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BARGAINING WITH A BUREAUCRAT

by

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Bargaining with a Bureaucrat*

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Abstract

We consider a bargaining problem where one of the players, the *bureaucrat*, has the power to dictate any outcome in a given set. The other players, the *agents*, negotiate with him which outcome to be dictated. In return, the agents transfer some part of their payoffs to the bureaucrat. We state five axioms and characterize the solutions which satisfy these axioms on a class of problems which includes as a subset all submodular bargaining problems. Every solution is characterized by a number α in the unit interval. Each agent in every bargaining problem obtains a weighted average of his individually rational level and his marginal contribution to the set of all players, where the weights are α and $1 - \alpha$, respectively. The bureaucrat obtains the remaing surplus. The solution when $\alpha = 1/2$ is the nucleolus of a naturally related game in characteristic form.

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1 Introduction

This paper considers bargaining problems between a *bureaucrat* and a few individuals (or *agents*). The bureaucrat has the power to dictate any outcome in a given set of feasible outcomes. An outcome is characterized by the payoffs it yields to the bureaucrat and to the agents. The agents may transfer some of their payoffs to the bureaucrat if he dictates a desirable outcome. The bureaucrat negotiates with the agents the outcome to be dictated and their transfers.

This setup can be applied to a broad class of problems where a decision-maker (a "bureaucrat") allocates resources among agents through rationing. For example, a bureaucrat has to make a policy choice which may benefit some lobby groups and may hurt others. It could also be applied to a patent-holder of an innovation in an oligopolistic market. The patent holder has a right to sell a licence to any subset of firms in the industry, thus increasing their competitive edge (see, e.g. Kamien and Tauman, 1986; Kamien, 1992; Kamien, Oren, and Tauman, 1992). Another application has to do with the value of information to its holder. An information holder exclusively owns a piece of information relevant to players engaged in a strategic conflict. The information holder has many ways to sophisticatedly transmit part of his information (or all of it) to some (or all) players (see Kamien, Tauman, and Zamir, 1990). He bargains with the players about the amount of information to be transmitted and the transfers to be received from the agents in return.

A solution is a mapping which associates with every bargaining problem a vector of net payoffs to all players. Indirectly, a solution determines the outcome to be dictated and the agents' transfers to the bureaucrat.

We study solutions which satisfy certain requirements (axioms). The framework resembles that of Buch and Tauman (1992) (thereafter, BT) who deal with similar bargaining problems. Their work, however, is confined to the special case where the bureaucrat has no payoff by himself, and his only source of income is the agents' transfers. The extension of BT to general bargaining problems turns out to be a nontrivial task. The BT problems do not apply, for instance, to patent licensing problems where the patent holder is an incumbent firm. Our axiomatic approach is very different from that of BT, and we argue that our solution is more appealing.

We state the following five axioms for a solution to satisfy. Our first axiom asserts that a solution should be undominated. Namely, for every subset of players including the bureaucrat, there is no outcome that makes every member of this subset strictly better off. The second axiom requires that a solution should not be affected by net payoff vectors which are dominated. That is, if two bargaining problems have the same sets of undominated payoff vectors, then they must have the same solution. The third axiom states that a solution should not depend on the unit of measurement. The fourth axiom requires that a solution should not depend on the names of the agents. The last axiom deals with bargaining problems that are composed of two independent problems with two different sets of agents. The axiom requires that in this case the net payoff of an agent should depend only on her bargaining problem.

We characterize the solutions which satisfy these five axioms on a certain class of bargaining problems, $\mathcal{X}^{SM}$. The property of every bargaining problem in $\mathcal{X}^{SM}$ is that the marginal contribution of every agent to a coalition is minimized for the grand coalition. This class includes, for instance, bargaining over a split of a cake where the bureaucrat has the exclusive power to dictate allocation, or problems involving a limited capacity technology. If some of the inputs are limited, a small coalition of players can increase its output by adding a player, more than a large coalition which has already used most of the available capacity. The special case where the marginal contribution of every agent decreases with the size of a coalition (with respect to inclusion) is the standard diminishing returns assumption. An example of such a bargaining problem is an interaction of a patent holder of a cost reducing innovation (or a quality innovation) and the firms in a oligopoly industry. The patent holder can sell licenses to use his technology to any number of firms via up-front fees, royalties, or combinations of the two. An additional licensee firm increases the total industry profit, but in a decreasing rate. The larger is the number of licenses sold, the smaller is the marginal value of an additional license.

We show that in every solution on $\mathcal{X}^{SM}$ the bureaucrat dictates an efficient outcome and every agent is awarded a weighted average of his individually rational level and his marginal contribution to the grand coalition. The bureaucrat obtains the remaining surplus. Furthermore, the weights are the same across all agents and across all bargaining problems in this class. The weights therefore can be used to measure the bargaining power of the bureaucrat. In other words, the bargaining power of the bureaucrat (and the agents) is endogenously characterized by the axioms. In fact, it is completely determined by the simple one agent problem, where the bureaucrat can dictate one out of two outcomes: In both cases the bureaucrat by himself can get only zero, and the agent can get one or zero, depending on the decision of the bureaucrat. Namely, the only source of income of the bureaucrat is the transfer that he obtains from the agent. This can be regarded as a symmetric problem: The bureaucrat and the agent can each achieve zero by themselves but could obtain one together. If the solution of this specific problem is symmetric, where the bureaucrat and the agent split the unit equally, then the weights are equal. That is, the solution of *every* bargaining problem with any number of agents awards every agent the simple average of his individually rational level and his marginal contribution to the grand coalition. It is shown that on $\mathcal{X}^{SM}$ this solution coincides with the *nucleolus* (Schmeidler, 1969) of a naturally related game in a characteristic form.

As for the general class of bargaining problems, we find that a solution which satisfy our five axioms is not unique. We construct a natural extension of our solution, which is based on a weighted average of the lexicographically maximal payoff vectors to players with respect to some random order of their locations on the unit interval. However, the nucleolus is also a solution which is different from the above.

A closely related work, but in a noncooperative setup, is Bernheim and Whinston (1986) (thereafter, BW). In BW every agent submits to the bureaucrat ("auctioneer" in their framework) a contingent plan ("menu") which specifies the transfer of the agent to the bureaucrat as a function of the dictated outcome. The contingent plans are selected simultaneously, and after observing these plans the bureaucrat dictates an outcome. The truthful[1] Nash equilibrium is the focus of BW. It turns out that on every problem in $\mathcal{X}^{SM}$ there is a unique truthful Nash equilibrium. This is one of our solutions, the extreme one, where the bargaining power of the bureaucrat is minimal. Namely, in the truthful Nash equilibrium every agent obtains his marginal contribution to the grand coalition (assigning zero weight to his individually rational level) and the remaining surplus goes to the bureaucrat.

2 Notations and Definitions

Consider a set of players $N^0 = N \cup \{0\}$, where $N = \{1, 2, \ldots, n\}$ is a set of *agents* and 0 is a *bureaucrat*. The players in N^0 are engaged in a bargaining problem. Let $X \subset \mathbb{R}_+^{N^0}$ be the set of all possible outcomes of this bargaining problem, where every outcome x in X is a *gross payoff* vector for the players in N^0. The bureaucrat (and only the bureaucrat) has the ability to dictate any outcome in X. The agents in N bargain with the bureaucrat about the outcome to be dictated and, as a result, transfer to the bureaucrat some parts of their gross payoffs. Thus, the bargaining is on both: the outcome

[1] A truthful strategy of an agent in BW is a contingent plan which is characterized by a real number x. The transfer to the bureaucrat is the difference between the payoff of the agent and x, as long as this difference is positive; otherwise, the transfer is zero. A truthful Nash equilibrium is a subgame perfect equilibrium of the game where every agent plays a truthful strategy.

in X and the transfers of the agents. It is assumed that only agreements with the bureaucrat are enforceable, and agents are not allowed to transfer payoffs from one to another.

For any subset $S \subset N$ let $S^0 = S \cup \{0\}$. The number of elements of S will be denoted by s.

An $(n+1)$-player bargaining problem is a pair (N^0, X) where $|N| = n$ and $N^0 = N \cup \{0\}$ is the set of players, and X is the set of outcomes, which is a nonempty and compact subset of $\mathbb{R}_+^{N^0}$. Denote by $\mathcal{X}_{n+1}$ the *class of all $(n+1)$-player bargaining problems.* Let $\mathcal{X} = \bigcup_{k=1}^{\infty} \mathcal{X}_k$. For convenience, we often identify $\mathcal{X}$ with the set of all X such that $(N^0, X) \in \mathcal{X}$.

For $X \in \mathcal{X}$, suppose that an outcome $x \in X$, $x = (x_0, x_1, \ldots, x_n)$, is dictated. Then every agent $i \in N$ obtains the gross payoff x_i and pays z_i, $0 \leq z_i \leq x_i$, to the bureaucrat, thus receiving the *net payoff* $y_i = x_i - z_i$. The bureaucrat receives the *net payoff* $y_0 = x_0 + \sum_{i \in N} z_i$. Let $y = (y_0, y_1, \ldots, y_n)$.

It is important to note that the bureaucrat must select an outcome in X no matter if he reaches an agreement with agents or not.

Definition Let $(N^0, X) \in \mathcal{X}$. An outcome $x^* \in X$ is *efficient for* $S^0 \subset N^0$ if

$$\sum_{i \in S^0} x_i^* = \max_{x \in X} \sum_{i \in S^0} x_i.$$

It is called *efficient* if it is efficient for N^0.

Let $S^0 \subset N^0$. Denote

$$E_{S^0}(X) = \{x \in X \mid x \text{ is efficient for } S^0\}$$

and let

$$E(X) = E_{N^0}(X).$$

By the *individually rational level* of a player $i \in N^0$, we shall understand the gross payoff in X that i can *guarantee* if she decides not to participate in negotiations. Formally, let d be a map which associates with every problem $(N^0, X) \in \mathcal{X}$ a payoff vector in $\mathbb{R}_+^{N^0}$. We shall call $d_i(X)$ the individually rational level of $i \in N^0$ with respect to X.

One possible definition for the individually rational level of the bureaucrat is the greatest gross payoff that he can attain on his own,

$$\bar{d}_0(X) = \max \{x_0 | x \in X\}. \tag{1}$$

As for the individually rational levels for the agents, suppose that whenever the bureaucrat reaches an agreement with some set of agents $S \subset N$ (possibly, $S = \varnothing$), he will dictate an outcome which is efficient for S^0. Thus, the individually rational level of $i \in N$ is the gross payoff that agent i is guaranteed to obtain among all outcomes that are efficient for all subsets $S^0 \subset N^0 - i$. Formally, for every $i \in N$ let

$$\bar{d}_i(X) = \min_{S \subset N - i} \left(\min_{x \in E_{S^0}(X)} x_i \right). \tag{2}$$

We assume that (1) and (2) are upper bounds on the individually rational levels of the players, that is, for every $(N^0, X) \in \mathcal{X}$ and every $i \in N^0$

$$d_i(X) \leq \bar{d}_i(X). \tag{3}$$

The statement of our result will not be affected by a particular choice of d, as long as it satisfies (3).

Definition Let $(N^0, X) \in \mathcal{X}$. A net payoff vector $y = (y_0, y_1, \ldots, y_n)$ is *feasible for* $S^0 \subset N^0$ *at* $x \in X$ if

(i) $y_i \geq d_i(X)$ for every $i \in S^0$

(ii) $y_i \leq x_i$ for every $i \in S$ and $y_j = x_j$ for every $j \in N - S$,

(iii) $\sum_{i \in S^0} y_i = \sum_{i \in S^0} x_i$.

A net payoff vector y is *feasible for* S^0 if it is feasible for S^0 at some $x \in X$. Clearly, if y is feasible for S^0, it is feasible for T^0 for all $N \supseteq T \supseteq S$. A net payoff vector y is *feasible* if it is feasible for N^0.

Condition (i) requires that every player in S^0 obtains at least his individually rational level; (ii) requires that only transfers *from the agents in S to the bureaucrat* are allowed (and agents not in S obtain their gross payoffs); condition (iii) requires that the total payoff of S^0 obtained from an outcome x is distributed entirely among the players in S^0, i.e., nothing is transferred to an outside party or wasted.

Let $(N^0, X) \in \mathcal{X}$ and $x \in X$. Denote by $Y(x)$ the set of net payoff vectors which are feasible at x and let $Y(X)$ be the set of net payoff vectors which are feasible for X, i.e., $Y(X) = \bigcup_{x \in X} Y(x)$.

3 Stability

Let (N^0, X) be a bargaining problem in $\mathcal{X}$.

Definition Let $y, y' \in Y(X)$. We say that y' *dominates* y *via* $S^0 \subset N^0$ if y' is feasible for S^0 and $y'_i > y_i$ for all $i \in S^0$.

Definition A payoff vector $y \in Y(X)$ is *stable* if it is undominated, that is, if for every $S^0 \subset N^0$ there is no $y' \in Y(X)$ which dominates y via S^0.

In other words, a payoff vector y is stable if the bureaucrat cannot find a subset S of agents and a feasible payoff vector y' for S^0 so that everyone in S^0 is strictly better off.

Definition A payoff vector $y \in Y(X)$ is *efficient* if it is feasible at some efficient outcome in X, i.e., if there is $x^* \in E(X)$ such that $y \in Y(x^*)$.

Proposition 1 *Let $(N^0, X) \in \mathcal{X}$. A payoff vector $y \in Y(X)$ is stable if and only if for every $S^0 \subset N^0$*

$$\sum_{i \in S^0} y_i \geq \max_{x \in X} \sum_{i \in S^0} x_i.$$

Proof. Let $y \in Y(X)$ be non-stable, that is, there is $S \subset N$ and y' feasible for S^0 such that $y_i < y'_i$ for all $i \in S^0$. Hence, there is $x \in X$ such that

$$\sum_{i \in S^0} y_i < \sum_{i \in S^0} y'_i = \sum_{i \in S^0} x_i.$$

Conversely, let $y \in Y(X)$ be stable. Suppose to the contrary that

$$\sum_{i \in S^0} y_i < \sum_{i \in S^0} \hat{x}_i \tag{4}$$

for some $S^0 \subset N^0$ and some $\hat{x} \in E_{S^0}(X)$. Let $T = \{j \in S^0 \mid y_j < \hat{x}_j\}$. Clearly, $T \neq \varnothing$ and $0 \notin T$ (if $y_0 < \hat{x}_0$, then y is dominated by $y' \in \operatorname*{argmax}_{x \in X} x_0$ for $\{0\}$) and define $z \in \mathbb{R}^{N^0}_+$ by

$$z_j = \begin{cases} y_j + \varepsilon, & j \in T, \\ \hat{x}_j, & j \in N - T, \\ \hat{x}_0 + \sum_{j \in S}(\hat{x}_j - y_j - \varepsilon), & j = 0. \end{cases}$$

Since $d_j(X) \leq z_j \leq \hat{x}_j$ for all $j \in T$ and ε small enough, and $\sum_{j \in T^0} z_j = \sum_{j \in T^0} \hat{x}_j$, z is feasible for T^0 at $\hat{x}$. But $z_j > y_j$ for all $j \in T$, and by (4) we have for ε small enough

$$z_0 = \hat{x}_0 + \sum_{j \in S}(\hat{x}_j - y_j) - |S|\,\varepsilon > y_0.$$

Hence, y is dominated by z for T^0, a contradiction. ∎

Corollary 1 *If $y \in Y(X)$ is stable, then it is efficient.*

Note that if y is stable, then $y_0 \geq \max\{x_0 | x \in X\} = \bar{d}_0(X)$, namely, y is also individually rational for the bureaucrat.

Denote by $ST(X)$ the set of all payoff vectors which are stable for X. An immediate consequence of Proposition 1 is the following characterization of $ST(X)$.

Proposition 2 *Let $(N^0, X) \in \mathcal{X}$ and $y \in \mathbb{R}^{N^0}$. Then $y \in ST(X)$ if and only if*

(i) $\sum_{i \in N^0} y_i = \max_{x \in X} \sum_{i \in N^0} x_i,$

(ii) $\sum_{i \in S^0} y_i \geq \max_{x \in X} \sum_{i \in S^0} x_i$ *for all* $S^0 \subset N^0$, *and*

(iii) $y_i \geq d_i(X)$ *for all* $i \in N^0$.

Note that (i) is implied by (ii) for every $y \in Y(X)$ (see Corollary 1).

Proof. If $y \in ST(X)$, then (i) and (ii) are satisfied by Proposition 2 and (iii) is satisfied because $y \in Y(X)$. Conversely, if $y \in Y(X)$ satisfies (i) – (iii), then $y \in ST(X)$ by Proposition 2. The only part which is left to prove is that $y \in Y(X)$ if it satisfies (i) – (iii). Let $x^* \in E(X)$ and $x^{N^0-i} \in E_{N^0-i}(X)$. By (i) and (ii), for all $i \in N$,

$$\begin{aligned} y_i &= \sum_{j \in N^0} x_j^* - \sum_{j \in N^0-i} y_j \leq \sum_{j \in N^0} x_j^* - \sum_{j \in N^0-i} x_j^{N^0-i} \\ &= x_i^* + \sum_{j \in N^0-i} x_j^* - \sum_{j \in N^0-i} x_j^{N^0-i} \leq x_i^*. \end{aligned}$$

Hence, $y \in Y(x^*) \subset Y(X)$. ■

Proposition 3 *For every* $X \in \mathcal{X}$, $ST(X)$ *is nonempty.*

Proof. Define

$$z_j = \begin{cases} d_j(X), & j \in N, \\ \sum_{i \in N^0} x_i^* - \sum_{i \in N} d_i(X), & j = 0. \end{cases}$$

Let $x^* \in E(X)$, $S^0 \subset N^0$, and $\hat{x} \in E_{S^0}(X)$. Then

$$\sum_{j \in S^0} z_j = \sum_{i \in N^0} x_i^* - \sum_{j \in N-S} d_j(X) \geq \sum_{j \in N^0} \hat{x}_j - \sum_{j \in N-S} d_j(X).$$

By definition, $\bar{d}_j(X) \leq \hat{x}_j$, and by assumption, $d_j(X) \leq \bar{d}_j(X)$, $j \in N - S$, hence,

$$\sum_{j \in S^0} z_j \geq \sum_{j \in S^0} \hat{x}_j + \sum_{j \in N-S} (\hat{x}_j - d_j(X)) \geq \sum_{j \in S^0} \hat{x}_j,$$

and, thus, z satisfies conditions (i) and (ii) of Proposition 2. Note that these conditions imply that $z_0 \geq \bar{d}_0(X) \geq d_0(X)$, hence z also satisfies condition (iii) of Proposition 2. ■

Corollary 2 *For every $X \in \mathcal{X}$ the set $ST(X)$ is nonempty, compact, and convex.*

The proof follows from Propositions 2 and 3.

4 An Axiomatic Approach

Let $\mathcal{X}'$ be any subset of $\mathcal{X}$. In this section we define a solution on $\mathcal{X}'$ and present five axioms for a solution to satisfy.

Definition A *solution on $\mathcal{X}'$* is a mapping, ϕ, which associates with every problem (N^0, X) in $\mathcal{X}'$ a payoff vector $\phi(X)$ in $Y(X)$.

We impose the following five axioms on ϕ.

The first axiom requires that a solution of every problem is stable.

Axiom 1 (Stability) *For every $(N^0, X) \in \mathcal{X}'$, $\phi(X) \in ST(X)$.*

This assumes that the bureaucrat will reject a payoff vector y if he can reach another settlement y' with some subset of agents $S \subset N$ such that every member of S^0 is strictly better off with y' than with y.

The second axiom asserts that only stable net payoff vectors are relevant for the solution. That is, any net payoff vector which is not stable is considered note to be a credible settlement for the bureaucrat, thus it should be ignored in negotiations.

Axiom 2 (Stability Dependence (STD)) *For every (N^0, X) and (N^0, X') in $\mathcal{X}'$, if $ST(X) = ST(X')$, then $\phi(X) = \phi(X')$.*

Next, we require that a solution does not depend on the unit of measurement.

Axiom 3 (Scale Covariance) *For every* $(N^0, X) \in \mathcal{X}'$*, every* $b = (b_0, b_1, \ldots, b_n) \in \mathbb{R}^{N^0}$*, and every scalar* $c > 0$*, if* $(N^0, cX + b) \in \mathcal{X}'$*, then*

$$\phi(cX + b) = c\phi(X) + b.$$

The next axiom requires that a solution does not depend on the names of the agents. Let $(N^0, X) \in \mathcal{X}'$ and let π be a permutation of $N = \{1, \ldots, n\}$. For every $x \in \mathbb{R}^n$, let $\pi x \in \mathbb{R}^n$ be such that $(\pi x)_i = x_{\pi(i)}$ for all $i \in N$ and let $\pi X = \{\pi x \mid x \in X\}$.

Axiom 4 (Anonimity) *Suppose that* $(N^0, X) \in \mathcal{X}'$*. For every permutation* π *of* N*, if* $(N^0, \pi X) \in \mathcal{X}'$*, then*

$$\phi_i(X) = \phi_{\pi(i)}(\pi X), \quad i \in N.$$

Finally, we require that in a solution the agents' payoffs are not affected if an independent (payoff-orthogonal) agent is added to the bargaining problem.

Axiom 5 (Separability) *Let* $(N^0, X) \in \mathcal{X}'$*, where* $N^0 = \{0, 1, \ldots, n\}$*. Denote* $N' = N^0 \cup \{n+1\}$ *and* $X' = X \times [a, b]$*,* $0 \leq a \leq b$*, If* $(N', X') \in \mathcal{X}'$*, then* $\phi_i(X') = \phi_i(X)$ *for all* $1 \leq i \leq n$*.*

Proposition 4 *Axioms 1 – 5 are independent on* $\mathcal{X}$*.*

Proof. Appears in the Appendix. ■

5 Related Games in Characteristic Form

A game (N^0, V) in characteristic (or coalitional) form consists of the set N^0 of players and a function $V : 2^{N^0} \to \mathbb{R}$ such that $V(\varnothing) = 0$. Every $S \subset N^0$ is called a coalition and N^0 is called the grand coalition. A game (N^0, V) is monotonic if $V(S) \leq V(T)$ for all $S \subset T \subset N^0$.

Denote by $\mathcal{G}$ the class of all monotonic games (N^0, V) where 0 is a *veto player*. That is, for every $S \subset N^0$, if $S \not\ni 0$ then $V(S) = 0$.

Two games, (N^0, V) and (N^0, V'), are called *strategically equivalent* if there exist numbers $a > 0$, $b_0, \ldots, b_n$, such that $V'(S) = aV(S) + \sum_{i \in S} b_i$ for all $S \subset N^0$.

The *core* of (N^0, V) is denoted by $\mathcal{C}_V$ and is defined to be the set of all $x \in \mathbb{R}^{N^0}$ such that $\sum_{i \in S} x_i \geq V(S)$ for all $S \subset N^0$ and $\sum_{i \in N^0} x_i = V(N^0)$.

Let (N^0, X) be a bargaining problem in $\mathcal{X}$. We associate with X the game V_X in characteristic form, for which the worth of every coalition S is the highest total payoff it can *guarantee* to its members,

$$V_X(S) = \begin{cases} \max\limits_{x \in X} \sum\limits_{i \in S} x_i, & S \ni 0, \\ \sum\limits_{i \in S} d_i(X), & S \not\ni 0. \end{cases} \tag{5}$$

Proposition 5 *For every $(N^0, X) \in \mathcal{X}$,*

(i) *the game (N^0, V_X) is strategically equivalent to a game in $\mathcal{G}$,*

(ii) $\mathcal{C}_{V_X} = ST(X) \neq \varnothing$.

Proof. Part (i) follows from (5) and the definition of $\mathcal{G}$. Part (ii) is an immediate consequence of Propositions 2 and 3. ■

6 Submodular Bargaining Problems

In this section we deal with bargaining problems where the marginal contribution of every agent $i \in N$ to a coalition $S \ni i$ is the smallest for $S = N$. Formally, let $(N^0, X) \in \mathcal{X}$ and for every $i \in N$ and every $S^0 \ni i$, denote

$$MC_i(S^0, X) = V_X(S^0) - V_X(S^0 - i).$$

Let $\mathcal{X}^{SM}$ be the set of all bargaining problems (N^0, X) such that for all

$S \subset N$ and all $i \in S$

$$MC_i(N^0, X) \leq MC_i(S^0, X). \tag{6}$$

Bargaining problems where the smallest marginal contribution of a player is to the grand coalition include splitting a cake, awarding licenses where the marginal value of an additional license decreases with the number of licenses, and the problems involving a limited capacity technology. The special case where the marginal contribution decreases with the size of a coalition (with respect to inclusion) is the standard diminishing returns assumption. Bargaining problems (N^0, X) for which $MC_i(S^0, X)$ is monotonically decreasing in S, namely,

$$MC_i(S^0, X) \leq MC_i(T^0, X),$$

for all $i \in N$ and all $S \supset T \ni i$, are called submodular (SM). Obviously, $\mathcal{X}^{SM}$ contains among others all the submodular bargaining problems.

We next characterize the solution on $\mathcal{X}^{SM}$ which satisfies the above five axioms.

Theorem 1 *A solution ϕ on $\mathcal{X}^{SM}$ satisfies Axioms 1 – 5 if and only if there exists α, $0 \leq \alpha \leq 1$, such that for all (N^0, X) in $\mathcal{X}^{SM}$*

$$\phi_i(X) = \phi_i^\alpha(X) = \alpha d_i(X) + (1-\alpha) MC_i(N^0, X) \quad \text{for all } i \in N, \tag{7}$$

$$\phi_0(X) = \phi_0^\alpha(X) = \max_{x \in X} \sum_{i \in N^0} x_i - \sum_{i \in N} \phi_i(X). \tag{8}$$

Proof. Appears in the Appendix. ■

The solution of every bargaining problem in $\mathcal{X}^{SM}$ awards every agent in N a weighted average of her individually rational level and her marginal contribution to the grand coalition. The bureaucrat extracts the remaining surplus. The weights, $(\alpha, 1-\alpha)$, are the same across all agents and across all bargaining problems in $\mathcal{X}^{SM}$. Thus, it is sufficient to determine α for one bargaining problem. The same α then applies to all bargaining problems in $\mathcal{X}^{SM}$. The parameter α measures the bargaining power of the bureaucrat: The larger is α, the larger is the payoff of the bureaucrat.

Example. Consider the following one-agent bargaining problem $\hat{X}_2 = \{(0,x) \in \mathbb{R}^2_+ \mid 0 \leq x \leq 1\}$. The bureaucrat and the agent, each can guarantee 0 on his own, and together they can achieve 1. By Theorem 1,

$$\begin{aligned} \phi_0^\alpha(\hat{X}_2) &= \alpha, \\ \phi_1^\alpha(\hat{X}_2) &= 1-\alpha. \end{aligned}$$

The theorem asserts that the bargaining power of the bureaucrat is completely determined by this simple bargaining problem. If for this example $\alpha = 1$, then the bureaucrat obtains the entire surplus of every bargaining problem, leaving the agents only with their individually rational levels. On the other hand, if for this example $\alpha = 0$, every agent in every bargaining problem in $\mathcal{X}^{SM}$ obtains his marginal contribution to the grand coalition, while the bureaucrat collects the smallest payoff in $ST(X)$. In $\hat{X}_2$ the bureaucrat and the agent may be regarded as symmetric players: Each can obtain zero by himself and together they can obtain one. Therefore, $\alpha = \frac{1}{2}$ could be regarded as a proper division of the surplus. In this case, by Theorem 1, $\alpha = \frac{1}{2}$ for all problems in $\mathcal{X}^{SM}$. The proposition below shows that, for all $X \in \mathcal{X}^{SM}$, $\phi^{1/2}(X)$ is actually the *nucleolus* of V_X.

Let (N^0, V) be a game in characteristic form. Denote by I_V the set of imputations of V,

$$I_V = \left\{ x \in \mathbb{R}^{N^0} \;\middle|\; \begin{array}{l} \sum_{i \in N^0} x_i = V(N^0), \\ x_i \geq V(i), \text{ all } i \in N^0. \end{array} \right\}.$$

The nucleolus of V is defined as follows (Schmeidler, 1969). For every nonempty $S \subsetneq N^0$ and every $y \in I_V$ define the excess of coalition S by

$$e_V(S,y) = V(S) - \sum_{j \in S} y_j. \tag{9}$$

Given $y \in I_V$ define the excess vector $\theta(y) \in \mathbb{R}^{2^{N^0}-2}$ whose components are the excesses $e_V(S, y)$, $S \neq N^0$ and $S \neq \varnothing$, arranged in an increasing order. The *nucleolus* of the game is the set of payoff vectors $\mathcal{N}_V \subset I_V$ which lexicographically minimizes $\theta(y)$ over y. The nucleolus is a singleton and it is in the core of V if the core is nonempty (Schmeidler, 1969).

Proposition 6 *The solution $\phi^{1/2}$ on $\mathcal{X}^{SM}$ is the nucleolus of V_X for every (N^0, X) in $\mathcal{X}^{SM}$.*

Proof. Appears in the Appendix. ■

7 Properties of the Solution

We discuss here some additional properties of the solution on $\mathcal{X}^{SM}$.

Property 1 (Dummy Agent) In $X \in \mathcal{X}^{SM}$ agent i is *dummy* if there is constant $c \geq 0$ such that $x_i = c$ for all $x \in X$. Then for every α, $0 \leq \alpha \leq 1$,

$$\phi_i^\alpha(X) = c.$$

Property 2 (Additivity) For every α, $0 \leq \alpha \leq 1$, the solution ϕ^α is additive on $\mathcal{X}^{SM}$. Namely, for (N^0, X) and (N^0, X') in $\mathcal{X}^{SM}$

$$\phi^\alpha(X + X') = \phi^\alpha(X) + \phi^\alpha(X').$$

This property follows from Theorem 1 and the additivity of V_X and $d_i(X)$ on $\mathcal{X}^{SM}$. Since $\mathcal{X}^{SM}$ is a cone, Proposition 6 implies that the nucleolus is additive on the class of games V_X, $X \in \mathcal{X}^{SM}$. Consequently, both the Shapley value and the nucleolus satisfy the Shapley axioms (Shapley, 1953) on this class of games. Note that the Shapley value does not constitute a solution $\mathcal{X}^{SM}$ since it violates stability. The nucleolus, on the other hand, is always an element of the core of V_X.

Next, we show that ϕ^α satisfies the consistency property.

Definition Let $X \in \mathcal{X}^{SM}$ and let $S \subset N$. The problem $X_{-S}(\phi)$ is called a *reduced problem by* S with respect to a solution ϕ on $\mathcal{X}^{SM}$ if

$$X_{-S}(\phi) = \left\{ x' \in \mathbb{R}_+^{N^0-S} \;\middle|\; \begin{array}{l} x'_i = x_i,\ i \in N - S, \\ x'_0 = x_0 + \sum_{j \in S} \left(x_j - \phi_j(X)\right), \\ x \in X. \end{array} \right\} \quad (10)$$

Property 3 (Consistency) For every $(N^0, X) \in \mathcal{X}^{SM}$, every $S \subset N$, and every $i \in N^0 - S$

$$\phi_i(X_{-S}(\phi)) = \phi_i(X).$$

Proof See the Appendix.

This consistency notion is due to Sobolev (1966).

8 Discussion of the General Case

In this section we extend the solution on $\mathcal{X}^{SM}$ to a solution on the general class of bargaining problems $\mathcal{X}$. First note that for $\alpha < 1$ the solution ϕ^α defined in Theorem 1 does not satisfy the Stability axiom for some problems in $\mathcal{X}$. Let $N^0 = \{0, 1, \ldots, n\}$ and let $X_n = \{(0, 1, \ldots, 1), (n-1, 0, \ldots, 0)\}$. Clearly, $(N^0, X_n) \notin \mathcal{X}^{MG}$. Note that $d_i(X_n) = 0$ and $MC_i(N^0, X) = 1$ for each $i \in N$. Hence, $\phi_i^\alpha(X) = 1 - \alpha$ for all $i \in N$ and $\phi_0^\alpha(X) = n\alpha$. Thus, whenever $n - 1 > n\alpha$ (or, equivalently, $\frac{n-1}{n} > \alpha$) $\phi^\alpha(X_n)$ is not stable. It can be easily verified that $\phi^{\alpha=1}$ (the solution where the bureaucrat has the entire bargaining power) is a solution on $\mathcal{X}$ which satisfies the five axioms.

Let $\mathcal{R}$ be the set of all orders of the players in N^0 and let $\mathcal{R}_j$ be the set of all orders where the bureaucrat is located in the j's place, $j = 1, \ldots, n+1$. For every $(N^0, X) \in \mathcal{X}$ and for every order $R \in \mathcal{R}$ let $\phi^R(X)$ be the lexicographically maximal element in $ST(X)$ with respect to the order R.

Notice that, by Proposition 2, for every problem $(N^0, X) \in \mathcal{X}^{SM}$, $\phi_i^R(X) = MC_i(N^0, X)$ if i precedes the bureaucrat in the order R and otherwise $\phi_i^R(X) = d_i(X)$. That is, for every $(N^0, X) \in \mathcal{X}^{SM}$, the solution $\phi^\alpha(X)$ is an average of the lexicographically maximal elements in $ST(X)$. The average is taken with respect to the distribution on $\mathcal{R}$ where the probability of every order in $\mathcal{R}_1$ is $\frac{\alpha}{n!}$ and the probability of every order in $\mathcal{R}_{n+1}$ is $\frac{1-\alpha}{n!}$. Every order in $\mathcal{R}_j$, $2 \leq j \leq n$, is selected with zero probability. Hence, we can rewrite $\phi^\alpha(X)$ for every $\mathcal{X}^{SM}$ as follows,

$$\phi^\alpha(X) = \frac{\alpha}{n!} \sum_{R_1 \in \mathcal{R}_1} \phi^{R_1}(X) + \frac{1-\alpha}{n!} \sum_{R_{n+1} \in \mathcal{R}_{n+1}} \phi^{R_{n+1}}(X). \tag{11}$$

This distribution on $\mathcal{R}$ has two properties: The probability, α, that the bureaucrat precedes an agent (i) is the same for all agents and (ii) it does not depend on the number n of agents.

We extend this idea to define solutions on $\mathcal{X}$. We will take the average of the ϕ^R's with respect to any probability distribution on $\mathcal{R}$ which satisfies the above two properties. These probability distributions are characterized as follows. The players in N^0 will be randomly located on $[0,1]$. The location of the bureaucrat is selected according to a measure μ on $[0,1]$ and the location of every agent in N is selected uniformly on $[0,1]$. The players' locations are selected independently. Every realization of locations defines an order $R \in \mathcal{R}$ of the players in N^0 and thus determines ϕ^R. Note that $\alpha = \int_0^1 t d\mu$ is the probability that an agent is preceded by the bureaucrat.

Define now a solution ϕ^μ on $\mathcal{X}$ as follows. For every $(N^0, X) \in \mathcal{X}$

$$\phi^\mu(X) = \sum_{R \in \mathcal{R}} \hat{\mu}(R) \phi^R(X), \tag{12}$$

where $\hat{\mu}(R)$ is the probability of an order $R \in \mathcal{R}$, given μ. By definition, the operator ϕ^R, and hence ϕ^μ, satisfies the axioms of Stability and STD. Clearly, ϕ^μ also satisfies the Anonymity axiom. The proof that ϕ^μ satisfies the other two axioms is similar to that for ϕ^α on $\mathcal{X}^{SM}$. We summarize the above in the next theorem.

Theorem 2 *For every measure μ on $[0,1]$ the solution ϕ^μ on $\mathcal{X}$ satisfies Axioms 1 – 5.*

Remark 1. Generally, the Shapley value of V_X is not in $\mathcal{C}_{V_X}$ and, therefore, by Proposition 5, it violates the Stability axiom. To illustrate this, consider the following example. Let $N^0 = \{0,1,2\}$ and $X^\varepsilon = \{(\varepsilon,0,0),(0,1,0),(0,0,1)\}$, where $0 \le \varepsilon \le 1$. Here, $ST(X^\varepsilon) = \{(1,0,0)\}$. It is easy to verify that the Shapley value of V_{X^ε} is $\left(\frac{2+\varepsilon}{3}, \frac{1-\varepsilon}{6}, \frac{1-\varepsilon}{6}\right) \notin ST(X^\varepsilon)$ for every $0 \le \varepsilon < 1$.

Remark 2. The solution $\phi^{\mathcal{N}}$ which is defined for every (N^0, X) as the nucleolus of V_X satisfies axioms 1 – 5, thus being an alternative to the solutions ϕ^μ described above. Axioms 1 and 3 – 5 are trivially satisfied by $\phi^{\mathcal{N}}$. The only nontrivial part is the STD axiom, since generally the nucleolus is not a function of the core (see Maschler, Peleg, and Shapley, 1979). But it turns out that on $\mathcal{G}$ the nucleolus *is* a function of the core and thus satisfies the STD axiom as well.

Proposition 7 (Arin and Feltkamp (1997)) *Let (N^0, V) and (N^0, W) be two games in $\mathcal{G}$ with the same core. Then their nucleoli coincide.*

Example. Consider a problem with $n+1$ players, where there are two groups of symmetric agents, N_1 and N_2, of the size n_1 and n_2 respectively, $n_1 + n_2 = n$. Suppose that there are two outcomes: outcome 1 grants every agent in N_1 the payoff of a/n_1 and the rest obtain zero; outcome 2 grants every agent in N_2 the payoff of b/n_2 and the rest obtain zero. Assume that $a \ge b \ge 0$. That is,

$$X = \left\{ \left(0, \frac{a}{n_1}, \ldots, \frac{a}{n_1}, 0, \ldots, 0\right), \left(0, 0, \ldots, 0, \frac{b}{n_2}, \ldots, \frac{b}{n_2}\right) \right\}.$$

This bargaining problem is not in $\mathcal{X}^{SM}$ for $a > b$. Consider three solutions: the nucleolus, $\phi^{\mathcal{N}}$, our solution with Lebesgue measure, ϕ^L, and our solution ϕ^0 where the bureaucrat's bargaining power is minimized, namely, the solution with the measure which assigns the entire mass to 1, that is, only the orders where bureaucrat is the last are considered. Then in all solutions agents in N_2 obtain zero, and every agent $i \in N_1$ obtains

$$
\begin{aligned}
\phi_i^{\mathcal{N}}(X) &= \begin{cases} \frac{a-b}{n_1+1}, & b \geq \frac{a}{2}\left(1-\frac{1}{n_1}\right), \\ \frac{a}{2n_1}, & \text{otherwise} \end{cases} \\
\phi_i^L(X) &= \frac{a-b}{n_1+1}\left(1+\frac{1}{2n_1}-\frac{a-b}{2a}\right), \\
\phi_i^0(X) &= \frac{a-b}{n_1}.
\end{aligned}
$$

Note that for every $i \in N_1$ if $0 < b < a\left(1-\frac{1}{n_1}\right)$, then $\phi_i^0(X) > \phi_i^{\mathcal{N}}(X) > \phi_i^L(X)$. On the other hand, if $a\left(1-\frac{1}{n_1}\right) < b < a$, then $\phi_i^0(X) > \phi_i^L(X) > \phi_i^{\mathcal{N}}(X)$ and if $a = b$, then $\phi_i^0(X) = \phi_i^L(X) = \phi_i^{\mathcal{N}}(X) = 0$.

9 A Comparison with the Buch-Tauman Model

Let $\mathcal{X}^0 \subset \mathcal{X}$ be the class of bargaining problems where for every $X \in \mathcal{X}^0$, $x_0 = 0$ for all $x \in X$, that is, the bureaucrat obtains a constant (zero) gross payoff in all outcomes.

Buch and Tauman (1992) (thereafter, BT) deal only with problems in $\mathcal{X}^0$. By an axiomatic approach BT find a unique solution, ϕ^{BT}. Let $(N^0, X) \in \mathcal{X}^0$. BT defines the individually rational level of an agent $i \in N$ by

$$\delta_i(X) = \min_{x \in E_{N^0-i}(X)} x_i.$$

That is, $\delta_i(X)$ is the payoff that i can guarantee if he *unilaterally* leaves the bargaining table (and provided that the ruler induces an efficient outcome for $N^0 - i$). They proved that

$$
\begin{aligned}
\phi_i^{BT}(X) &= \delta_i(X) \quad \text{for all} \quad i \in N, \\
\phi_0^{BT}(X) &= V_X(N^0) - \sum_{i \in N} \delta_i(X).
\end{aligned}
$$

Namely, each agent receives his individually rational level, and the ruler obtains the surplus. Note that the solution ϕ^{BT} coincides with ϕ^{α} for $\alpha = 1$ on bargaining problems (N^0, X) such that $\delta_i(X) = d_i(X)$ for all $i \in N$. The BT axiomatic approach omits the stability and STD axioms and instead it imposes the well known axiom of independence of irrelevant alternatives. We demonstrate the difference between the BT approach and ours by the following two examples.

Example 1. Let $N^0 = \{0, 1, 2\}$ and $X = \{0\} \times [0, 1] \times [0, 1]$. That is, the bureaucrat may dictate any gross payoff in $[0, 1]$ for agent i independently of agent j, $i, j = 1, 2$, $i \neq j$. Then $\phi_0^{BT}(X) = 2$, $\phi_1^{BT}(X) = \phi_2^{BT}(X) = 0$, that is, both agents end up with zero net payoff. This is reasonable only if the bureaucrat has the entire bargaining power. Our solution varies with the bargaining power of the bureaucrat α, $0 \leq \alpha \leq 1$. It is $\phi_0^{\alpha}(X) = 2\alpha$, $\phi_1^{\alpha}(X) = \phi_2^{\alpha}(X) = 1 - \alpha$. In a special case when all players have the same bargaining power, namely, $\alpha = 1/2$, the solution is $\left(1, \frac{1}{2}, \frac{1}{2}\right)$.

Example 2. Let $N^0 = \{0, 1, 2\}$ and $X = \{(0, 0, 0), (0, 1, 1)\}$. The BT solution is $\phi_0^{BT}(X) = 0$, $\phi_1^{BT}(X) = \phi_2^{BT}(X) = 1$, though it is a credible threat of the bureaucrat to dictate $(0, 0, 0)$. This is reasonable only if the bureaucrat has no bargaining power, the opposite extreme to Example 1. Our solution yields $\phi_0^{\alpha}(X) = 2\alpha$, $\phi_1^{\alpha}(X) = \phi_2^{\alpha}(X) = 1 - \alpha$, the same as in Example 1.

10 Conclusion

In this paper we provide solutions to general bargaining problems with bureaucrats. We impose five axioms and construct solutions which satisfy these axioms. On a specific class of bargaining problems, $\mathcal{X}^{SM}$, we fully characterize the solution satisfying the five axioms. It assigns every agent an average of her individually rational level and her marginal contribution to the other players. The weights defining this average are the same for all agents and for all problems in $\mathcal{X}^{SM}$. Thus, they can be used to measure the bargaining power of the bureaucrat. The higher is the weight assigned to the individually rational level of an agent, the higher is the bargaining power of the bureaucrat. When he has the full bargaining power, every agent obtains her individually rational level only, and the bureaucrat, who dictates an efficient outcome, obtains the rest of the "cake". If the bureaucrat has no bargaining power, every agent obtains her marginal contribution.

We provide various solutions on the general class of bargaining problems $\mathcal{X}$ which satisfy Axioms 1 – 5, but we were not able to characterize all the solutions on $\mathcal{X}$. It is a challenging project.

Another possible direction which we find interesting to explore is bargaining with several bureaucrats. The bureaucrats can dictate any outcome (for instance, by unanimity or by majority vote). Even the case of a single agent and multiple bureaucrats seems to be nontrivial.

Appendix

Lemmata

We make use of the following four lemmata (the last two are straightforward).

Lemma 1 *Let $(N^0, X) \in \mathcal{X}^{SM}$ and $y \in Y(X)$. If $y_i \leq MC_i(N^0, X)$ for all $i \in N$, then $\sum_{i \in S} y_i \leq V_X(N^0) - V_X(N^0 - S)$ for all $S \subset N$.*

Proof. Let $S = \{i_1, \ldots, i_s\} \subset N$. Then by (6)

$$\begin{aligned} \sum_{i \in S} y_i &\leq \sum_{j=1}^{s} MC_{i_j}(N^0, X) \\ &\leq MC_{i_1}(N^0, X) + MC_{i_2}(N^0 \backslash \{i_1\}, X) + \ldots + MC_{i_s}(N^0 \backslash \{i_1, \ldots, i_{s-1}\}, X) \\ &= V_X(N^0) - V_X(N^0 - S). \end{aligned}$$

■

Lemma 2 *Let* $(N^0, X) \in \mathcal{X}^{SM}$*. Then* $y \in ST(X)$ *if and only if*

(i) $d_i(X) \leq y_i \leq MC_i(N^0, X)$ *for all* $i \in N$,

(ii) $y_0 = V_X(N^0) - \sum_{i \in N} y_i$.

Proof. Suppose that $y \in ST(X)$. Then, by Corollary 1, y is efficient, namely, $\sum_{i \in N^0} y_i = V_X(N^0)$. By Proposition 2, for all $i \in N$,

$$\sum_{j \in N^0 - i} y_j \geq \max_{x \in X} \sum_{j \in N^0 - i} x_j = V_X(N^0 - i).$$

Consequently, $y_i \leq V_X(N^0) - V_X(N^0 - i) = MC_i(N^0, X)$.

Conversely, suppose that y satisfies (i) and (ii). To prove that $y \in ST(X)$ it suffices to show that for every $S \subset N$ $y_0 + \sum_{i \in S} y_i \geq V_X(S^0)$. By (i) and (ii),

$$y_0 + \sum_{i \in S^0} y_i = V_X(N^0) - \sum_{j \in N-S} y_j \geq V_X(N^0) - \sum_{j \in N-S} MC_j(N^0, X),$$

and since $X \in \mathcal{X}^{SM}$ we have

$$\begin{aligned} \sum_{j \in N-S} MC_j(N^0, X) &\leq MC_{j_1}(N^0, X) + MC_{j_2}(N^0 - j_1, X) \\ &\quad + \ldots + MC_{j_{n-s}}(N^0 - \{j_1, \ldots, j_{n-s-1}\}, X) \\ &= V_X(N^0) - V_X(S^0), \end{aligned}$$

where $\{j_1, j_2, \ldots, j_{n-s}\} = N - S$. ■

Lemma 3 *Let* (N^0, X) *and* (N^0, X') *be in* $\mathcal{X}$*. Suppose that for some* $a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{N^0}$ *and* $c \in \mathbb{R}_{++}$, $X' = cX + a$*. Then for every* $K^0 \subset N^0$,

$$\begin{aligned} V_{X'}(K^0) &= cV_X(K^0) + \sum_{j \in K^0} a_j, \text{ and} \\ E_{K^0}(X') &= cE_{K^0}(X) + a. \end{aligned}$$

Lemma 4 *Let* $(N^0, X) \in \mathcal{X}$*, where* $N^0 = \{0, 1, \ldots, n\}$*. Let* $N' = N^0 \cup \{n + 1\}$ *and* $X' = X \times [a, a']$*, where* $0 \leq a \leq a'$*. Then for every* $S^0 \subset N^0$,

$$\begin{aligned} V_{X'}(S^0) &= V_X(S^0), \\ E_{S^0}(X') &= E_{S^0}(X) \times [a, a']. \end{aligned}$$

Proof of Proposition 4

The proof uses notations and definitions introduced in Sections 5 – 6. It suffices to show that Axioms 1 – 5 are independent on the class $\mathcal{X}^{SM}$.

If we drop just the requirement that the solution is stable, the following solution on $\mathcal{X}$

$$\begin{aligned} \phi_0(X) &= \min_{y \in ST(X)} y_0, \\ \phi_i(X) &= d_i(X) \text{ for all } i \in N \end{aligned}$$

satisfies axioms 2 – 5 for every $(N^0, X) \in \mathcal{X}^{SM}$.

To show that the STD axiom is independent of the others, consider the following solution. For every $(N^0, X) \in \mathcal{X}^{SM}$ and every $i \in N$, $\phi_0(X) = V_X(N^0) - \sum_{i \in N} \phi_i(X)$, where

$$\phi_i(X) = \begin{cases} \bar{d}_i(X), & MC_i(N^0, X) = MC_i(N^0 - j, X) \text{ for some } j \in N - i \\ MC_i(N^0, X), & MC_i(N^0, X) < MC_i(N^0 - j, X) \text{ for all } j \in N - i \end{cases}$$

Clearly, $\phi(X) \in ST(X)$ and it satisfies axioms 3 – 5. As for the STD axiom, consider the following problems. Let $N^0 = \{0, 1, 2\}$ and consider $X = \{(0,3,0), (0,0,3), (0,2,2)\}$ and $X' = \{(2,1,0), (2,0,1), (2,1,1)\}$. Clearly, both $\left(N^0, X\right)$ and $\left(N^0, X'\right)$ are in $\mathcal{X}^{SM}$, and $ST(X) = ST(X')$ by Lemma 2. Now, $MC_1(N^0, X) = 1$ is less than $MC_1(N^0 - \{2\}, X) = 3$, hence $\phi_1(X) = 1$. However, $MC_1(N^0, X') = MC_1(N^0 - \{2\}, X') = 1$, hence $\phi_1(X') = \bar{d}_1(X') = 0$, which violates the STD axiom.

Next, we show that the Scale Covariance axiom is independent of the other axioms. For every $(N^0, X) \in \mathcal{X}$ and every $i \in N$, define $\phi_0(X) = V_X(N^0) - \sum_{i \in N} \phi_i(X)$ and for all $i \in N$

$$\phi_i(X) = \begin{cases} d_i(X), & \text{if } d_i(X) = 0, \\ V_X(N^0) - V_X(N^0 - i), & \text{otherwise.} \end{cases}$$

It is easy to verify that ϕ satisfies Axioms 1,2,4, and 5, but not Axiom 3.

Now, we show that the Separability axiom is independent. For every $(N^0, X) \in \mathcal{X}^{SM}$, let $K_X = \underset{j \in N}{\text{argmax}}\, V_X(N^0) - V_X(N^0 - j)$, and let for every $i \in N$

$$\phi_i(X) = \begin{cases} V_X(N^0) - V_X(N^0 - i), & \text{if } i \in K_X, \\ d_i(X), & \text{otherwise,} \end{cases}$$

and $\phi_0(X) = V_X(N^0) - \sum_{i \in N} \phi_i(X)$. It is easy to verify that ϕ satisfies axioms 1 – 4. As for the Separability axiom, consider bargaining problem $(\{0,1\}, X)$ with $X = \{0\} \times [0,1]$. Then $\phi_1(X) = 1$. Consider next $(\{0,1,2\}, X')$ with $X' = X \times [0,2]$. Here, $1 \notin K_{X'}$ and hence $\phi_1(X) = 0$, which violates the Separability axiom.

Finally, we demonstrate the independence of the Anonymity axiom. Let $(N^0, X) \in \mathcal{X}^{SM}$. Player n is called *separable* if $X = X_{N^0 - 1} \times X_n$, where X_S denotes the projection of X on the coordinates of S. Let $\phi_0(X) = V_X(N^0) - \sum_{i \in N} \phi_i(X)$ and let

$$\phi_i(X) = \begin{cases} V_X(N^0) - V_X(N^0 - i), & \text{if } i = n \text{ and } i \text{ is not separable,} \\ d_i(X), & \text{otherwise.} \end{cases}$$

Clearly, ϕ satisfies Axioms 1 – 3 and 5, but not 4. Indeed, consider $X = \{(0,3,0), (0,0,3), (0,2,2)\}$. A permutation of agents 1 and 2 leaves X unchanged, so the Anonymity axiom requires $\phi_1(X) = \phi_2(X)$. But we obtain $\phi_1(X) = 0$ and $\phi_2(X) = 1$.

Proof of Theorem 1

Existence. By Lemma 2, ϕ satisfies Stability and STD axioms. To verify the Scale Covariance axiom, let (N^0, X) and (N^0, X') be in $\mathcal{X}^{SM}$ such that for some $\hat{b} \in \mathbb{R}^{N^0}$ and $\hat{c} \in \mathbb{R}_{++}$, $X' = \hat{c}X + \hat{b}$. By Lemma 3, for all $i \in N$, $d_i(X') = \hat{c}d_i(X) + \hat{b}_i$, $b_i(X') = \hat{c}b_i(X) + \hat{b}_i$, and $V_{X'}(N^0) = \hat{c}V_X(N^0) + \sum_{j \in N^0} \hat{b}_j$. Therefore, for all $i \in N$, $\phi_i(X') = \hat{c}\phi_i(X) + \hat{b}_i$, and $\phi_0(X') = \hat{c}V_X(N^0) + \sum_{j \in N^0} \hat{b}_j - \sum_{j \in N^0} (\hat{c}\phi_j(X) + \hat{b}_j) = \hat{c}\left(V_X(N^0) - \sum_{j \in N^0} \phi_j(X)\right) + \hat{b}_0 = \hat{c}\phi_0(X) + \hat{b}_0$. The Anonymity axiom is trivially satisfied. Finally, we verify the separability axiom. Let $(N^0, X) \in \mathcal{X}^{SM}$, where $N^0 = \{0, 1, \ldots, n\}$.

Let $N' = N^0 \cup \{n+1\}$ and $X' = X \times [a, a']$, where $0 \leq a \leq a'$. Clearly, $(N', X') \in \mathcal{X}^{SM}$. By Lemma 4, for all $i \in N$, $d_i(X') = d_i(X)$, $MC_i(N^0, X') = MC_i(N', X)$, and $V_{X'}(N^0) = V_X(N^0)$, implying that for all $i \in N$, $\phi_i(X') = \phi_i(X)$.

Uniqueness (up to the parameter α). Let ϕ be a solution on $\mathcal{X}^{SM}$ which satisfies Axioms 1 – 5. Let

$$\hat{X}_2 = \{(0, x) \mid 0 \leq x \leq 1\}$$

and let $\phi_0(\hat{X}_2) = \alpha$. Since $ST(\hat{X}_2) = \{y \in \mathbb{R}^2_+ \mid y_0 + y_1 = 1\}$, it must be that $\phi_1(\hat{X}_2) = 1 - \alpha$. We shall show that $\phi(X)$ is uniquely determined, given α, for all $X \in \mathcal{X}^{SM}$.

Consider next the bargaining problem in $\mathcal{X}_2^{SM}$ defined by

$$X_{(d,b)} = \{d_0\} \times [d_1, b_1],$$

where $d = (d_0, d_1) \in \mathbb{R}^2_+$ and $b_1 \geq d_1$. Clearly, $X_{(d,b)} = d + (b_1 - d_1)\hat{X}$, and, by the Scale Covariance axiom,

$$\phi(X_{(d,b)}) = d + (b_1 - d_1)(\alpha, 1 - \alpha),$$

and $\phi(X_{(d,b)})$ is uniquely determined. Next, consider the bargaining problem $(N^0, \bar{X}_{(d,b)}) \in \mathcal{X}^{SM}$, where $d = (d_0, d_{1,} \ldots, d_n) \in \mathbb{R}^{N^0}_+$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^N_+$ such that $b_i \geq d_i$ for all $i \in N$, and

$$\bar{X}_{(d,b)} = \{d_0\} \times [d_1, b_1] \times \ldots \times [d_n, b_n].$$

By the Separability and Anonymity axioms, for every $i \in N$,

$$\phi_i(\bar{X}_{(d,b)}) = \alpha d_i + (1 - \alpha) b_i.$$

This, together with the fact that $\phi(\bar{X}_{(d,b)})$ is efficient, uniquely determines $\phi(\bar{X}_{(d,b)})$. Also observe that

$$ST(\bar{X}_{(d,b)}) = \left\{ y \in R^{N^0}_+ \;\middle|\; \begin{array}{l} d_i \leq y_i \leq b_i \text{ for all } i \in N, \\ y_0 = d_0 + \sum_{i \in N}(b_i - y_i) \end{array} \right\}.$$

Let (N^0, X) be an arbitrary bargaining problem in $\mathcal{X}^{SM}$. Let $\hat{d}_i = d_i(X)$ and $\hat{b}_i = MC_i(N^0, X)$ for all $i \in N$. Also, let $\hat{d}_0 = V_X(N^0) - \sum_{i \in N} \hat{b}_i$. Then, by Lemma 2,

$$ST(X) = \left\{ y \in R_+^{N^0} \;\middle|\; \begin{array}{l} \hat{d}_i \leq y_i \leq \hat{b}_i \text{ for all } i \in N, \\ y_0 = \hat{d}_0 + \sum_{i \in N} (\hat{b}_i - y_i) \end{array} \right\} = ST(\bar{X}_{(\hat{d}, \hat{b})}).$$

Since $ST(X) = ST(\bar{X}_{(\hat{d}, \hat{b})})$, by the STD axiom, $\phi(X) = \phi(\bar{X}_{(\hat{d}, \hat{b})})$, and $\phi(X)$ is uniquely determined for every $X \in \mathcal{X}^{SM}$. This completes the proof.

Proof of Proposition 6

Let $(N^0, X) \in \mathcal{X}^{SM}$. Then for every $S \subset N$ and every $i \in N - S$, $V_X(N^0) - V_X(N^0 - i) \leq V_X(N^0 - S) - V_X(N^0 - S - i)$, or

$$\sum_{j \in S} V_X(N^0) - V_X(N^0 - j) \leq V_X(N^0) - V_X(N^0 - S). \qquad (1)$$

For every $y \in Y(X)$ and every $S \subset N^0$ define

$$e_X(y, S) = V_X(S) - \sum_{i \in S} y_i.$$

First, note that for every $S \subset N$, $V_X(S) = \sum_{i \in S} d_i(X)$, hence, for all $y \in Y(X)$,

$$e_X(y, S) = \sum_{i \in S} e_X(y, \{i\}). \qquad (2)$$

Next, for every $S \subset N$ and every $y \in Y(X)$, by (1),

$$\begin{aligned} e_X(y, N^0 - S) &= V_X(N^0 - S) - \sum_{i \in N^0 - S} y_i = V_X(N^0 - S) - V_X(N^0) + \sum_{i \in S} y_i \\ &\leq \sum_{i \in S} \big(V_X(N^0 - i) - V_X(N^0) + y_i\big) \\ &= \sum_{i \in S} \left(V_X(N^0 - i) - \sum_{j \in N^0 - i} y_j \right) = \sum_{i \in S} e_X(y, N^0 - i). \qquad (3) \end{aligned}$$

By (2) and (3), for all $y \in Y(X)$ and all $S \subset N$,

$$\begin{aligned}\sum_{i \in S} e_X(y, \{i\}) &\geq e_X(y, S),\\ \sum_{i \in S} e_X(y, N^0 - i) &\geq e_X(y, N^0 - S).\end{aligned}$$

Therefore, the nucleolus y^* of V_X is defined for every $i \in N$ by

$$y_i^* = \underset{y \in ST(X)}{\operatorname{argmin}} \left[\max\left\{e_X(y, \{i\}), e_X(y, N^0 - i)\right\}\right].$$

Since $e_X(y, \{i\}) = d_i(X) - y_i$ and $e_X(y, N^0 - i) = V_X(N^0 - i) - V_X(N^0) + y_i$, y_i^* is the solution of

$$d_i(X) - y_i = V_X(N^0 - i) - V_X(N^0) + y_i.$$

Thus,

$$y_i^* = \frac{V_X(N^0) - V_X(N^0 - i) + d_i(X)}{2} = \frac{MC_i(N^0, X) + d_i(X)}{2}, \quad i \in N.$$

Proof of Property 3

Let $X \in \mathcal{X}^{SM}$ and let $S \subset N$. Using (10) we have for every $T \subset N^0 - S$

$$V_{X_{-S}(\phi^\alpha)}(T) = \begin{cases} V_X(N^0) - \sum_{j \in S} \phi^\alpha(X), & T \ni 0 \\ \sum_{j \in T} d_j(X), & T \not\ni 0 \end{cases}$$

Clearly, $X_{-S}(\phi) \in \mathcal{X}^{SM}$, since $V_{X_{-S}(\phi^\alpha)}(N^0 - S - T) - V_{X_{-S}(\phi^\alpha)}(N^0 - S - T - i) = V_X(N^0 - T) - V_X(N^0 - T - i)$ for all $T \subset N - S$, and $X \in \mathcal{X}^{SM}$ by assumption. Then, by Theorem 1, for some $x^* \in E(X)$

$$\phi_i^\alpha(X_{-S}(\phi^\alpha)) = d_i(X) + (1 - \alpha) MC_i(N^0, X) = \phi_i^\alpha(X) \quad \text{for all } i \in N - S,$$

$$\phi_0^\alpha(X_{-S}(\phi^\alpha)) = \left(\sum_{i \in N^0 - S} x_i^* - \sum_{i \in N - S} \phi_i^\alpha(X)\right) + \sum_{i \in S} (x_i^* - \phi_i^\alpha(X)) = \phi_0^\alpha(X).$$

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