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AVERAGE SPEED BUMPS: FOUR PERSPECTIVES ON AVERAGING SPEEDS

by

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Average Speed Bumps

Four Perspectives on Averaging Speeds

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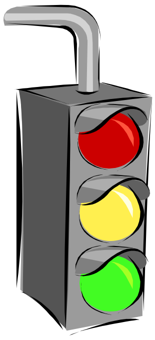

Introduction

Suppose you drive on a long one-way highway, where each car travels at a constant speed. Assume that the distribution of the speeds is the same throughout the length of the highway.

You adjust your speed so that during a given time unit you overtake the same number of cars as the number of cars that overtake you. Does this mean that your speed is the median of the speeds of the cars on the highway? Surprisingly, the answer is no! (Clevenson, Schilling, Watkins, & Watkins, 2001).

Imagine further a radar device at the side of the highway, measuring and recording the speeds of all the cars that pass this point within a fixed time interval. Again, contrary to lay expectations, the arithmetic mean of these recordings would generally not reproduce the arithmetic mean of the speeds of all the cars on the highway (Stein & Dattero, 1985).

The above examples illustrate that identifying the correct average may have its difficulties (the correct answers for both cases will be detailed later). Average speed, in general, is not all that self-evident a concept. The apparently simple question "what is the average speed of the cars that drive on the highway?" is equivocal. As students of introductory statistics know, the term average may be interpreted in various ways and hence may assume several different forms. One needs to know in what sense the average is supposed to represent a set of observations.

Four Faces of the Average Speed

We consider four (mathematical) requirements that may be imposed on the average speed, each of which has its own answer, thus resulting in the definitions of the median, the harmonic mean, the arithmetic mean, and the self-weighted mean. In parallel, the four averages are presented from the point of view of an observer on the highway, whether in a static position, or on the move.

We do not purport to offer practical traffic applications. Our aim is to highlight several distinctions that are instructive in principle for students of chance. Moreover, this understanding may spare us all some intuitively compelling fallacies.

The basic well-known relation connecting speed, or velocity (denoted V), time (t), and the distance (S) traveled during that time (in a constant-speed motion) is $V = S/t$. Speed is obtained by dividing the distance by the time it took to be traversed. Assume that all the cars on a very long highway drive in the same direction, each at some constant speed.

<u>Halving the Line</u>

Consider the case of n cars on the highway, and the following requirement:

> A car traveling at the average speed should have equal number of cars traveling faster and slower than itself. Evidently, the **median** speed, M_e, satisfies this requirement.

Some convention should be adopted in the problematic case of tied values at the location of the median (e.g., Downie & Heath, 1970, p. 36). However, accurate speed measurements are quite unlikely to be identical.

> On the highway: Suppose the n cars travel for a long time until the order of the cars stabilizes (i.e. no car passes another car, only gaps between cars traveling at different speeds increase with time). If you adjust your speed so that, when achieving that stability, the number of cars driving ahead of you equals the number of cars driving behind you, then your speed is the **median**, M_e, of all the speeds.

This description (which disregards technical details such as the parity of n) captures the gist of the definition of the median.

<u>Accompanying a Relay Race</u>

Let n cars travel with speeds $V_1, V_2, \ldots, V_n$ (not all of which are necessarily different from each other), so that they traverse a fixed distance, S, during times $t_1, t_2, \ldots, t_n$, respectively.

Imagine a relay race in which all n cars drive consecutively, each covering the same distance S and starting to move at full speed when its predecessor arrives. Altogether, the cars will cross a distance nS. You start moving (in a parallel lane) with the first car (see Figure 1). If your constant speed is such that you reach the final point synchronously with the nth car, then your speed is the **harmonic mean**, H, of all the speeds.

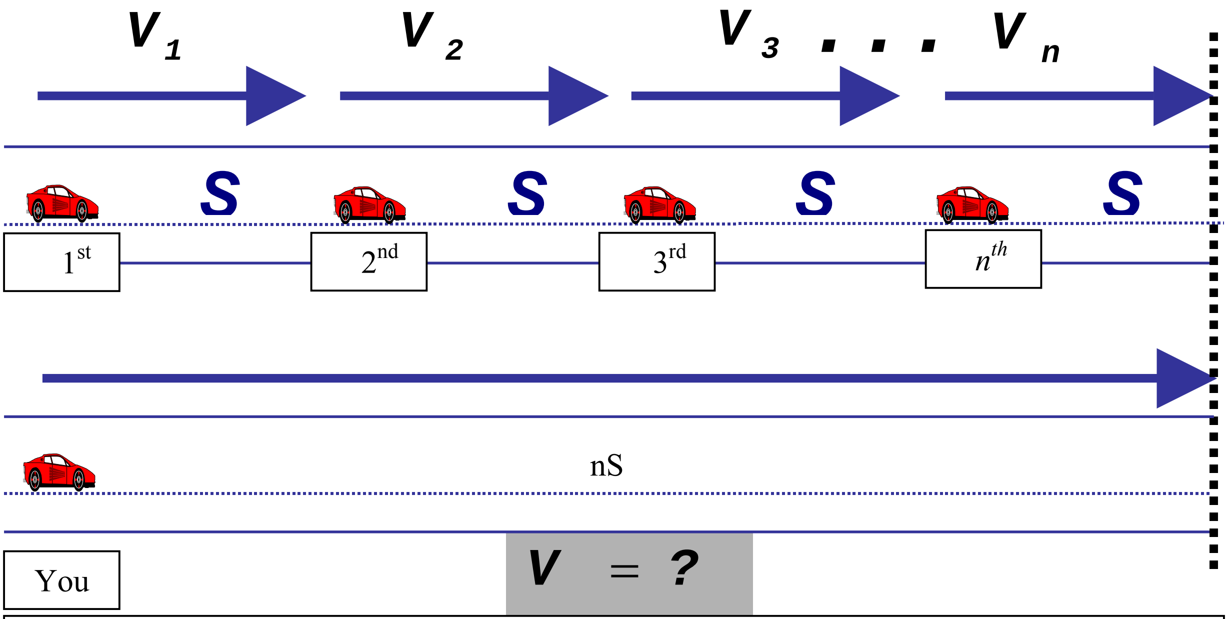

Figure 1 A relay race: n cars drive consecutively, each covering a distance S. You start moving with the first car.

According to this description, you cover a total distance nS during a time interval that equals the sum of the driving times of the n cars. Therefore, your speed is:

$$H = \frac{nS}{t_1 + t_2 + \dots + t_n} = \frac{S}{\dfrac{t_1 + t_2 + \dots + t_n}{n}} \qquad (1)$$

If we replace every t_i in the right hand of (1) by S/V_i (for $i = 1, 2, \dots, n$), and then divide both numerator and denominator by S – we obtain:

$$H = \frac{1}{\dfrac{\dfrac{1}{V_1}+\dfrac{1}{V_2}+\ldots+\dfrac{1}{V_n}}{n}} = \frac{n}{\dfrac{1}{V_1}+\dfrac{1}{V_2}+\ldots+\dfrac{1}{V_n}} = \frac{\dfrac{1}{V_1}V_1+\dfrac{1}{V_2}V_2+\ldots+\dfrac{1}{V_n}V_n}{\dfrac{1}{V_1}+\dfrac{1}{V_2}+\ldots+\dfrac{1}{V_n}} \qquad (1')$$

The right-hand part of equality (1) conveys an equivalent (mathematical) definition of the harmonic mean:

> Let all *n* cars traverse a fixed distance *S*. A car moving at the average speed is required to cover the distance *S* during a time interval that equals the arithmetic mean of the traveling times of the *n* cars. The speed that satisfies this requirement is the **harmonic mean**, *H*, of the *n* speeds.

Averaging the speeds according to this definition imparts greater weights to lower speeds, for which more time is needed to cross the same distance. It makes sense, therefore, that the harmonic mean will be smaller than *M*, the arithmetic mean of the same set of speeds.

We learn from (1') first, that the harmonic mean is the reciprocal of the arithmetic mean of the reciprocals of the averaged values. This is, in fact, the most common definition of the harmonic mean in the literature (e.g., Downie & Heath, 1970, p. 50; Hoehn & Niven, 1985; Yule & Kendall, 1953, pp. 120-121). Second, *H* equals (see the right-hand expression) the weighted mean of the speeds in which each speed is weighted by its reciprocal. This confirms the conclusion that for a given set of speeds, the harmonic mean, *H*, is less than (or equal to) the arithmetic mean, *M*, because in *H* the weights of low speeds are greater than those of high speeds, whereas in *M* all the speeds are weighted equally.

Apparently, the harmonic mean is hardly people's primary intuitive choice, even when appropriate. In a typical word problem, high-school and college students are told that a car goes from city *A* to city *B* at, say, 30 mph, and back from *B* to *A* at 50 mph, and they are asked about the car's average speed for the round trip. The prevalent preference of students is the answer *M*=40 mph, whereas the correct answer is the harmonic mean of the two speeds, *H*=37.5 mph. Average is too often identified with the arithmetic mean.

On the Way Back

Gardner's (1982, p. 142) young skier is impatient with the slow cable car climbing up the slope at 5 km/h. By skiing faster downhill, he wants to raise his mean speed for the round trip up and down the slope to 10 km/h. How fast must he ski down? Most readers' first attempted answer is 15 km/h, whereas in fact no speed could accomplish this goal. In order for the mean speed for the whole trip to be double the speed up the slope, the skier should make the distances up and down in the same time it took him to go up. But all this time has already been consumed on the way up. Formally, let the respective speeds up and down be *a* and *b*. The mean speed for the round trip is the harmonic mean of these speeds, $H_{a,b} = \frac{1}{\frac{1/a + 1/b}{2}} = \frac{2ab}{a+b} = 2a\frac{b}{a+b} < 2a$. The skier's wish that $H_{a,b}$ would equal $2a$ can never be achieved, because for positive *a* and *b* it is always true that $H_{a,b} < 2a$.

In contrast, given a value $a>0$, one can easily find a value *b* so that the arithmetic mean of *a* and *b* would equal $2a$. The solution is $b=3a$.

Monitoring Overtakings

Consider now *n* cars per mile. In each mile, the cars' speeds are $V_1, V_2, \ldots, V_n$ mph (not all these speeds are necessarily different from each other). The requirement for the average speed is:

> During a fixed time interval, a car traveling at the average speed should overtake the same number of cars as the number of cars by which it is overtaken. The **arithmetic mean**, *M*, of the *n* speeds, that is, the familiar mean obtained by adding all the speeds and dividing the sum by the number of speeds, is this average.

The procedure, from the point of view of a driver on the highway, as an observer, is:

> You have to count how many cars you pass and how many pass you, during, say, an hour. If you adjust your speed so that these two numbers are equal, your speed is the **arithmetic mean**, *M*, of all the speeds.

This assertion is not trivial. Upon first encounter, many would opt for the median as the average that satisfies the above requirement. The misleading first impression can, however, be counteracted by careful analysis.

Suppose you drive at speed A=30 mph. Let us look at cars that drive constantly at speed V_i = 20 mph (recall that there is a car of speed V_i per every mile). Figure 2 presents your starting position, at milestone 0, and your position after one hour, at milestone 30, as well as the positions (indicated by the black rectangle), at the same points of time, of all cars traveling at 20 mph that, at the beginning, are located within 30 miles ahead of you.

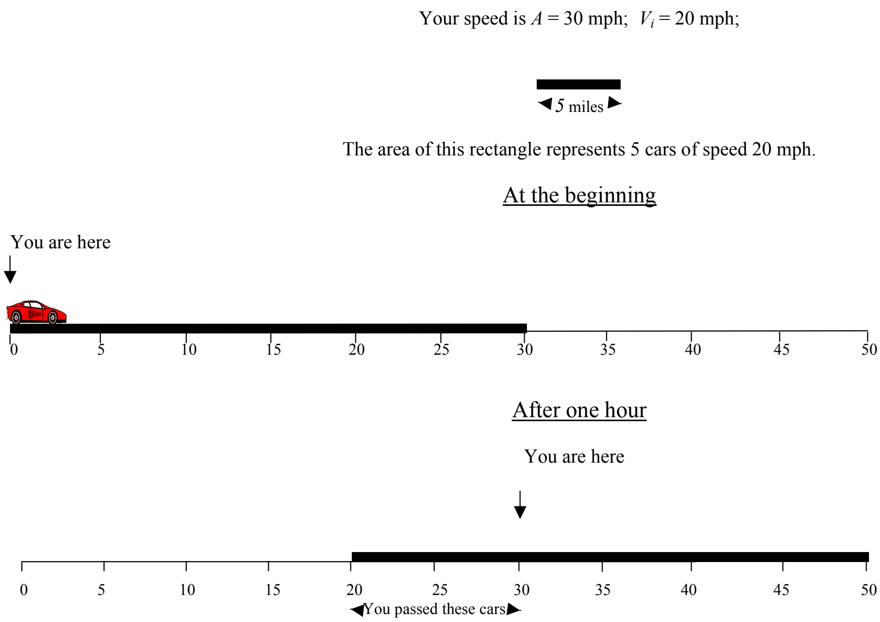

Figure 2 The positions of your car, and of all the cars that drive 20mph within a range of 30 miles, at two points of time, one hour apart.

The rectangular black area between two milestone marks, 5 miles apart, represents 5 cars of speed 20 mph. The cars that you pass during one hour are indicated by the rectangular black area between milestones 20 and 30. Hence you overtake (30 - 20) = 10 = A - V_i such cars in an hour. By the same token, it is easy to see that if $V_i > A$, then the number of cars of speed V_i that overtake you, in an hour, is V_i - A.

If your speed, A, is such that the number of cars that you pass is equal to the number of cars that pass you, in one hour, we have $\sum_{V_i>A} (V_i - A) = \sum_{V_i<A} (A - V_i)$.

Solving for A, we obtain $A=M$, where M is the arithmetic mean of all the speeds. This establishes that your speed is the arithmetic mean, M, of the speeds on the highway. Indeed, M can be defined as the value from which the sum of all deviations is zero and which satisfies $\sum_{V_i>M}(V_i - M) = \sum_{V_i<M}(M - V_i)$. This equality does not generally imply that the number of speeds greater than M equals the number of speeds that are less than M, confirming what is well known, that the median, M_e, may be different from M.

Mean Versus Median

The following is a much simplified numerical example for comparing the number of cars that overtake you with the number that you overtake, in an hour, when you drive either at the median or at the mean speed (see also Clevenson et al., 2001).

Let three cars of constant speeds 15 mph, 20 mph, 55 mph drive (in respective parallel lanes) in each mile along the highway. Our averages of interest are:

$$M_e=20 \text{ mph} \;;\; M=30 \text{ mph}$$

If you drive at the median speed, the number of cars that you overtake in an hour: 20–15=5, differs from the number of cars that overtake you: 55–20=35.

If you drive at the mean speed, the number of cars that you overtake in an hour: (30–15)+(30–20)=25, equals the number of cars that overtake you: 55–30=25.

Observing From the Side

Consider again n cars per mile traveling in the same direction with speeds $V_1, V_2, \ldots, V_n$, as before. The following procedure for obtaining the average speed is described in terms of observations taken on the highway. At the same time, this method entails a mathematical definition of that average:

> Suppose a radar gun, hidden at a fixed arbitrary location along the highway, measures the speeds of all the cars that pass that point during a fixed time interval. The arithmetic mean of these measurements is the **self-weighted mean**, SW, of the speeds of all the cars that drive on the highway. SW is a

weighted mean in which the weight of each speed equals (or is proportional to) the speed itself (Haight, 1963, pp. 114-116):

$$SW = \frac{V_1^2 + V_2^2 + \ldots + V_n^2}{V_1 + V_2 + \ldots + V_n} \qquad (2)$$

This result is rather counterintuitive (try your uncultivated tendencies). It might appear paradoxical that one computes the arithmetic mean of what, on the face of it, seems a random (perfectly representative) sample of observations of speeds on the highway and obtains (an estimate of) the self-weighted mean of the population of speeds. However, this sampling method is biased in favor of higher values.To justify formula (2), we may let our stationary point of measurement play the role of a car traveling at speed A=0. We learned from the analysis of the previous section that the number of cars traveling at V_i mph that overtake a car of speed $A<V_i$ (during an hour) is $V_i–A$. Therefore, when A=0, the number of cars of speed V_i that pass the radar point is V_i. Each speed is thus weighted by its own magnitude.

As can easily be seen from (2), weighting a speed by itself implies $M \leq$ SW (for the same set of speeds), because, in *SW*, higher values are weighted more heavily than lower values. Note that both (1’) and (2) present weighted means of the speeds V_1, V_2, …, V_n, but with inverse respective weights.

In Absence of a Timetable

The following story is about waiting times, not speeds. But our averages are involved.

In a certain Middle-Eastern country, two trains per hour were regularly going from city A to city B. But the public often complained about waiting too long in the station. In response, the management added a third train per hour. For security reasons no timetable was published. Passengers were expected to come to the station and wait for the forthcoming train.

Curiously, the number of complaints about excessive waiting rose after the introduction of the third train. Apparently, the mean waiting time increased, instead of decreasing, as a consequence of that addition. How did this happen?

This ostensible-paradox can easily be resolved if one considers, not only the number of trains per hour, but also the variability among the time intervals between successive trains (see van Dijk, 1997). To illustrate, consider an extreme (admittedly contrived) example:

Suppose that initially the trains came exactly each half hour. A passenger, arriving at a random moment, would have to wait 15 min, on the average. Suppose, further, that after introducing a third train per hour, the intervals between trains were, in turn, 6 min, 50 min, and 4 min. Arriving at a random point of time during any of these intervals, a passenger would wait, on the average, for half its length, but the chances of arriving during a given interval are proportional to its length. Therefore, although the arithmetic mean of the half-intervals {3, 25,2} is 10 min (i.e., less than 15 min), a passenger's mean waiting time (in minutes) would be greater after the addition of the third train:

$$\frac{3\times 3+25\times 25+2\times 2}{3+25+2}=21.27>15.00$$

Note that the mean waiting time is the self-weighted mean of the half-intervals. The mean waiting time increased (after adding the third train) from 15 min to over 21 min, despite the decline of the mean interval from 30 min to 20 min.

This method of collecting observations is known in the literature as size-biased or length-biased sampling (Patil, Rao, & Zelen, 1988). It makes sense that fast cars are measured by this method more frequently than slow cars, even when there are equal numbers of cars per mile of both kinds (see Figure 3). The chances that a car's speed will be recorded by the radar are proportional to the distance traveled by that car in an hour, and this, in turn, is proportional to the car's speed. Hence, in the set of measured speeds, the frequency of each speed is proportional to its own size, and the mean of all these values is, according to (2), *SW* of the original set of speeds.

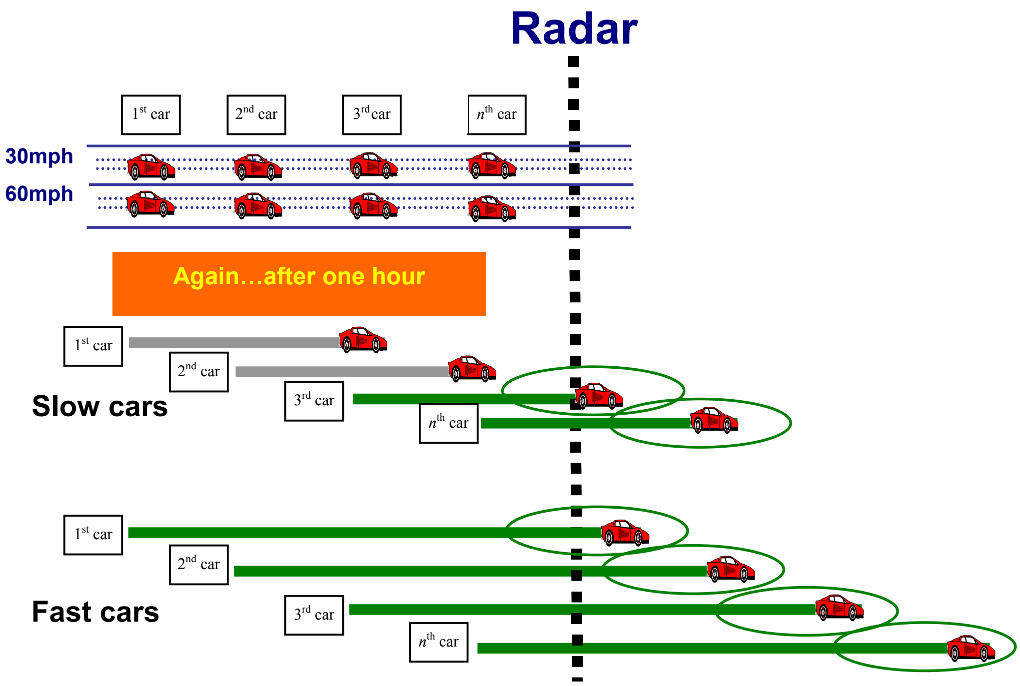

Figure 3 A radar gun measures the speeds of fast and slow cars that pass its location during an hour. The number of recorded (circled) cars is proportional to their speed.

Reckless drivers should thus be doubly cautioned not to exceed the speed limit. Against the feeling that by going fast one may evade the trap altogether, one could argue that: If there is a radar somewhere, operating during an hour, the faster you go the greater your chances of being caught by this method of sampling. You are "self weighting" your probability of detection.

Data From Old Faithful

Real data sets of wait times between eruptions of the Old Faithful geyser in Yellowstone National Park are presented by Shaughnessy and Pfannkuch (2002). For example, one list (for one day) of minutes between eruptions is (p. 253):

51	82	58	81	49	92	50	88	62
93	56	89	51	79	58	82	52	88

Suppose some friends were planning to visit Yellowstone. The authors ask "how long should we tell them to expect to wait between eruptions of Old Faithful?" (p. 253). Their answer is the **arithmetic mean** of these 18 values, M=70.06 min.

The arithmetic mean indeed answers the above question as phrased. However, the more relevant, or practical question would be: "how long should a prospective visitor, *arriving at a random moment*, expect to wait until the next eruption?"

A visitor who arrives randomly during a certain interval would have to wait for half its length, on the average. However, the chances of arriving during a given interval are proportional to its length, just as the probability that the radar will sample a car's speed is proportional to that speed. The expected waiting time would thus be the **self-weighted mean** of the half-intervals. The answer for the above data is 36.99 min. This is greater than half the arithmetic mean: 70.06/2 = 35.03 min.

Authors' Compromise

Perfectionists would protest that our background assumptions, on which the derivation of the formulas of the averages relied, are not strict enough. They would be right.

Indeed, for the definition of M to be tight, the equalities that should hold "during a fixed time interval" or "during an hour" should hold for any duration. The same is true for the measurements of speeds "during a fixed time interval" at an arbitrary point on the highway, in the definition of SW. Likewise, the "fixed distance" that all the cars traverse, in the definition of H, should ideally be a segment of any length on the highway.

In order for our analysis to be independent of choice of units of time and/or distance, that may get as small as one pleases, there seems to be no way out of assuming that all traveling cars – whatever their distribution of speeds per mile (or per any unit of distance) – are spread continuously along the highway. This means that fractions of cars might be counted if necessary.

Assumptions of this kind would have created conceptual and technical difficulties. They bring up the age-old dilemma of crossing the unbridgeable gap between the discrete and the continuous. Cars on the road are definitely discrete entities, hence their behavior might only approximate a continuous model. The denser the traffic on the highway, the closer it would be to an assumed continuous model (Haight, 1963, pp. 125-126). But presuming continuity would have considerably interfered with the fluency, readability, and acceptability of the exposition.

Rigor

"Mathematical rigor is like clothing: in its style it ought to suit the occasion, and it diminishes comfort and restricts freedom of movement if it is either too loose or too tight."

From the introduction to a book on differential equations by G. F. Simmons.

To promote understanding and ease the reading, we had to skip some mathematical rigor and to compromise over "elegance" as well. For instance, we considered a total of n cars when introducing the median and the harmonic mean, whereas for the sake of presenting the arithmetic and the self-weighted means we looked at n cars per mile all through the highway.

An ideal, flawless derivation of the arithmetic mean and the self-weighted mean should have presupposed an unbounded linear highway (or, as in Haight, 1963, p. 114, a circular highway). To be realistic, we had again to compromise over a long highway.

Highway Contemplations

Daughter, D. (4-years old) while traveling on the highway in a long trip:

"Mom, does the road go to infinity?"

Mother, M: "No, the road has an end."

D. (pondering the answer for a while):

"Yes, the road has an end, but the *direction* goes to infinity."

We thank Mara Beller and her daughter, Dana, for letting us use their dialogue.

Quasi-Realistic Representation

Trains, made up of contiguous wagons, may better fit the image of a continuum of vehicles along a line. Although nonexistent in reality, such trains are imaginable. Envision *n* immensely long trains composed of many adjoining wagons of the same length. The (continuous) wagons now take the place of cars on the highway in the previous presentations. The trains travel on parallel rails in the same direction, each at its own constant speed. Let the speeds of the *n* trains be V_1, V_2, …, V_n. These trains satisfy the continuity assumption. The number of wagons in *any* rail segment, and/or *any* time interval, may be easily determined, counting fractions of wagons if necessary. Moreover, all the trains are so long, that viewed from our observation point(s), no problem of reaching the ends during the observation period arises.

This representation may aid in visualizing the real-life interpretations of *M* (versus M_e) and *SW*:

> You travel at a constant speed in a train on one of these rails, and you observe all the other parallel rails. Your speed is the **arithmetic mean**, *M*, of the speeds of all the trains, iff the number of wagons that pass you equals the number of wagons that you pass, during some time interval. Note that when moving at a constant speed, it feels after a while as if you are immobile. In that case, your speed is the **median** of the speeds of all the trains when you see equal numbers of trains that move forward and backward.
>
> Suppose you are stationed at a fixed arbitrary point on the side of the parallel rails. You record the speeds of all the wagons that go by that point during some time interval. The arithmetic mean of these measurements is the **self-weighted mean**, *SW*, of the speeds of all the trains that travel on the rails.

Evidently, more wagons of fast trains pass your station than wagons of slow trains. In the set of measurements, the frequency of wagons of a given speed is proportional to that speed, which is the speed of the train to which the wagons belong.

In Conclusion

Averaging speeds (or any other values) requires a careful approach to sort out which mean does one mean. Assumptions should be explicitly spelled out. Extra care should be taken to clarify what is the method of collecting observations and what consequences it has for the weights of the values that should be averaged. It turns out that the good advice “drive carefully” applies not only to moving on the roads but also to manipulating this situation mathematically.

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