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by

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Bargaining Sets of Voting Games*

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Abstract

Let A be a finite set of $m \geq 3$ alternatives, let N be a finite set of $n \geq 3$ players and let R^N be a profile of linear preference orderings on A of the players. Throughout most of the paper the considered voting system is the majority rule. Let u^N be a profile of utility functions for R^N . Using α -effectiveness we define the NTU game V_{u^N} and investigate its Aumann-Davis-Maschler and Mas-Colell bargaining sets. The first bargaining set is nonempty for $m = 3$ and it may be empty for $m \geq 4$. Moreover, in a simple probabilistic model, for fixed m , the probability that the Aumann-Davis-Maschler bargaining set is nonempty tends to one if n tends to infinity.

The Mas-Colell bargaining set is nonempty for $m \leq 5$ and it may be empty for $m \geq 6$. Moreover, we prove the following startling result: The Mas-Colell bargaining set of any simple majority voting game derived from the k -th replication of R^N is nonempty, provided that $k \geq n + 2$.

We also compute the NTU games which are derived from choice by plurality voting and approval voting, and we analyze some interesting examples.

Keywords: NTU game, bargaining set, majority rule, plurality voting, approval voting

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1 Introduction

The Voting Paradox prevents us from applying the majority voting rule to choice problems with more than two alternatives. The standard way to avoid the paradox is to assume that the preferences of the voters are restricted so that the method of decision by majority yields no cycles (see Gaertner (2001) for a recent comprehensive survey). In this paper we follow a different path. It is well-known that the Voting Paradox is equivalent to the emptiness of the core of the corresponding cooperative majority voting game. We investigate various bargaining sets which include the core.

We shall now review our results. At the end of the review we shall present our main conclusions.

In Section 2 we derive the exact form of the cooperative NTU games which correspond to simple majority voting, plurality voting, and approval voting (see Brams and Fishburn (1983)). We also recall the definitions of the Aumann-Davis-Maschler and Mas-Colell bargaining sets of cooperative NTU games. Throughout our study we focus, almost exclusively, on the foregoing two bargaining sets of simple majority voting games.

The Voting Paradox with three voters and three alternatives is analyzed in Section 3. Existence of the two bargaining sets holds for the simple majority voting game. Only the Mas-Colell bargaining set is also nonempty for the plurality and approval voting games.

The bargaining sets of simple majority voting games with three alternatives are almost completely characterized in Section 4. They are always nonempty (for three alternatives).

Section 6 contains our first non-existence result and our first asymptotic result. The Aumann-Davis-Maschler bargaining set of a simple majority voting game with four alternatives may be empty. Nevertheless, in a simple probabilistic model, if the number of alternatives is fixed, then the probability that the Aumann-Davis-Maschler bargaining set is nonempty tends to one as the number of voters tends to infinity.

Our main existence theorem is contained in Section 6: The Mas-Colell bargaining set of a simple majority voting game with at most five alternatives is non-empty. For six or more alternatives the Mas-Colell bargaining set may be empty.

We conclude in Section 7 with the following result: If R^N is a profile of preferences of the members of the set N of voters, kR^N is the k -th replication of R^N , $k \geq n + 2$, where n is the number of voters, then the Mas-Colell bargaining set of any simple majority voting game that is derived from kR^N is nonempty.

Let (N, V) be a simple majority voting game and let x be an individually rational payoff vector. The vector x is in a bargaining set if (i) x is (weakly) Pareto optimal and if (ii) for every objection (in the sense of the bargaining set) there is a counter objection. Our study proves

that the tension between (i) and (ii) is so strong that for six or more alternatives all bargaining sets may be empty. This is our first conclusion. Our second conclusion is more vague: If the number of players tends to infinity and the number of alternatives is held fixed, then the bargaining set of (simple majority) voting games are likely to be non-empty.

2 Preliminaries

Let $N = \{1, \dots, n\}$, $n \geq 3$, be a set of voters, also called players, and let $A = \{a_1, \dots, a_m\}$, $m \geq 3$, be a set of m alternatives. For $S \subseteq N$ we denote by \mathbb{R}^S the set of all real functions on S . So \mathbb{R}^S is the $|S|$ -dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D .) If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x^i \geq y^i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$ and we write $x \gg y$ if $x^i > y^i$ for all $i \in S$. Denote $\mathbb{R}_+^S = \{x \in \mathbb{R}^S \mid x \geq 0\}$. A set $C \subseteq \mathbb{R}^S$ is comprehensive if $x \in C$, $y \in \mathbb{R}^S$, and $y \leq x$ imply that $y \in C$. An *NTU game* with the player set N is a pair (N, V) where V is a function which associates with every *coalition* S (that is, $S \subseteq N$ and $S \neq \emptyset$) a set $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \emptyset$, such that

- (1) $V(S)$ is closed and comprehensive;
- (2) $V(S) \cap (x + \mathbb{R}_+^S)$ is bounded for every $x \in \mathbb{R}^S$.

We shall focus on choice by simple majority voting, by plurality voting, and by approval voting. The corresponding three strategic game forms leading to three kinds of NTU voting games may be described as follows. The first game form consists of the voters selecting an element of A . If a strict majority of voters agrees on $\alpha \in A$, then the outcome is α ; otherwise no alternative is selected. The second game form is a multi-valued game form which differs from the first game form only inasmuch as the set of all alternatives that are announced by a maximal number of voters is selected. In the third game form each voter has to announce a nonempty subset – a ballot – of alternatives. The outcome is the set of alternatives that are members of a maximal number of ballots.

We shall now assume that each $i \in N$ has a linear preference R^i on A . Thus, for every $i \in N$, R^i is a complete, transitive, and antisymmetric binary relation on A . Moreover, let u^i , $i \in N$, be a utility function that represents R^i . We shall always assume that

$$\min_{\alpha \in A} u^i(\alpha) = 0 \text{ for all } i \in N. \quad (2.1)$$

As we are going to break ties by even-chance lotteries, we shall further assume that the utilities are *weakly cardinal*, that is, they satisfy the expected utility hypothesis for even-chance lotteries (see Fishburn (1972)). For each of the three strategic game forms any utility profile $u^N = (u^i)_{i \in N}$

that satisfies the foregoing assumptions determines its corresponding strategic game. These considerations motivate to define the cooperative NTU *voting games* that are associated (via α -effectiveness) with our strategic games. Indeed, let u^N be a utility profile that satisfies (2.1). The NTU game (N, V_{u^N}) associated with choice by simple majority voting and called *simple majority voting game* (see Aumann (1967)) is defined by

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid x \leq 0\} \text{ if } S \subseteq N, 1 \leq |S| \leq \frac{n}{2}; \quad (2.2)$$

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid \exists \alpha \in A \text{ such that } x \leq u^S(\alpha)\} \text{ if } S \subseteq N, |S| > \frac{n}{2}. \quad (2.3)$$

The coalition function of the *plurality voting game*, that is, the NTU game associated with choice by plurality voting, is denoted by $V_{u^N}^{pl}$ and it may differ from V_{u^N} only for coalitions $S \subseteq N$ such that $|S| = n/2$ and for the grand coalition N . Indeed, we define

$$V_{u^N}^{pl}(S) = \left\{ x \in \mathbb{R}^S \mid \exists \alpha \in A \text{ such that } x \leq \frac{1}{2}u^S(\alpha) \right\} \text{ for all } S \subseteq N, |S| = \frac{n}{2}, \quad (2.4)$$

and

$$V_{u^N}^{pl}(N) = \left\{ x \in \mathbb{R}^N \mid \begin{array}{l} \exists B \subseteq A \text{ such that } 1 \leq |B| \leq n, \\ \left(\left[\frac{n}{|B|} \right] - 1 \right) |A| + |B| \geq n, \text{ and } x \leq \frac{\sum_{\beta \in B} u^N(\beta)}{|B|} \end{array} \right\}, \quad (2.5)$$

where $[r]$ denotes the largest integer less than or equal to r . Indeed, if $|S| = n/2$ and all members of S select the same alternative α , then a player $i \in S$ cannot be prevented from the utility $u^i(\alpha)/2$ even if all members of $N \setminus S$ select i 's worst alternative (see (2.1)). Moreover, if B is the set of alternatives that are announced by a maximal number t of voters, then $0 \leq n - t|B| \leq (t-1)(|A| - |B|)$ and, hence, $t \leq [n/|B|]$ and

$$n - |B| \leq ([n/|B|] - 1)|A|. \quad (2.6)$$

If $B \subseteq A$ satisfies (2.6), then there exists a profile of strategies that results in the outcome B .

Now, if approval voting is employed, if $S \subseteq N$ satisfies $|S| = n/2$, and if each member j of S selects a ballot B^j , then the strategies of the players in $N \setminus S$ may induce the following sets of outcomes: (1) Any subset of $\bigcup_{j \in S} B^j$ and (2) any superset of $\bigcap_{j \in S} B^j$. Hence, if $i \in S$, then $N \setminus S$ may prevent i from receiving more than the utility

$$\min \left\{ \min_{\beta \in \bigcup_{j \in S} B^j} u^i(\beta), \min_{C \supseteq \bigcap_{j \in S} B^j} \sum_{\gamma \in C} \frac{u^i(\gamma)}{|C|} \right\} \leq \min \left\{ \min_{\beta \in B^j} u^i(\beta), \min_{C \supseteq B^j} \sum_{\gamma \in C} \frac{u^i(\gamma)}{|C|} \right\} \quad \forall j \in S.$$

Also, if all members of the grand coalition select $B \subseteq A$, then the resulting utility profile is $\sum_{\beta \in B} u^N(\beta)/|B|$. Hence, the NTU game associated with choice by approval voting, $(N, V_{u^N}^{ap})$, called *approval voting game*, differs from (N, V_{u^N}) only inasmuch as for any $S \subseteq N, |S| = \frac{n}{2}$,

$$V_{u^N}^{ap}(S) = \left\{ x \in \mathbb{R}^S \mid \exists \emptyset \neq B \subsetneq A \text{ such that } x^i \leq \min \left\{ \min_{\beta \in B} u^i(\beta), \min_{\emptyset \neq C \subseteq A \setminus B} \frac{\sum_{\beta \in B \cup C} u^i(\beta)}{|B| + |C|} \right\} \quad \forall i \in S \right\}, \quad (2.7)$$

and

$$V_{u^N}^{ap}(N) = \left\{ x \in \mathbb{R}^N \mid \exists \emptyset \neq B \subseteq A \text{ such that } x \leq \frac{\sum_{\beta \in B} u^N(\beta)}{|B|} \right\}. \quad (2.8)$$

Hence, for each coalition S , $V_{u^N}(S)$ (or $V_{u^N}^{pl}(S)$, $V_{u^N}^{ap}(S)$, respectively) consists of all vectors $x \in \mathbb{R}^S$ that S can get at least, regardless of the strategies chosen by the members of $N \setminus S$, with respect to choice by simple majority voting (or plurality voting, approval voting, respectively). Note that the selection of no alternative in the context of choice by simple majority voting is assumed to result in the utility 0 for each voter.

Notation 2.1 In the sequel let $L = L(A)$ denote the set of linear preferences on A . If $R^N \in L^N$, then denote

$$\mathcal{U}^{R^N} = \{(u^i)_{i \in N} \mid u^i \text{ is a representation of } R^i \text{ satisfying (2.1)} \forall i \in N\}.$$

Remark 2.2 Let $R^N \in L^N$. Then the associated simple majority voting games are derived from each other by ordinal transformations. The associated plurality voting games and the associated approval voting games may not be derived from each other by an ordinal transformation, because weakly cardinal utilities may not be covariant under monotone transformations.

Let (N, V) be an NTU game. The pair (N, V) is *zero-normalized* if $V(\{i\}) = -\mathbb{R}_+^{\{i\}} (= \{x \in \mathbb{R}^i \mid x \leq 0\})$ for all $i \in N$. Also, (N, V) is *superadditive* if for every pair of disjoint coalitions S, T , $V(S) \times V(T) \subseteq V(S \cup T)$. It should be remarked that the three foregoing NTU games are zero-normalized and superadditive.

Now we shall recall the definitions of two bargaining sets introduced by Davis and Maschler (1967) and by Mas-Colell (1989). Let (N, V) be a zero-normalized NTU game and $x \in \mathbb{R}^N$. We say that x is

- *individually rational* if $x \geq 0$;
- *Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if $y \in V(N)$ and $y \geq x$ imply $x = y$;
- *weakly Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if for every $y \in V(N)$ there exists $i \in N$ such that $x^i \geq y^i$;
- a *preimputation* if x is weakly Pareto optimal in $V(N)$;
- an *imputation* if x is an individually rational preimputation.

A pair (P, y) is an *objection* at x if $\emptyset \neq P \subseteq N$, y is Pareto optimal in $V(P)$, and $y > x^P$. An objection (P, y) is *strong* if $y \gg x^P$. The pair (Q, z) is a *weak counter objection* to the objection (P, y) if $Q \subseteq N$, $Q \neq \emptyset, P$, if $z \in V(Q)$, and if $z \geq (y^{P \cap Q}, x^{Q \setminus P})$. A weak counter

objection (Q, z) is a *counter objection* to the objection (P, y) if $z > (y^{P \cap Q}, x^{Q \setminus P})$. A strong objection (P, y) is *justified in the sense of the bargaining set* if there exist players $k \in P$ and $\ell \in N \setminus P$ such that there does not exist any weak counter objection (Q, z) to (P, y) satisfying $\ell \in Q$ and $k \notin Q$. The *bargaining set* of (N, V) , $\mathcal{M}(N, V)$, is the set of all imputations x that do not have strong justified objections at x in the sense of the bargaining set (see Davis and Maschler (1967)). An objection (P, y) is *justified in the sense of the Mas-Colell bargaining set* if there does not exist any counter objection to (P, y) . The *Mas-Colell bargaining set* of (N, V) , $\mathcal{MB}(N, V)$, is the set of all imputations x that do not have a justified objection at x in the sense of the Mas-Colell bargaining set (see Mas-Colell (1989)).

Notation 2.3 If $R^N \in L^N$ and $\alpha, \beta \in A$, $\alpha \neq \beta$, then α *dominates* β (abbreviated $\alpha \succ_{R^N} \beta$) if $|\{i \in N \mid \alpha R^i \beta\}| > \frac{n}{2}$. For $R \in L$ and for $k \in \{1, \dots, m\}$, let $t_k(R)$ denote the k -th alternative in the order R . Also, for $B \subseteq A$ let $R|_B$ denote the restriction of R to B .

Remark 2.4 Let $u^N \in \mathcal{U}^{R^N}$, let $B \subsetneq A$, let $i \in N$, and let

$$(t_1(R^i|_{A \setminus B}), \dots, t_{m-|B|}(R^i|_{A \setminus B})) = (\alpha_1, \dots, \alpha_{m-|B|})$$

be the vector of alternatives in $A \setminus B$ ordered by R^i . For $j = 1, \dots, m - |B|$, define

$$z_j = \frac{1}{m - j + 1} \left(\sum_{\beta \in B} u^i(\beta) + \sum_{k=j}^{m-|B|} u^i(\alpha_k) \right).$$

It can be deduced that the sequence $(z_j)_{j=1}^{m-|B|}$ is unimodal, i.e., there exists $t \in \{1, \dots, m - |B|\}$ such that $z_k > z_{k+1}$ for $k \leq t - 1$, $z_k < z_{k+1}$ for $k > t$, and $z_t \leq z_{t+1}$ if $t < m - |B|$. We conclude that

$$\min_{\emptyset \neq C \subseteq A \setminus B} \sum_{\beta \in B \cup C} \frac{u^i(\beta)}{|B| + |C|} = \min_{j=1, \dots, m-|B|} z_j = z_t.$$

This remark enables us to easily compute (2.7), taking (2.1) into account, that is,

$$t_m(R^i) \in B \Rightarrow \min_{\beta \in B} u^i(\beta) = 0 \leq z_t, \quad (2.9)$$

$$t_m(R^i) \notin B \Rightarrow u^i(\alpha_{m-|B|}) = u^i(t_m(R^i)) = 0. \quad (2.10)$$

We shall say that an alternative $\alpha \in A$ is a *weak Condorcet winner* (with respect to R^N) if $\beta \not\succeq_{R^N} \alpha$ for all $\beta \in A$.

3 An Example: The Voting Paradox

In this section we shall compute the bargaining sets of the Voting Paradox and interpret the results.

Table 3.1: Preference Profile of the Voting Paradox

R^1	R^2	R^3
a	c	b
b	a	c
c	b	a

Let $A = \{a, b, c\}$, let $n = 3$, and let $R^N \in L^N$ be given by Table 3.1.

For $i \in N$ let u^i be a utility representation of R^i satisfying (2.1) and let $V = V_{u^N}$ (see (2.2) and (2.3)).

We claim that $\mathcal{M}(N, V) = \{0\}$. Indeed, it is straightforward to verify that $0 \in \mathcal{M}(N, V)$. In order to show the opposite inclusion let $x \in \mathcal{M}(N, V)$. Then there exists $\alpha \in A$ such that $x \leq u^N(\alpha)$. Without loss of generality we may assume that $\alpha = a$. Assume, on the contrary, that $x > 0$. If $x^1 > 0$, then $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection of 3 against 1 at x in the sense of the bargaining set. If $x^1 = 0$ and, hence, $x^2 > 0$, then $(\{1, 3\}, u^{\{1,3\}}(b))$ is a justified objection of 1 against 2.

As we shall see in Section 4, if $n > 3$ then \mathcal{M} may contain many non-zero vectors. In our present example there are not enough partners for counter objections.

In order to compute the Mas-Colell bargaining set, we define $x = (u^1(b), u^2(a), 0)$ and claim that $x \in \mathcal{MB}(N, V)$. Indeed, let (P, y) be an objection at x . Then $|P| \geq 2$. As y is Pareto optimal in $V(P)$, $y \in \{u^P(\alpha) \mid \alpha \in A\}$. If $y = u^P(a)$, then (P, y) is countered by $(\{2, 3\}, u^{\{2,3\}}(c))$. If $y = u^P(b)$, then $y > x^P$ implies that $P = \{1, 3\}$. In this case (P, y) is countered by $(\{1, 2\}, u^{\{1,2\}}(a))$. Finally, if $y = u^P(c)$, then $y > x$ implies that $P = \{2, 3\}$ and that (P, y) is countered by $(\{1, 3\}, u^{\{1,3\}}(b))$.

Now, we shall show that x is Pareto optimal in $\mathcal{MB}(N, V)$. Indeed, if $\tilde{x} \in V(N)$ satisfies $\tilde{x} > x$, then $\tilde{x} \leq u^N(a)$, because $x \not\leq u^N(b)$ and $x \not\leq u^N(c)$. Hence, $x^1 = u^1(b) < \tilde{x}^1$ and $x^{\{2,3\}} = u^{\{2,3\}}(a) = \tilde{x}^{\{2,3\}}$. Thus, $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection at \tilde{x} .

In order to show that every $\hat{x} \in \mathbb{R}^N$ satisfying $0 \leq \hat{x} \leq x$ is an element of $\mathcal{MB}(N, V)$, it should be noted that each objection at \hat{x} is also an objection at x if $\hat{x}^1 > 0$ and $\hat{x}^2 > 0$. If $\hat{x}^1 = 0$ and $\hat{x}^2 > 0$, then the additional objections are of the form $(P, u^P(c))$ for some $P \subseteq N$ and these objections can be countered by $(\{1, 3\}, u^{\{1,3\}}(b))$. Similarly, if $\hat{x}^1 > 0$ and $\hat{x}^2 = 0$, then the additional objections can be countered by $(\{1, 2\}, u^{\{1,2\}}(a))$. Finally, if $\hat{x} = 0$, then each additional objection can be countered by one of the foregoing pairs $(\{1, 3\}, u^{\{1,3\}}(b))$ or

$(\{1, 2\}, u^{\{1,2\}}(a))$.

Two other Pareto optimal members of $\mathcal{MB}(N, V)$ are

$$y = (u^1(b), 0, u^3(c)) \text{ and } z = (0, u^2(a), u^3(c)).$$

Also, if $\tilde{x} \in V(N)$ satisfies $\tilde{x}^1 > u^1(b)$ and $\tilde{x}^2 = \tilde{x}^3 = 0$, then $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection in the sense of the Mas-Colell bargaining set at \tilde{x} . Similarly, for any $\tilde{y}, \tilde{z} \in \mathcal{MB}(N, V)$ such that $\tilde{y}^{\{1,2\}} = 0$ and $\tilde{z}^{\{1,3\}} = 0$ we may deduce that $\tilde{y} \leq y$ and $\tilde{z} \leq z$. We conclude that $\mathcal{MB}(N, V)$ is the intersection of \mathbb{R}_+^N and the comprehensive hull of $\{x, y, z\}$.

Discussion: The singleton $\mathcal{M}(N, V)$ tells us that in order to achieve (coalitional) stability the players have to give up any profit above their individually protected levels of utility. There is no hint how an alternative of A will be chosen. The message of $\mathcal{MB}(N, V)$ is much more detailed. For example, the element $x = (u^1(b), u^2(a), 0)$ tells us that the alternative a may be chosen provided player 1 disposes of $u^1(a) - u^1(b)$ utiles. Thus, we also see here that lower utility levels guarantee stability. Actually, x implies that there is an agreement between 1 and 2, the alternative a is chosen as a result of the agreement, and the utility of 1 is reduced (because of the agreement) from $u^1(a)$ to $u^1(b)$. Note that cooperative game theory does not specify the details of agreements that support stable payoff vectors.

We shall now show that, for any utility representation $u^N \in \mathcal{U}^{R^N}$, the Mas-Colell bargaining sets of the associated plurality and approval voting games are nonempty. Indeed, we distinguish two possibilities. If $u^i(t_1(R^i)) \leq 2u^i(t_2(R^i))$ for some $i \in N$, let us say for $i = 1$, then we claim that $x = (u^1(b), u^2(a), 0)$ is an element of $\mathcal{MB}(N, V_{u^N}^{pl})$ and of $\mathcal{MB}(N, V_{u^N}^{ap})$. Indeed, x is weakly Pareto optimal, because if $z \in V_{u^N}^{pl}(N)$ or $z \in V_{u^N}^{ap}(N)$ such that $z^1 > x^1$, then $z \leq u^N(a)$. Moreover, let (P, y) be an objection at x . If $y \in V(P)$, then we may proceed as in the proof that $x \in \mathcal{MB}(N, V)$. If $y \notin V(P)$, then $P = N$ and $y \leq \frac{1}{3}(u^N(a) + u^N(b) + u^N(c))$ or $y \leq \frac{1}{2}(u^N(a) + u^N(c))$. In both subcases $(\{1, 3\}, u^{\{1,3\}}(b))$ is a counter objection to (P, y) . Now, if $u^i(t_1(R^i)) > 2u^i(t_2(R^i))$ for all $i \in N$, then $\frac{1}{3}(u^N(a) + u^N(b) + u^N(c))$ is Pareto optimal and it belongs to the Mas-Colell bargaining sets and to the bargaining sets of the plurality and of the approval voting game, because it has no objection at all.

Finally we shall show that $\mathcal{M}(N, V_{u^N}^{pl}) = \mathcal{M}(N, V_{u^N}^{ap}) = \emptyset$ for any $u^N \in \mathcal{U}^{R^N}$ such that $u^i(t_1(R^i)) < 2u^i(t_2(R^i))$ for all $i \in N$. Let $V^{pl} = V_{u^N}^{pl}$ and $V^{ap} = V_{u^N}^{ap}$. Assume that $x \in \mathcal{M}(N, V^{pl})$. We distinguish the following 2 possibilities. If $x \leq u^N(\alpha)$ for some $\alpha \in A$, then we may assume that $\alpha = a$. As in the proof of the inclusion $\mathcal{M}(N, V) \subseteq \{0\}$, we conclude that $x = 0$ which now contradicts the assumption of weak Pareto optimality. If $x \leq \frac{1}{3}(u^N(a) + u^N(b) + u^N(c))$, then our assumption on the utility representations imply that $x^i \ll (u^1(b), u^2(a), u^3(c))$. Hence, $(\{1, 2\}, u^{\{1,2\}}(a))$ is a strong objection of 2 against 3. We conclude that $x^3 = 0$. Similarly we may deduce that $x^1 = x^2 = 0$ and, so, we have derived the desired contradiction. Now assume that $x \in \mathcal{M}(N, V^{ap})$. The cases $x \leq u^N(\alpha)$ and

$x \leq \frac{1}{3}(u^N(a) + u^N(b) + u^N(c))$ may be treated as before. Hence, just the case $x \leq \sum_{\beta \in B} \frac{u^N(\beta)}{|B|}$ for some $B \subseteq N$, $|B| = 2$, has to be considered. We may assume that $B = \{a, b\}$. Then $x \ll (u^1(a), u^2(a), u^3(c))$ and, so, $(\{1, 2\}, u^{\{1,2\}}(a))$ is a strong objection of 2 against 3. As there is a weak counter objection, $x^3 = 0$. Similarly, the fact that $(\{2, 3\}, u^{\{2,3\}}(c))$ is a strong objection of 2 against 1 implies that $x^1 = 0$. Now, $(\{1, 3\}, u^{\{1,3\}}(b))$ is a strong objection of 1 against 2 and, hence, $x^2 = 0$ and the desired contradiction has been obtained.

4 Three Alternatives

Throughout this section let $R^N \in L(A)^N$, $u^N \in \mathcal{U}^{R^N}$ (see Notation 2.1), $V = V_{u^N}$ (see (2.2) and (2.3)) and let $\succ = \succ_{R^N}$ (see Notation 2.3).

Theorem 4.1 *If $|A| = 3$, then $\mathcal{M}(N, V_{u^N}) \neq \emptyset$.*

Proof: Let $A = \{a, b, c\}$. If there exists a weak Condorcet winner $\alpha \in A$, then $u^N(\alpha) \in \mathcal{M}(N, V)$. Otherwise we may assume without loss of generality that $a \succ b$, $b \succ c$, and $c \succ a$. We claim that

$$\text{for any } \alpha \in A \text{ there exists } i \in N \text{ such that } t_3(R^i) = \alpha. \quad (4.1)$$

Indeed, if $\alpha \in \{t_1(R^i), t_2(R^i)\}$ for all $i \in N$ and if $\beta \succ \alpha$, then $|\{i \in N \mid \beta = t_1(R^i)\}| > \frac{n}{2}$ and β is a Condorcet winner which was excluded. We conclude that $0 \in \mathbb{R}^N$ is weakly Pareto optimal. Hence $0 \in \mathcal{M}(N, V)$. **q.e.d.**

Lemma 4.2 *Let $|A| = 3$ and $x \in \mathcal{M}(N, V)$. If there is no weak Condorcet winner, then $x^i \leq u^i(t_2(R^i))$ for all $i \in N$.*

Proof: Let $i \in N$ and $R^i = (\alpha, \beta, \gamma)$. Assume, on the contrary, that $x^i > u^i(\beta)$. We distinguish the following cases:

(1) $\gamma \succ \alpha$. Let $S = \{j \in N \mid \gamma R^j \alpha\}$. Then $|S| > \frac{n}{2}$ and $i \notin S$. If $k \in S$, then $(S, u^S(\gamma))$ is a strong objection of k against i . Any weak counter objection must be of the form (T, y) where $i \in T \not\prec k$ and $y^i \geq x^i$. We conclude that $y \leq u^T(\alpha)$ which is impossible.

(2) $\beta \succ \alpha$. This case may be treated similarly. **q.e.d.**

We are now ready for a partial characterization of the bargaining set. For $\alpha, \beta \in A$, $\alpha \neq \beta$, let

$$D_{\alpha\beta}(R^N) = D_{\alpha\beta} = \{i \in N \mid \alpha R^i \beta\}.$$

Theorem 4.3 Let $A = \{a, b, c\}$. Assume that $a \succ b$, $b \succ c$, $c \succ a$, and that

$$|D_{\alpha\beta}| > \frac{n}{2} + 1 \text{ for all } (\alpha, \beta) \in \{(a, b), (b, c), (c, a)\}.$$

If $x \in \mathbb{R}^N$ satisfies

$$0 \leq x \leq u^N(\alpha) \text{ for some } \alpha \in A \quad (4.2)$$

and

$$x^i \leq u^i(t_2(R^i)) \text{ for all } i \in N, \quad (4.3)$$

then $x \in \mathcal{M}(N, V)$.

Proof: By (4.1), x is weakly Pareto optimal. Assume without loss of generality that $x \leq u^N(a)$. Let (P, y) be a strong objection of k against ℓ at x . As $P \neq \emptyset$, as $y \gg x^P \geq 0$, and as y is Pareto optimal in $V(P)$, we conclude that $|P| > \frac{n}{2}$ and that there is some $\beta \in A$ such that $u^P(\beta) = y$. If $\beta = a$, then define $Q = (P \setminus \{k\}) \cup \{\ell\}$ and observe that $(Q, u^Q(a))$ is a weak counter objection to $(P, u^P(a))$. If $\beta = b$, then define $Q = (D_{ab} \setminus \{k\}) \cup \{\ell\}$ and observe that $(Q, u^Q(a))$ is a counter objection to $(P, u^P(b))$. If $\beta = c$, then the following two cases may occur. Case 1: $u^\ell(b) \geq x^\ell$. Define $Q = (D_{bc} \setminus \{k\}) \cup \{\ell\}$. We claim that in this case $(Q, u^Q(b))$ is a weak counter objection to $(P, u^P(c))$. Indeed, for all $i \in P \cap Q$, $u^i(b) > u^i(c) > x^i$. Also, if $i \in Q \setminus P$, $i \neq \ell$, then $u^i(b) > u^i(c)$ and, hence, by (4.3), $u^i(b) \geq x^i$. Case 2: $u^\ell(c) \geq x^\ell$. Define $Q = (P \setminus \{k\}) \cup \{\ell\}$ and observe that $(Q, u^Q(c))$ is a weak counter objection to $(P, u^P(c))$ in this case. **q.e.d.**

Remark 4.4 A careful inspection of the foregoing proof shows that $|D_{ca}| > \frac{n}{2} + 1$ is not used when $x \leq u^N(a)$. Thus, we have proved the following stronger result.

Corollary 4.5 Let $A = \{a, b, c\}$. Assume that $x \in \mathbb{R}^N$ satisfies $0 \leq x \leq (u^i(t_2(R^i)))_{i \in N}$ and assume that $a \succ b$, $b \succ c$, and $c \succ a$. Then $x \in \mathcal{M}(N, V)$ in each of the following three cases:

$$\begin{aligned} & (x \leq u^N(a) \text{ and } |D_{ab}|, |D_{bc}| > \frac{n}{2} + 1), \quad \text{or} \\ & (x \leq u^N(b) \text{ and } |D_{bc}|, |D_{ca}| > \frac{n}{2} + 1), \quad \text{or} \\ & (x \leq u^N(c) \text{ and } |D_{ca}|, |D_{ab}| > \frac{n}{2} + 1). \end{aligned}$$

Example 4.6 Let $N = \{1, \dots, 9\}$ and let R^N be given by Table 4.1. Then $a \succ b \succ c \succ a$. Also, $x = (\min\{u^i(a), u^i(b)\})_{i \in N} \in \mathcal{M}(N, V)$. Indeed, by (4.1), x is weakly Pareto optimal. Let $\emptyset \neq P \subsetneq N$, let $k \in P$ and $\ell \in N \setminus P$. Assume that (P, y) is a strong objection of k against ℓ at x . If $y \leq u^P(a)$, then $(N \setminus \{k\}, u^{N \setminus \{k\}}(a))$ is a weak counter objection. If $y \not\leq u^P(a)$, then $P = \{4, 5, 6, 8, 9\}$. If $k \neq 4$, then define $Q = \{1, 2, 3, 4, 7\}$, and if $k = 4$, then define $Q = \{1, 2, 3, 7, 9\}$. Observe that $(Q, u^Q(b))$ is a weak counter objection to (P, y) . Let $z = (0, 0, 0, 0, u^5(a), 0, 0, 0, 0)$. By (4.1), z is weakly Pareto optimal. Also, with $P = \{1, 2, 3, 4, 6, 7, 9\}$,

Table 4.1: Preference Profile of a 9-Person Voting Problem

R^1	R^2	R^3	R^4	R^5	R^6	R^7	R^8	R^9
a	a	b	b	c	c	a	c	b
b	b	a	c	a	b	b	a	c
c	c	c	a	b	a	c	b	a

$(P, u^P(b))$ is a justified objection at z from 1 against 5. We conclude that $z \notin \mathcal{M}(N, V)$, precisely because $|D_{ab}| = 5 < \frac{n}{2} + 1$.

Now we shall consider \mathcal{MB} . It is straightforward to verify that, for any weak Condorcet winner α with respect to R^N , $u^N(\alpha)$ belongs to $\mathcal{MB}(N, V)$ (regardless of the number of alternatives). Moreover, the following “opposite” result is true for $m = 3$.

Remark 4.7 Let $m = 3$, $R^N \in L(A)^N$, and $u^N \in \mathcal{U}^{R^N}$. If there exists a weak Condorcet winner with respect to R^N , then $\mathcal{MB}(N, V_{u^N})$ is the set of the utility profiles of all weak Condorcet winners. Indeed, let $x \in \mathcal{MB}(N, V_{u^N})$ and assume that x is not the utility profile of any weak Condorcet winner. Then $0 \leq x \leq u^N(\alpha)$ for some $\alpha \in A$. If α is a Condorcet winner, then $x < u^N(\alpha)$ by our assumption. Hence, $(N, u^N(\alpha))$ is a justified objection in this case. If α is not a weak Condorcet winner, then there exists $\beta \in A$ such that $\beta \succ_{R^N} \alpha$. Hence, let $P = \{i \in N \mid \beta R^i \alpha\}$ and observe that $(P, u^P(\beta))$ is an objection at x . As $x \in \mathcal{MB}(N, V_{u^N})$, there exist a counter objection $(Q, u^Q(\gamma))$ to $(P, u^P(\beta))$. We conclude that $\gamma \neq \alpha$ and $\gamma \succ_{R^N} \beta$. According to our assumption there is a weak Condorcet winner. Hence γ is a - the unique - weak Condorcet winner and $(Q, u^Q(\gamma))$ is a justified objection at x in the sense of the Mas-Colell bargaining set which is impossible.

The following example shows that the foregoing remark is not valid if $m \geq 4$.

Example 4.8 Let $n = 6$, let $A = \{a, b, c, d\}$, and let R^N be given by Table 4.2.

Similarly as in Section 3 we may deduce that $(u^1(c), u^2(b), u^3(a), 0, 0, 0) \in \mathcal{MB}(N, V_{u^N})$. However d is the unique weak Condorcet winner. Moreover, this example may easily be generalized to $m > 4$ by adding alternatives a_5, \dots, a_m such that $t_j(R^i) = a_j$ for all $i \in N$ and $j = 5, \dots, m$.

We shall now return to three alternatives.

Example 4.9 Let $n = 4$ and let R^N be given by Table 4.3.

Table 4.2: Preference Profile of a 6-Person Voting Problem

R^1	R^2	R^3	R^4	R^5	R^6
a	c	b	d	d	d
b	a	c	a	c	b
c	b	a	b	a	c
d	d	d	c	b	a

Table 4.3: Preference Profile of a 4-Person Voting Problem

R^1	R^2	R^3	R^4
a	a	c	c
b	b	b	b
c	c	a	a

Then $x = (\min\{u^i(b), u^i(a)\})_{i \in N} \in \mathcal{M}(N, V)$, because there is no strong objection at x . In view of Remark 4.7, $x \notin \mathcal{MB}(N, V) = \{u^N(a), u^N(b), u^N(c)\}$.

Remark 4.7 and the following result partially characterize \mathcal{MB} in the case of three alternatives.

Theorem 4.10 *If $|A| = 3$ and if there is no weak Condorcet winner with respect to R^N and if $x \in \mathbb{R}^N$ satisfies*

$$0 \leq x^i \leq u^i(t_2(R^i)) \text{ for all } i \in N; \tag{4.4}$$

$$\text{there exists } \alpha \in A \text{ such that } x \leq u^N(\alpha), \tag{4.5}$$

then $x \in \mathcal{MB}(N, V)$.

Proof: Let $A = \{a, b, c\}$. Without loss of generality we assume that $a \succ b \succ c \succ a$ and that $x \leq u^N(a)$. Let $(P, u^P(\alpha))$ be an objection at x . If $\alpha = a$, then $(D_{ca}, u^{D_{ca}}(c))$ is a counter objection. If $\alpha = b$, then $(D_{ab}, u^{D_{ab}}(a))$ is a counter objection. Finally, if $\alpha = c$, then we claim that $(Q, u^Q(b))$, where $Q = D_{bc}$, is a counter objection. Indeed, if $i \in P \cap Q$, then $u^i(b) > u^i(c) \geq x^i$. If $i \in Q \setminus P$, then $u^i(b) > u^i(c)$ and, hence, by (4.4), $u^i(b) \geq x^i$. As $|P|, |Q| > n/2$, $P \cap Q \neq \emptyset$. Thus, $u^Q(b) > (u^{P \cap Q}(c), x^{Q \setminus P})$ and $(Q, u^Q(b))$ is a counter objection and the proof is complete. **q.e.d.**

Corollary 4.11 *If $|A| = 3$ and there is no weak Condorcet winner with respect to R^N , then $\mathcal{M}(N, V) \subseteq \mathcal{MB}(N, V)$.*

The following example shows that the inclusion in the foregoing corollary may be strict.

Example 4.12 (Example 4.6 continued) We claim that

$$\hat{x} = (u^1(b), u^2(b), u^3(b), u^4(c), 0, u^6(b), u^7(b), 0, u^9(c)) \in \mathcal{MB}(N, V).$$

Indeed, if $(P, u^P(\alpha))$ is an objection at \hat{x} , then there exists $\beta \succ \alpha$. The reader may easily check that $(D_{\beta\alpha}, u^{D_{\beta\alpha}}(\beta))$ is a counter objection. Thus \mathcal{MB} may violate (4.4).

5 The Bargaining Set for Four and More Alternatives

By means of an example we shall show that $\mathcal{M}(N, V_{u^N})$ may be empty for any $u^N \in \mathcal{U}^{R^N}$, provided $|A| \geq 4$.

Example 5.1 Let $A = \{a, b, c, d\}$, let $n = 3$, let R^N be given by Table 5.1, let $u^N \in \mathcal{U}^{R^N}$, and

Table 5.1: Preference Profile of a 4-Alternative Voting Problem

R^1	R^2	R^3
a	c	b
b	a	c
d	d	d
c	b	a

let $V = V_{u^N}$. We claim that $\mathcal{M} = \emptyset$. Let x be an imputation of (N, V) . In order to show that $x \notin \mathcal{M}(N, V)$ we may assume without loss of generality that $x^1 \geq u^1(d)$. We distinguish the following possibilities:

- (1) $x \leq u^N(a)$ or $x \leq u^N(d)$. Then $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection (in the sense of the bargaining set) of 3 against 1.
- (2) $x \leq u^N(b)$. If $x^3 < u^3(c)$, then we may use the same justified strong objection as in the first possibility. If $x^3 \geq u^3(c)$, then $(\{1, 2\}, u^{\{1,2\}}(a))$ is a justified objection of 2 against 3.

Example 5.1 shows the tension between (weak) Pareto optimality and stability (à la Aumann and Maschler (1964)) may result in an empty bargaining set.

Example 5.1 may be generalized to any number $m \geq 4$ of alternatives. Indeed, let $A = \{a, b, c, d_1, \dots, d_k\}$, where $k = m - 3$, and define R^N by

$$R^1 = (a, b, d_1, \dots, d_k, c),$$

$$R^2 = (c, a, d_1, \dots, d_k, b),$$

$$R^3 = (b, c, d_1, \dots, d_k, a),$$

and note that $\mathcal{M}(N, V_{u^N}) = \emptyset$ for any $u^N \in \mathcal{U}^{R^N}$. More interestingly, Example 5.1 can be generalized to yield an empty bargaining set for simple majority voting games on four alternatives with infinitely many numbers of voters.

Example 5.2 (Example 5.1 generalized) Let

$$R_1 = (a, b, d, c), \quad R_2 = (a, c, d, b), \quad R_3 = (b, a, d, c),$$

$$R_4 = (b, c, d, a), \quad R_5 = (c, a, d, b), \quad R_6 = (c, b, d, a),$$

and let $k \in \mathbb{N}$. Let $N = \{1, \dots, 6k - 3\}$ and let $R^N \in L^N$ satisfy

$$|\{j \in N \mid R^j = R_i\}| = \begin{cases} k & , \text{ if } i = 1, 4, 5, \\ k - 1 & , \text{ if } i = 2, 3, 6. \end{cases}$$

Then $\mathcal{M}(N, V_{u^N}) = \emptyset$ for any $u^N \in \mathcal{U}^{R^N}$. Indeed, $k = 1$ coincides with Example 5.1. The reader may check e.g. the case $k = 2$ (see Table 5.2) by repeating the arguments of Example 5.1.

Table 5.2: Preference Profile for $k = 2$

R^1	R^2	R^3	R^4	R^5	R^6	R^7	R^8	R^9
a	a	b	b	c	c	a	c	b
b	c	a	c	a	b	b	a	c
d	d	d	d	d	d	d	d	d
c	b	c	a	b	a	c	b	a

Notwithstanding Example 5.1, there is a simple probabilistic model in which most preference profiles lead to a nonempty bargaining set \mathcal{M} as the number of players becomes large. Let $|A| = m \geq 4$ and let $L(A) = L$. Assume that each $R \in L$ appears with positive probability $p_R > 0$ in the population of potential voters, where $\sum_{R \in L} p_R = 1$. Now let $(\mathcal{R}^i)_{i \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables such that $Pr(\{\mathcal{R}^i = R\}) = p_R$ for all $i \in \mathbb{N}$, $R \in L$. Call $R^N \in L^N$ *good* if for all $\alpha \in A$ there exists $i \in N$ such that $\alpha = t_m(R^i)$. If R^N is good, then $(u^i(t_m(R^i)))_{i \in N} \in \mathcal{M}(N, V_{u^N})$ for any $u^N \in \mathcal{U}^{R^N}$. By the law of large numbers, $\lim_{n \rightarrow \infty} Pr(\{\mathcal{R}^N \text{ is good}\}) = 1$, where $\mathcal{R}^N = (\mathcal{R}^1, \dots, \mathcal{R}^n)$. Hence, $\lim_{n \rightarrow \infty} Pr(\{\mathcal{M}(N, V(\mathcal{R}^N)) \neq \emptyset\}) = 1$, where $(N, V(\mathcal{R}^N))$ is a random NTU game which is a simple majority voting game V_{u^N} , $u^N \in \mathcal{U}^{R^N}$, for any realization R^N of \mathcal{R}^N .

6 \mathcal{MB} for Four and More Alternatives

We shall show that \mathcal{MB} is nonempty for any simple majority voting game on less than six alternatives. Also, we shall show that there is a simple majority voting game on six alternatives whose Mas-Colell bargaining set is empty. We shall always assume that $|A| = m \geq 3$, $R^N \in L(A)^N$, and $\succ = \succ_{R^N}$. We start with the following simple lemma.

Lemma 6.1 *Let $u^N \in \mathcal{U}^{R^N}$ and $x \in \mathbb{R}_+^N$. Assume that there is no weak Condorcet winner. If $x^i \leq u^i(t_{m-1}(R^i))$ for all $i \in N$ and if x is weakly Pareto optimal in $V_{u^N}(N)$, then $x \in \mathcal{MB}(N, V_{u^N})$.*

Proof: If (S, y) is an objection at x , then $|S| > n/2$ and there exists $\alpha \in A$ such that $u^S(\alpha) = y$. Choose $\beta \in A$ such that $\beta \succ \alpha$. Then there exists $T \subseteq N$, $|T| > n/2$ such that $u^T(\beta) \gg u^T(\alpha)$. Thus, $(T, u^T(\beta))$ is a counter objection. **q.e.d.**

Theorem 6.2 *If $m \leq 5$, then $\mathcal{MB}(N, V_{u^N}) \neq \emptyset$ for all $u^N \in \mathcal{U}^{R^N}$.*

Proof: Let $V = V_{u^N}$. By Remark 4.7 and Theorem 4.10 our claim is valid for $m = 3$. In order to prove the theorem For $m = 4$ we may, by Remark 4.7, assume that there is no weak Condorcet winner. Then, for each $\alpha \in A$,

$$\text{there exists } i \in N \text{ such that } \alpha \in \{t_3(R^i), t_4(R^i)\}. \quad (6.1)$$

Indeed, if for some $\alpha \in A$, $\alpha \in \{t_1(R^i), t_2(R^i)\}$ for all $i \in N$, then $\beta \succ \alpha$ implies that β is a Condorcet winner which was excluded. For $\alpha \in A$, define $x_\alpha = (\min\{u^i(\alpha), u^i(t_3(R^i))\})_{i \in N}$. By Lemma 6.1, $x_\alpha \in \mathcal{MB}(N, V)$, if x_α is weakly Pareto optimal. Hence, in order to complete the proof for $m = 4$, it suffices to show that there exists $\alpha \in A$ such that x_α is weakly Pareto

optimal. Two possibilities may occur: If there exists $\alpha \in A$ such that $\alpha \neq t_4(R^i)$ for all $i \in N$, then, by (6.1), x_α is weakly Pareto optimal. Otherwise, any x_α is weakly Pareto optimal.

Now, let $m = 5$, let $A = \{a_1, \dots, a_5\}$, and assume that $\mathcal{MB}(N, V) = \emptyset$. Then, for each $\alpha \in A$

- (1) there exists $\beta \in A$ such that $\beta \succ \alpha$ (by Remark 4.7);
- (2) $u^N(\alpha)$ is Pareto optimal (because \mathcal{MB} is nonempty when we restrict our attention to the game corresponding to the restriction of u^N to $A \setminus \{\alpha\}$).

For $\alpha \in A$ denote $\ell(\alpha) = \max\{k \in \{1, \dots, 5\} \mid \exists i \in N : t_k(R^i) = \alpha\}$. Let $\ell_{min} = \min_{\alpha \in A} \ell(\alpha)$.

We distinguish cases:

(i) $\ell_{min} \geq 4$: Then there exists a weakly Pareto optimal $x \in V(N)$ such that $x^i \leq u^i(t_4(R^i))$ for all $i \in N$ which is impossible by Lemma 6.1.

(ii) $\ell_{min} \leq 2$: Let $\alpha, \beta \in A$ such that $\ell(\alpha) = \ell_{min}$ and $\beta \succ \alpha$. Then β is a Condorcet winner, which is impossible by (1).

(iii) $\ell_{min} = 3$: Let $B = \{\beta \in A \mid \ell(\beta) = 3\}$. If $|B| = 3$, then any $\alpha \in A \setminus B$ violates Pareto optimality. If $|B| = 2$, let us say $B = \{\alpha, \beta\}$, then we may assume without loss of generality that $\alpha \neq \beta$. Let $\gamma \in A$ such that $\gamma \succ \beta$. Then none of the remaining $\delta \in A \setminus (\{\gamma\} \cup B)$ dominates any of the elements α, β, γ . By (1) we conclude that $\gamma \succ \beta \succ \alpha \succ \gamma$. Then $(\min\{u^i(\alpha), u^i(\beta)\})_{i \in N} \in \mathcal{MB}(N, V)$.

Now we turn to the case $|B| = 1$, let us say $B = \{a_3\}$. Let $\widehat{S} = \{i \in N \mid t_3(R^i) = a_3\}$. For any $k \in \widehat{S}$ there exists $x_k \in \mathbb{R}^N$ such that x_k is Pareto optimal, $x_k^k = u^k(a_3)$, and $x_k^i \leq u^i(t_4(R^i))$ for all $i \in N \setminus \{k\}$. As $x_k \notin \mathcal{MB}(N, V)$, there exists a justified objection $(S, u^S(\alpha))$ for some $S \subseteq N$, $|S| > n/2$, and some $\alpha \in A$. Let $\beta \in A$ such that $\beta \succ \alpha$. Then there exists $T \subseteq N$, $|T| > n/2$, such that $u^{S \cap T}(\beta) \gg u^{S \cap T}(\alpha)$ and $u^{T \setminus S}(\beta) \geq (u^i(t_4(R^i)))_{i \in T \setminus S}$. As $(T, u^T(\beta))$ is not a counter objection, we conclude that $k \in T$, $t_4(R^k) = \beta$, and $t_5(R^k) = \alpha$. We conclude that for any $k \in \widehat{S}$ the alternative $t_5(R^k)$ is only dominated by $t_4(R^k)$ and $|\{i \in N \mid t_4(R^k) R^i t_5(R^k)\} \setminus \{k\}| \leq n/2$. If n is odd, we may now easily finish the proof by the observation that α dominates all but one alternative. Hence we may assume from now on that n is an even number. As $a_3 \neq \alpha$, $\{i \in N \mid u^i(\alpha) > u^i(a_3)\} \cap \{i \in N \mid u^i(\beta) > u^i(\alpha)\} \neq \emptyset$. Thus, there exists $j \in \widehat{S}$ such that $t_1(R^j) = \beta$ and $t_2(R^j) = \alpha$. So far we have for any $k \in \widehat{S}$, where $\alpha = t_5(R^k)$, $\beta = t_4(R^k)$:

$$\alpha \text{ is only dominated by } \beta; \tag{6.2}$$

$$\text{There exists } j \in \widehat{S} \text{ such that } t_1(R^j) = \alpha, t_2(R^j) = \beta; \tag{6.3}$$

$$|\{i \in N \mid u^i(\alpha) > u^i(a_3)\}| \geq \frac{n}{2}. \tag{6.4}$$

Now, let $k, j \in \widehat{S}$ have the foregoing properties, let us say $k = 1$ and $j = 2$. We also may assume that $t_4(R^1) = a_4$, $t_5(R^1) = a_5$, $t_4(R^2) = a_1$, $t_5(R^2) = a_2$ (hence $R^2 = (a_4, a_5, a_3, a_1, a_2)$). So, for

any $k \in \widehat{S}$, we have

$$\{t_4(R^k), t_5(R^k)\} = \{a_4, a_5\} \Rightarrow t_4(R^k) = a_4 \quad (6.5)$$

$$t_5(R^k) = a_5 \Rightarrow t_4(R^k) = a_4 \quad (6.6)$$

$$\{t_4(R^k), t_5(R^k)\} = \{a_1, a_2\} \Rightarrow t_4(R^k) = a_1 \quad (6.7)$$

$$t_5(R^k) = a_2 \Rightarrow t_4(R^k) = a_1 \quad (6.8)$$

We are now going to show that there exists $k \in \widehat{S}$ such that $t_5(R^k) \notin \{a_5, a_2\}$. Assume the contrary. Then $\{i \in N \mid u^i(a_5) > u^i(a_3)\} \cap \{i \in N \mid u^i(a_2) > u^i(a_3)\} = \emptyset$ and, by (6.4), $a_5 \succ a_3$ and $a_2 \succ a_3$. Hence, by (1), $a_1 \succ a_3$ or $a_4 \succ a_3$. However, note that by our assumption $u^i(a_1) > u^i(a_3)$ implies $u^i(a_1) > u^i(a_5)$ for all $i \in N$. Thus, if $a_1 \succ a_3$, then $a_1 \succ a_5$ which contradicts (6.2). Similarly, $a_4 \succ a_3$ can be excluded.

Hence, we may assume without loss of generality, that there exists $k \in \widehat{S}$ such that $t_5(R^k) = a_1$. We now claim that there exists $j \in \widehat{S}$ such that $t_5(R^j) = a_4$. By (6.2) and the fact that $a_1 \succ a_2$, $t_4(R^k) \in \{a_4, a_5\}$. If $t_4(R^k) = a_4$, then by (6.3) there exists $j \in \widehat{S}$ such that $\{t_4(R^j), t_5(R^j)\} = \{a_2, a_5\}$. By (6.6), $a_5 \neq t_5(R^j)$, and by (6.8), $a_2 \neq t_5(R^j)$. Hence this possibility is ruled out. We conclude that $t_4(R^k) = a_5$. By (6.3) there exists $j \in \widehat{S}$ such that $\{t_4(R^j), t_5(R^j)\} = \{a_2, a_4\}$. By (6.8), $t_5(R^j) = a_4$. So our claim has been shown.

So far we have deduced there exist $k_j \in \widehat{S}$, $j = 1, 2, 4, 5$, such that $t_5(R^{k_j}) = a_j$. By (6.4), $|\{i \in N \mid u^i(a_j) > u^i(a_3)\}| \geq \frac{n}{2}$ for all $j = 1, 2, 4, 5$. We conclude that $a_3 = t_3(R^i)$ for all $i \in N$ and $|\{i \in N \mid u^i(a_j) > u^i(a_3)\}| = \frac{n}{2}$ for all $j = 1, 2, 4, 5$. Therefore a_3 is not dominated by any alternative, which contradicts (1). **q.e.d.**

We shall now present an example of a simple majority voting game on six alternatives whose Mas-Colell bargaining set is empty.

Table 6.1: Preference Profile leading to an empty \mathcal{MB}

R^1	R^2	R^3	R^4
a_1	a_4	a_3	a_2
a_2	a_1	a_4	a_3
c	c	c	b
b	b	b	a_4
a_3	a_2	a_1	c
a_4	a_3	a_2	a_1

Example 6.3 Let $n = 4$, $A = \{a_1, \dots, a_4, b, c\}$, let $R^N \in L^N$ be given by Table 6.1 and let $u^N \in \mathcal{U}^{R^N}$. We claim that $\mathcal{MB}(N, V_{u^N}) = \emptyset$. The proof of the claim in Example 6.4 also shows our current claim. Note that the proof is similar to the proof of the emptiness of an extension of the Mas-Colell bargaining set of a game derived from a 4-person voting problem on ten alternatives (see Section 3 of Peleg and Sudhölter (2004)).

Example 6.3 may be generalized to any number $m \geq 6$ of alternatives. Also, if $R_i = R^i$ for $i = 1, \dots, 4$, if

$$R_5 = (a_2, a_1, c, b, a_3, a_4), R_6 = (a_4, a_3, c, b, a_1, a_2),$$

if $n = 4 + 2k$ for some $k \in \mathbb{N}$, if $\tilde{R}^N \in L^N$ such that

$$|\{j \in N \mid \tilde{R}^j = R_i\}| = \begin{cases} k & , \text{ if } i = 5, 6, \\ 1 & , \text{ if } i = 1, 2, 3, 4, \end{cases}$$

then $\mathcal{MB}(N, V_{u^N}) = \emptyset$ for all $u^N \in \mathcal{U}^{\tilde{R}^N}$.

In what follows we shall show that a suitable choice of utilities in Example 6.3 shows that the Mas-Colell bargaining set of a plurality or of a approval voting game on six alternatives may be empty.

Example 6.4 (Example 6.3 continued) We now specify a utility representation $u^N \in \mathcal{U}^{R^N}$ by

$$u^i(t_j(R^i)) = 6^5 - 6^{j-1} \text{ for all } i \in N \text{ and } j = 1, \dots, 6.$$

Let (N, V) the corresponding plurality or approval voting game, that is, $V \in \{V_{u^N}^{pl}, V_{u^N}^{ap}\}$.

Claim: $\mathcal{MB}(N, V) = \emptyset$.

As $V_{u^N}(S) \subseteq V(S)$ for all $S \subseteq N$ satisfying $|S| = n/2$ or $|S| = n$ and by Remark 2.2, the following proof may also be used to show that \mathcal{MB} specifies the empty set when applied to any simple majority game corresponding to R^N .

Proof: For any $B \subseteq A$ and each $i \in N$,

$$\sum_{\beta \in B} \frac{u^i(\beta)}{|B|} < u^i(\alpha) \text{ for all } \alpha \in A \text{ such that } u^i(\alpha) > \min_{\beta \in B} u^i(\beta) \quad (6.9)$$

and

$$\sum_{\beta \in B} \frac{u^i(\beta)}{|B| + 1} < 6^5 - 6^4 = u^i(t_5(R^i)). \quad (6.10)$$

For $x \in V(N)$, $x \geq 0$, define $\tilde{x} \in \mathbb{R}^N$ by $\tilde{x}^i = \max\{u^i(\alpha) \mid \alpha \in A, u^i(\alpha) \leq x^i\}$ for all $i \in N$. By (6.9) (see (2.5) or (2.8), respectively),

$$\tilde{x} \text{ is weakly Pareto optimal in } V_{u^N}(N) \text{ for all weakly Pareto optimal } x \in V(N), x \geq 0. \quad (6.11)$$

Now assume that there exists $x \in \mathcal{MB}(N, V)$. Let $\alpha \in A$ such that $\tilde{x} \leq u^N(\alpha)$. Let

$$S_1 = \{1, 2, 3\}, S_2 = \{1, 2, 4\}, S_3 = \{1, 3, 4\}, S_4 = \{2, 3, 4\}.$$

We distinguish the following possibilities:

- (1) $\tilde{x} \leq u^N(a_1)$. As $u^{S_4}(a_4) \gg u^{S_4}(a_1) \geq \tilde{x}^{S_4}$, the definition of \tilde{x} implies that $(S_4, u^{S_4}(a_4))$ is an objection at x . In view of (6.10) this objection has no counter objection of the form (Q, z) satisfying $|Q| \leq 2$. As $u^N(a_4)$ is Pareto optimal and $u^1(a_4) = 0$, there does not exist any counter objection of the form (N, z) . We conclude that $(S_3, u^{S_3}(a_3))$ is an objection at x . Hence, $x^1 \leq u^1(a_3)$. Moreover, by similar reasons as before, this objection has no counter objection which uses a coalition of size less than or equal to 2 or which uses the coalition N . We conclude that $(S_2, u^{S_2}(a_2))$ is a counter objection. Hence, $x \ll u^N(b)$ and the desired contradiction has been obtained in this case.
- (2) The possibilities $\tilde{x} \leq u^N(\alpha)$ for $\alpha \in \{a_2, a_3, a_4\}$ may be treated similarly.
- (3) $\tilde{x} \leq u^N(b)$. Then $(S_1, u^{S_1}(c))$ is an objection at x which cannot be countered with the help of coalitions of size less than or equal to 2. If there exists $z \in \mathbb{R}^N$ such that (N, z) is a counter objection, then $x^4 \leq u^4(c)$. Hence, $(S_4, u^{S_4}(a_4))$ is an objection at x and $x^4 \leq u^4(c)$. Now we conclude that $(S_3, u^{S_3}(a_3))$ must be a counter objection and, hence, an objection at x . We continue by concluding that $(S_2, u^{S_2}(a_2))$ must be an objection and that, hence $(S_1, u^{S_1}(a_1))$ is a counter objection. Therefore, $x \ll u^N(b)$ and the desired contradiction has been obtained.
- (4) $\tilde{x} \leq u^N(c)$. We consecutively deduce that $(S_4, u^{S_4}(a_4)), \dots, (S_1, u^{S_1}(a_1))$ are objections. The desired contradiction again is obtained by the observation that $x \ll u^N(b)$. q.e.d.

Remark 6.5 It is possible to modify the utility profile u^N of the foregoing example in such a way that the Mas-Colell bargaining sets of the approval or the plurality voting game are nonempty. Indeed, if we just replace u^i , $i = 1, 2$, by \tilde{u}^i which differs from u^i only inasmuch as $\tilde{u}^i(t_j(R^i)) = 12 - 2j$ for $j = 4, 5$, then

$$x = \left(\frac{\tilde{u}^1(a_3) + \tilde{u}^1(a_4)}{2}, \frac{u^2(a_1)}{2}, u^{\{3,4\}}(a_4) \right) = (1, 3885, 7770, 7560) \in \mathcal{MB}(N, V).$$

Indeed, weak Pareto optimality follows from the observations that $x \leq \frac{u^N(a_3) + u^N(a_4)}{2}$ and that $x \leq \sum_{\beta \in B} \frac{u^N(\beta)}{|B|}$ implies $B = \{a_3, a_4\}$. Furthermore, there are only objections which use N , S_4 , S_3 , and $\{1, 2\}$. These objections can be countered by $(S_4, u^{S_4}(a_4))$, $(S_3, u^{S_3}(a_3))$, $(\{1, 2\}, \frac{u^{\{1,2\}}(a_1)}{2})$, and $(S_4, u^{S_4}(a_4))$, respectively.

7 Replication

Let $N = \{1, \dots, n\}$, let $A = \{a_1, \dots, a_m\}$, let $R^N \in L^N$, and let $u^N \in \mathcal{U}^{R^N}$. In order to replicate the simple majority voting game (N, V_{u^N}) , let $k \in \mathbb{N}$ and denote

$$kN = \{(j, i) \mid i \in N, j = 1, \dots, k\}.$$

Furthermore, let $R^{(j,i)} = R^i$ and $u^{(j,i)} = u^i$ for all $i \in N$ and $j = 1, \dots, k$. Then $(kN, V_{u^{kN}})$ is the k -fold replication of (N, V_{u^N}) .

Remark 7.1 *If α is a weak Condorcet winner with respect to R^N , then $u^{kN}(\alpha) \in \mathcal{MB}(kN, V_{u^{kN}})$.*

Theorem 7.2 *If $k \geq \left\{ \begin{array}{l} n+2, \text{ if } n \text{ is odd,} \\ \frac{n}{2} + 2, \text{ if } n \text{ is even,} \end{array} \right\}$ then $\mathcal{MB}(kN, V_{u^{kN}}) \neq \emptyset$.*

Proof: Let $V = V_{u^{kN}}$. By Remark 7.1 we may assume that for every $\alpha \in A$ there exists $\beta(\alpha) \in A$ such that $\beta(\alpha) \succ_{R^N} \alpha$ and such that $u^N(\beta(\alpha))$ is Pareto optimal in $V_{u^N}(N)$. Let $\tilde{x} \in \mathbb{R}_+^N$ be any weakly Pareto optimal element in $V_{u^N}(N)$. We define $x \in \mathbb{R}^{kN}$ by $x^{(1,i)} = \tilde{x}^i$ and $x^{(j,i)} = 0$ for all $i \in N$ and $j = 2, \dots, k$ and claim that $x \in \mathcal{MB}(kN, V)$. Let (P, y) be an objection at x . Then there exists $\alpha \in A$ such that $y \leq u^P(\alpha)$. Let $\beta = \beta(\alpha)$ and let $T = \{i \in N \mid \beta R^i \alpha\}$. Then

$$|T| \geq \left\{ \begin{array}{l} \frac{n+1}{2}, \text{ if } n \text{ is odd,} \\ \frac{n}{2} + 1, \text{ if } n \text{ is even.} \end{array} \right. \quad (7.1)$$

Let $Q = \{(j, i) \mid i \in T, j = 2, \dots, k\}$ and define $z \in \mathbb{R}^Q$ by $z^{(j,i)} = u^i(\beta)$ for all $i \in T$ and $j = 2, \dots, k$. Then $|Q| = (k-1)|T|$ and $z > (y^{P \cap Q}, x^{Q \setminus P})$. By (7.1), $|Q| \geq \frac{kn+1}{2}$. So, z is a Pareto optimal element in $V(Q)$ and (Q, z) is a counter objection to (P, y) . **q.e.d.**

It should be remarked that the foregoing theorem remains valid for any $u^{kN} \in \mathcal{U}^{R^{kN}}$.

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