value we have $\underline{b}_1 = \underline{v}_1$. Consequently, $\underline{b}_1 \leq \underline{b}_2$ and $\underline{b}_1 \leq \max\{\underline{b}_2, m\}$.)

First, solve for $\underline{b}$ when $m = \underline{v}_1$. In the interval $[\underline{b}, \overline{b}]$, buyer 2 with value $v_2$ solves the following maximization problem

$$\max_b \left(\frac{v_1(b) - \underline{v}_1}{\overline{v}_1 - \underline{v}_1}\right)(v_2 - b)$$

Below $\underline{b}$ buyer 1 bids his value, thus when $v_2 = v_2(\underline{b})$, the following must hold for buyer 2's choice of $\underline{b}$. Buyer 2 with value $v_2(\underline{b})$ must not benefit from bidding less than $\underline{b}$:

$$(\underline{b} - \underline{v}_1)(v_2(\underline{b}) - \underline{b}) \geq (b - \underline{v}_1)(v_2(\underline{b}) - b),\ \forall\ b \leq \underline{b}.$$

This is true only if $\underline{b} \leq \frac{\underline{v}_1 + v_2(\underline{b})}{2}$. Similarly buyer 2 with value $v_2(\underline{b})$ does not benefit from bidding more than $\underline{b}$:

[3] Without this assumption a bidder with value $v$, who in equilibrium, has a zero probability of winning, can sometimes be bidding more than his value. Formally, this could still be part of a Bayes-Nash equilibrium and can have a different allocation than other Bayes-Nash equilibria. Such equilibria can be eliminated, for example, by a trembling-hand argument: assuming that each bidder $i$ bids with positive density on $[\underline{v}_i, \overline{v}_i]$. While a bidder bidding below his value when he has a zero probability of winning can also be supported in a Bayes-Nash equilibrium, the allocation is the same as the Bayes-Nash equilibrium where they bid their value. Hence, we may eliminate these for simplicity.

$$(\underline{b} - \underline{v}_1)(v_2(\underline{b}) - \underline{b}) \geq (v_1(b) - \underline{v}_1)(v_2(\underline{b}) - b), \; \forall \; b \geq \underline{b}.$$

However since $v_1(b) \geq b$, we have

$$(\underline{b} - \underline{v}_1)(v_2(\underline{b}) - \underline{b}) \geq (b - \underline{v}_1)(v_2(\underline{b}) - b), \; \forall \; b \geq \underline{b}.$$

This can happen only if $\underline{b} \geq \frac{\underline{v}_1 + v_2(\underline{b})}{2}$, therefore $\underline{b} = \frac{\underline{v}_1 + v_2(\underline{b})}{2}$. Since $m = \underline{v}_1$, $\underline{b}_2 \geq \underline{v}_1 = m$ implying $\underline{b}_2 = \underline{b}$, we have $v_2(\underline{b}) = v_2(\underline{b}_2) = \underline{v}_2$. Thus,

$$\underline{b} = \frac{\underline{v}_1 + \underline{v}_2}{2}. \tag{1}$$

With a minimum bid $m$, by definition $\underline{b} \geq m$. If $m \leq \frac{\underline{v}_1 + \underline{v}_2}{2}$, the above still holds. If $m \geq \frac{\underline{v}_1 + \underline{v}_2}{2}$, then we have $\underline{b} = m$ (the first constraint from above is not necessary and the second constraint is satisfied). Therefore,

$$\underline{b} = \max\{\frac{\underline{v}_1 + \underline{v}_2}{2}, m\}. \tag{2}$$

In the interval $[\underline{b}, \bar{b}]$, the functions $v_1(b)$ and $v_2(b)$ must satisfy (by the first-order conditions of the maximization problems)

$$\begin{aligned} v_1'(b)(v_2(b) - b) &= v_1(b) - \underline{v}_1 \\ v_2'(b)(v_1(b) - b) &= v_2(b) - \underline{v}_2 \end{aligned} \tag{3}$$

Adding these equations together yields

$$v_1'(b)v_2(b) + v_2'(b)v_1(b) = [(v_1(b) + v_2(b) - (\underline{v}_1 + \underline{v}_2))b]'$$

$$v_1(b) \cdot v_2(b) = b(v_1(b) + v_2(b)) - (\underline{v}_1 + \underline{v}_2) \cdot b + c \tag{4}$$

Let us look now at the boundary conditions. As we noted above, $\underline{b}$ belongs to $[\underline{v}_1, \overline{v}_1]$. Furthermore, if $m \geq \underline{v}_2$, then $\underline{b} = m$. We must have, in equilibrium, the following

B1 $v_1(\underline{b}) = \underline{b}$ (recall that a buyer bids his value when his probability of winning is zero).

B2 $v_2(\underline{b}) = \max\{\underline{v}_2, m\}$ (this is the mininum value giving buyer 2 a positive probability of winning).

B3 $v_1(\bar{b}) = \bar{v}_1$ and $v_2(\bar{b}) = \bar{v}_2$ (the highest bid of each buyer is reached for his highest value.)

Substituting the lower boundary conditions B1 into (4), yields

$$v_2(\underline{b})\underline{b} = \underline{b}(v_2(\underline{b}) + \underline{b}) - (\underline{v}_1 + \underline{v}_2)\underline{b} + c$$

$$c = (\underline{v}_1 + \underline{v}_2)\underline{b} - \underline{b}^2 \tag{5}$$

From $\underline{b} = \max\{\frac{\underline{v}_1+\underline{v}_2}{2}, m\}$, we have

$$c = \begin{cases} \frac{(\underline{v}_1+\underline{v}_2)^2}{4} & \text{if } \frac{\underline{v}_1+\underline{v}_2}{2} \geq m \\ (\underline{v}_1 + \underline{v}_2)m - m^2 & \text{otherwise} \end{cases} \tag{6}$$

(Note that $c$, as a function of $m$, reaches its peak at $m = \frac{\underline{v}_1+\underline{v}_2}{2}$). Using B3 and (4) we have

$$\bar{v}_1 \cdot \bar{v}_2 = \bar{b}(\bar{v}_1 + \bar{v}_2) - (\underline{v}_1 + \underline{v}_2) \cdot \bar{b} + c$$

$$\bar{b} = \frac{\bar{v}_1 \cdot \bar{v}_2 - c}{(\bar{v}_1 - \underline{v}_1) + (\bar{v}_2 - \underline{v}_2)} \tag{7}$$

We can use (4) to find $v_2(b)$ in terms of $v_1(b)$ as follows.

$$v_2(b) = \frac{bv_1(b) - (\underline{v}_1 + \underline{v}_2)b + c}{v_1(b) - b}$$

Finally, we can rewrite the differential equation (3) as

$$v_1'(b) \cdot (\frac{bv_1(b) - b(\underline{v}_1 + \underline{v}_2) + c}{v_1(b) - b} - b) = v_1(b) - \underline{v}_1$$

or

$$v_1'(b) \cdot (-b(\underline{v}_1 + \underline{v}_2) + c + b^2) = (v_1(b) - \underline{v}_1)(v_1(b) - b) \tag{8}$$

This equation and boundary condition $v_1(\bar{b}) = \bar{v}_1$ is sufficient to find a solution for $v_1(b)$, which we will do now.

## 2.1 Auction without a mimimum bid.

The auction without a minimum bid has the same solution as an auction with a minimum bid $m$ that satisfies $m \leq \frac{\underline{v}_1+\underline{v}_2}{2}$.

In this case, we can solve the differential equation as follows.

As (by (6) and (7)) $c = \frac{(\underline{v}_1+\underline{v}_2)^2}{4}$ and

$$\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - (\frac{\underline{v}_1+\underline{v}_2}{2})^2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}, \tag{9}$$

we can rewrite equation (8) as

$$v_1'(b) \cdot (\underline{v}_1 + \underline{v}_2 - 2b)^2 = 4(v_1(b) - \underline{v}_1)(v_1(b) - b)$$

We now define $\alpha \equiv \underline{v}_1 + \underline{v}_2 - 2\underline{v}_1 = \underline{v}_2 - \underline{v}_1$, $x \equiv b - \underline{v}_1$ and $D(x)$ such that

$$v_1(b) = \frac{\alpha^2}{D(x)} + \underline{v}_1. \tag{10}$$

We then have $v_1'(x) = -\frac{\alpha^2}{D(x)^2} D'(x)$, and equation (8) becomes

$$\begin{aligned} D'(x) \cdot (\alpha - 2x)^2 &= 4(D(x)x - \alpha^2) \\ D'(x) \cdot (\alpha - 2x)^2 &= 4D(x)x - 16x(\alpha - x) - 4(\alpha - 2x)^2 \\ (D'(x) + 4) \cdot (\alpha - 2x)^2 &= 4x(D(x) - 4(\alpha - x)) \end{aligned}$$

$$\begin{aligned} \frac{D'(x) + 4}{D(x) - 4(\alpha - x)} &= \frac{4x}{(\alpha - 2x)^2} \\ &= \frac{2\alpha}{(\alpha - 2x)^2} - \frac{2}{\alpha - 2x} \end{aligned}$$

By integrating both sides, we obtain

$$\ln(D(x) - 4(\alpha - x)) = \frac{\alpha}{\alpha - 2x} + \ln(\alpha - 2x) + \ln c_1,$$

and taking the exponent of both sides yields

$$\begin{aligned} D(x) - 4(\alpha - x) &= (\alpha - 2x) c_1 e^{\frac{\alpha}{\alpha - 2x}} \\ D(x) &= (\alpha - 2x) c_1 e^{\frac{\alpha}{\alpha - 2x}} + 4(\alpha - x) \end{aligned} \tag{11}$$

The upper boundary condition $v_1(\overline{b}) = \overline{v}_1$ determines $c_1$. When $b = \overline{b}$, we have $x = \overline{x} \equiv \overline{b} - \underline{v}_1$.

From our definition we have $D(\overline{x}) = \frac{\alpha^2}{\overline{v}_1 - \underline{v}_1}$. Hence our boundary condition becomes

$$c_1 = \frac{\frac{\alpha^2}{\overline{v}_1 - \underline{v}_1} - 4(\alpha - (\overline{b} - \underline{v}_1))}{(\alpha - 2(\overline{b} - \underline{v}_1))} e^{-\frac{\alpha}{\alpha - 2(\overline{b} - \underline{v}_1)}}$$

which can be rewritten as (recall that in this case $\underline{b} = \frac{\underline{v}_1+\underline{v}_2}{2}$)

$$c_1 = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\overline{v}_1-\underline{v}_1} + 4(\overline{b} - \underline{v}_2)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_2-\underline{v}_1}{2(\overline{b}-\underline{b})}}$$

Note that this depends only on the constants of the game $\underline{v}_i, \overline{v}_i$, since

$$\overline{b} - \underline{v}_2 = \frac{\overline{v}_1 \cdot \overline{v}_2 - \frac{(\underline{v}_1+\underline{v}_2)^2}{4}}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)} - \underline{v}_2$$

and

$$\overline{b} - \underline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \frac{(\underline{v}_1+\underline{v}_2)^2}{4}}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)} - \frac{\underline{v}_1 + \underline{v}_2}{2}$$

From our definitions of $\alpha, x$ and equations (10) and (11), we find $v_1(b)$

$$v_1(b) = \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_2 + \underline{v}_1 - 2b)c_1 e^{\frac{\underline{v}_2-\underline{v}_1}{\underline{v}_2+\underline{v}_1-2b}} + 4(\underline{v}_2 - b)} + \underline{v}_1 \quad (12)$$

where

$$c_1 = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\overline{v}_1-\underline{v}_1} + 4(\overline{b} - \underline{v}_2)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_2-\underline{v}_1}{2(\overline{b}-\underline{b})}} \quad (13)$$

Note that $v_2(b)$ is obtained from $v_1(b)$ by reversing the roles of $\underline{v}_1, \overline{v}_1$ with those of $\underline{v}_2, \overline{v}_2$. Hence,

$$v_2(b) = \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_1 + \underline{v}_2 - 2b)c_2 e^{\frac{\underline{v}_1-\underline{v}_2}{\underline{v}_1+\underline{v}_2-2b}} + 4(\underline{v}_1 - b)} + \underline{v}_2 \quad (14)$$

where

$$c_2 = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\overline{v}_2-\underline{v}_2} + 4(\overline{b} - \underline{v}_1)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_1-\underline{v}_2}{2(\overline{b}-\underline{b})}} \quad (15)$$

## 2.2 A limit case where buyer 2's value is known.

As a test of the above result let us relate it to the asymmetric situation treated by Kaplan and Zamir (KZ) (2000), namely the situation in which the valuation of one of the two buyers is common knowledge. For instance, assume that $[\underline{v}_1, \overline{v}_1] = [0, 1]$ and $\underline{v}_2 = \overline{v}_2 = \beta$ where $0 < \beta < 2$ (when $\beta > 2$, the equilibrium is that buyer 2 bids 1 and wins with certainty).

For this situation, KZ found that in the equilibrium of the first-price auction, buyer 1's inverse bid function is

$$v_1(b) = \frac{\beta^2}{4(\beta - b)}$$

while buyer 2, whose value is known to be $\beta$ uses a mixed strategy given by the following cummulative probability distribution (with support from $\underline{b} = \frac{\beta}{2}$ to $\overline{b} = \beta - \frac{\beta^2}{4}$) :

$$F(b) = \frac{(2-\beta)\beta}{(2b-\beta)2} e^{-\frac{\beta}{2b-\beta} - \frac{2}{\beta-2}} \tag{16}$$

Let us view this situation as a limiting case of our model where $[\underline{v}_1, \overline{v}_1] = [0,1]$,$[\underline{v}_2, \overline{v}_2] = [\beta, \beta+\varepsilon]$ and $\varepsilon \to 0$. Now, the probability distribution of the bids of buyer 2 is given by (we use $V_2$ for the random valuation of buyer 2, denote $b_i(v,\varepsilon)$ as the bid function for bidder $i$ when the distribution is $[\beta, \beta+\varepsilon]$, and denote $v_i(b,\varepsilon)$ as the respective inverse bid function).

$$P(b_2(V_2,\varepsilon) \leq b) = P(V_2 \leq v_2(b,\varepsilon)) = \frac{v_2(b,\varepsilon) - \beta}{\varepsilon}$$

If the bid distribution is continuous in this limiting process, we should have

$$\lim_{\varepsilon \to 0} \frac{v_2(b,\varepsilon) - \beta}{\varepsilon} = F(b).$$

First, we observe that for $[\underline{v}_1, \overline{v}_1] = [0,1]$ and $\underline{v}_2 = \overline{v}_2 = \beta$ we obtain from our above equations for $\underline{b}$ and $\overline{b}$ ((1) and (9)) the correct range of bids: $\underline{b} = \frac{\beta}{2}$ and $\overline{b} = \beta - \frac{\beta^2}{4}$. Next notice, that $\overline{b} > \underline{b}$ whenever $\beta - \frac{\beta^2}{4} > \frac{\beta}{2}$ or $\beta < 2$. Assuming this is indeed the case, we have a range of bids even when one buyer's value is known with almost certainty. (This makes sense since it converges to a mixed-strategy equilibrium.) Now using our analytical solution for buyer 1's inverse bid function , (12) and (13), with the distributions of $[\underline{v}_1, \overline{v}_1] = [0,1]$,$[\underline{v}_2, \overline{v}_2] = [\beta, \beta+\varepsilon]$, we have

$$\begin{aligned} v_1(b,\varepsilon) &= \frac{\beta^2}{(\beta - 2b)c_1 e^{\frac{\beta}{\beta-2b}} + 4(\beta - b)} \\ c_1(\varepsilon) &= \frac{\beta^2 - 4(\beta - \overline{b})}{(\beta - 2\overline{b})} e^{-\frac{\beta}{\beta - 2\overline{b}}} \end{aligned}$$

where $\overline{b} = \overline{b}(\varepsilon) = \frac{\beta + \varepsilon - \frac{\beta^2}{4}}{1+\varepsilon}$.

We have

$$\lim_{\varepsilon\to 0} v_1(b,\varepsilon) = \frac{\beta^2}{(\beta-2b)\lim_{\varepsilon\to 0} c_1(\varepsilon) e^{\frac{\beta}{\beta-2b}} + 4(\beta-b)} = \frac{\beta^2}{4(\beta-b)}$$

since

$$\lim_{\varepsilon\to 0} c_1(\varepsilon) = \lim_{\varepsilon\to 0} \frac{\beta^2 - 4(\beta-\bar{b}(\varepsilon))}{(\beta-2\bar{b}(\varepsilon))} e^{-\frac{\beta}{\beta-2\bar{b}(\varepsilon)}} = 0$$

Furthermore, using the analytical solution for buyer 2's inverse bid function, (14) and (15), we have

$$v_2(b,\varepsilon) = \frac{\beta^2/\varepsilon}{\left(\frac{4-\beta+\frac{\beta}{\varepsilon}}{\frac{1}{2}\beta-1}\right)(\beta-2b)\, e^{-\frac{\beta}{\beta-2b}} e^{-\frac{2}{2-\beta}} - 4b} \tag{17}$$

And finally it can be verified (by straightforward calculation using (17) and (16)) that indeed

$$\lim_{\varepsilon\to 0} \frac{v_2(b,\varepsilon)-\beta}{\varepsilon} = F(b).$$

## 2.3 Auction with a minimum bid

When the minimum bid is binding, as in the case when $m \geq (\underline{v}_1+\underline{v}_2)/2$, equation (6) becomes $c = (\underline{v}_1+\underline{v}_2)m - m^2$ and (7) becomes $\bar{b} = \frac{\bar{v}_1\cdot\bar{v}_2-(\underline{v}_1+\underline{v}_2)m+m^2}{(\bar{v}_1-\underline{v}_1)+(\bar{v}_2-\underline{v}_2)}$. Now, we can rewrite the differential equation (8) as

$$v_1'(b)\cdot(b-m)(b+m-\underline{v}_1-\underline{v}_2) = (v_1(b)-\underline{v}_1)(v_1(b)-b) \tag{18}$$

Notice that since $b \geq m$ and $2m \geq \underline{v}_1+\underline{v}_2$, the coefficient of $v_1'(b)$ on the left hand side of the above equation is positive.

The solution to this equation is

$$v_1(b) = \underline{v}_1 + \frac{(m-\underline{v}_1)(m-\underline{v}_2)}{b-\underline{v}_2+(b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}c_1} \tag{19}$$

$$c_1 = -\frac{(\bar{v}_1-m)(\bar{v}_2-\underline{v}_2)\left(\frac{(m-\underline{v}_2+\bar{v}_1-\underline{v}_1)(m-\underline{v}_1+\bar{v}_2-\underline{v}_2)}{(\bar{v}_1-m)(\bar{v}_2-m)}\right)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}{(\bar{v}_1-\underline{v}_1)(m-\underline{v}_1+\bar{v}_2-\underline{v}_2)} \tag{20}$$

The derivation of this solution is in the appendix. Again $v_2(b)$ is obtained from $v_1(b)$ by interchanging the roles of $\underline{v}_1, \bar{v}_1$ and $\underline{v}_2, \bar{v}_2$.

**Example 1** $\underline{v}_1 = 0, \underline{v}_2 = 0.$

Substituting these values into our solution yields

$$v_1(b) = \frac{m^2}{b + \sqrt{b^2 - m^2 c_1}}$$

$$c_1 = -\frac{(\overline{v}_1 - m)(\overline{v}_2)\left(\frac{(m+\overline{v}_1)(m+\overline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)}\right)^{1/2}}{\overline{v}_1(m + \overline{v}_2)}$$

Taking $\lim_{m\to 0} v_1(b)$ and applying L'Hopital's rule yields

$$v_1(b) = \frac{2b\overline{v}_1^2\overline{v}_2^2}{\overline{v}_1^2\overline{v}_2^2 + b^2(\overline{v}_2^2 - \overline{v}_1^2)}$$

Reversing the roles of $\overline{v}_1$ and $\overline{v}_2$ gives us

$$v_2(b) = \frac{2b\overline{v}_1^2\overline{v}_2^2}{\overline{v}_1^2\overline{v}_2^2 - b^2(\overline{v}_2^2 - \overline{v}_1^2)}$$

Setting $\overline{v}_1 = 1$ and $\overline{v}_2 = \beta$ to find $v_1(b)$ and $v_2(b)$ yields the Griesmer et al. (1967) result.

Furthermore setting $\overline{v}_1 = \overline{v}_2 = 1$ yields

$$v_1(b) = \frac{m^2}{b + \sqrt{b^2 - m^2 c_1}}$$

$$c_1 = -\frac{(1 - m)\frac{(m+1)}{(1-m)}}{(m + 1)} = -1$$

The limit as $m \to 0$ is $v_1(b) = 2b$ which agrees with the standard result.

## 2.4 Limit when $m \searrow (\underline{v}_1 + \underline{v}_2)/2$.

So far we found the equilibrium bidding functions on two regions of the minimum bid $m$:

(1) For $m \leq (\underline{v}_1 + \underline{v}_2)/2$. This was the case of 'no minimum bid', that is the minimum bid is not binding in equilibrium. This equilibrium thus does not depend upon $m$.

(2) For $m > (\underline{v}_1 + \underline{v}_2)/2$. The minimum bid is binding in equilibrium which in fact does depend upon $m$.

Here we check the continuity of the equilibrium as a function of the minimum bid $m$ at the critical value of $m = (\underline{v}_1 + \underline{v}_2)/2$ (when $m$ approaches this value from above). First we verify that:

$$\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} (b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}} (b+m-\underline{v}_1-\underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}} = \frac{1}{2} e^{-\frac{\underline{v}_2-\underline{v}_1}{2b-\underline{v}_1-\underline{v}_2}} (2b-\underline{v}_1-\underline{v}_2)$$

$$\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} \left( \frac{(m-\underline{v}_2+\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)} \right)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}} = e^{2\frac{(\overline{v}_1-\underline{v}_1+\overline{v}_2-\underline{v}_2)(\underline{v}_2-\underline{v}_1)}{(2\overline{v}_1-\underline{v}_2-\underline{v}_1)(2\overline{v}_2-\underline{v}_2-\underline{v}_1)}}$$

Using these in our solution for $v_1(b)$ and $c_1$ in equations (19) and (20), we have

$$\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} v_1(b) = \underline{v}_1 + \lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} \frac{(m-\underline{v}_1)(m-\underline{v}_2)}{b-\underline{v}_2+\frac{1}{2}e^{-\frac{\underline{v}_2-\underline{v}_1}{2b-\underline{v}_1-\underline{v}_2}}(2b-\underline{v}_1-\underline{v}_2)\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} c_1}$$
$$= \underline{v}_1 + \frac{-(\underline{v}_2-\underline{v}_1)^2/4}{b-\underline{v}_2+\frac{1}{2}e^{-\frac{\underline{v}_2-\underline{v}_1}{2b-\underline{v}_1-\underline{v}_2}}(2b-\underline{v}_1-\underline{v}_2)\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} c_1} \quad (21)$$

$$\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} c_1 = -\frac{(2\overline{v}_1-(\underline{v}_1+\underline{v}_2))(\overline{v}_2-\underline{v}_2)e^{2\frac{(\overline{v}_1-\underline{v}_1+\overline{v}_2-\underline{v}_2)(\underline{v}_2-\underline{v}_1)}{(2\overline{v}_1-\underline{v}_2-\underline{v}_1)(2\overline{v}_2-\underline{v}_2-\underline{v}_1)}}}{(\overline{v}_1-\underline{v}_1)(2\overline{v}_2-(\underline{v}_1+\underline{v}_2))} \quad (22)$$

We now see that indeed this limit yields the equilibrium bid functions for the case of no minimum bid. Note that the range of bids is as follows:

$$\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \frac{(\underline{v}_1+\underline{v}_2)^2}{4}}{(\overline{v}_1+\overline{v}_2)-(\underline{v}_1+\underline{v}_2)}$$

$$\underline{b} = \frac{\underline{v}_1+\underline{v}_2}{2}$$

Notice by (9) and (1), we have $\overline{b}-\underline{b} = \frac{1}{4}\frac{(2\overline{v}_1-\underline{v}_2-\underline{v}_1)(2\overline{v}_2-\underline{v}_2-\underline{v}_1)}{(\overline{v}_1+\overline{v}_2)-(\underline{v}_1+\underline{v}_2)}$ and that

$$\frac{(\underline{v}_2-\underline{v}_1)^2}{\overline{v}_1-\underline{v}_1} + 4(\overline{b}-\underline{v}_2) = \frac{(\overline{v}_2-\underline{v}_2)(2\overline{v}_1-(\underline{v}_1+\underline{v}_2))}{(\overline{v}_1-\underline{v}_1)(\overline{v}_1-\underline{v}_1+\overline{v}_2-\underline{v}_2)}$$

Using these two in equations (21) and (22) yields the equilibrium bid function without a minimum bid; namely, it establishes the equality between (21), (22) and (12), (13), respectively.

## 2.5 Limit when $m \to \underline{v}_2$.

Looking at the solution for the case of a minimum bid, the expressions $(m - \underline{v}_1)$ and $(m - \underline{v}_2)$ appear in the denominator (in the constant). Since we are in the case when $m \geq (\underline{v}_1 + \underline{v}_2)/2$ and $\underline{v}_2 \geq \underline{v}_1$, we have $m = \underline{v}_1$ only when $\underline{v}_1 = \underline{v}_2 = m$ which reduces to the case of no minimum bid. This leaves us to check the limit of our solution with a minimum bid as $m \to \underline{v}_2$. By doing so we find that the limit of our solution is

$$v_1(b) = \frac{\left(-\underline{v}_1 c_1 + \underline{v}_2 \frac{(\underline{v}_2 - \underline{v}_1)}{(b - \underline{v}_2)}\right) + \underline{v}_1 (\log(\frac{b - \underline{v}_1}{b - \underline{v}_2}))}{-c_1 - \underline{v}_1 + \underline{v}_2 - (\log(\frac{b - \underline{v}_1}{b - \underline{v}_2}))}$$

$$c_1 = \frac{(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)} + \frac{(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_2 - \underline{v}_2)} - \log\left(\frac{(\overline{v}_1 - \underline{v}_1)(\overline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_2)(\overline{v}_2 - \underline{v}_2)}\right)$$

which is precisely the solution to the differential equation (18) (for $m = \underline{v}_2$)

$$v_1'(b) \cdot (b - \underline{v}_2)(b - \underline{v}_1) = (v_1(b) - \underline{v}_1)(v_1(b) - b)$$

with boundary condition B3.

## 2.6 Some New Examples

In this section, we provide a few examples of interest that were not solved analytically before. In looking at these examples, we note the minimum bid m provides a way to model distribution of values with atoms at the lower end of the intervals. In fact when $V_i \sim U[\underline{v}_i, \overline{v}_i]$ and $m$ is in $(\underline{v}_i, \overline{v}_i)$ then this is equivalent to a distribution with an atom $\delta_i = \frac{(m - \underline{v}_i)}{(\overline{v}_i - \underline{v}_i)}$ at $m$ and uniform distribution on $[m, \overline{v}_i]$ with the remaining probability. (For that, in the distribution with atoms, we have to relax the assumption that a buyer bids his value when he has zero probability of winning.)

Thus, our analytical solution for the general uniform case with a minimum bid covers also the case with two buyers with distribution which are uniform on an interval with an atom at the lower end of the interval.

In this section, we generated the examples using the solution with a minimum bid given by equations (19) and (20).

**Example 2** $\underline{v}_1 = 0, \underline{v}_2 = 1, m = 2, \overline{v}_2 = 3, \overline{v}_1 = 4.$

Here, we have

$$v_1(b) = \frac{2}{b - 1 + (b - 2)^{\frac{2}{3}}(b + 1)^{\frac{1}{3}} c_1}$$

$$c_1 = \frac{(10)^{\frac{2}{3}}}{(-4)}$$

and

$$v_2(b) = \frac{2}{b + (b-2)^{\frac{1}{3}}(b+1)^{\frac{2}{3}} c_2} + 1$$

$$c_2 = \frac{2\,(10)^{\frac{1}{3}}}{(-5)}$$

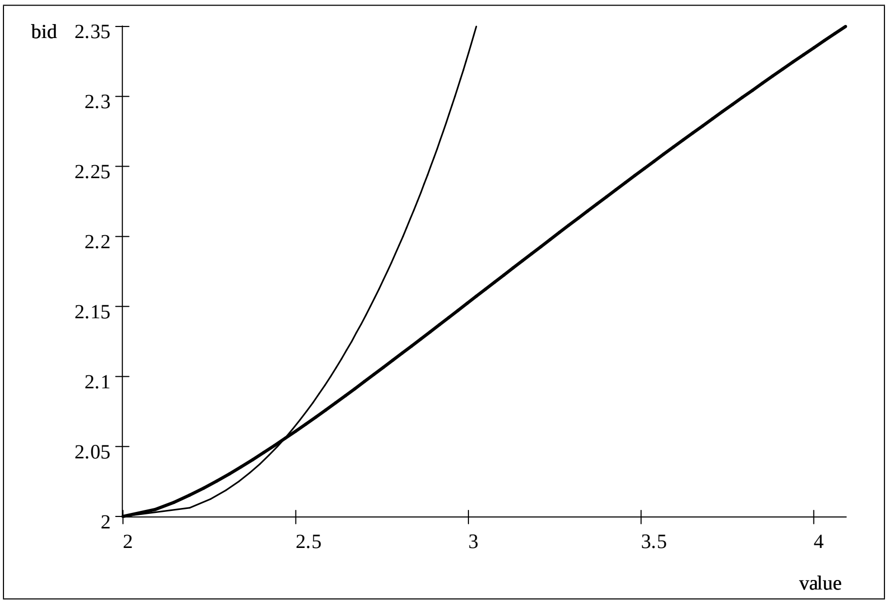


**Figure 1**: Solution when $\underline{v}_1 = 0, \underline{v}_2 = 1, m = 2, \overline{v}_2 = 3, \overline{v}_1 = 4$. The thick line is $v_1(b)$.

We note that conditional distribution of $V_1$ above the minimum bid $m = 2$ stochastically dominates that of $V_2$. Nevertheless, there is no dominance of the bid functions in this region (see Figure 1). As a matter fact, this is the first case of interstecting bid functions that we are aware of.

It is interesting to compare this with the same conditional value distributions above 2 (without the atoms at $m = 2$), namely $V_1 \sim U[2, 4]$ and $V_2 \sim U[2, 3]$. This is given in Figure 2 (and it is a shift of the Griesmer et al. (1967) result).

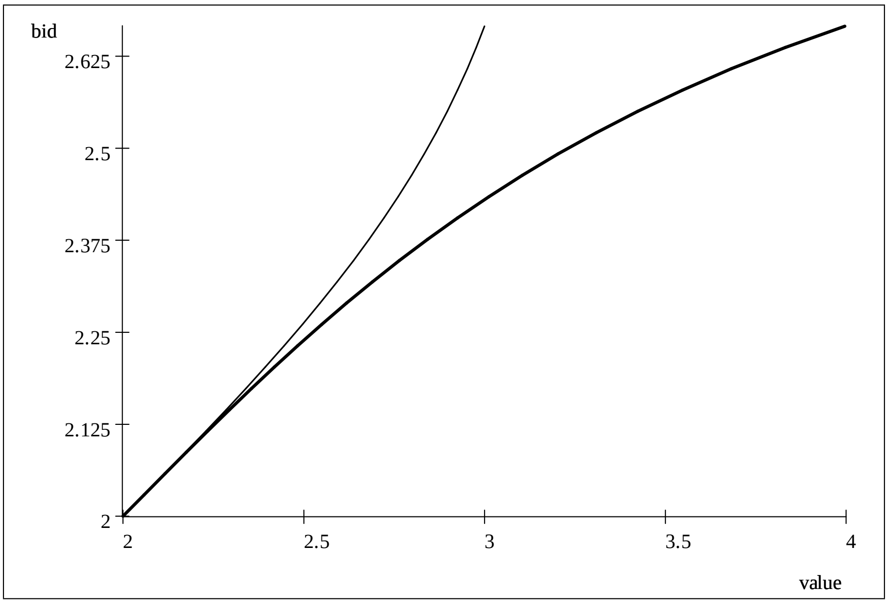


**Figure 2**: Solution when $\underline{v}_1 = 2, \underline{v}_2 = 2, \overline{v}_2 = 3, \overline{v}_1 = 4$. The thicker line is $v_1(b)$.

As we see, the presence of a minimum bid, even though it is at the center of both distributions, changes the equilibrium qualitatively by introducing the crossing of the bid functions. This example generalizes to whole range of minimum bids.

**Example 3** $\underline{v}_1 = 0, \underline{v}_2 = 1, 1/2 < m < 3, \overline{v}_2 = 3, \overline{v}_1 = 4$

By (19) and (20), we have

$$v_1(b) = \frac{m(m-1)}{b - 1 + (b-m)^{\frac{m}{2m-1}}(b+m-1)^{\frac{m-1}{2m-1}}c_1}$$

$$c_1 = -\frac{(4-m)\left(\frac{(m+3)(m+2)}{(3-m)(4-m)}\right)^{\frac{m}{2m-1}}}{2(m+2)}$$

$$\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 + m^2 - m(\underline{v}_1 + \underline{v}_2)}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)} = \frac{12 + m^2 - m}{6}$$

$$v_2(b) = 1 + \frac{m(m-1)}{b + (b-m)^{\frac{m-1}{2m-1}}(b+m-1)^{\frac{m}{2m-1}}c_2}$$

$$c_2 = -\frac{2(3-m)\left(\frac{(m+2)(m+3)}{(3-m)(4-m)}\right)^{\frac{m}{2m-1}}}{m+3}$$

We have found by numerical computation of the solution that the crossing occurs for all values of $m$ in the range.

In the following example we characterize a family of auctions with uniform distributions with linear equilibrium bid functions.

**Example 4** $\underline{v}_1 = 0, \overline{v}_1 = m + z, \underline{v}_2 = 3m/2, \overline{v}_2 = 3m/2 + z$ *(where* $z > 0$*).*

Here we obtain from (19) and (20),

$$\begin{aligned} v_1(b) &= 2(b-m) + m = 2b - m \\ v_2(b) &= 2(b-m) + 3m/2 = 2b - m/2 \end{aligned}$$

$$\begin{aligned} b_1(v) &= \frac{v+m}{2} \\ b_2(v) &= \frac{v}{2} + \frac{m}{4} \end{aligned}$$

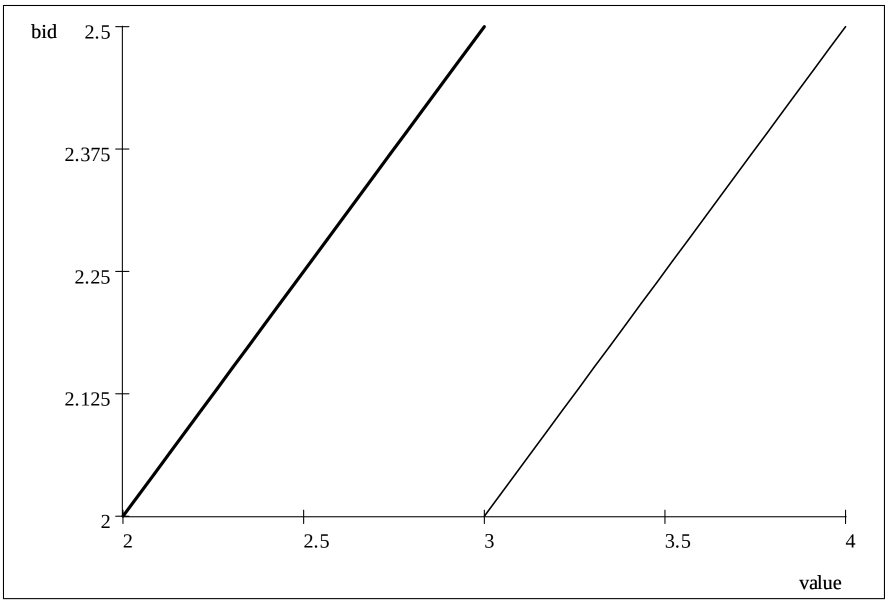


**Figure 3**: Solution when $\underline{v}_1 = 0, \underline{v}_2 = 3, m = 2, \overline{v}_1 = 3, \overline{v}_2 = 4$. The thicker line is $v_1(b)$.

Notice that these bid functions are independent of $z$ and linear. Furthermore, the measure of values where a bid is submitted above the minimum is the same for both buyers, namely $z$. Also notice that when $m \to 0$, this goes to the standard symmetric uniform case of uniformly distributed values on $[0, z]$.

It turns out that linear bid functions appear only in the special case where $m - \underline{v}_1 = 2(\underline{v}_2 - m)$ and $\overline{v}_1 - m = \overline{v}_2 - \underline{v}_2$. (See the appendix for the proof.)

We note that in this class of auctions, the revenue for the first-price auction is

$$R_{FP} = \frac{12m^2 + 15mz + 4z^2}{12(m+z)}$$

and the revenue for the second-price auction is

$$R_{SP} = \begin{cases} \frac{m^3+42m^2z+60mz^2+16z^3}{48z(m+z)} & \text{if } z > m/2 \\ \frac{2m^2+2mz+z^2}{2(m+z)} & \text{if } z \leq m/2 \end{cases}$$

In both cases, the first-price auction has higher revenue (it is higher by $\frac{m^2(6z-m)}{48z(m+z)}$ when $z > m/2$ and by $\frac{(3m-2z)z}{12(m+z)}$ when $z \leq m/2$).

To illustrate that no other linear solution exists. The following example demonstrates that this linearity is lost by stretching the upper range.

**Example 5** $\underline{v}_1 = 0, \overline{v}_1 = 3, \underline{v}_2 = 4, \overline{v}_2 = 6, m = 2,$

Here we obtain:

$$\begin{aligned} v_1(b) &= \frac{8(b-1)}{(8+b(b-4))} \\ v_2(b) &= 3 + \frac{10(b-2)}{(4+2b-b^2)} \end{aligned}$$

By inverting the functions, we get the following non-linear bid functions (see Figure 4):

$$\begin{aligned} b_1(v) &= \frac{2\left(2+v-\sqrt{4+2v-v^2}\right)}{v} \\ b_2(v) &= \frac{v-8+\sqrt{5}\sqrt{8-4v+v^2}}{(v-3)} \end{aligned}$$

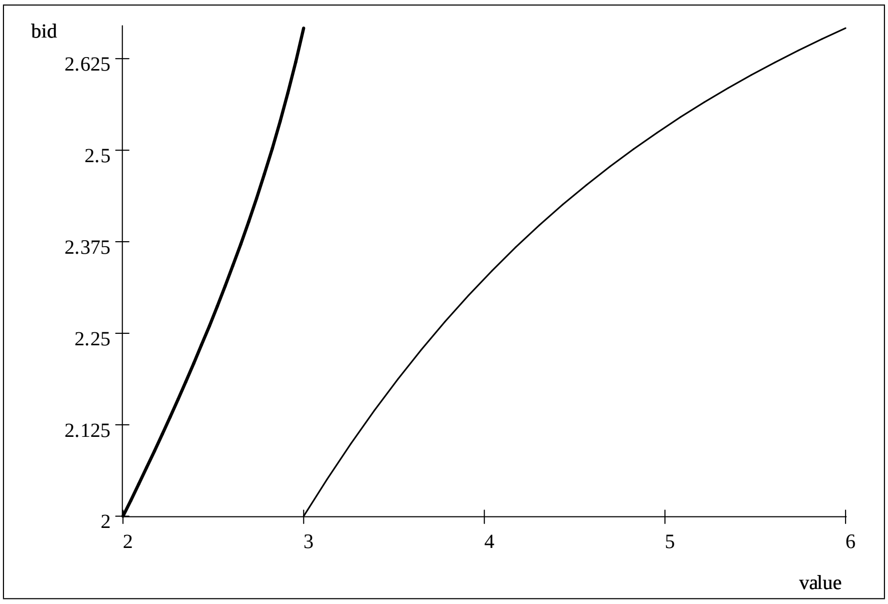

**Figure 4**: Solution when $\underline{v}_1 = 0, \underline{v}_2 = 3, m = 2, \overline{v}_1 = 3, \overline{v}_2 = 6$.The thicker line is $v_1(b)$.

# 3 Concluding Remarks

In this paper, we have analytically solved the general uniform case for two bidders. The uniform distribution is one of the simplest and it is useful to know more than just the existence of the equilibrium but also have an explicit analytical expression of the bid functions. This may be helpful in comparative statics and in detecting interesting features of asymmetric auctions. Future work would be to search for analytical solutions for other environments such as extending our solution to N bidders. On the other direction of research, it is useful to find environments where simple solutions exist. The simplest being of course the linear solution. We have work in progress that shows a linear solution exists when the values are drawn from power distributions (not necessarily the same) and any risk aversions (also not necessarily the same) Together these should provide a useful set of examples for researchers and students as well as suggest a set of parameters for additional experiments (see Guth et al. 2005) on asymmetric auctions.

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# 4 Appendix

## 4.1 Second order conditions.

Here we show that second-order conditions are satisfied for our solution. (This is adapted from Wolfstetter, 1996.) Buyer $j$ with value $v$ and bid $b$ has probility of winning $Pwin^j(b)$ and expected profit $\pi^j(v,b)$ where

$$\pi^j(v,b) = Pwin^j(b)(v-b)$$

Define $b^j(v)$ as a bid function that is monotonic and solves the first-order conditions, namely $\pi^j_b(v,b) = 0$.

Assume these bid functions are monotonic. If so, then second-order conditions are satisfied. Since $\pi_b^j(v,b) = Pwin^{j\prime}(b)(v-b) - Pwin^j(b)$, we have

$$\pi_{bv}^j(v,b) = Pwin^{j\prime}(b) > 0 \tag{23}$$

Take $b^* = b^j(v^*)$. If $\widehat{b} < b^*$, then by monotonicity of the bid function, we $\widehat{v} \equiv (b^j)^{-1}(\widehat{b}) < v^*$. Hence, by (23) we have $\pi_b^j(v^*,b) > \pi_b^j(\widehat{v},b)$ for all $b$. This includes $\pi_b^j(v^*,\widehat{b}) > \pi_b^j(\widehat{v},\widehat{b}) = 0$. Thus, $\pi_b^j(v,\widehat{b}) > 0$ for all $\widehat{b} < b^j(v)$. Likewise, $\pi_b^j(v,\widehat{b}) < 0$ for all $\widehat{b} > b^j(v)$. Hence, second-order conditions are satisfied (as long as our solution is monotonic).

## 4.2 Solution with minimum bids.

The solution that we presented with minimum bids is

$$v_1(b) = \underline{v}_1 + \frac{(m-\underline{v}_1)(m-\underline{v}_2)}{b-\underline{v}_2 + (b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}c_1} \tag{24}$$

$$c_1 = -\frac{(\overline{v}_1-m)(\overline{v}_2-\underline{v}_2)\left(\frac{(m-\underline{v}_2+\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)}\right)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}{(\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}$$

To derive this solution we divide both sides of equation (18) by

$$(v_1(b)-\underline{v}_1)^2(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}$$

to obtain

$$\frac{v_1'(b)}{(v_1(b)-\underline{v}_1)^2(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}} =$$

$$\frac{(v_1(b)-b)}{(v_1(b)-\underline{v}_1)(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}} \tag{25}$$

The RHS can be broken into two expressions:

$$\frac{1}{(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}} +$$

$$\frac{(\underline{v}_1-b)}{(v_1(b)-\underline{v}_1)(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}$$

Observe that

$$\int \frac{1}{(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}db =$$

$$\frac{1}{(b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}} \cdot \frac{\underline{v}_2-b}{(m-\underline{v}_1)(m-\underline{v}_2)}+C$$

and

$$\int \left[ \begin{array}{c} \frac{v_1'(b)}{(v_1(b)-\underline{v}_1)^2(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}} - \\ \frac{(\underline{v}_1-b)}{(v_1(b)-\underline{v}_1)(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}} \end{array} \right] db =$$

$$-\frac{1}{(b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}} \cdot \frac{1}{v_1(b)-\underline{v}_1}+C$$

Hence, we can integrate (25). From this we can obtain $v_1(b)$ as in (19) and the expression for $c_1$ is obtained by the boundary condition B3.

## 4.3 Linear solutions.

We know in the symmetric case that linear bid functions are possible for the uniform distribution. Here we now ask what conditions are necessary for linear solutions to exist in general (for the uniform asymmetric case).

Recall our two differential equations from the first order conditions (3):

$$\begin{aligned} v_1'(b)(v_2(b)-b) &= v_1(b)-\underline{v}_1 \\ v_2'(b)(v_1(b)-b) &= v_2(b)-\underline{v}_2 \end{aligned}$$

Assume a linear solution for both inverse bid functions are as follows

$$v_i(b) = \alpha_i b + \beta_i \text{ where } \alpha_i > 0$$

This implies that

$$v_i'(b) = \alpha_i$$

Substituting this into our two equations yields

$$\begin{aligned} \alpha_1(\alpha_2 b + \beta_2 - b) &= \alpha_1 b + \beta_1 - \underline{v}_1 \\ \alpha_2(\alpha_1 b + \beta_1 - b) &= \alpha_2 b + \beta_2 - \underline{v}_2 \end{aligned}$$

Since this is true for all $b$, the derivative of both sides must also be equal. Hence,

$$\begin{aligned} \alpha_1(\alpha_2 - 1) &= \alpha_1 \\ \alpha_2(\alpha_1 - 1) &= \alpha_2 \end{aligned}$$

This implies $\alpha_1 = \alpha_2 = 2$. Substituting this into the equations yields

$$\begin{aligned} 2\beta_2 &= \beta_1 - \underline{v}_1 \\ 2\beta_1 &= \beta_2 - \underline{v}_2 \end{aligned}$$

Combining this shows that

$$\beta_1 = -\frac{1}{3}\underline{v}_1 - \frac{2}{3}\underline{v}_2$$

By boundary condition B1, $v_1(\underline{b}) = \underline{b}$, we have $\underline{b} = 2\underline{b} + \beta_1$. This implies $\beta_1 = -\underline{b}$ and $\underline{b} = \frac{1}{3}\underline{v}_1 + \frac{2}{3}\underline{v}_2$. Since $\underline{b} > (\underline{v}_1 + \underline{v}_2)/2$, it must be, by (2), that there is a binding minimum bid $m = \underline{b}$.

Now rewriting, $m = \frac{1}{3}\underline{v}_1 + \frac{2}{3}\underline{v}_2$ yields $m - \underline{v}_1 = 2(\underline{v}_2 - m)$ (or $\underline{v}_2 = \frac{3}{2}m - \frac{1}{2}\underline{v}_1$). Finally, we use the upper boundary conditions B3,

$$\overline{v}_1 = 2\overline{b} - m$$

$$\overline{v}_2 = 2\overline{b} - m/2 - \underline{v}_1/2$$

to find that $\overline{v}_1 = \overline{v}_2 + \underline{v}_1/2 - m/2$ (or $\overline{v}_1 - m = \overline{v}_2 - \underline{v}_2$). Thus, if define $z$ such that $\overline{v}_1 = m + z$, we have $\overline{v}_2 = \frac{3}{2}m + z - \underline{v}_1/2$.