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OPTIMAL TWO CHOICE STOPPING ON AN EXPONENTIAL SEQUENCE

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Optimal Two Choice Stopping on an Exponential Sequence*†

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Abstract

Asymptotic results for the problem of optimal two choice stopping on an n element long i.i.d. sequence $X_n, \ldots, X_1$ have previously been obtained for two of the three domains of attraction. An asymptotic result is proved for the exponential distribution, a representative from the remaining, Type I domain, and it is conjectured that the same behavior obtains for all Type I distributions.

1 Introduction and Summary

In recent years the optimal stopping problem, when the "statistician" is given $k > 1$ choices among the random variables $X_1, \ldots, X_n$, using k consecutive stopping rules, has been considered. The "reward" to the statistician is the expected value of the maximum of the k chosen X-values.

Kennedy and Kertz (1991) study the asymptotic behavior of the reward sequence for i.i.d. random variables, with distribution F, when $k = 1$, as $n \to \infty$. They show that the asymptotic behavior is determined by the domain of attraction, for the maximum, of F, and is closely related to the asymptotic behavior of the maximum.

Here we study the case $k = 2$. Judging by the asymptotic behavior of the maximum, and by the case $k = 1$, it is clear that the asymptotic behavior of the reward sequence for $k = 2$ will also depend on the corresponding domain of attraction.

In two recent papers, the following two of the three domains of attraction,

$$F \in \mathcal{D}(\exp(-(-x)^\alpha)I\{x \le 0\} + I\{x > 0\}), \quad \alpha > 0 \qquad (III)$$

and

$$F \in \mathcal{D}(\exp(-x)^{-\alpha}I\{x > 0\}) \text{ and } \alpha > 1 \qquad (II)$$

are treated in some generality (Assaf, Goldstein and Samuel-Cahn (2004a and 2004b)).

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For F in the remaining domain,

$$F \in \mathcal{D}(\exp(-e^{-x})I(-\infty < x < \infty)) \tag{I}$$

we conjecture, with V_n^2 denoting the optimal reward with two choices when there are n i.i.d. observations, that

$$\lim_{n\to\infty} n(1 - F(V_n^2)) = 1 - e^{-1}. \tag{1}$$

Unfortunately we have not been able to establish this conjecture for the full domain (I). In the present note we prove (1) for the special, exponential case where $F(x) = (1 - e^{-\theta x})I\{x \geq 0\}$, where without loss of generality, we set $\theta = 1$.

In the papers considering domains (III) and (II), asymptotic results were proven for large subclasses of the domains by first obtaining the results using a specific representative distribution, and then extending to nearly the full domain. For class (I), however, it seems that the methods developed there for extension cannot be applied. However, in contrast to the previous two cases, the present results for class (I) are in some sense more explicit.

2 Preliminaries and Heuristics

Let V_n^1 denote the one-choice reward sequence and $M_n = \max(X_1, \ldots, X_n)$. Then, for the exponential distribution, by Kennedy and Kertz (1991) and Leadbetter (1983),

$$\lim_{n\to\infty}(V_n^1 - \log n) = 0 \quad \text{and} \quad \lim_{n\to\infty}(EM_n - \log n) = \gamma, \tag{2}$$

where $\gamma = .5772\ldots$ is Euler's constant. It therefore makes sense to conjecture that

$$\lim_{n\to\infty}(V_n^2 - \log n) = c, \text{ where } 0 < c < \gamma. \tag{3}$$

In fact, we prove the following

Theorem 2.1 *Let $X_1, \ldots, X_n$ be i.i.d. exponential random variables and V_n^2 the optimal two choice value. Then (3) holds with $c = 1 - \log(e-1) = .4586\ldots$, and hence (1) holds in the exponential case.*

For simplicity of notation we shall write V_n instead of V_n^2. Also, let $V_j^1(x)$ denote the expected reward when there are j observations and one choice left, but the value x is already guaranteed. Clearly

$$V_1^1(x) = E(x \vee X_1) = x + e^{-x} = g(x) = g_1(x)$$

and

$$V_{n+1}^1(x) = E(V_n^1(x) \vee X_{n+1}) = g_{n+1}(x) = g_1(g_n(x)), \quad n \geq 1.$$

The recursion relation for V_n is as follows:

$$\begin{aligned} V_2 &= E[X_1 \vee X_2] \\ V_{n+1} &= E[V_n^1(X_{n+1}) \vee V_n], \quad n \geq 2, \end{aligned} \tag{4}$$

where the present observation X_{n+1} is chosen when it is large enough, and then one is left with one additional choice only, and otherwise X_{n+1} is passed and two choices remain. Equation (4) can be written as

$$V_{n+1} = \int_0^\infty [g_n(x) \vee V_n] e^{-x} dx, \tag{5}$$

or, if we let $b_n > 0$ denote the unique value such that $g_n(b_n) = V_n$ (also called the "indifference value") then (5) can be rewritten as

$$V_{n+1} = (1 - e^{-b_n}) V_n + \int_{b_n}^\infty g_n(x) e^{-x} dx. \tag{6}$$

Set

$$h_n(x) = g_n(x + \log n) - \log n, \quad a_n = \log\left((n+1)/n\right), \tag{7}$$

$$B_n = b_n - \log n, \quad \text{and} \quad W_n = V_n - \log n = h_n(B_n). \tag{8}$$

Then (6) can be rewritten as

$$W_{n+1} = \left(1 - \frac{1}{n} e^{-B_n}\right) W_n + \frac{1}{n} \int_{B_n}^\infty h_n(x) e^{-x} dx - a_n \tag{9}$$

or

$$n(W_{n+1} - W_n) = -e^{-B_n} W_n + \int_{B_n}^\infty h_n(x) e^{-x} dx - n a_n. \tag{10}$$

To motivate our result, consider the following heuristics. Assume that for some B and h,

$$n(W_{n+1} - W_n) \to 0, \; B_n \to B \quad \text{and} \quad h_n(y) \to h(y) \quad \text{as} \quad n \to \infty.$$

Then under regularity (10) yields

$$0 = -W e^{-B} + \int_B^\infty h(x) e^{-x} dx - 1 \tag{11}$$

where $\lim_{n\to\infty} W_n = W$. Since $g_n(b_n) = V_n$ implies $W_n = h_n(B_n)$, we anticipate, in the limit,

$$W = h(B). \tag{12}$$

Substituting (12) in (11) gives an equation for the unknown B, thus yielding W if h were known.

Here is a heuristic for determining h. By (2),

$$\lim_{n\to\infty} [V_n^1 - \log(n+1)] = 0.$$

Since V_n^1 is the value when nothing is guaranteed, we have $V_n^1 = g_n(0)$, and thus $g_n(0) \approx \log(n+1)$. Suppose that for large enough n and a fixed guaranteed value x, there is t such that

$$g_n(x) = g_{n+t}(0) = g_n(g_t(0)). \tag{13}$$

That is, there is some number of 'extra observations' t such that the statistician is indifferent to having $n+t$ variables from which to chose, or the guaranteed x and n variables.

Equation (13) implies $x = g_t(0) \approx \log(t+1)$, yielding $t+1 \approx e^x$. But on the other hand $g_{n+t}(0) \approx \log(n+t+1) \approx \log(n+e^x) \approx g_n(x)$. Using (7), we have

$$h_n(x) \approx \log(n + e^{x+\log n}) - \log n = \log(1+e^x).$$

This suggests

$$\lim_{n\to\infty} h_n(x) = h(x) = \log(1+e^x) = x + \log(1+e^{-x}), \quad -\infty < x < \infty. \tag{14}$$

From (11), (12) and (14), B solves

$$1 = -\log(1+e^B)e^{-B} + e^{-B}\int_0^\infty \log(1+e^{B+u})e^{-u}du. \tag{15}$$

Letting $s = e^{-u}$, the integral in (15) can be evaluated as

$$\int_0^\infty \log(1+e^{B+u})e^{-u}du = \int_0^1 \log(1+\frac{e^B}{s})ds = e^B\left((1+e^{-B})\log(1+e^B) - B\right),$$

and now substitution back into (15) yields $1+B = \log(1+e^B)$, the unique solution of which is

$$B = -\log(e-1) = -.54132\ldots,$$

and now, from (12) and (14), $W = 1 - \log(e-1) = .45867\ldots$, implying the conclusions of the Theorem subject to the heuristics being made rigorous.

3 Properties of h_n and the limiting h

Lemma 3.1 *$h_n(x)$ is strictly monotone increasing for $-\log n \le x < \infty$.*

Proof. We have that $g(x)$, and hence $g_n(x)$, are strictly monotone increasing for $x \ge 0$, and now the result follows by (7). □

Lemma 3.2 *Let $h(x)$ be given in (14). Then $h_n(x) > h(x)$ for $x \ge -\log n$, $n = 1, 2, \ldots$*

Proof. For $n = 1$ the claim is simply that $h_1(x) = x + e^{-x} > x + \log(1+e^{-x}) = h(x)$ for all $x \ge 0$, which is immediate. Now suppose the claim holds for n. We show that it holds for $n+1$. By the induction hypothesis

$$\begin{aligned} g_n(x) &= \log n + h_n(x - \log n) > \log n + h(x - \log n) \\ &= \log n + \log(1 + e^{x-\log n}) = \log(n + e^x). \end{aligned} \tag{16}$$

Thus, since g is increasing

$$\begin{aligned} g_{n+1}(x) &= g(g_n(x)) > g(\log(n+e^x)) = \log(n+e^x) + e^{-\log(n+e^x)} \\ &= \log(n+e^x) + \frac{1}{n+e^x}. \end{aligned} \tag{17}$$

Thus, similar to (16) it suffices to show that the right hand side of (17) is greater than $\log(n+1+e^x)$. The latter statement is equivalent to $\frac{1}{n+e^x} > \log(1+\frac{1}{n+e^x})$ which clearly holds. □

Lemma 3.3 *Let $\varepsilon_n(x) = h_n(x) - h(x)$. Then $\varepsilon_n(x) < e^{-x}/\sqrt{n}$, for $x \geq -\log n$.*

Proof. For $n = 1$ we have $\varepsilon_1(x) = e^{-x} - \log(1+e^{-x})$, so clearly the statement holds for $n = 1$. Now, using (7),

$$\begin{aligned} h_{n+1}(x) &= g_{n+1}(x+\log(n+1)) - \log(n+1) \\ &= g_n(x+\log(n+1)) + e^{-g_n(x+\log(n+1))} - \log(n+1) \\ &= h_n(x+a_n) - a_n + e^{-[h_n(x+a_n)+\log n]} \\ &= h_n(x+a_n) + \frac{1}{n}e^{-h_n(x+a_n)} - a_n. \end{aligned} \tag{18}$$

In particular, for $n = 1$,

$$\begin{aligned} h_2(x) &= h_1(x+a_1) + e^{-h_1(x+a_2)} - a_1 \\ &= x + a_1 + e^{-(x+a_1)} + e^{-(x+a_1+e^{-(x+a_1)})} - a_1. \end{aligned}$$

We shall show directly that the lemma is true for $n = 2$, for which

$$\varepsilon_2(x) = \frac{1}{2}e^{-x}(1+e^{-\frac{1}{2}e^{-x}}) - \log(1+e^{-x}). \tag{19}$$

For $-\log 2 \leq x \leq 0$ we shall show

$$\varepsilon_2(x) - \frac{e^{-x}}{\sqrt{2}} < 0, \quad \text{that is,}$$

$$\frac{1}{2}e^{-x}(1-\sqrt{2}+e^{-\frac{1}{2}e^{-x}}) - \log(1+e^{-x}) < 0. \tag{20}$$

Differentiation shows that the left hand side of (20) is increasing in x for $x \leq 0$. Thus we shall show that for $x = 0$ inequality (20) holds, that is, that $\frac{1}{2}(1-\sqrt{2}+e^{-\frac{1}{2}}) - \log 2 < 0$, which is equivalent to $1-\sqrt{2}+e^{-\frac{1}{2}} - \log 4 < 0$, which clearly holds.

Now for $x > 0$ the inequality $\log(1+e^{-x}) > e^{-x} - \frac{e^{-2x}}{2}$ holds. Substituting this in (19) we have

$$\varepsilon_2(x) < \frac{1}{2}e^{-x}(-1+e^{-\frac{1}{2}e^{-x}}+e^{-x}). \tag{21}$$

We shall show that the right hand side of (21) is less than $e^{-x}/\sqrt{2}$, which is equivalent to

$$-1+e^{-\frac{1}{2}e^{-x}}+e^{-x}<\sqrt{2}. \tag{22}$$

Now the left hand side of (22) is decreasing in x, thus it suffices to show (22) for $x=0$, where the inequality simplifies to $-1+e^{-\frac{1}{2}}+1<\sqrt{2}$, which clearly holds. Thus the lemma holds for $n=2$.

Suppose the lemma holds for $n\geq 2$. We shall show that it holds for $n+1$. By (18), for $x\geq -\log n$

$$h_{n+1}(x-a_n)=h_n(x)+\frac{1}{n}e^{-h_n(x)}-a_n. \tag{23}$$

We show that, for $x\geq -\log n$, $\varepsilon_{n+1}(x-a_n)<\frac{n+1}{n}e^{-x}/\sqrt{n+1}=\frac{\sqrt{n+1}}{n}e^{-x}$ by a Talylor expansion of $h(x-a_n)$. Note

$$h'(x)=e^x/(1+e^x),\quad h''(x)=e^x/(1+e^x)^2>0,$$

thus for some $\theta\in(0,1)$

$$h(x-a_n)=h(x)-a_n\frac{e^{x-\theta a_n}}{1+e^{x-\theta a_n}}>h(x)-a_n\frac{e^x}{1+e^x}. \tag{24}$$

Thus by (23) and (24),

$$\begin{aligned}
\varepsilon_{n+1}(x-a_n) &< \varepsilon_n(x)+\frac{1}{n}e^{-h_n(x)}-a_n+a_n\frac{e^x}{1+e^x}\\
&< \varepsilon_n(x)+\frac{1}{n(1+e^x)}-\frac{a_n}{1+e^x}\\
&< \varepsilon_n(x)+\frac{1}{n(1+e^x)}-\left(\frac{1}{n}-\frac{1}{2n^2}\right)\frac{1}{1+e^x}\\
&= \varepsilon_n(x)+\frac{1}{2n^2(1+e^x)}\\
&< \frac{e^{-x}}{\sqrt{n}}+\frac{e^{-x}}{2n^2(1+e^{-x})}
\end{aligned}$$

where the second inequality uses $h_n(x)>h(x)$ by Lemma 3.2, the third inequality uses $\log(1+y)>y-\frac{y^2}{2}$ for $0<y<1$ and the last inequality uses the induction hypothesis. Thus we must show that for $x\geq -\log n$ we have $\frac{1}{\sqrt{n}}+\frac{1}{2n^2(1+e^{-x})}<\frac{\sqrt{n+1}}{n}$, and hence it is sufficient to show $1+\frac{1}{2n^{3/2}}<\sqrt{\frac{n+1}{n}}$. But $(1+\frac{1}{n})^{1/2}>1+\frac{1}{2n}-\frac{1}{8n^2}$, hence sufficient to show $\frac{1}{2n^{3/2}}<\frac{1}{2n}-\frac{1}{8n^2}$, or equivalently that $1<\sqrt{n}-\frac{1}{4\sqrt{n}}$, which holds for $n\geq 2$. □

4 Proof of Theorem 2.1

Lemma 4.1 *For some constant A_q, let q be a continuous and strictly monotone increasing function in the interval $[A_q,\infty)$ such that for all $y\geq A_q$ the integral $\int_y^\infty q(x)e^{-x}dx$ is finite.*

Further, defining

$$Q(y) = \int\limits_{y}^{\infty} q(x)e^{-x}dx - q(y)e^{-y} - 1, \tag{25}$$

suppose $Q(A_q) > 0$. *Then*

$$\lim_{y\to\infty} Q(y) = -1, \tag{26}$$

$Q(y)$ *is monotone decreasing, and there exists a unique value* $\beta \in [A_q, \infty)$ *such that* $Q(\beta) = 0$.

Proof. The assumption the the integral in (25) be finite and q increasing implies that $q(y)e^{-y} \to 0$ as $y \to \infty$, thus (26) holds. The function Q is differentiable with $dQ(y)/dy = -q'(y)e^{-y} < 0$, thus Q is monotone decreasing. Since $Q(A_q) > 0$, $Q(y)$ is continuous and negative for y sufficiently large, the root β exists and is unique in $[A_q, \infty)$. □

Theorem 4.1 *Let* A_q *and* q *be as in Lemma 4.1. Then there exists* n_0 *such that for any* $r \geq n_0$ *and* $\beta_r \in [A_q, \infty)$, *the sequence* β_n *for* $n \geq r$ *is well defined by the recursion*

$$q(\beta_{n+1}) = q(\beta_n)\left(1 - \frac{1}{n}e^{-\beta_n}\right) + \frac{1}{n}\int\limits_{\beta_n}^{\infty} q(x)e^{-x}dx - a_n, \tag{27}$$

and satisfies

$$\lim_{n\to\infty} \beta_n = \beta,$$

where β *is the root of (25) whose existence and uniqueness in* $[A_q, \infty)$ *is guaranteed in Lemma 4.1.*

Proof. First, rewrite (27) as

$$q(\beta_{n+1}) - q(\beta_n) = \frac{Q(\beta_n)}{n} + (\frac{1}{n} - a_n). \tag{28}$$

Note that for all $n \geq 1$

$$0 < \frac{1}{n} - a_n < \frac{1}{2n^2}, \tag{29}$$

and that

$$\frac{Q(c)}{n} + (\frac{1}{n} - a_n)$$

is positive and decreasing in n with limit 0 for all $c \leq \beta$, and is decreasing in n and negative for all n sufficiently large with limit 0 for $c > \beta$.

We show that for any $\underline{\beta}$ and $\overline{\beta}$ with $A_q < \underline{\beta} < \beta < \overline{\beta}$, for all n sufficiently large β_n is well defined and $\underline{\beta} < \beta_n < \overline{\beta}$; clearly the theorem follows.

Let $A_q < \underline{\beta} < \beta < \overline{\beta}$ be given, and let n_0 be so large that for all $n \geq n_0$

$$\begin{array}{ll} (i) & \frac{Q(A_q)}{n} + (\frac{1}{n} - a_n) < q(\overline{\beta}) - q(\beta) \\ (ii) & (\frac{1}{n} - a_n) < q(\overline{\beta}) - q(\frac{\beta + \overline{\beta}}{2}) \\ (iii) & \frac{1}{n} Q(\frac{\beta + \overline{\beta}}{2}) + (\frac{1}{n} - a_n) < 0 \\ (iv) & a_n < q(\beta) - q(\underline{\beta}). \end{array} \tag{30}$$

We first show that if $A_q < \underline{\beta} < \beta_n < \overline{\beta}$ for $n \geq n_0$, then β_{n+1} is well defined and satisfies $\underline{\beta} < \beta_{n+1} < \overline{\beta}$; thus the sequence β_n remains in the interval $(\underline{\beta}, \overline{\beta})$ for all $n \geq n_0$. We show this fact by considering the following cases.

Case (A):

$$\underline{\beta} < \beta_n \leq \beta.$$

By (28), (29) and (30(i)), and the fact that q is increasing and Q decreasing,

$$\begin{aligned} q(\beta_n) &< q(\beta_n) + \frac{Q(\beta_n)}{n} + (\frac{1}{n} - a_n) = q(\beta_{n+1}) \\ &< q(\beta_n) + \frac{Q(A_q)}{n} + (\frac{1}{n} - a_n) \\ &< q(\beta_n) + (q(\overline{\beta}) - q(\beta)) \leq q(\overline{\beta}); \end{aligned} \tag{31}$$

thus β_{n+1} exists uniquely by the strict monotonicity of q and satisfies

$$\underline{\beta} < \beta_n < \beta_{n+1} < \overline{\beta}.$$

Case (B):

$$\beta < \beta_n < \frac{\beta + \overline{\beta}}{2}.$$

There are two subcases

B1)

$$\frac{Q(\beta_n)}{n} + (\frac{1}{n} - a_n) > 0, \quad \text{which may happen for small } n \text{, and}$$

B2)

$$\frac{Q(\beta_n)}{n} + (\frac{1}{n} - a_n) \leq 0. \tag{32}$$

In subcase B1), by (30(ii)),

$$\begin{aligned} q(\beta) &< q(\beta_n) < q(\beta_n) + \frac{Q(\beta_n)}{n} + (\frac{1}{n} - a_n) = q(\beta_{n+1}) \\ &< q(\beta_n) + (\frac{1}{n} - a_n) < q(\beta_n) + (q(\overline{\beta}) - q(\frac{\beta + \overline{\beta}}{2})) < q(\overline{\beta}), \end{aligned}$$

so again β_{n+1} is well defined and $\underline{\beta} < \beta_{n+1} < \overline{\beta}$. The subcase B2) can be combined with Case (C).

Case (C):

$$\frac{\beta + \overline{\beta}}{2} \leq \beta_n < \overline{\beta}.$$

In this case by (30(iii)), and in B2) by (32), if β_{n+1} exists, it must be smaller than β_n, thus $q(\overline{\beta}) > q(\beta_{n+1})$, but also

$$q(\beta_{n+1}) = q(\beta_n) + \frac{Q(\beta_n)}{n} + (\frac{1}{n} - a_n) > q(\beta_n) - a_n > q(\beta_n) - (q(\beta) - q(\underline{\beta})) > q(\underline{\beta}), \quad (33)$$

where the inequalities are justified by (26), (30(iv)), and $\beta_n > \beta$, this last which holds for (C) as well as for (B), so in particular for $B2$). Thus again β_{n+1} exists and $\underline{\beta} < \beta_{n+1} < \overline{\beta}$.

It remains to show that for any $r \geq n_0$ and any starting value $\beta_r \in [A_q, \infty) \cap (\underline{\beta}, \overline{\beta})^c$, β_n is well defined and that β_n will eventually enter the interval $(\underline{\beta}, \overline{\beta})$. First suppose $\beta_r \in [A_q, \underline{\beta}]$. Then the sequence will be well defined and start out monotone increasing, and (31) and its subsequent inequalities continue to hold as long as $\beta_n \leq \beta$, and for all such n one has $\beta_{n+1} < \overline{\beta}$. There are two possibilities: (a) Either for some k the inequality

$$\underline{\beta} < \beta_k < \overline{\beta}$$

holds, in which case we have shown that $\underline{\beta} < \beta_n < \overline{\beta}$ for all $n > k$.

(b) The sequence β_n is monotone increasing throughout with $\lim \beta_n = \beta_0$, which necessarily satisfies $\beta_0 \leq \underline{\beta}$. We show that (b) leads to a contradiction. Clearly $Q(\beta_0) > 0$. By (28),

$$q(\beta_{n+1}) - q(\beta_n) > \frac{Q(\beta_0)}{n} + (\frac{1}{n} - a_n)$$

thus for n arbitrarily large and $m > n$,

$$q(\beta_m) - q(\beta_n) > Q(\beta_0) \sum_{k=n}^{m-1} \frac{1}{k} + \sum_{k=n}^{m-1} (\frac{1}{k} - a_k).$$

Now the right hand side tends to infinity as $m \to \infty$, thus the value $q(\beta_m)$ must also tend to infinity, contradicting the fact that $\beta_m \leq \underline{\beta}$.

Now consider a starting value β_r for $r \geq n_0$ satisfying $\overline{\beta} \leq \beta_r < \infty$. By (30(iii)) the sequence will be well defined and decreasing, as long as $\beta_n \geq (\beta + \overline{\beta})/2$, and (33) continues to hold, thus $\beta_{n+1} > \underline{\beta}$. Again there are two possibilities. Either (a), for some n we have $\overline{\beta} > \beta_n > \underline{\beta}$, in which case the theorem holds, or (b) the sequence is monotone decreasing for all n, with $\beta_n \geq \overline{\beta}$, and thus the limit $\beta^0 \geq \overline{\beta}$ exists, and clearly satisfies $Q(\beta^0) < 0$. We suppose (b) and show that this leads to a contradiction. By (28) and (29)

$$q(\beta_{n+1}) - q(\beta_n) < \frac{Q(\beta^0)}{n} + \frac{1}{2n^2},$$

thus for m arbitrarily large,

$$q(\beta_m) - q(\beta_n) < Q(\beta^0) \sum_{k=n}^{m-1} \frac{1}{m} + \frac{1}{2} \sum_{k=n}^{m-1} \frac{1}{k^2}. \quad (34)$$

Now the last summand on the right hand side of (34) converges to a finite limit, while the first term there tends to $-\infty$ as $m \to \infty$. Thus $q(\beta_m)$ must also tend to $-\infty$, contradicting the fact that $q(\beta_m) \geq q(\overline{\beta})$. □

Let H be given by (25) with q replaced by h of (14); the change of variable $x = B + u$ in the integral in this definition of H shows that (15) is the equation $H(B) = 0$, and using Lemma 4.1 we conclude that the solution $-\log(e-1)$ is unique. Let $-\infty < A \leq -1$ be some constant, and define

$$\tilde{h}_j(x) = h(x) + \frac{e^{-x}}{\sqrt{j}} \quad \text{for } A \leq x < \infty.$$

Then by Lemmas 3.2 and 3.3, for all $j > j(A) = e^{-A}$ we have

$$h(x) < h_j(x) < \tilde{h}_j(x) \quad \text{for } A \leq x < \infty. \tag{35}$$

Also, since there is some $j_0(A) \geq j(A)$ such that for all $j \geq j_0(A)$

$$\frac{d\tilde{h}_j(x)}{dx} = \frac{e^x}{1+e^x} - \frac{e^{-x}}{\sqrt{j}} > 0,$$

the functions $\tilde{h}_j(x), j \geq j_0(A)$ are strictly increasing in $[A, \infty)$.

Lemma 4.2 *Let $\tilde{H}_j(x)$ be defined as in (25), with $q(x)$ replaced by $\tilde{h}_j(x)$. Then for all $j \geq j_0(A)$ there exists a value $\tilde{\beta}_j \in [A, \infty)$ such that $\tilde{H}_j(\tilde{\beta}_j) = 0$,*

$$\lim_{j\to\infty} \tilde{\beta}_j = -\log(e-1), \quad \textit{and} \quad \lim_{j\to\infty} \tilde{h}_j(\tilde{\beta}_j) = h(-\log(e-1)) = 1 - \log(e-1). \tag{36}$$

Proof. Since $\tilde{H}_j(x) \to H(x)$ uniformly on $[A, \infty)$, in particular $\lim_{j\to\infty} \tilde{H}_j(A) = H(A) > H(-\log(e-1)) = 0$. Thus for all $j > j_0(A)$ the value $\tilde{\beta}_j$ exists uniquely in $[A, \infty)$. Now (36) follows from the uniform convergence of $\tilde{H}_j(x)$ and $\tilde{h}_j(x)$ to $H(x)$ and $h(x)$, respectively, on $[A, \infty)$. □

Note (9) can be rewritten as

$$W_{n+1} = h_{n+1}(B_{n+1}) = \frac{1}{n} \int_{-\log n}^{\infty} [h_n(B_n) \vee h_n(y)] e^{-y} dy - a_n, \tag{37}$$

whereas (27) can be rewritten, with h instead of q, (keeping the β_n notation) as

$$h(\beta_{n+1}) = \frac{1}{n} \int_{-\log n}^{\infty} [h(\beta_n) \vee h(y)] e^{-y} dy - a_n. \tag{38}$$

Comparing (37) and (38) we see that the only difference between the two expressions is that in (37) the function in the integral depends on n, whereas in (38) this function is fixed.

We can now prove our main result:

Proof of Theorem 2.1. We apply Theorem 4.1 to (38) for $n \geq n_0$ with starting value $\beta_{n_0} = B_{n_0}$ as in (8), where n_0 is the value given by Theorem 4.1 for A and h, after which recursion (38) is well defined. For all $j > j_0(A)$ let $r_j = \max\{n_0, j\}$, and for $n \geq r_j$, define the sequence $\tilde{\beta}_{j,n}$ through (38) with h replaced by $\tilde{h}_j$, and initial value $\tilde{\beta}_{j,r_j} = B_{r_j}$. Then by (35), (37) and (38), the inequality

$$h(\beta_n) < W_n < \tilde{h}_j(\tilde{\beta}_{j,n})$$

holds for all $n > r_j$, noting that the right hand side of (38) say, is made larger by replacing h by a larger function. Thus as $n \to \infty$,

$$1 - \log(e-1) = \lim h(\beta_n) \leq \liminf W_n \leq \limsup W_n \leq \lim \tilde{h}_j(\tilde{\beta}_{j,n}) = \tilde{h}_j(\tilde{\beta}_j). \tag{39}$$

Now by Lemma 4.1, if we let $j \to \infty$, from (39)

$$1 - \log(e-1) \leq \liminf W_n \leq \limsup W_n \leq 1 - \log(e-1),$$

from which the first claim of the theorem follows. The other assertion follows as an immediate consequence. □

Remark 4.1 *The limiting one-choice value can be obtained in a similar, but simpler way. Let* $\{V_n^1\}$ *denote the sequence of one-choice optimal values and let* $W_n^1 = V_n^1 - \log n$*. Since* $V_{n+1}^1 = E[X_{n+1} \vee V_n^1]$ *it follows that the* $\{W_n^1\}$ *sequence satisfies (27) with* $q(x) = x$ *and* $\beta_n = W_n^1$*. By Theorem 4.1 it therefore follows that* $\lim_{n\to\infty} W_n^1 = W^1$ *is the solution* β *of* $Q(\beta) = 0$*, where*

$$Q(y) = \int_y^\infty x e^{-x} ds - y e^{-y} - 1, \quad i.e. \quad Q(y) = e^{-y} - 1,$$

which implies $W^1 = 0$*. This clearly agrees with the more general result of Kennedy and Kertz (1991), see (2) above.*

Remark 4.2 *A measure of the limiting effectiveness of having a second choice is the value* $\lim_{n\to\infty} (V_n^2 - V_n^1)/(EM_n - V_n^1)$*. It compares the relative advantage of having two choices over having only one choice, divided by the similar advantage for the "prophet", whose value is* EM_n*. For the exponential distribution we have*

$$\lim_{n\to\infty} \frac{V_n^2 - V_n^1}{EM_n - V_n^1} = \frac{1 - \log(e-1)}{\gamma} = .7946\ldots, \tag{40}$$

where γ *is the Euler constant. For the large subclasses of distributions of Types III and II, treated in Assaf, Goldstein and Samuel-Cahn (2004 a and b), the corresponding minimal values over all* α*-values is (40) and .7880... respectively. Thus the minimal saving in all the known cases is near 80%.*

5 References

1. Assaf, D., Goldstein, L., and Samuel-Cahn E. (2004a). Optimal Two Choice Stopping. *Adv. Appl. Prob.* **36**

2. Assaf, D., Goldstein, L., and Samuel-Cahn E. (2004b). Maximizing expected value with two stage stopping rules. *Discussion paper #351, Center for the study of rationality, Hebrew University of Jerusalem.*

3. Kennedy, D.P and Kertz, R.P. (1991). The asymptotic behavior of the reward sequence in the optimal stopping of i.i.d random variables. *Ann. Prob.* **19**, pp. 329-341.

4. Leadbetter, M.R., Lindgren, Georg and Rootzén, Holger.(1983). *Extremes and Related Properties of Random Sequences and Processes.* Springer-Verlag, New York, Heidelberg, Berlin