

Epistemic Conditions for Equilibrium in Beliefs without Independence*

Oscar Volij

*Department of Economics, Brown University,
Providence, Rhode Island 02912*

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Aumann and Brandenburger [*Econometrica* 63 (1995), 1161–1180.] provide sufficient conditions on the knowledge of the players in a game for their beliefs to constitute a Nash equilibrium. They assume, among other things, mutual knowledge of rationality. By rationality of a player, it is meant that the action chosen by him maximizes his expected utility, given his beliefs. There is, however, no need to restrict the notion of rationality to expected utility maximization. This paper shows that their result can be generalized to the case where players' preferences over uncertain outcomes belong to a large class of non-expected utility preferences. *Journal of Economic Literature* Classification Numbers: C72, D81.

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1. INTRODUCTION

Expected utility theory of choice under uncertainty rests upon several axioms, among which the most controversial is independence. The normative appeal of this theory stems from the fact that the independence axiom can be regarded as a combination of two normatively appealing assumptions: the compound independence and the reduction of compound lotteries axioms. Compound independence requires for any positive probability α and lotteries X , Y , and Z that X is preferred to Y if and only if a α : $(1 - \alpha)$ chance of getting either X or Z is preferred to the same chance of getting Y or Z . The reduction axiom, on the other hand, requires indifference between any two-stage lottery and the one-stage lottery that yields the same prizes with the corresponding multiplied probabilities. Despite its normative appeal, the descriptive usefulness of expected utility theory has been challenged by some systematic violations observed in experiments. Machina [5] shows, however, that many of the results of expected utility theory are robust to certain kinds of violations of independence.

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In the last few years some effort has been spent in extending game theory to non-expected utility preferences. Part of the importance of these developments lies in the fact that they allow us to check how much of our understanding of strategic behavior survives the relaxation of the experimentally violated independence. Since, as said before, independence is a combination of two different axioms, the failure to satisfy independence can be explained alternatively as a failure to satisfy either of the two axioms above, and each of the two explanations leads to a different theory of choice under uncertainty in interactive situations. Crawford [3] extends the concept of Nash equilibrium to games in which the players' preferences do not satisfy the compound independence axiom. He also shows, however, that in this context a Nash equilibrium may fail to exist in a finite normal form game. Partly for this reason, he proposes a new equilibrium concept, called equilibrium in beliefs. Roughly speaking, an equilibrium in beliefs is a pair of probability assessments (the beliefs), one for each player, over the other's strategic choices, that can each be expressed as a mixture of best responses to the other belief. Crawford [3] proves that in any finite normal form game, an equilibrium in beliefs always exists. Dekel *et al.* [4] choose to drop the reduction axiom and to maintain the compound independence axiom; within this framework they prove existence of Nash equilibrium.

The traditional interpretation of mixed strategies views them as random devices that select actions according to some probability distribution. Another interpretation of mixed strategies maintains that players never randomize, that they always choose a specific action, but that the mixed strategy of player i represents the uncertainty in the other players' minds about his own action. When the second interpretation is adopted, a Nash equilibrium in mixed strategies is an equilibrium in beliefs. Equilibrium in beliefs is however, a more general concept because beliefs may be in equilibrium without constituting a Nash equilibrium.

In a recent paper, Aumann and Brandenburger [2] describe sufficient conditions about the knowledge of the players in a game for their beliefs at some state to constitute an equilibrium in beliefs. One of the key assumptions is mutual knowledge of rationality, namely the players must know that everybody is rational. By rationality of a player, the authors mean that the action actually chosen by that player maximizes his expected utility, given his beliefs at that state. There is, however, no need to restrict the notion of rationality to expected utility maximization. Rationality refers to making a choice that brings one closest to a desired end. It has nothing to do with the specific functional form that may represent that end. In this paper we will show that Aumann and Brandenburger's result can be generalized to the case where the players' preferences over uncertain outcomes do not necessarily satisfy the independence axiom of the expected utility theory. In this broader setting, rationality of a player means that a

player chooses a mixed strategy (in the traditional sense) that maximizes his preferences, given his beliefs. The criterion for rationality is given in terms of mixed strategies, because when the preferences are not necessarily linear in the probabilities, a player may well strictly prefer to randomize rather than to pick any specific action.

One remark is in order. Aumann and Brandenburger [2] state their results in terms of Nash equilibrium. Specifically, they found sufficient conditions for the beliefs of the players to constitute a Nash equilibrium. They did not use the concept of equilibrium in beliefs. When we deal with expected utility maximizers there is an equivalence between the concept of Nash equilibrium and the concept of equilibrium in beliefs. Since they focused on games with expected utility maximizing agents, Aumann and Brandenburger were able to state their results in terms of Nash equilibrium. When the players do not satisfy the independence axiom, Nash equilibrium and equilibrium in beliefs are no longer equivalent and the results must be stated in terms of the latter. We will show sufficient conditions for the beliefs of the players to constitute an equilibrium in beliefs.

In Section 2 we define two-stage lotteries and state some properties that preferences over them may satisfy. The space of two-stage lotteries is a natural environment to deal with non-expected utility preferences. Section 3 adapts the Nash equilibrium concept to games in which players' preferences over simple lotteries do not satisfy the independence axiom. Section 4 uses an interactive belief system to define a notion of rationality that is consistent with non-expected utility preferences and states the generalization of Aumann and Brandenburger's result. Section 5 concludes.

2. TWO-STAGE LOTTERIES AND PREFERENCES ON THEM

This section is based on Segal [7, 8]. Let L_1 be the set of lotteries with outcomes in a bounded interval $[-M; M] \subseteq \mathbb{R}$. That is

$$L_1 = \left\{ (x_1, p_1; \dots; x_n, p_n) \mid x_1, \dots, x_n \in [-M; M], \right. \\ \left. p_1, \dots, p_n \geq 0 \text{ and } \sum_{i=1}^n p_i = 1 \right\}.$$

Elements of L_1 are denoted by X, Y , etc. and are sometimes called simple or one-stage lotteries. In some cases it will be convenient to denote simple lotteries by $\{(x_i, p_i)\}_{i \in I}$, where I is some finite set of indexes. For $X \in L_1$, define the cumulative distribution function $F_X(x) = \Pr(X \leq x)$. A preference relation over simple lotteries is a complete and transitive relation on $(L_1)^2$.

By continuity of a preference relation \succsim_1 over simple lotteries we mean that if $\{Y_i\}_{i=1}^\infty, Y, X \in L_1$ are such that at each continuity point x of $F_Y, F_{Y_i}(x)$ converges to $F_Y(x)$ and if for all i $X \succsim_1 Y_i$, then $X \succsim_1 Y$. And similarly, if for all i $Y_i \succsim_1 X$ then $Y \succsim_1 X$.

A two-stage lottery is a lottery whose outcomes are themselves lotteries in L_1 . The set of two-stage lotteries is then

$$L_2 = \left\{ (X_1, q_1; \dots; X_m, q_m) \mid X_1, \dots, X_m \in L_1, \right. \\ \left. q_1, \dots, q_m \geq 0 \text{ and } \sum_{j=1}^m q_j = 1 \right\}.$$

Elements of L_2 are denoted A, B , etc. and are sometimes called compound lotteries. Again, it will sometimes be convenient to denote compound lotteries by $\{(X_j, q_j)\}_{j \in J}$, where J is some finite set of indices and $X_j \in L_1$.

Two subsets of L_2 are of special interest:

$$\mathcal{A} = \{(X, 1) \mid X \in L_1\} \subset L_2; \\ \Gamma = \{ \{(x_i, 1), p_i\}_{i=1}^n \mid \{(x_i, p_i)\}_{i=1}^n \in L_1 \} \subset L_2.$$

\mathcal{A} is the set of all two-stage lotteries that have no uncertainty in the first stage. Lotteries in \mathcal{A} give a specific lottery $X \in L_1$ with probability 1. All the uncertainty is resolved in the second stage. Γ is the set of all lotteries with no uncertainty in the second stage. All the uncertainty is resolved in the first stage. Let $X = \{(x_i, p_i)\}_{i \in I}$ be a simple lottery. Based on X define the two lotteries $\gamma_X \in \Gamma$ and $\delta_X \in \mathcal{A}$, as follows:

$$\gamma_X = \{(x_i, 1), p_i\}_{i \in I}; \\ \delta_X = \{(x_i, p_i)_{i \in I}, 1\}.$$

Any complete and transitive preference relation \succsim on L_2 induces two, possibly different, complete and transitive preference relations on L_1 in the following way:

$$X \succsim_\Gamma Y \quad \text{if and only if} \quad \gamma_X \succsim \gamma_Y; \\ X \succsim_{\mathcal{A}} Y \quad \text{if and only if} \quad \delta_X \succsim \delta_Y.$$

We shall say that a preference relation \succsim over two-stage lotteries is time neutral if for every simple lottery $X \in L_1, \gamma_X \sim \delta_X$.

Time neutrality means that the decision maker is indifferent about the timing of resolution of the uncertainty, as long as all the uncertainty is resolved at one and the same time.

Segal [7] shows the following:

THEOREM 2.1. *Let \succcurlyeq be a preference relation over compound lotteries whose induced preference relations \succcurlyeq_I and \succcurlyeq_A are continuous. The induced preference relations \succcurlyeq_I and \succcurlyeq_A are identical if and only if \succcurlyeq satisfies time neutrality.*

Given a time neutral preference relation \succcurlyeq on L_2 , this theorem allows us to define its induced preference relation on simple lotteries to be $\succcurlyeq_I = \succcurlyeq_A$.

There are some properties that a preference relation over compound lotteries may or may not satisfy. One commonly assumed axiom is the following:

Reduction of Compound Lotteries Axiom. For any compound lottery $A = \{X_j, q_j\}_{j \in J}$ where for each $j \in J$, $X_j = \{(x_{ij}, p_{ij})\}_{i_j \in I_j}$ is a simple lottery, define the following lottery in L_1 :

$$X_A = \{(x_{ij}, p_{ij}q_j)\}_{i_j \in I_j, j \in J}.$$

We shall say that \succcurlyeq satisfies the reduction of compound lotteries axiom if and only if for all $A \in L_2$, $A \sim \gamma_{X_A}$.

This axiom says that a decision maker is indifferent between any two lotteries that are actuarially equivalent (that can be reduced to the same simple lottery by application of the rules of probability). Segal [7] showed that the reduction axiom implies time neutrality.

Another property is:

Compound Independence Axiom. Let X, Y be two simple lotteries and let $A = (Z_1, q_1; \dots; X, q_i; \dots; Z_m, q_m)$ and $B = (Z_1, q_1; \dots; Y, q_i; \dots; Z_m, q_m)$ be two compound lotteries. We shall say that \succcurlyeq satisfies the compound independence axiom if for all such lotteries, $A \succcurlyeq B$ if and only if $\delta_X \succcurlyeq \delta_Y$.

This axiom says that if the individual prefers a simple lottery X to a simple lottery Y , then he must also prefer to replace Y with X in any compound lottery containing Y as an outcome, and vice versa.

Let $X = (x_1, p_1; \dots; x_n, p_n)$ and $Y = (y_1, q_1; \dots; y_m, q_m)$ be two simple lotteries and let $\alpha \in (0, 1]$. We denote by $\alpha X + (1 - \alpha) Y$ the lottery $(x_1, \alpha p_1; \dots; x_n, \alpha p_n; y_1, (1 - \alpha) q_1; \dots; y_m, (1 - \alpha) q_m)$. It can be shown (see Segal [7]) that if \succcurlyeq satisfies the reduction of compound lotteries and the compound independence axioms, then the induced preference relation over simple lotteries satisfies the following property:

Independence. Let X, Y , and Z be simple lotteries and let $\alpha \in (0, 1]$. We shall say that a preference relation \succcurlyeq_1 over simple lotteries satisfies

independence if for all such lotteries

$$X \succcurlyeq_1 Y \quad \text{if and only if} \quad \alpha X + (1 - \alpha) Z \succcurlyeq_1 \alpha Y + (1 - \alpha) Z.$$

Segal [8] proved the following important result:

THEOREM 2.2. *Each continuous preference relation \succcurlyeq_1 on L_1 has a unique extension to L_2 satisfying $\succcurlyeq'_R \equiv \succcurlyeq'_A \equiv \succcurlyeq_1$ and the compound independence axiom and another unique extension to L_2 satisfying $\succcurlyeq'_R \equiv \succcurlyeq'_A \equiv \succcurlyeq_1$ and the reduction of compound lotteries axiom.*

This theorem points out to two different ways to deal with non-expected utility maximizers in general and within game theory in particular. Crawford [3] chose to keep the reduction axiom while Dekel *et al.* [4] chose to keep the compound independence axiom.

3. GAME THEORY WITHOUT INDEPENDENCE

Consider a 2-person game form $G^* = \langle N; (S^i)_{i \in N}; (h^i)_{i \in N} \rangle$ in strategic form, where $N = \{1, 2\}$, and for each player $i \in N$, S^i is his finite set of pure actions and h^i is i 's monetary payoff function, i.e., a function that assigns to every n -tuple of pure actions a monetary payoff to player i . In order to transform G^* into a game G we need to specify for each player a preference relation over lotteries on the set of monetary outcomes. We endow each player i with a complete and transitive preference relation \succcurlyeq_i over compound lotteries on money. We also assume that each preference relation satisfies time neutrality and that the induced preference relation over simple lotteries is continuous.

A mixed strategy for player i in this game is, as usual, a probability distribution over pure actions. We shall denote the set of mixed strategies of player i by Σ^i and specific elements of it by σ^i, τ^i , etc. The mixed strategy that assigns probability 1 to action $s^i \in S^i$ will be denoted by s^i . If $\sigma = (\sigma^1, \sigma^2)$ is a pair of mixed strategies, then for all $i, i = 1, 2, \sigma^{-i}$ denotes $\sigma^j, j \neq i$.

Each pair $\sigma = (\sigma^1, \sigma^2)$ of mixed strategies defines for each player i the following simple lottery over monetary outcomes

$$h^i(\sigma) := \{h^i(s); \sigma^1(s^1) \times \sigma^2(s^2)\}_{s \in S},$$

where $S := X_{i \in N} S^i$. That is, σ defines the lottery that assigns the amount $h^i(s)$ with probability $\sigma^1(s^1) \times \sigma^2(s^2)$. If player i 's preference relation does not satisfy the reduction axiom, he will not evaluate the optimality of his mixed strategy σ^i against the mixed strategy of his opponent σ^{-i} according to his evaluation of the lottery $h^i(\sigma)$. Although we know the way he

evaluates compound lotteries, we do not know how he translates pairs of mixed strategies into compound lotteries. In other words, it is not clear what is the relevant compound lottery that represents the given pair of strategies. In this paper we are going to assume that the relevant compound lottery with which a player evaluates a pair of strategies is the one defined by the timing of revelation of the information and not the one defined by the actual or objective resolution of the uncertainty. No matter who randomizes first, each player will first be informed of the result of his own randomization following which the uncertainty about the other player's action will be revealed to him. This point of view is what Dekel *et al.* [4] call "the first perceptual hypothesis." Therefore, for any realization s^i of i 's randomizing device he will face the simple lottery $h^i(s^i, \sigma^{-i})$. And since he chooses s^i with probability $\sigma^i(s^i)$, he will in fact be facing, ex ante, the following two-stage lottery:

$$\{h^i(s^i, \sigma^{-i}), \sigma^i(s^i)\}_{s^i \in S^i}.$$

Taking this into account the best reply correspondences $BR^i: \Sigma^i \rightarrow \Sigma^i$, $i = 1, 2$, are defined as

$$BR^i(\sigma^{-i}) = \{\sigma^i \in \Sigma^i: \{h^i(s^i, \sigma^{-i}); \sigma^i(s^i)\}_{s^i \in S^i} \succcurlyeq_i \{h^i(s^i, \sigma^{-i}); \tau^i(s^i)\}_{s^i \in S^i} \text{ for all } \tau^i \in \Sigma^i\}.$$

DEFINITION. A Nash equilibrium in G is a pair of mixed strategies $\sigma = (\sigma^1, \sigma^2)$ such that

$$\sigma^i \in BR^i(\sigma^{-i}) \quad \text{for all } i \in N. \tag{1}$$

Remark 1. Dekel *et al.* [4] showed that assuming preferences which satisfy compound independence, a Nash equilibrium always exists in any finite game.

Remark 2. The above definition takes into account that players may not satisfy the reduction of compound lotteries axiom. If, however, they do satisfy this axiom, condition (1) could be replaced by the more familiar

$$\{h^i(\sigma^i, \sigma^j), 1\} \succcurlyeq_i \{h^i(\tau^i, \sigma^j), 1\} \quad \text{for all } \tau^i \in \Sigma^i, \quad \text{for all } i \in N.$$

In order to define the notion of equilibrium in beliefs, we need some notation. Let σ_1^i and σ_2^i be a pair of mixed strategies of player $i \in N$ and let $\alpha \in [0, 1]$. The mixture $\alpha \sigma_1^i \oplus (1 - \alpha) \sigma_2^i$ is the mixed strategy of i that assigns to each pure action $s^i \in S^i$ the probability $\alpha \sigma_1^i(s^i) + (1 - \alpha) \sigma_2^i(s^i)$. For any set $T \subset \Sigma^i$, $D(T)$ denotes the intersection of all the supersets of T

that are closed under the mixture operation. Clearly, each member of $D(T)$ can be expressed as a mixture of elements of T .

DEFINITION. An equilibrium in beliefs is a pair of strategies (σ^1, σ^2) such that

$$\sigma^i \in D[BR^i(\sigma^{-i})], \quad \forall i \in N.$$

If (σ^1, σ^2) is an equilibrium in beliefs, σ^i is not to be interpreted as the actual mixed strategy chosen by player i ; rather, it must be understood as the equilibrium beliefs held by j ($\neq i$) about the action taken by i . Equilibrium in beliefs requires each player's beliefs about the other's actions to be expressible as a probability distribution over the other's best replies to his own beliefs.

Though different in nature, equilibrium in beliefs and Nash equilibrium are related concepts. It follows directly from the definitions that a Nash equilibrium is an equilibrium in beliefs as well. As for the converse, Crawford [3] showed that if the players' preferences satisfy the reduction of compound lotteries axiom and if the induced preference relation over simple lotteries is convex, i.e., it can be represented by a quasi-concave utility function, then equilibria in beliefs are also Nash equilibria. Further, another relation is given by the following observation.

OBSERVATION. Let $G = \langle N; (S^i, h^i, \succsim_i)_{i \in N} \rangle$ be a two player game and assume that for all i in N , \succsim_i satisfies the compound independence axiom. Let $\sigma = (\sigma_1, \sigma_2)$ be an equilibrium in beliefs. Then σ is a Nash equilibrium as well.

Proof. The observation will follow from the following lemmas, which are proved in the Appendix.

LEMMA 3.1. Let σ^i be a best response to σ^j and let s^* be a pure action which is assigned positive probability by σ^i . Then s^* is a best response to σ^j as well.

LEMMA 3.2. Let $\{s_1^i, \dots, s_n^i\}$ be pure actions which are all of them best responses to σ^j . Any mixed strategy involving only elements of this set is a best response to σ^j as well.

Now pick $i \in N$. Since σ is an equilibrium in beliefs $\sigma^i \in D[BR^i(\sigma^{-i})]$. Let s^i be an action to which σ^i assigns positive probability. Then there must be a best response to σ^{-i} that gives positive probability to s^i . By Lemma 3.1 s^i must itself be a best response to σ^j . By Lemma 3.2 σ^i is a best response to σ^{-i} . ■

4. RATIONALITY WHEN INDEPENDENCE FAILS

Let us again be given a 2-player game $G = \langle N; (S^i, h^i, \succsim_i)_{i \in N} \rangle$, where as in the previous section $N = \{1, 2\}$ and for each $i \in N$, S^i is his set of available actions, h^i is his monetary outcome function, and \succsim_i is his preference relation over compound lotteries which is assumed to satisfy time neutrality and whose induced preference relation is continuous. As pointed out by Crawford [3], if one player has preferences that satisfy the reduction of compound lotteries axiom and his induced preferences over simple lotteries are convex, it may well be that, given two distinct lotteries among which he is almost indifferent, he strictly prefers a convex combination of them (a compound lottery) to either of them separately. This may seem peculiar, because after the first-stage lottery has taken place, the player may end up with the worst lottery. This phenomenon may seem at first sight to be a case of preference reversal which contradicts transitivity, but this interpretation cannot be correct because the preferences were assumed from the outset to satisfy transitivity. The correct interpretation of convex preferences is that, when faced with the choice between two distinct lotteries among which he is indifferent, the agent prefers to choose by means of a random device rather than choosing directly; when the device chooses one specific lottery, the agent strictly prefers to stick to this choice, i.e., the player's preferences satisfy dynamic consistency (see Machina [6]). Another interpretation is that, though not dynamically consistent, the player can commit himself to abide by the outcome of his randomizing device. This commitment ability makes his planned behavior coincide with his actual or ex post behavior. In either case, the player's best response to any combination of mixed strategies of the other players may be a non-degenerate mixed strategy, and this best response may even be unique. Hence, in order to establish the rationality of a given player, the action actually chosen is no longer relevant, as it was in the case of linear preferences. Now, the mixed strategy he chooses is the relevant variable. We shall have to take this last remark into account when we define the event "player i is rational".

In order to evaluate the rationality of an agent, we need to know what he does and what he believes, and in order to be able to express what a player does and believes within a game, we are going to use the formal tool of an *interactive belief system*.

An interactive belief system for G consists of the following elements:

- (1) for each player $i \in N$, a finite set T^i of types and for each type t^i of i
- (2) a mixed strategy $\sigma^i \in \Sigma^i$ and
- (3) a probability distribution on the set T^j of types of his opponent (t^i 's theory).

Interactive belief systems were first introduced by Aumann [1] and thoroughly discussed in Aumann and Brandenburger [2]. The differences between the interactive belief system used here and the one in Aumann and Brandenburger [2] are discussed in the concluding section.

Set $T := T^1 \times T^2$. Call each of the elements $t = (t^1, t^2)$ of T a state of the world. An event is a subset of T . Denote by $p(\cdot; t^i)$ the probability distribution on T induced by t^i ; formally, for any event E , $p(E; t^i)$ is defined to be the probability assigned by t^i 's theory to $\{t^j \in T^j: (t^i, t^j) \in E\}$. We say that player i knows an event E at state t if at t he ascribes probability 1 to E , i.e., if $p(E; t^i) = 1$. We say that an event E is mutual knowledge at $t = (t^1; t^2)$ if both players know E at t , i.e., if $p(E; t^i) = 1$, $i = 1, 2$. We denote by σ_t^i the mixed strategy chosen by i at state t , namely the mixed strategy of t^i . Functions defined on T (like σ^i), which are going to be typed in boldface, can be used to define events. If \mathbf{x} is such a function and x is one of its values, then $[\mathbf{x} = x]$ will denote the event $\{t \in T: \mathbf{x}(t) = x\}$. A conjecture Ψ of i about the mixed strategy of player j ($\neq i$) is a simple probability distribution on Σ^j . The theory of t^i at state t yields a conjecture, denoted Ψ_t^i and called the conjecture of i at t which is given by

$$\Psi_t^i(\sigma^j) := p([\sigma^j = \sigma^j]; t^i).$$

The conjecture Ψ_t^i represents i 's assessments at t concerning j 's choice of mixed strategy. Similarly, a belief φ^i of player i about the other players' actions, or more shortly a belief of player i , is a probability distribution on S^j . The conjecture of i at t yields a belief of i about j 's actions in the following manner:

$$\varphi_t^i(s^j) = \sum_{\sigma \in M^j} \Psi_t^i(\sigma) \sigma(s^j),$$

where $M^j \subseteq \Sigma^j$ is the set of mixed strategies assigned positive probability by Ψ_t^i .

The probability that i assigns at state t to j 's taking action s^j is the average over all j 's mixed strategies of the probabilities given by those mixed strategies to s^j , where the averaging weights are given by i 's conjecture at t concerning j 's choice of mixed strategy. The fact that player i applies the laws of probabilities in order to calculate the chances of j playing s^j does not contradict the fact that i 's preferences over compound lotteries may not satisfy the reduction axiom. The basic assumption is that i knows how to calculate probabilities while he may not evaluate compound lotteries according to the ultimate distribution of monetary outcomes.

Player i is called rational at t if his mixed strategy at t maximizes his preferences given his belief at t . Formally, if

$$\sigma_t^j \in BR^i(\phi_t^i).$$

In other words, a player is rational at state t , if what he really does at t is the best he can do given his beliefs about his opponent's choice of action. We denote by R^i the event "player i is rational," i.e., $R^i = \{t \in T: \sigma_t^i \in BR^i(\phi_t^i)\}$.

Within the interactive belief system, there is a natural relation between a player's beliefs and his knowing his own beliefs. This relation is stated in the following lemma.

LEMMA 4.1. *Let Ψ be a conjecture about player j 's mixed strategy and assume i knows at t that his own conjecture at t is Ψ . Then his conjecture at t is indeed Ψ , i.e., $\Psi_t^i = \Psi$.*

This lemma is a corollary of Lemma 2.6 in Aumann and Brandenburger [2]. We give a proof for completeness.

Proof. Let E be the event " i 's conjecture is Ψ ," i.e., $E = [\Psi = \Psi]$. By assumption, i knows E at t , i.e., $p(E; t^i) = 1$. Let Π be i 's conjecture at t and let F be the event " i 's conjecture is Π ." Since i 's conjecture at t depends only on i 's type, we have that $\{t^j \in T^j: (t^i, t^j) \in F\} = T^j$. Therefore $p(F; t^i) = 1$. But then E and F have a non-empty intersection which means that $\Pi = \Psi$. ■

We can now state the main result of this paper.

THEOREM 4.2. *Let (Ψ^1, Ψ^2) be a pair of conjectures. Suppose that at some state t it is mutually known both that the players are rational and that $(\Psi^1, \Psi^2) = (\Psi^1, \Psi^2)$. Then (ϕ_t^2, ϕ_t^1) is an equilibrium in beliefs of G .*

Proof. Pick player i . By Lemma 4.1 $\Psi_t^i = \Psi^i$. Then by the definition of i 's belief at t about j 's actions,

$$\phi_t^i(s^j) = \sum_{\sigma \in M^j} \Psi_t^i(\sigma) \sigma(s^j).$$

Now choose $\sigma^j \in M^j$. By assumption then, i assigns positive probability at t to $[\sigma^j = \sigma^j]$. Also, i attributes probability 1 at t to the event " j is rational" and to $[\Psi^j = \Psi^j]$. There must be a state t' at which all the three events obtain, which means that $\sigma^j \in BR^j(\phi_{t'}^j)$. Therefore ϕ_t^j is a mixture of best responses to $\phi_{t'}^j$. ■

We have seen in the previous section that if the players' preferences satisfy the compound independence axiom, then all equilibria in beliefs are Nash equilibria as well. As a corollary then, we have that mutual knowledge of rationality and of beliefs imply that those beliefs constitute a Nash equilibrium. For this case, however, we can reach the same conclusion with a conceptually simpler interactive belief system. A basic ingredient of the interactive belief system used in the previous section was a mixed strategy for every type. This was needed because when a player's preferences do not satisfy the independence axiom, it may be the case that he strictly prefers certain mixed strategy to any of his pure actions. This possibility, in turn, imposed on us the definition of rationality: a player is rational if his choice of mixed strategy (and not the actual realization of his randomizing device) yields his most preferred lottery. If we restrict attention to preferences that satisfy the compound independence axiom, it is always the case that an optimal mixed strategy is a mixture of pure actions which are themselves optimal (see Lemma 3.1). This property allows us to define rationality in terms of actions actually chosen instead of mixed strategies. With this idea in mind we can formulate the model in the following way.

Let G be a game as before but in which the preferences of the players satisfy the compound independence axiom. An interactive belief system for G is composed of the following elements:

- (1) for each player $i \in N$, a set T^i of types and for each type t^i of i
- (2) and action $s^i \in S^i$ and
- (3) a probability distribution on the set T^j of types of his opponent (t^i 's theory).

Again $T := T^1 \times T^2$ is the set of states of the world, with generic element t , and $p(\cdot; t^i)$ is the probability distribution on T induced by t^i . A belief φ^i of i is a probability distribution on S^j . Now, the theory of i at t yields i 's belief at t which is given by

$$\varphi_t^i(s^j) = p([s^j(t) = s^j]; t^i).$$

Accordingly, player i is called rational at t if his choice of action at t is a best response given his conjecture, namely if $s^i(t) \in BR^i(\varphi_t^i)$. Now we have the following theorem.

THEOREM 4.3 *Let (φ^1, φ^2) be a pair of beliefs. Suppose that at state t it is mutually known that the players are rational and that $(\varphi^1, \varphi^2) = (\varphi^1, \varphi^2)$. Then (φ^2, φ^1) is a Nash equilibrium of G .*

Proof. Let s^j be an action to which φ^j assigns positive probability. Since by assumption i knows that his conjecture at t is φ^i , by an argument analogous to the one in Lemma 1 it can be shown that i 's conjecture at t is indeed φ^i . Hence at t i assigns positive probability to $[s^j = s^j]$. Further, i ascribes probability one to the events "j is rational" and to $[\varphi^j = \varphi^j]$. Therefore, there must be a state at which all the three events obtain, which means that $s^j \in BR^j(\varphi^j)$. By Lemma 3.2 $\varphi^i \in BR^i(\varphi^j)$. ■

5. CONCLUDING REMARKS

Extensions of the Nash equilibrium concept have been provided by Crawford [3] and Dekel *et al.* [4] for the cases in which the compound independence and the reduction of compound lotteries axioms, respectively, are not satisfied. Using these generalizations we showed that Aumann and Brandenburger's [2] Theorem A can be extended to games in which players' preferences do not satisfy the independence axiom of expected utility maximization. The theorem states a relation between the rationality of the players and an equilibrium concept and supports the idea that the notion of rationality should not be restricted to expected utility maximization. It would be interesting to know whether other results concerning the relation between rationality and different solution concepts can also be generalized to the case of non-expected utility preferences.

Since whenever players have convex preferences, every equilibrium in beliefs is also a Nash equilibrium in the game (see Crawford [3]), it follows that in this case mutual knowledge of rationality and of conjectures imply that the beliefs associated to the conjectures are a Nash equilibrium in a 2-person game. Moreover, we could have also proved Theorem 4.3, using the appropriate belief system, for a game where players' preferences satisfy the reduction axiom and the induced preference relation over simple lotteries is convex. The proof would be identical to the Proof of Theorem 4.3 with the only difference that the conclusion would follow by invoking the convexity of the preferences rather than Lemma 3.2. Conceptually, the problem is that there might be games for which the conditions of the theorem cannot be satisfied even though a Nash equilibrium exists. This will happen when all Nash equilibria are in mixed strategies and those mixed strategies are the unique best response to each other. In this case, the definition of rationality cannot be fulfilled. That is, there is no pure action that is best response against i 's beliefs, hence the impossibility of being rational.

Aumann and Brandenburger [2] proved another theorem concerning finite n -player games with expected utility preferences. Their result states that if in an n -player game all players share a common prior and there is

mutual knowledge of rationality and common knowledge of each players conjectures (beliefs) about the other players' actions, then for each player j , the rest of the players agree on their beliefs about j 's choice of action and the profile of agreed beliefs constitutes a Nash equilibrium. In this theorem, the assumptions of a common prior and common knowledge of the players' conjectures are used only to show that for each player j , (i) j 's beliefs over the others' choice of actions is given by a product probability measure on the product of the strategy spaces of his rivals, and (ii) all the other players agree on their beliefs about j 's action, thus singling out a unique belief of j . Once the existence of a unique common belief about j 's actions is proved for all j , the proof of the fact that the profile of common conjectures constitutes a Nash equilibrium follows the lines of the 2-player case, but using the fact that conjectures are a product measure. Since in the proof of the existence of a unique profile of beliefs about each player's choice of action and its independence it is nowhere used the fact that the players' preferences are of any particular kind, adding the assumptions of the existence of a common prior and of common knowledge of the conjectures to the hypothesis of Theorem 4.2 leads to a generalization of Aumann and Brandenburger's [2] Theorem B for the preferences that do not satisfy the independence axiom. Clearly, in order to state the result for the n -player case, we need to extend the definitions of Nash equilibrium and equilibrium of beliefs for this case as well, but this is done in a straightforward manner. We have chosen not to deal with this case because the main point of this paper is completely made by the 2-player case. Similarly, for the sake of simplicity, we have chosen to implicitly assume that payoff functions are common knowledge and not to introduce any uncertainty about payoff functions. This would only have distracted attention from the main point.

APPENDIX

In this Appendix we prove Lemmas 3.1 and 3.2 used in the Proof of the Observation in Section 3. For this purpose we need first to show the following.

CLAIM. Let \succsim be a preference relation over compound lotteries that satisfies time neutrality and compound independence and whose induced preference relation over simple lotteries is continuous. Then for any simple lottery X , $\{X, q_k\}_{k=1}^n \sim (X, 1)$, where $q_k \geq 0$ $k = 1, \dots, n$ and $\sum_k q_k = 1$.

Proof. Let X be a simple lottery. By continuity, there exists a number C such that

$$(X; 1) \sim \{(C, 1); 1\}. \tag{A.1}$$

By continuity again, $\{(C, 1); 1\} \sim \{(C, q_k)_{k=1}^n; 1\}$. By time neutrality, $\{(C, q_k)_{k=1}^n; 1\} \sim \{(C, 1); q_k\}_{k=1}^n$. Taking into account (A.1) and by repeated application of compound independence, we have $\{(C, 1); q_k\}_{k=1}^n \sim \{X; q_k\}_{k=1}^n$. The desired result follows by transitivity. ■

LEMMA 3.1. *Let σ^i be a best response to σ^j and let s^* a pure action which is assigned positive probability by σ^i . Then s^* is a best response to σ^j as well.*

Proof. Since $\sigma^i \in BR^i(\sigma^j)$ and s^* is assigned positive probability by σ^i , by compound independence we must have

$$\{h^i(s^*; \sigma^j), 1\} \succsim_i \{h^i(s^i; \sigma^j), 1\} \quad \text{for all } s^i \in S^i.$$

By compound independence again

$$\{h^i(s^*; \sigma^j), \tau^i(s^i)\}_{s^i \in S^i} \succsim_i \{h^i(s^i; \sigma^j), \tau^i(s^i)\}_{s^i \in S^i} \quad \text{for all } \tau^i \in \Sigma^i.$$

But since by the previous claim

$$\{h^i(x^*; \sigma^j), \tau^i(s^i)\}_{s^i \in S^i} \sim_i \{h^i(s^*; \sigma^j), 1\},$$

s^* is a best response to σ^j . ■

LEMMA 3.2. *Let $\{s_1^i, \dots, s_n^i\}$ be pure actions which are all best responses to σ^j . Any mixed strategy involving only elements of this set as a best response to σ^j as well.*

Proof. By the Claim and compound independence we have

$$\begin{aligned} \{h^i(s_1^i, \sigma^j), 1\} &\sim_i \{h^i(s_1^i, \sigma^j), q_1; \dots; h^i(s_1^i, \sigma^j), q_n\} \\ &\sim_i \{h^i(s_1^i, \sigma^j), q_1; \dots; h^i(s_n^i, \sigma^j), q_n\}, \end{aligned}$$

where $q_k \geq 0$ $k = 1, \dots, n$ and $\sum_k q_k = 1$.

Hence, any mixed strategy involving only pure best replies is a best reply. ■

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