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TIGHT CORRELATED EQUILIBRIUM

by

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Tight Correlated Equilibrium*

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Abstract

A correlated equilibrium of a strategic form n-person game is called *tight* if all the incentive constraints are satisfied as equalities. The game is called *tight* if all of its correlated equilibria are tight. This work shows that the set of tight games has positive measure.

1 Introduction

A correlated equilibrium of a strategic form n-person game is called *strict* if for every pure strategy that is played with positive probability, the associated inequalities (indicating that a player will not deviate from that strategy) are strict. A game is *strict* if it possesses at least one strict correlated equilibrium. We call a correlated equilibrium *tight* if all the associated inequalities are satisfied as equalities. The game is called *tight* if all of its correlated equilibria are tight. Tightness is the "opposite" of strictness.

Aumann and Drèze [2005] characterize all possible expected payoffs of a player in a game G, when common priors and common knowledge of rationality are assumed. The characterization has a particularly simple form when G is strict.

The question thus arises how general strictness is. Specifically, is strictness generic?

To begin with, consider "matching pennies." "Matching pennies" is a tight game, and so is any game in a sufficiently small neighborhood of "matching pennies," in the 8-dimensional space of payoffs in two-person 2×2 games.

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Thus for two-person 2×2 games, the set of tight games has a positive measure, so strictness cannot be generic. But this could be an artifact of the small size of these games; for larger games, one might conjecture that strictness *is* generic and in particular that the tight games have measure 0. The main objective of this study is to show that the tight games have positive measure, even in the case of arbitrarily large games. Specifically, we prove:

Theorem A. *Let G be a two-person $m \times m$ game with a unique correlated equilibrium, and suppose that this equilibrium has full support.*[1] *Then there exists a neighborhood of G where all games are tight.*

The proof of this theorem uses properties of the orthogonal space, uppersemicontinuity of the Nash correspondence, and theorems of the alternative.

It remains to show that there are arbitrarily large games satisfying the conditions of theorem A. For each m, no matter how large, we construct two-person $m \times m$ games of this kind. An example of the 3×3 case is "Rock, Scissors, Paper," where the unique Nash equilibrium is $((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$, and this is also the unique correlated equilibrium. Specifically, we prove:

Theorem B. *For every natural m, there exists a two-person $m \times m$ game G with a unique correlated equilibrium, and this equilibrium has full support.*

The next section presents the background and motivation for this work. Section 3 clarifies notations used in the proofs, and Sections 4 and 5 provide proofs of the two main theorems.

2 Background

Correlated equilibria were first introduced by Aumann [1974]. A correlated equilibrium of a two-person game G in strategic form [2] is a probability distribution p over the set of pure strategy profiles, satisfying the following requirement: If a pure strategy profile is chosen in accordance with p, say by a mediator, and each player is informed by the mediator only of his component of the chosen profile, then it is optimal for each player to play that component, i.e., to obey the mediator.

Correlated equilibria are described by a finite number of linear inequalities in rk variables. Apart from the inequalities saying that the probabilities are non-negative and sum to 1, there are $r(r-1)+k(k-1)$ inequalities indicating

[1] Such an equilibrium is tight, and so the game is tight.
[2] G is represented by a bimatrix game with r rows and k columns.

that when a player is informed of his component in the chosen strategy profile, then he has no incentive to deviate from that strategy. In other words, the player is willing to follow the mediator's recommendation. These inequalities are called *incentive constraints*.

A correlated equilibrium is called *strict* if for every pure strategy appearing with positive probability, the incentive constraints indicating that a player will not deviate from that strategy are satisfied as strict inequalities. In other words, if a strategy is recommended by the mediator, then a player has a strong incentive to obey the mediator. A game is *strict* if it possesses at least one strict correlated equilibrium. On the other hand, we call a correlated equilibrium *tight* if all the incentive constraints are satisfied as equalities, and a game is *tight* if all of its correlated equilibria are tight.

The notion of strict correlated equilibrium leads us to the motivation for this work. Aumann and Drèze [2005] define a *rational expectation* of a player i in a strategic form game G to be a possible expected payoff, assuming common priors and common knowledge of rationality. They then characterize all rational expectations of i; the characterization has an especially simple form when G is strict. In particular, it follows from their results that in a strict game, if M is the highest payoff to i with positive probability in some strict correlated equilibrium of G, then M is a rational expectation of i.

The question thus arises how general strictness is. This work provides an answer to this question: strictness is *not* generic. Indeed, we establish a stronger result, namely that the set of tight games has positive measure.

3 Definitions and Notations

Let G be a two-person $m \times m$ game. We refer to G as a vector in the r-dimensional space, where r is the number of payoffs defining G. Precisely, $r = 2m^2$. The set $CE(G)$ of correlated equilibria of G is defined by a finite number of linear inequalities on the set[3] $\triangle^{m^2}$. Specifically, it is defined by $2m(m-1)$ inequalities ($m(m-1)$ for each player). $CE(G)$ is a nonempty polytope containing the convex-hull of the Nash equilibria set $NE(G)$. Let A be the matrix defining $CE(G)$. That is, $CE(G) = \{x \in \triangle(\mathbb{R}^{m^2}) | Ax \geq 0\}$, where A belongs to the matrix space $\mathrm{M}_{2m(m-1)\times m^2}$. We say that G determines A. Correlated equilibria as well as Nash equilibria are referred to as vectors in $\mathbb{R}^{m^2}$.

A property of games is *generic* if the closure of the set of all games that do not possess this property is of measure 0 in the Euclidean space of strategic form payoff vectors.

[3] $\triangle^k$ denotes the $k-1$-dimensional unit simplex.

0 denotes either the real number zero or a vector in $\mathbb{R}^n$ for which all components are zero. For any $x, y \in \mathbb{R}^n$ we shall use the following conventions for equalities and inequalities.

$$\begin{array}{lcll} x = y & \Leftrightarrow & x_i = y_i & i = 1, ..., n \\ x \geq y & \Leftrightarrow & x_i \geq y_i & i = 1, ..., n \\ x \gneqq y & \Leftrightarrow & x_i \geq y_i & \text{and} \quad x \neq y \\ x > y & \Leftrightarrow & x_i > y_i & i = 1, ..., n \end{array}$$

4 Proof of Theorem A

Let A be the matrix defining $CE(G)$. $A \in \mathrm{M}_{2m(m-1)\times m^2}$. By the conditions of the theorem, G has a unique correlated equilibrium and this equilibrium is a completely mixed Nash equilibrium.

Lemma 4.1. *Let G be a two-person $m \times m$ game. Let A be the matrix defining $CE(G)$. If x is a completely mixed Nash equilibrium of G, then $Ax = 0$.*

Proof. In a mixed-strategy Nash equilibrium, a player must receive the same expected payoff from every pure strategy to which he assigns positive probability. □

By the conditions of the theorem and by Lemma 4.1, there exists a probability vector $x_0 \in \mathbb{R}^{m^2}, x_0 > 0$, satisfying $Ax_0 = 0$. As mentioned above, $A \in \mathrm{M}_{2m(m-1)\times m^2}$. Let $s = m^2$. For $m > 2$, $2m(m-1) \geq m^2$, and therefore $rank(A) \leq s$. Since $x_0 \neq 0$ satisfies $Ax_0 = 0$, we obtain that $rank(A) \leq s - 1$. The following claim establishes that this latter inequality reduces to an equality.

Lemma 4.2. $rank(A) = s - 1$.

Proof. Suppose that $rank(A) < s - 1$. In such a case, the dimension of the subspace $S = \{x \in \mathbb{R}^{m^2} | \; Ax = 0\}$ is at least 2. Thus S contains a line l passing through x_0, but not through the origin (This is why $\dim S \geq 2$ rather than $\dim S \geq 1$ is needed). Since $x_0 > 0$, there exists on l a point $z_0 \neq x_0$, close enough to x_0, such that $z_0 > 0$. Now let us define $z = \frac{z_0}{\|z_0\|} \neq x_0$. z is a probability vector satisfying $Az = 0$. Thus z is a correlated equilibrium, and we have a contradiction to the uniqueness of x_0. □

By Lemma 4.2, there are $s - 1$ linearly independent rows in A. Without loss of generality, the first $s - 1$ rows are linearly independent. Since the determinant is a continuous function, a sufficiently small perturbation of A leaves the first $s-1$ rows of A linearly independent. Therefore, there exists a sufficiently small neighborhood U of G, in which every game G' determines a matrix A' with $s-1$ linearly independent first rows, that is, $rank(A') \geq s-1$. (Notice that if G' is in ε-neighborhood of G then A' is in 2ε-neighborhood of the matrix A.) The following claim establishes that this latter inequality reduces to an equality in a whole neighborhood of G.

Lemma 4.3. *There exists a neighborhood V of the game G in which for every game, the matrix that defines its correlated equilibria is of rank $s-1$.*

Proof. As mentioned above, there exists a neighborhood of G, in which every game G' determines a matrix A' satisfying $s-1 \leq rank(A') \leq s$. Therefore, it suffices to show that in a sufficiently small neighborhood, every game G' possesses a vector $x' \in \mathbb{R}^{m^2}, x' \neq 0$, satisfying $A'x' = 0$. The game G has a unique, completely mixed Nash equilibrium x_0. The Nash correspondence is uppersemicontinuous. Therefore, in a sufficiently small neighborhood of G, for every game G', all Nash equilibria are completely mixed. Each G' in the neighborhood has at least one Nash equilibrium, say x'. By Lemma 4.1, $A'x' = 0$. As $s-1 \leq rank(A')$, we obtain that $rank(A') = s-1$. □

In order to complete the proof, we need the following theorem and the consequent lemmas.

Stiemke's Theorem. *For each given matrix B, either $Bx \gneqq 0$ has a solution x, or $yB = 0, y > 0$ has a solution y, but not both.*

Proof. See, e.g., Mangasarian [1969], p. 32. □

Lemma 4.4. *There exists $y \in \mathbb{R}^{2m(m-1)}$, $y > 0$, such that $yA = 0$.*

Proof. According to Stiemke's Theorem, either such a y exists or there exists $x \in \mathbb{R}^{m^n}$ satisfying $Ax \gneqq 0$. Suppose that such an x exists. As $Ax_0 = 0$ then necessarily x is not proportional to x_0 and for every $0 < \alpha < 1$, the vector $x_\alpha = \alpha x + (1-\alpha)x_0$ satisfies $Ax_\alpha \gneqq 0$. Since $x_0 > 0$, there exists a small enough α_0 , $0 < \alpha_0 < 1$ such that $x_{\alpha_0} > 0$. The vector $z = \frac{x_{\alpha_0}}{\|x_{\alpha_0}\|} \neq x_0$ is a probability vector satisfying $Az \gneqq 0$, i.e., z is a correlated equilibrium, which contradicts the uniqueness of x_0. Hence, such a y exists. □

Lemma 4.5. *There exists a neighborhood W of G, in which for every game G', there is $y' > 0$ such that $y'A' = 0$.*

Proof. For simplicity assume that $A \in \mathrm{M}_{k\times l}$. Since $rank(A) = s-1$, A contains a regular sub-matrix $A_0 \in \mathrm{M}_{s-1\times s-1}$. W.l.o.g. $A = \begin{pmatrix} A_0 & \cdot \\ \cdot & \cdot \end{pmatrix}$. Let $A = \begin{pmatrix} A_0 & \cdot \\ A_1 & \cdot \end{pmatrix}$, where $A_1 \in \mathrm{M}_{k-s+1\times s-1}$. By Lemma 4.4, there exists $y \in \mathbb{R}^k$, $y > 0$, such that $yA = 0$. Let $y_0 = (y_1, ..., y_{s-1})$, $y_1 = (y_s, ..., y_k)$. That is, $y = (y_0, y_1)$. Since $yA = 0$, we have in particular $y_0A_0 + y_1A_1 = 0$, which implies that $y_0 = -y_1A_1{A_0}^{-1}$.

The continuity of the determinant function together with Lemma 4.3 imply that there exists a sufficiently small neighborhood U of A, in which every matrix A' satisfies the following: $rank(A') = s-1$ and A' is of the form $A' = \begin{pmatrix} A'_0 & \cdot \\ A'_1 & \cdot \end{pmatrix}$, where $A'_0 \in \mathrm{M}_{s-1\times s-1}$ is *regular* and $A'_1 \in \mathrm{M}_{k-s+1\times s-1}$. Let $A' \in U$. The regularity of A'_0 implies that the first $s-1$ columns of A' are linearly independent and form a basis for the column space of A'. I.e., each one of the last $l-s+1$ columns of A' is a linear combination of the first $s-1$ columns. Thus a vector $v \in \mathbb{R}^k$ satisfies $vA' = 0$ iff v is orthogonal to the first $s-1$ columns of A'. Define $y'_1 = y_1$ and $y'_0 = -y_1A'_1{A'_0}^{-1}$. The following holds: $y'_0A'_0 + y'_1A'_1 = 0$. In other words, the vector $y' = (y'_0, y'_1)$ is orthogonal to the first $s-1$ columns of A' and therefore satisfies $y'A' = 0$. As y' depends continuously on A', we obtain that for sufficiently small $\delta > 0$, if $\|A - A'\| < \delta$ then $y' > 0$. Thus there is a sufficiently small neighborhood W of G, in which for every game G', there is $y' > 0$ such that $y'A' = 0$. □

We can now, using the above lemmata, complete the proof of Theorem A.

Proof of Theorem A. According to Stiemke's Theorem, for every game G' in W specified in Lemma 4.5, there exists no $x \in \mathbb{R}^{m^n}$ satisfying $A'x \gneqq 0$. Therefore, for G' in W, if x' is a correlated equilibrium of G' and A' is the matrix defining $CE(G')$, then $rank(A') = s-1$ and $A'x' = 0$. In particular, x' is a unique correlated equilibrium of G'. This completes the proof of Theorem A. □

5 Proof of Theorem B

We first show the examples in the cases where $m = 3, 4$. The example in the 3×3 case is "Rock, Scissors, Paper":

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

where the unique Nash equilibrium is $((\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3}))$, and this is also the unique correlated equilibrium. The example in the 4×4 case is the following zero-sum game:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

where the unique Nash equilibrium and the unique correlated equilibrium is $((\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}),(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}))$. For general m, we now construct a zero-sum game, represented by a game matrix $G \in \mathrm{M}_{m\times m}$:

$$G_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j-1, \\ 0 & \text{otherwise;} \end{cases}$$

i.e., G is of the form:

$$G = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

In order to prove that G has a unique correlated equilibrium we shall observe the inequalities defining $CE(G)$. For the distribution

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,m} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,m} \\ \dots & \dots & \dots & \dots \\ p_{m,1} & p_{m,2} & \cdots & p_{m,m} \end{pmatrix}$$

to be a correlated equilibrium distribution, there are $2m(m-1)$ inequalities that it must satisfy. Let us look at the $m-1$ inequalities that specify the conditions under which the row player is not to deviate from his first strategy:

$$\begin{aligned} p_{1,1} - p_{1,2} - p_{1,2} + p_{1,3} &\geq 0, \\ p_{1,1} - p_{1,2} - p_{1,3} + p_{1,4} &\geq 0, \\ p_{1,1} - p_{1,2} - p_{1,4} + p_{1,5} &\geq 0, \\ &\vdots \\ p_{1,1} - p_{1,2} - p_{1,m-1} + p_{1,m} &\geq 0, \\ p_{1,1} - p_{1,2} - p_{1,m} + p_{1,1} &\geq 0. \end{aligned} \tag{5.1}$$

If we sum these $m-1$ inequalities, we obtain: $mp_{11} - mp_{12} \geq 0$, which implies that $p_{1,1} \geq p_{1,2}$. Similarly, we obtain:

$$\begin{array}{r} \forall i = 1, ..., m-1, \quad p_{i,i} \geq p_{i,i+1} \\ p_{m,m} \geq p_{m,1} \end{array}.$$

A similar calculation for the column player implies that:

$$\begin{array}{r} \forall i = 2, ..., m, \quad p_{i-1,i} \geq p_{i,i} \\ p_{m,1} \geq p_{1,1} \end{array}.$$

We obtain a chain of inequalities:

$$p_{1,1} \geq p_{1,2} \geq p_{2,2} \geq p_{2,3} \geq p_{3,3} \geq ... \geq p_{m-1,m} \geq p_{m,m} \geq p_{m,1} \geq p_{1,1}.$$

As the two ends of the chain coincide, this is a chain of equalities:

$$z := p_{1,1} = p_{1,2} = p_{2,2} = p_{2,3} = p_{3,3} = ... = p_{m-1,m} = p_{m,m} = p_{m,1} = p_{1,1},$$

which indicates that all the inequalities in 5.1 are equalities. Our objective now is to show that for every $i, j = 1, ..., m, \quad p_{i,j} = z$. The first equality in 5.1 implies that $p_{1,3} = z$, the second equality in 5.1 implies that $p_{1,4} = z$, and so forth. The $m-2$ equality in 5.1 implies that $p_{1,m} = z$, and thus all the elements in the first row are equal to z. By repeating the same calculations for each row, we obtain that for every $i, j = 1, ..., m, \quad p_{i,j} = z$. Hence, necessarily for every $i, j = 1, ..., m, \quad p_{i,j} = 1/m^2$. Since this is a unique correlated equilibrium, it is also a unique Nash equilibrium. This equilibrium is a completely mixed one; hence the result follows.[4]

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