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ADDING THE NOISE: A THEORY OF COMPENSATION-DRIVEN EARNINGS MANAGEMENT

by

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Abstract

Adding the Noise: A Theory of Compensation-Driven Earnings Management

Empirical evidence suggests that the distribution of earnings reports is discontinuous. This is puzzling since the distribution of true earnings is likely to be continuous. We present a model that rationalizes this phenomenon. In our model, managers report their earnings to rational investors, who price the stock accordingly. We assume that misreporting is costly, but since managers' compensation is based on the stock price, they may want to manipulate the reported earnings. The model fits into the general framework of signaling games with a continuum of types. The conventional equilibrium in this game is fully revealing (e.g. Stein 1989), and does not explain the observed discontinuity of earnings reports. We show that a partially pooling equilibrium exists in such games as well, and it generates an endogenous discontinuity in reports. By pooling reports of different types, the informed manager introduces "home-made" noise into his report. The resulting vagueness enables the manager to reduce the manipulation costs. While *a priori* pooling looks manipulative, it is actually a way to reduce earnings management. The empirical implications of our model relate earnings management and price reaction to price- and earnings-based compensation, growth opportunities of the firm, underlying volatility, and the stringency of accounting rules. We show that this equilibrium arises due to stock-based compensation of the managers, and does not arise when they are paid based on their earnings directly. Finally, we present a general version of this model describing the behavior of biased experts in many real-life situations.

1 Introduction

Managers of publicly traded firms possess information that is crucial to the valuation of their firms. Lacking access to this information, investors must determine the stock price primarily based on public information. Consequently, mandatory periodical earnings announcements by the managers serve as the predominant source of information for the investors. Earnings announcements have been studied extensively, and were recently the subject of an intense scrutiny by the public and the regulators. We know that managers have some leeway in reporting the earnings within the accounting conventions, and that managerial compensation is frequently related to earning either directly, or indirectly via the stock price. Thus managers have the incentives and the ability to "manage" (manipulate) earnings, and the empirical evidence suggests that they frequently do so.[1] There is also evidence that the compensation of managers is a major determinant of the extent of earnings management.[2]

While it is reasonable to assume that the true earnings of a firm are drawn from a continuous distribution, the empirical distribution of earnings reports is discontinuous. Burgstahler and Dichev (1997) and Degeorge, Patel and Zeckhauser (1999) provide evidence on such discontinuity. They show that managers manage earnings as if to meet exogenously pre-specified targets, such as avoiding losses, meeting analyst earnings forecasts (see also Abarbanell and Lehavy (2000)), or meeting the last quarter earnings benchmark. The resulting discontinuity manifests itself as a dent in the distribution of reports slightly below certain predetermined levels. Thus, the unconditional distribution of reports often takes a bell shape with a dent located to the left of the mean. From a rational point of view, this behavior is puzzling. If such a manipulative conduct persists, then rational investors are not fooled, and discount the stock price of manipulating firms. So what is the incentive for managers to engage in such an exercise in the first place? Consequently, the conjectures offered to explain this phenomenon are primarily based on behavioral arguments either on the investor's utility (see Burgstahler and Dichev 1997) or on their information processing heuristics (see Degeorge, Patel and Zeckhauser 1999).[3]

We feel that it is important to understand this discontinuity in a fully rational

[1] The empirical literature on earnings management is voluminous. See Healy and Wahlen (1999) for a review.

[2] See for instance: Healy (1985), Bergstresser and Philippon (2002), and Kedia (2003).

[3] The extant theoretical literature on earnings manipulations (see Verrecchia (2003) for an extensive survey) focuses on equilibria in which the reported earnings distribution is continuous.

model to gain insights into the manager's overall reporting behavior. In this paper we propose a game-theoretic framework in which the discontinuity phenomenon arises endogenously in a game played between an informed manager and uninformed investors. In our model the manager trades off the costs of earnings management imposed by a third party, against his private benefits from such management in the face of a rational response by uninformed investors. This trade-off determines the optimal level of manipulation. We believe that our model contributes to a deeper understanding of the earnings reporting process, and the resulting stock price reactions. We then argue that this framework fits many real-life situations in which potentially biased experts are hired to observe the state of the world and report to their clients.

We use the framework of a signalling game, in which the private information of the manager is considered to be his "type". The manager provides his report (a costly signal) and the investors value the stock conditional on the report. If we assume a continuous "type" space to capture the fact that the true earnings come from a continuous distribution, then a standard equilibrium in this setting is a perfectly separating one (see Riley (1979)). In this equilibrium each manager "type" inflates his report, however investors correctly interpret this action; the equilibrium is fully revealing and the manager gains no benefit from this action. Tragically, the manager cannot avoid this manipulation; indeed, were he to report truthfully, the investors would mistake him for a lower "type" and his payoff would be even lower. Within this equilibrium, developed by Stein (1989) in a dynamic setting, the manager inflates his report, reveals the true earnings, but still incurs the costs of the earnings distortion.

The perfectly separating equilibrium creates a continuous distribution of reports, and cannot explain the observed discontinuity of reported earnings. We show that another equilibrium exists in this game, where the earnings report discontinuity arises endogenously. We claim that the following is an equilibrium: if the earnings are either low or high, the manager publishes a report that reveals his type, as in the fully revealing equilibrium.[4] However, if the manager observes earnings in the intermediate range, he always reports the same amount; i.e., his report is not fully revealing. The pooling behavior of the manager creates an endogenous discontinuity in the distribution of reports. We also show that this partially pooling equilibrium ex-ante Pareto dominates the standard fully revealing equilibrium. Intuitively, the manager intro-

[4] The manager does not report the truth, but his report does reveal the true earnings to the investors.

duces “home-made” noise into his report by hiding among other intermediate types. This camouflage enables the manager to moderate his manipulation activity and incur lower (on average) manipulation costs. It follows that in this newly suggested equilibrium, vagueness serves as an antidote to inherent inefficiency. Moreover, this vagueness implies lower (on average) earnings management relative to the benchmark separating equilibrium.

The empirical implications of this equilibrium relate the earnings' volatility and the incentives of the manager to the extent of earnings management, and the extent of pooling behavior. We show that the pooling behavior is more pronounced, and *hence* earnings management is lower in firms with low earnings volatility. We also show that the pooling behavior is highly pronounced and *still* earnings management is high in firms (also industries or countries) with a high level of stock based compensation, high growth opportunities, and less stringent accounting rules.

We specifically make earnings manipulation costly in the model, assuming that the marginal cost is increasing in the degree of earnings management. Managers can distort the true earnings either by using discretionary accruals, or by taking real actions (e.g., making suboptimal investments). The former may carry regulatory, legal, or reputational cost, while the latter reduces the future earnings of the firm: in either case earnings management is costly for the manager. This assumption places our model within the *costly* signalling framework. A similar approach is taken by Fischer and Verrecchia (2000), who study earnings bias: they limit their attention to a continuous equilibrium, but allow for uncertainty regarding the incentives of the manager. We propose a Pareto-dominant alternative that introduces noise into reporting.[5] Another relevant paper is Milgrom and Roberts (1986), who consider a game where a manager attempts to increase the stock price by convincing investors that the firm has favorable earnings prospects. The result is again a fully revealing equilibrium, even in a non-competitive environment.

A different approach found in the “cheap talk” models pioneered by Crawford and Sobel (1982), is to assume that managers bear no cost of earnings manipulation. In these models the default equilibrium is a perfectly pooling one (“babbling” equilibrium), where no one pays any attention to the managerial report. However, Crawford and Sobel (1982) show that other equilibria also exist, in which the report

[5] A partially pooling equilibrium with a continuous type space also appears in Harrington (1987) in the context of a limit-pricing model.

takes a form of the increasing step function of earnings. These equilibria reduce the uncertainty, yielding welfare improvement over the perfectly pooling one due to noise reduction (exactly the opposite of our model). We feel that our assumption of costly misreporting better fits the context of managerial incentives, and is also applicable to a wide range of real-life situations where individuals reveal their true information when it is extreme, while they try to be vague when the information quality is intermediate. Such examples include analyst reports, expert opinions, recommendation letters, and various political, administrative and regulatory processes.

Our model is a “one shot” game, thus does not address earnings smoothing across periods. The theoretical literature on earnings smoothing (e.g. Dye (1988), Trueman and Titman (1988), Fundenberg and Tirole (1995), and Goel and Thakor (2003)), does not explain the discontinuity of earnings reports. Thus, the contribution of our model is in explaining earnings management within a reporting period and not across periods.[6]

The paper is organized as follows. In Section 2 we present the model, and the benchmark fully revealing equilibrium. In Section 3 we study the partially pooling equilibrium and compare it to the conventional fully revealing one. In Section 4 we provide several extensions to the model. In Section 5 we investigate the comparative statics of the partially pooling equilibrium, and point out the empirical implications. Section 6 shows the model's applicability to the case of an informed expert and an uninformed decision-maker, and presents numerous examples. In Section 7 we study the robustness of the partially pooling equilibrium, while Section 8 concludes. Technical proofs are in the Appendix.

2 Model

We assume that the true earnings of the firm, x, are drawn from a normal distribution with mean x_0 and variance σ^2. The cumulative distribution is denoted by F, and the density is denoted by f. The parameters of the distribution are common knowledge; however, only the manager observes the realization x.[7] The manager is mandated to publish an earnings report, x^R, which the investors observe and use to price the

[6] Nevertheless, the managerial utility in our static model can be considered as a reduced from of a broader model, which could include earnings smoothing considerations.

[7] While x denotes the realization of earnings known only to the manager, we denote by $\tilde{x}$ the random variable of earnings observed by the uninformed investores.

stock. The utility of a manager who reports x^R after observing x is given by

$$U^M(x, x^R) = \alpha P(x^R) - \beta(x - x^R)^2, \tag{1}$$

where $\alpha \neq 0$, $\beta > 0$, and $P(x^R)$ is the market price of the firm given the report.

The second term of the manager's utility function represents the cost of manipulation, which is central to our paper. Positive β implies that higher deviations from the truth carry higher legal, regulatory, or reputational penalties for the manager. An alternative interpretation of this term is that the manager can take real actions (e.g., asset sales, suboptimal investments, aggressive sales efforts) to generate earnings that are different from the true earnings under the optimal policy. These actions are costly in terms of the future earnings of the firm, and therefore are costly for the manager.[8] Parameter β is likely to be fairly stable over time for firms in the same industry, since it is governed by accounting conventions and practices in the industry, but it may also vary across managers depending on their reputations.

The first term represents the fact that managerial compensation depends on the stock price, which makes the manager potentially biased.[9] Usually, managers prefer to see higher stock prices, but it well may be that temporarily they may be interested in reducing the price of the stock, e.g., to improve the price of awarded options, to buy back shares, or to make the next period price increase more dramatic. Thus, α may be positive or negative, and is likely to vary over time and across firms. For clarity of presentation we provide a detailed analysis of the case $\alpha > 0$ (a positive bias). The case $\alpha < 0$ yields parallel results, and is presented in Section 4. All parameters are common knowledge.

We assume that investors are risk neutral and study the consequences of risk aversion in Section 4. It follows that investors price the stock proportionally to the expected value conditional on all the available information. We denote by $c > 0$ the Price-Earnings (P/E) ratio for this firm. Thus, the price of the stock prior to the manager's report is $p_0 = cx_0$ - the prior mean, while the post-report price is

[8] This interpretation implies that the value of the firm is reduced due to these actions. The investors must take this into account when pricing the stock. We do not model this effect for expositional clarity; however, we conjecture that putting it in would only reinforce the attarctiveness of the proposed equilibrium, since it reduces the manipulation cost (see below).

[9] The pay-for-performance sensitivity of compensation has been increasing over the years. See Hall and Liebman (1998) compared to the findings of Jensen and Murphy (1990).

$p_1 = P(x^R) = cE(\tilde{x}|x^R)$.[10] We conclude that for all pairs x and x^R we have

$$U^M(x, x^R) = \alpha c E(\tilde{x}|x^R) - \beta(x - x^R)^2. \tag{2}$$

The manager has two conflicting interests: on the one hand, he would like to boost (if $\alpha > 0$) the stock price by manipulating his report; on the other hand, he does not want to manipulate his report too much, because the marginal cost is increasing. The relative weight that the manager assigns to each of these two incentives is determined by the ratio $\frac{\alpha}{\beta}$. The higher this ratio is, the more inclined is the manager to deviate from the truth. The combination of the normal prior distribution of earnings with a quadratic cost function yields a tractable model.

A reporting strategy for the manager is a real function $\rho : \mathbb{R} \to \mathbb{R}$ that maps true earnings into reports: $x^R = \rho(x)$. A pricing function for the investors is a function $P : \mathbb{R} \to \mathbb{R}$ that maps the manager's report into a price. A perfect Bayesian equilibrium is defined as a reporting strategy ρ^* for the manager, joint with a pricing function P^* for the investors such that:

1. The pricing function P^* is consistent with the strategy ρ^*, by applying Bayes rule whenever possible.

2. For all $x \in \mathbb{R}$, $\rho^*(x) \in \arg\max_{x^R} U^M(x, x^R)$.

Observe first that truthful reporting, i.e., $\rho(x) = x$, is not an equilibrium. Indeed, if $\rho(x) = x$ for all $x \in \mathbb{R}$, then investors adjust their beliefs to reflect this strategy; thus, $P(x^R) = x^R$ for all reports x^R. If a manager who observes real earnings of x reports truthfully, he obtains $\alpha c x$. If this manager raises his report to $x + \varepsilon$ ($\varepsilon > 0$), he obtains $\alpha c(x + \varepsilon) - \beta\varepsilon^2$. Thus, deviation is beneficial for all sufficiently small ε. It is also easy to verify that a perfect pool (a babbling equilibrium), i.e., $\rho(x)$ *is constant* for all $x \in \mathbb{R}$, cannot be an equilibrium for any pricing function.

We show below the existence of two types of equilibria in this model. The first and conventional equilibrium is the perfectly separating (fully revealing) one. This equilibrium serves as a benchmark for our analysis.

[10] Note that in our model, c stands for the "true" P/E ratio of the firm - the ratio between price and *true* earnings. Earnings management renders this ratio somewhat different from the implied P/E ratio - the ratio between price and *reported* earnings.

Proposition 1 *There exists a unique perfectly separating, continuously differentiable equilibrium. The equilibrium strategy of the manager is linear in x:* $\rho_s^*(x) = x + \frac{\alpha c}{2\beta}$ *for all* $x \in \mathbb{R}$*. The pricing function is linear in the report:* $P_s^*(x^R) = c\left(x^R - \frac{\alpha c}{2\beta}\right)$ *for all* $x^R \in \mathbb{R}$*.*

Proof: In the Appendix.

This perfectly separating equilibrium is characterized by a constant earnings manipulation. Regardless of the realization of true earnings, the manager inflates his report by $\frac{\alpha c}{2\beta}$. Thus, a high level of stock based compensation, a high P/E ratio, and a low level of accounting standards imply a higher level of earnings management.[11] Naturally, the investors are not fooled and price the stock correctly. This kind of equilibrium is standard in the continuous type, costly signalling literature (e.g., Riley (1979) and Stein (1989)). Notice that this equilibrium is tragically inefficient. The manager cannot avoid the costly information management, and pays the costs that it imposes on him. Unfortunately, he gains nothing from this behavior, because it is correctly interpreted by the investors.[12] In the next section we show that there exists another equilibrium in this model such that the manager is able to reduce his manipulation cost.

3 Pooling Reports and Home-Made Noise

We transform the fully separating equilibrium in Proposition 1 into a partially pooling equilibrium. We conjecture the existence of an interval $[a, b]$ such that the following partially pooling strategy is optimal for the manager:

$$\rho_p^*(x) = \begin{cases} b & x \in [a, b] \\ x + \frac{\alpha c}{2\beta} & \text{otherwise} \end{cases} \tag{3}$$

The conjectured strategy $\rho_p^*(x)$ is a simple modification of the fully revealing equilibrium strategy $\rho_s^*(x)$. For high or low values of earnings outside the interval $[a, b]$ the manager sticks to the same strategy as in Proposition 1: $\rho_p^*(x) = \rho_s^*(x) = x + \frac{\alpha c}{2\beta}$.

[11] This is consistent with the findings of Leuz, Nanda, and Wysocki (2003).

[12] The empirical evidence on whether investors are actually fooled by earnings management is mixed. Rangan (1998) and Teoh, Welch and Wong (1998) claim that managers succeed in fooling investors by manipulating reports. On the contrary, Shivakumar (2000) concludes that investors are not misled and account correctly for the manipulative behavior of managers. Some of the evidence may be driven by the investor's uncertainty about the manager's bias as in Fisher and Verrecchia (2000).

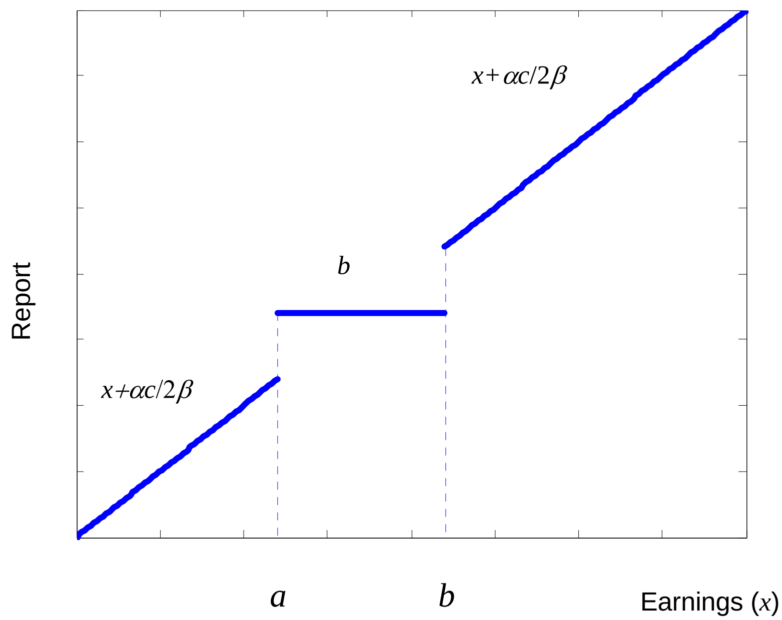

Figure 1: The Partially Pooling Equilibrium Reporting Strategy

He does not report truthfully (this is suboptimal), but does reveal his true type. For the intermediate values that fall inside the interval $[a, b]$, the manager always reports b, the upper bound of the interval. Figure 1 depicts this partially pooling strategy. We show below that an equilibrium of this kind exists, is unique (in the sense that the interval $[a, b]$ is determined uniquely), and ex-ante Pareto dominates the separating equilibrium $\rho_s^*(x)$.

The first step in proving the existence of equilibrium is to find necessary conditions for $\rho_p^*(\cdot)$ to be an equilibrium. First, note that if $\rho_p^*(\cdot)$ is an equilibrium then the type is fully revealed for all reports $x^R < a + \frac{\alpha c}{2\beta}$, or $x^R > b + \frac{\alpha c}{2\beta}$. On the contrary, when the investors observe a report of b, they can only deduce that the type is somewhere in the interval $[a, b]$. Using Bayes rule, it follows that conditional on a report of b, the posterior beliefs of the investors are distributed according to a truncated normal distribution on $[a, b]$.[13] Thus, the first necessary condition for a partially pooling

[13] This means that for all $s \in [a, b]$, $\Pr(\tilde{x} \le s | \pi^R = b) = \Pr(\tilde{x} \le s | x \in [a, b]) = \frac{F(s)-F(a)}{F(b)-F(a)}$.

equilibrium is that the pricing function of the investors must satisfy

$$P_p^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a + \frac{\alpha c}{2\beta} \;\; or \;\; x^R > b + \frac{\alpha c}{2\beta} \\ \\ cd(a,b) & x^R = b, \end{cases} \tag{4}$$

where $d(a,b) \equiv E(\tilde{x}|\tilde{x} \in [a,b])$ is the mean of true earnings conditional on the information that they are in $[a,b]$. Next, notice that in order for $\rho_p^*(\cdot)$ to be an equilibrium, a manager with type $x = a$ must be indifferent between reporting $a + \frac{\alpha c}{2\beta}$ and reporting b. Similarly, the manager with type $x = b$ must be indifferent between reporting $b + \frac{\alpha c}{2\beta}$ and reporting b. Evaluating the manager's utility given by (2) at these points, and using (4) we obtain a system of equations:

$$\begin{aligned} \alpha ca - \beta(\frac{\alpha c}{2\beta})^2 &= \alpha cd(a,b) - \beta(b-a)^2 \\ \alpha cb - \beta(\frac{\alpha c}{2\beta})^2 &= \alpha cd(a,b). \end{aligned} \tag{5}$$

Solving it yields

$$b = a + \frac{\alpha c}{\beta}, \tag{6}$$

and

$$d(a,b) \equiv E(\tilde{x}|\tilde{x} \in [a,b]) = a + \frac{3\alpha c}{4\beta}. \tag{7}$$

The intuition behind these necessary conditions is as follows. Both the 'a' and the 'b' types must be indifferent between the two alternatives they face. While the 'a' type increases his earnings manipulation from $\frac{\alpha c}{2\beta}$ to $\frac{\alpha c}{\beta}$ (he reports $b = a + \frac{\alpha c}{\beta}$ instead of $a + \frac{\alpha c}{2\beta}$), the '$b$' type gains by reducing his earnings manipulation costs to zero (he reports truthfully, instead of reporting $b + \frac{\alpha c}{2\beta}$). Thus, the increase in earnings manipulation by the 'a' type is exactly identical to the decrease in earnings manipulation by the 'b' type. However, because the manipulation cost is convex, the compensation in terms of price required by the 'a' type exceeds the price concession made by the 'b' type. Thus, $cd(a,b)$, the price given a report of b, must be strictly higher than c times the midpoint of the interval $[a,b]$, namely $c\frac{a+b}{2}$. This implies that $d(a,b)$ lies to the right of the midpoint of the interval $[a,b]$. Given the quadratic cost function assumption, this conditional expectation lies exactly three quarters of the way between a and b (see (7)).

The pricing function in (4) is formed using Bayes rule given the conjectured equilibrium $\rho_p^*(\cdot)$. However, there are potential reports that never appear in this equilibrium. Specifically, no type will ever publish a report that lies in $[a+\frac{\alpha c}{2\beta}, b) \cup (b, b+\frac{\alpha c}{2\beta}]$.

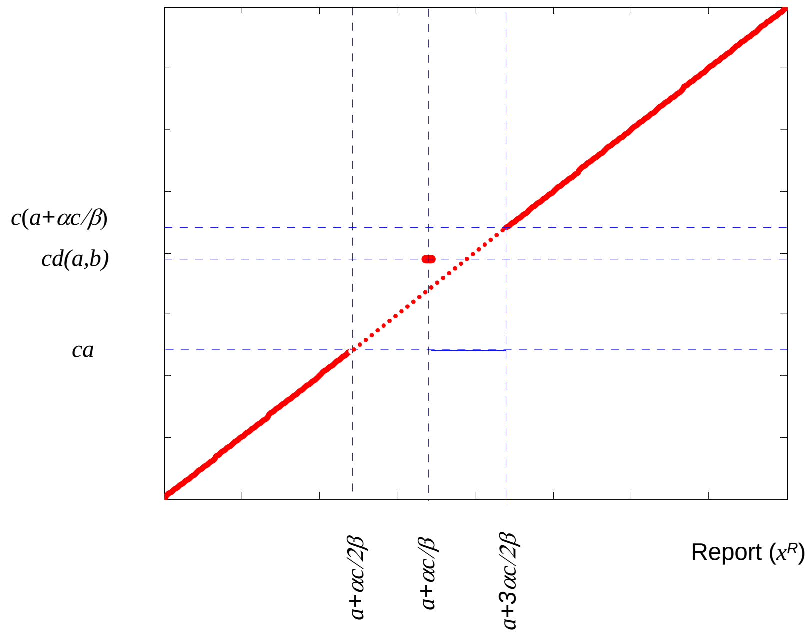

Figure 2: The Pricing Function in the Partially Pooling Equilibrium

To complete the picture, and fully delineate the partially pooling equilibrium, we have to specify the out-of-equilibrium pricing. Since Bayes rule does not apply we have some leeway in this choice. Actually, there exists a continuum of out-of-equilibrium pricing functions that support this equilibrium. By way of introduction we use the following: for all reports $x^R \in [a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta}]$, the price is $P(x^R) = c\left(x^R - \frac{\alpha c}{2\beta}\right)$. Thus, if investors observe an unexpected report, they conclude that the manager is "mistakenly" playing the benchmark separating equilibrium $\rho_s^*(\cdot)$. These out-of-equilibrium beliefs are fairly weak, and are only used here for simplicity. In Section 7 we show that the partially pooling equilibrium is robust to much stricter out-of-equilibrium pricing that satisfies a monotonicity requirement. Figure 2 depicts the pricing function of the investors in the partially pooling equilibrium. The dotted line describes the out-of-equilibrium pricing, while the bold dot is $cd(a, b)$: the price conditional on observing a report of $b = a + \frac{\alpha c}{\beta}$.

Below we prove the existence and uniqueness of the conjectured partially pooling equilibrium.

Proposition 2 *There exists a unique interval* $[a,b]$ *such that the reporting strategy*

$$\rho_p^*(x) \equiv \begin{cases} b & x \in [a,b] \\ x + \frac{\alpha c}{2\beta} & \text{otherwise} \end{cases}$$

joint with the pricing function

$$P_p^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a + \frac{\alpha c}{2\beta} \ \text{ or } \ x^R > b + \frac{\alpha c}{2\beta} \\ cd(a,b) = c\left(a + \frac{3\alpha c}{4\beta}\right) & x^R = b \\ c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R \in [a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta}] \end{cases}$$

constitute an equilibrium.

Proof: In the Appendix we prove that there exists a unique interval $[a,b]$ such that the necessary conditions (6) and (7) are satisfied. In particular, this interval makes the types $x = a$ and $x = b$ indifferent between reporting b and reporting $x + \frac{\alpha c}{2\beta}$. We claim that $\rho_p^*(\cdot)$ applied to this interval is an equilibrium strategy. The pricing function on the equilibrium path satisfies Bayes rule, given the manager's strategy by construction. Since $\rho_s^*(\cdot)$ is an equilibrium, and $\rho_p^*(\cdot)$ differs from $\rho_s^*(\cdot)$ only on $[a,b]$, we only have to rule out deviations to and from the pooling interval $[a,b]$.

Consider first a type $\hat{x} \in (a,b)$. The conjectured equilibrium strategy $\rho_p^*(\cdot)$ specifies that he should report b. Since the out-of-equilibrium pricing is identical to the pricing given $\rho_s^*(\cdot)$, Proposition 1 implies that his best possible deviation is to report $\rho_s^*(\hat{x}) = \hat{x} + \frac{\alpha c}{2\beta}$. However, using the facts that $d(a,b) = a + \frac{3\alpha c}{4\beta}$, and $b = a + \frac{\alpha c}{\beta}$ we obtain

$$\begin{aligned} U^M(\hat{x}, \rho_p^*(\hat{x})) - U^M(\hat{x}, \rho_s^*(\hat{x})) &= U^M(\hat{x}, b) - U^M(\hat{x}, \hat{x} + \frac{\alpha c}{2\beta}) \\ &= \alpha c d(a,b) - \beta(b - \hat{x})^2 - [\alpha c \hat{x} - \beta(\frac{\alpha c}{2\beta})^2] \\ &= \beta(\hat{x} - a)(b - \hat{x}) > 0, \end{aligned} \tag{8}$$

where the inequality follows since $\hat{x} \in (a,b)$. Therefore, type $\hat{x}$ is better off reporting b as required.

Consider now a type $\hat{x} \notin [a,b]$. According to $\rho_p^*(\cdot)$ he should report $\hat{x} + \frac{\alpha c}{2\beta}$. Since the out-of-equilibrium pricing is identical to the pricing function given $\rho_s^*(\cdot)$, Proposition 1 implies that this type would not deviate to any report in $[a + \frac{\alpha c}{2\beta}, b) \cup$

$(b, b+\frac{\alpha c}{2\beta}]$. Thus, his only potential beneficial deviation is to report b. However, using the facts that $d(a,b) = a + \frac{3\alpha c}{4\beta}$, and $b = a + \frac{\alpha c}{\beta}$ we obtain:

$$\begin{aligned} U^M(\hat{x}, \hat{x} + \frac{\alpha c}{2\beta}) - U^M(\hat{x}, b) &= \alpha c\hat{x} - \beta(\frac{\alpha c}{2\beta})^2 - \alpha c d(a,b) + \beta(\hat{x} - b)^2 \\ &= \alpha c\hat{x} - \beta(\frac{\alpha c}{2\beta})^2 - \alpha c(a + \frac{3\alpha c}{4\beta}) + \beta(\hat{x} - a - \frac{\alpha c}{\beta})^2 \\ &= \beta(\hat{x} - a)(\hat{x} - b) > 0, \end{aligned}$$

where the inequality follows since $\hat{x} \notin [a, b]$. Therefore, no deviation is beneficial. ■

A direct application of Proposition 2 yields

Corollary 1 *The partially pooling equilibrium satisfies the following properties:*

a. $b = a + \frac{\alpha c}{\beta}$.

b. $d(a,b) \equiv E(\tilde{x} | \tilde{x} \in [a,b]) = a + \frac{3\alpha c}{4\beta}$.

c. $d(a,b) < x_0$, and $cd(a,b) < p_0$.

Properties (a) and (b) are satisfied by construction given the choice of the interval $[a, b]$. As for property (c), it follows directly from property (b), unimodality, and symmetry of the normal distribution. Indeed, given that $d(a, b) = a + \frac{3\alpha c}{4\beta}$, it must be that the distribution mass on the right hand side of the interval outweighs the distribution mass on the left hand side of the interval. Under the normal distribution (as well as under any unimodal symmetric distribution) this is possible only if the conditional mean lays strictly to the left of the unconditional mean. Figure 3 illustrates this argument.

By pooling all types in the interval $[a, b]$, the manager introduces vagueness into his reports. This home-made noise enables him to reap ex-post economic rents compared to the fully separating equilibrium. Indeed, if the real earnings x fall outside of the pooling interval, then both equilibria are identical. However, if $x \in [a, b]$, the pooling report dominates. Formally:

Lemma 1 *For all* $x \in [a, b]$, $U^M(x, \rho_p^*(x)) > U^M(x, \rho_s^*(x))$.

Proof: Follows directly from equation (8). ■

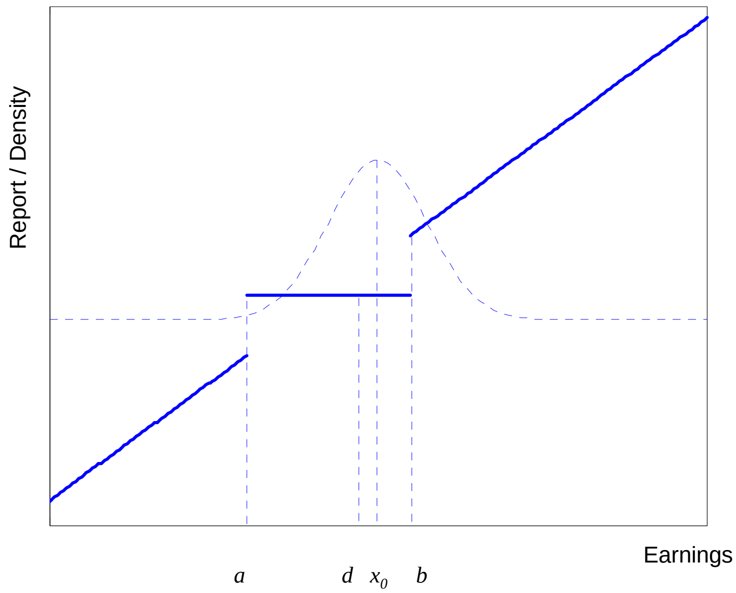

Figure 3: The Partially Poolling Equilibrium and the Underlying Distribution

Intuitively, the vagueness in the manager's report has two effects on his utility compared to the fully separating equilibrium. On the one hand, it changes the extent of the earnings management by the manager. On the other hand it affects the pricing of the stock given this manipulation. The net effect is always positive within the pooling interval $[a, b]$ and is zero outside of this interval. If the manager is forced to manipulate more, then he is more than compensated by price increase, while if he manipulates less, then the price decline is not sufficient to offset the reduction in cost.

From the point of view of the uninformed investors, this pooling behavior can be either ex-post beneficial or harmful, compared to the fully separating equilibrium. Specifically, if $x \in [a, d(a, b))$, the investors end up paying $cd(a, b)$ for a stock that is worth less, but for $x \in (d(a, b), b]$ the investors pay $cd(a, b)$ and get a stock that is worth more. The probability mass on the right half of the pooling interval is larger than the probability mass on the left half to such an extent that the investors are ex-ante indifferent between the two reporting strategies. We obtain

Proposition 3 *The partially pooling equilibrium ex-ante Pareto dominates the separating equilibrium.*

Proof: For the manager: from Lemma 1, we have $U^M(x, \rho_p^*(x)) > U^M(x, \rho_s^*(x))$ for all $x \in [a, b]$. As the probability mass of $[a, b]$ is positive we obtain immediately that: $E_x U^M(x, \rho_p^*(x)) > E_x U^M(x, \rho_s^*(x))$.

For the investors: the investors' payoff is the price they pay less the actual value of the stock: $P(\rho(x)) - cx$. In the fully separating equilibrium we have $P(\rho_s^*(x)) = cx$; hence the payoff is identically 0. In the partially pooling equilibrium we have $P(\rho_p^*(x)) = cE(\tilde{x}|\rho_p^*(x))$. Hence, from an ex-ante point of view, the payoff to the investors is

$$E_x(P(\rho_p^*(x)) - cx) = cE_x(E(\tilde{x}|\rho_p^*(x)) - x) = cE_x E(\tilde{x}|\rho_p^*(x)) - cE_x(x) = 0.$$

Thus, ex-ante the risk neutral investors are indifferent between these two equilibria. ■

Grossman and Stiglitz (1980) argue that residual noise must remain in equilibrium with costly information acquisition. Our model suggests that noise may arise endogenously even when the information can be acquired costlessly. The informed party - the managers, may reap information rents by pooling their signals and creating home-made noise. The investors get the same payoff on average in both equilibria, and risk neutrality makes them indifferent between the two. In Section 4 we discuss the implication of investor risk aversion on our results.

At a first glance, the endogenous noise demonstrated in our partially pooling equilibrium seems highly manipulative. However, it is important to notice that *costly earnings management is inevitable in equilibrium.* Hence, the pooling behavior should be viewed as a way to *reduce* the extent of earnings management and its cost as is shown in the next proposition.

Proposition 4 *The following holds:*

1. *The expected earnings management in the partially pooling equilibrium is lower than the expected earnings management in the separating equilibrium.*

2. *The expected cost of earnings management in the partially pooling equilibrium is lower than the expected cost of earnings management in the separating equilibrium.*

Proof: The expected earnings management in the separating equilibrium is $\frac{\alpha c}{2\beta}$. The expected earnings management in the pooling equilibrium is:

$$\begin{aligned}
&\int_{-\infty}^{a} \frac{\alpha c}{2\beta} f(x)dx + \int_{a}^{b} (b-x) f(x)dx + \int_{b}^{\infty} \frac{\alpha c}{2\beta} f(x)dx \qquad (9)\\
= &\ \frac{\alpha c}{2\beta} - \frac{\alpha c}{2\beta}(F(b)-F(a)) + \int_{a}^{b} (b-x) f(x)dx \\
= &\ \frac{\alpha c}{2\beta} - \frac{\alpha c}{2\beta}(F(b)-F(a)) + b(F(b)-F(a)) - d(a,b)(F(b)-F(a)) \\
= &\ \frac{\alpha c}{2\beta} - \frac{\alpha c}{4\beta}(F(b)-F(a)),
\end{aligned}$$

where the second equality follows since by definition: $d(a,b) = \frac{\int_a^b xf(x)dx}{F(b)-F(a)}$, and the last equality follows from the fact that $b - d(a,b) = \frac{\alpha c}{4\beta}$ (Equations (6) and (7)). This implies that the expected earnings management in the pooling equilibrium is strictly lower than in the separating equilibrium.

As for the second part of the proposition. Proposition 3 implies that the total expected utility of the manager is strictly higher in the partially pooling equilibrium. From the law of iterated expectations we obtain that the expected price is identical in the two equilibria. It follows, then, that the expected manipulation costs are strictly lower in the pooling equilibrium. ■

In summary, the benchmark in our model is not the truthful reporting, because it is not an equilibrium; instead, the benchmark is the inefficient separating equilibrium. Introducing noise into the reports, while a priori looks like a manipulative behavior, is actually a way to reduce the extent of manipulation relative to the benchmark. This renders the partially pooling equilibrium "less inefficient" than the relevant benchmark.

4 Extensions

In this section we analyze several extensions of our base model. We start by considering earnings based compensation, then we introduce risk aversion on the side of the investors, and finally we consider the case of $\alpha < 0$, representing a downward bias by managers.

4.1 Earnings vs. Stock-Based Compensation

Managers are frequently paid bonuses based directly on their reported earnings. We would like to know how does this compensation scheme affect the incentive of the managers to manipulate the reported earnings.[14] We assume for simplicity that such bonuses are linear in reported earnings: the manager maximizes the following utility function

$$U^M(x, x^R) = \alpha P(x^R) + \gamma x^R - \beta(x - x^R)^2,$$

where $\gamma > 0$ represents the strength of the earnings-based compensation. The fully revealing equilibrium is similar: the optimal earnings report is

$$x^R = x + \frac{\alpha c + \gamma}{2\beta}.$$

The addition of the earnings-based compensation affects the incentive to inflate earnings, and changes the manipulation cost for the manager.

Using the same reasoning as in our base model we can show that the partially pooling equilibrium is determined by the two equations:

$$\begin{aligned} \alpha ca + \gamma a + \gamma \frac{\alpha c + \gamma}{2\beta} - \beta(\frac{\alpha c + \gamma}{2\beta})^2 &= \alpha c d(a,b) + \gamma b - \beta(b-a)^2 \\ \alpha cb + \gamma b + \gamma \frac{\alpha c + \gamma}{2\beta} - \beta(\frac{\alpha c + \gamma}{2\beta})^2 &= \alpha c d(a,b) + \gamma b \end{aligned}$$

Solving these yields the following generalization to Equations (6) and (7):

$$\begin{aligned} b &= a + \frac{\alpha c + \gamma}{\beta}, \text{ and} \\ d(a,b) &\equiv E(\tilde{x} | \tilde{x} \in [a,b]) = a + \frac{(3\alpha c + \gamma)}{4\alpha c}(b-a). \end{aligned}$$

It is important to note that the pooling equilibrium exists if and only if the conditional expectation $d(a,b)$ is located in the pooling interval (between a and b). This implies that

$$\alpha c > \gamma > -3\alpha c. \tag{10}$$

Condition (10) imposes a bound on the extent of earnings based compensation that is consistent with the pooling equilibrium. To understand this condition suppose first that $\alpha = 0$, namely that compensation depends on reported earnings only. In this case, Condition (10) implies that the pooling equilibrium cannot exist. The reason is that the pooling equilibrium depends on investors beliefs, represented by the pricing function. In equilibrium, investors beliefs must support the strategy of the manager. When the manager's compensation does not depend on the stock price,

[14] The effect of earnings based compensation on earnings management has been studied empirically in several papers. See for instance Healy (1985).

investors beliefs are of no interest to him, and the first order condition makes the separating equilibrium the only one. In a less polar case, the manager's compensation is based on both earnings reports and stock price. Condition (10) then tells us that in order to get pooling, the extent of earnings based compensation relative to stock based compensation cannot be too extreme. Thus, the manager's compensation must strongly depend on the stock price for him to pool. Incidentally, the reliance on stock-based compensation has been steadily increasing over time (see Jensen and Murphy (1990) and Hall and Liebman (1998)), which adds relevance to the proposed equilibrium.

4.2 Risk Averse Investors

So far we made investors risk neutral. How does risk aversion affect our results? We assume that the utility of a manager who observes x and reports x^R is still given by

$$U^M(x, x^R) = \alpha P(x^R) - \beta(x - x^R)^2,$$

where $\alpha, \beta > 0$. However, we assume that investors are risk averse and are not fully diversified, thus they demand a risk premium for the variance of the stock value. In particular, investors maximize the certainty equivalent (CE) of their expected utility with respect to the number of shares they want to hold at the beginning of the period, n. We assume that the CE takes a well-known form:

$$\max_n CE(x^R) \equiv (E(\tilde{x}|x^R) - P(x^R))n - 0.5zn^2 Var(x|x^R), \tag{11}$$

where $z > 0$ is a proxy for the investors' coefficient of risk aversion.

The per-capita demand for stock is obtained by writing the first order condition of (11):

$$n^*(p_1) = \frac{E(\tilde{x}|x^R) - P(x^R)}{zVar(x|x^R)}.$$

As usual in such models, the investor's demand increases in the expected value, but declines in the price of the stock, the coefficient of risk aversion, and the variance. Market clearing condition requires that the per-capita demand must equal the per-capita supply of the stock, $S > 0$, which is given exogenously. This yields the pricing function of the stock

$$P^*_{ra}(x^R) = E(\tilde{x}|x^R) - SzVar(\tilde{x}|x^R), \tag{12}$$

where the subscript ra represents the model with risk-averse investors. Thus, the price for any given report must equal to the price in the model with risk-neutral investors minus a risk premium that increases in the posterior variance.

Substituting the price (12) into the certainty equivalent, and using the fact that $n^* = S$ we obtain:

$$CE(x^R = b) = 0.5S^2 zVar(\tilde{x}|x^R \in [a,b]). \tag{13}$$

Thus, the certainty equivalent increases in the variance of the value conditional on the report. This implies that if the same two types of equilibrium exist in this case, the investors and the manager ex-ante strictly prefer the partially pooling equilibrium. The intuition of this result is quite straightforward. Higher variance makes the investors' demand for shares less elastic, and since the holdings of the investors remain the same (market clears) the price declines dramatically. This increases the consumer surplus of the investors, as seen in equation (13). Those who bear the cost of the price reduction are the owners of private firms, trying to go public. They get lower prices for their shares in the pooling equilibrium relative to the fully revealing one. However, they do not take part in the reporting game in the secondary market, and thus are not likely to influence the choice of the equilibrium. Their utility must be taken into account in the overall welfare analysis, but is not relevant here.

The final question is whether the same two equilibria exist in the model with risk-averse investors. The fully separating equilibrium needs no modifications; after all, $Var(x|x^R) = 0$, when $\rho(x) = x + \frac{\alpha}{2\beta}$. The existence of a partially pooling equilibrium in this case depends on the existence of an interval $[a, b]$ that satisfies:

$$\begin{gathered} b = a + \tfrac{\alpha}{\beta}, \text{ and} \\ d(a,b) \equiv E(\tilde{x}|x \in [a,b]) = a + \tfrac{3\alpha}{4\beta} + SzVar(x|x \in [a,b]). \end{gathered} \tag{14}$$

To show the existence of equilibrium in this case we must show that the system of equations (14) has a solution. Unfortunately, we have not been able to provide an analytical proof as we did in the risk-neutral case. At the same time, we have performed a large number of numerical calculations for a variety of parameter values, and in each case obtained a partially pooling equilibrium with the same features as in the risk-neutral case. All these equilibria ex-ante Pareto dominate the fully separating equilibrium according to the above arguments.

4.3 Downward Bias

So far we have restricted our attention to the case of $\alpha > 0$. If $\alpha < 0$ we obtain symmetric results. The perfectly separating equilibrium in Proposition 1 is unaffected; however, since $\alpha < 0$, managers bias their earnings downwards by a constant $\frac{\alpha c}{2\beta}$. As for the partially pooling equilibrium we obtain the following parallel to Proposition 2:

Corollary 2 *Suppose $\alpha < 0$. There exists a unique interval $[b, a]$ such that the reporting strategy*

$$\rho_p^*(x) \equiv \begin{cases} b & x \in [b, a] \\ x + \frac{\alpha c}{2\beta} & \text{otherwise} \end{cases}$$

joint with the pricing function

$$P_p^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R > a + \frac{\alpha c}{2\beta} \text{ or } x^R < b + \frac{\alpha c}{2\beta} \\ cd(a,b) = c\left(a + \frac{3\alpha c}{4\beta}\right) & x^R = b \\ c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R \in [b, a + \frac{\alpha c}{2\beta}) \cup (b + \frac{\alpha c}{2\beta}, b] \end{cases}$$

is an equilibrium.

In this case, the price conditional on observing a report of b is higher than the unconditional price, namely, $cd(a, b) > p_0 = cx_0$.

5 Empirical Implications

In this section we study the comparative statics of the partially pooling equilibrium, and provide empirical implications. In some cases we were not able to derive the comparative statics analytically and used numerical methods instead. These are indicated explicitly in the text.

5.1 Comparative Statics

We study the comparative statics of the partially pooling equilibrium with respect to four parameters: the volatility of earnings, σ^2; the P/E ratio, c; the degree of managerial stock-price-based compensation, α; and the cost of misrepresentation, β.[15] We then derive cross-sectional and time-series empirical predictions.

In the separating equilibrium, a change in the volatility of earnings has no effect on the equilibrium. In particular, the earnings management is equal to $\frac{\alpha c}{2\beta}$ regardless of σ. On the contrary, in the partially pooling equilibrium, a change in σ moves the pooling interval $[a, b]$ and hence affects the probability of pooling and the earnings management we expect to observe.[16] In order to study the impact of σ on the location of the pooling interval we consider a, the left hand side of the interval, as a function of σ: $a(\sigma)$. The location effect is studied in the next proposition.

Proposition 5 *A decrease in the underlying volatility moves the pooling interval towards the unconditional mean:* $\frac{\partial a(\sigma)}{\partial \sigma} < 0$. *Moreover,* $\lim_{\sigma \to 0} a(\sigma) = x_0 - \frac{3\alpha c}{4\beta}$, *and* $\lim_{\sigma \to \infty} a(\sigma) = -\infty$.

Proof: In the Appendix.

Proposition 5 shows that a's distance from the unconditional mean is sensitive to changes in the underlying information asymmetry. Higher σ moves the pooling interval to the far left tails of the distribution, thus the probability of observing a pooling report always goes to zero in the limit. On the other hand, as σ declines, the pooling interval moves closer to the unconditional mean, eventually straddling it. In the limit, as σ tends to 0, the unconditional mean x_0 and the conditional mean $d(a, b) = a + \frac{3\alpha c}{4\beta}$ coincide.

While we have established the impact of the change in σ on the location of the pooling interval, it is not sufficient in order to establish that the probability of observing pooling behavior declines in σ. This probability is given by $F_\sigma(b(\sigma)) - F_\sigma(a(\sigma))$. The subscript σ reminds us that a change in σ affects the probability of pooling in two ways: first, it affects the location of $a(\sigma)$ as described in Proposition 5. But it also affects the probability distribution itself. A higher variance puts more mass in the tails of the distribution. The two effects act in opposite directions, thus the overall effect is ambiguous. We have not been able to derive the net effect analytically, but in all of our extensive numerical calculations the location effect dominates:

[15] Any shift in the expected earnings form x_0 to $x_0 + \Delta$ just shifts the location of the pooling interval from $[a, b]$ to $[a + \Delta, b + \Delta]$. Thus, it has no material effect on the equilibrium since the probability of pooling is not changed. For this reason we do not study in details the comparative statics with respect to x_0, the prior mean.

[16] Notice that the size of the pooling interval is not affected by the volatility of earnings, and is equal to $\frac{\alpha c}{\beta}$.

the probability of observing the pooling behavior, $F_\sigma(b(\sigma)) - F_\sigma(a(\sigma))$, declines in σ. Recall from Equation (9) that the expected earnings management observed in the pooling equilibrium for any given σ is given by: $\frac{\alpha c}{2\beta} - \frac{\alpha c}{4\beta}(F_\sigma(b(\sigma)) - F_\sigma(a(\sigma)))$. This implies that the expected earnings management is increasing in σ.

To demonstrate this result, Figure 4 plots the effect of a change in σ calculated numerically. We use the following parameter values: $\frac{\alpha c}{\beta} = 1$, $x_0 = 0$, and let σ vary between 0.1 and 1.5. The left box depicts the probability of pooling as a function of σ. When σ is very low the probability of pooling is almost 1. This happens since the pooling interval contains the unconditional mean x_0 (Proposition 5), and the underlying distribution is highly concentrated around this mean. On the other hand, when σ is large, the probability of pooling is almost 0, and we are essentially back in the perfectly separating equilibrium. The right box depicts the expected earnings management. When σ is low, it is close to $\frac{\alpha c}{4\beta} = 0.25$, while when σ is large it converges to $\frac{\alpha c}{2\beta} = 0.5$, which is the expected earnings management in the separating equilibrium.

To interpret these findings recall that the pooling behavior, while seemingly manipulative, is actually a way to reduce the degree of earnings management (Proposition 4). When the underlying uncertainty is low, the manager is able to reap significant benefits from this pooling behavior, since a large portion of the distribution falls inside the pooling interval. Thus, in these cases, although the report is quite noisy, the extent of earnings management is low on average. Actually, the low expected manipulation is the direct consequence of the "home made" endogenous noise.

The incentive parameters α and β, and the P/E ratio, c, enter the model only as a ratio $\frac{\alpha c}{\beta}$. A higher $\frac{\alpha c}{\beta}$ has three effects: (i) it induces a larger pooling interval, which tends to increase the probability of pooling and hence lower the expected earnings management; (ii) it moves the position of the pooling interval; and (iii) it increases the earnings management outside the pooling interval. The net effect of these three is hard to sign analytically. However, extensive numerical calculations show that an increase in $\frac{\alpha c}{\beta}$ *always* increases the probability of pooling, but also increases the expected earnings management. Thus, while an increase in $\frac{\alpha c}{\beta}$ induces more pooling and less earnings management inside the pooling interval, it has a more pronounced effect outside the pooling interval, hence induces higher manipulation overall. Thus, a higher extent of stock based compensation, less stringent accounting

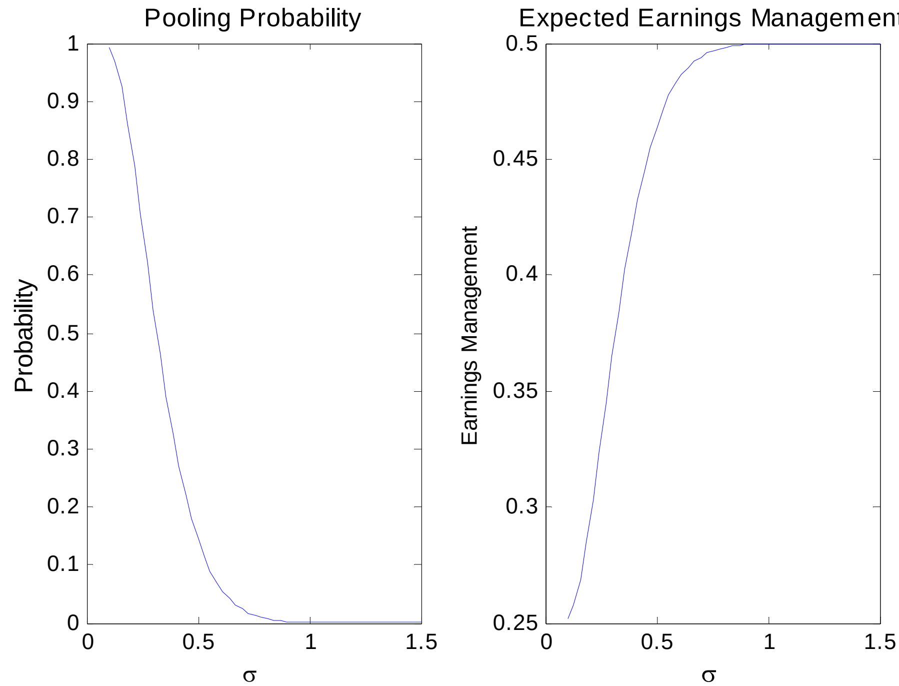

Figure 4: The Effect of a Change in the Volatility of Earnings

rules, and a higher P/E ratio all induce more pronounced pooling and yet a higher expected earnings management. We demonstrate this result in Figure 5. We use $x_0 = 0$, and $\sigma = 0.7$ and let $\frac{\alpha c}{\beta}$ vary between 0.5 and 3. The left box depicts the probability of pooling. This probability is close to 0 when the ratio $\frac{\alpha c}{\beta}$ is small, making the partially pooling equilibrium essentially identical to the separating one. As $\frac{\alpha c}{\beta}$ increases, the probability of pooling increases. The right box depicts the expected earnings management. When $\frac{\alpha c}{\beta} = 0.5$ the probability of pooling is essentially zero, hence the expected earnings management is almost equal to $\frac{\alpha c}{2\beta} = 0.25$, which is the expected earnings management in the separating equilibrium. As $\frac{\alpha c}{\beta}$ increases the probability of pooling increases, however, the earnings management outside the pooling interval increases as well. The latter effect dominates and the overall expected earnings management increases.

The above comparative statics lead to the following list of empirical predictions:

- Controlling for the incentive schemes, the P/E ratio, and the accounting rules, we expect to observe more pronounced pooling behavior and less earnings man-

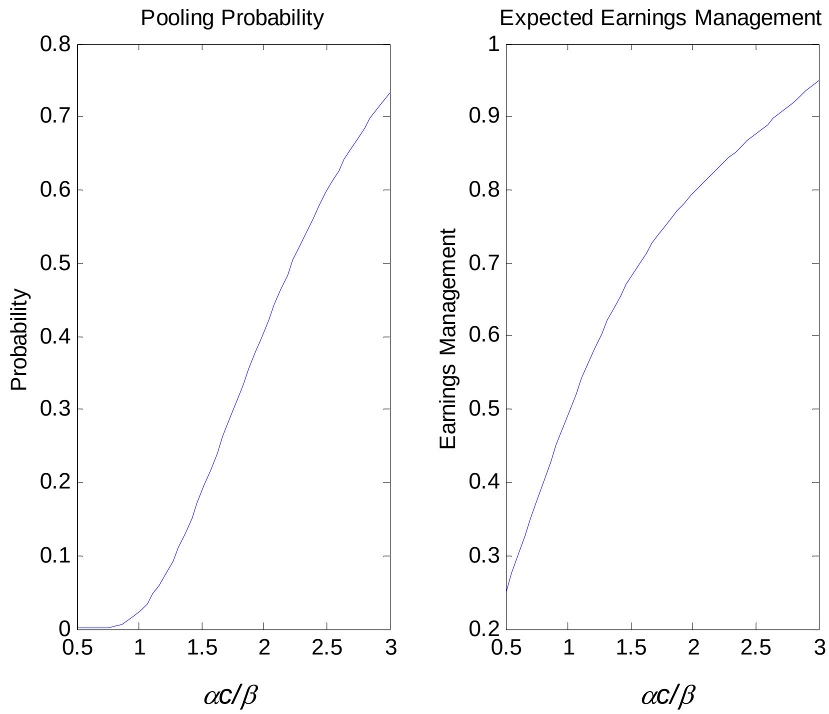

Figure 5: The Effect of a Change in $\frac{\alpha c}{\beta}$

agement in firms and industries with lower degree of information asymmetry between managers and investors (lower volatility of earnings). The same prediction applies to time periods with relatively predictable earnings.

- Controlling for the volatility of earnings, the P/E ratio, and the accounting rules, stronger stock-based incentives imply a larger degree of pooling behavior and a higher level of earnings management. Firms with a higher proportion of managerial ownership should exhibit this effect. Reliance on options and stock compensation differs across countries and industries; and in the US the use of stock-based compensation has been rapidly increasing over time. Cross-sectional variations in the P/E ratio generate the same predictions as variations in α.

- More stringent accounting rules, *ceteris paribus* imply less pooling and less earnings management in general. This prediction is consistent with the findings of Luez, Nanda and Wysocki (2003), and Bhattacharya, Daouk and Welker (2003). They show that more stringent accounting rules are associated with less earnings management around the world.

Example: High tech firms vs. public utilities.

High tech firms rely on stock-based compensation to a large extent (high α); while utilities have weak stock-based compensation. The accounting rules for high tech firms with mostly human capital allow them to manipulate earnings to a larger extent (low β). Utilities have been around for over a century, and they are regulated, which implies that their earnings accounting numbers are scrutinized by many authorities and are harder to manipulate. High tech promises a bright future; thus the P/E ratio, c, of these firms is very high: even after the 2000-2001 bear markets, the P/E ratio on Nasdaq 100 is around 30. Public utilities earnings grow approximately at the rate of population growth (low c). All these suggest a higher level of earnings management and more pronounced pooling in the high tech firms. Moreover, the volatility of earnings in high tech firms tends to be high, inducing an even higher level of expected earnings management due to reduction in pooling.

5.2 Meeting Targets

The behavior of the manager in the pooling interval resembles an attempt to meet a target. Indeed, looking at the data, one might be tempted to consider b, the upper bound of the pooling interval, as an exogenously determined target. All the types above b stick to the perfectly revealing strategy and ignore this target since their reports exceed it anyway. Types lower than a do not pay attention to it either, since it is too costly for them to attain it. Intermediate types in the interval $[a, b]$ manage their report in a way that will make them meet the target and report b exactly. Naturally, the investors interpret this correctly, and price the stock at the expected value conditional on being in the interval $[a, b]$.

We must stress however, that there is no exogenously imposed target in this model, and b, which looks like a target, is endogenously determined in the partially pooling equilibrium. Moreover, in our model the behavior of the manager depends only on the investors' beliefs. To the extent an exogenous "target" affects these beliefs, it also affects the manager's actions. However, when managers are said to be striving to "meet targets", as in Degeorge, Patel and Zeckhauser (1999), it is implied that a change in the exogenous target by itself can alter the behavior of the manager. This is not the case in our model. For example, suppose that analysts (who set the targets) always add 2 cents to every earnings forecast, and this is a common knowledge. This behavior raises the target, yet does not alter the investors' beliefs. In our model the manager's actions are unaffected by it, while it should change the report if the manager truly wants to meet the target.

We do not claim that exogenous targets are not important in driving managerial behavior. In particular, the roles of analysts as information providers and target setters require an extensive study that is outside the scope of this paper. Guttman, Kadan, and Kandel (2003) incorporate the analysts into the manager-investor game and study equilibria corresponding to information provision and target setting.

6 Biased Experts and Recommendation Letters

All of us are constantly asked to provide letters of recommendation for students and colleagues. Suppose, for the sake of argument, that we do know the true quality of these individuals. We may have preference for the direction of their advancement, which introduces a potential bias in our letters (let's assume that the bias is always positive). On the other hand, an exaggeration imposes an externality on us in the sense that our future recommendations (judgement) will be more heavily discounted. Thus we balance the desire to promote a student, or a colleague, and the cost of this action. Our model applied to this scenario, predicts that recommendation letters for extremely good or extremely bad students will reveal their true type (in many cases we will abstain from writing the letter altogether in the latter case, which introduces another pooling, which is not modeled here). For intermediate-value students, recommendation letters will be vague and will pool many student types.

This is just one example of a biased agent reporting to a decision-maker. We can generalize this example to include many real-life situations. Formally, suppose that the true state of nature x is drawn from a normal distribution with mean x_0 and variance σ^2 (we denote the cumulative distribution by F, and the density by f). A decision-maker, who knows this prior distribution, but does not know the true state of nature, has to take an action $q \in \mathbb{R}$. Her utility from taking an action q given the state of nature x is given by

$$U^D(q, x) = -\psi(q - x)^2,$$

where $\psi > 0$ is a scaling parameter. Thus, the best action is $x = q$, and any deviation is costly. Given that the true state of nature is not known, it is easy to see that the optimal choice for the decision maker is $q^* = Ex = x_0$. Consequently, the expected utility of the uninformed decision-maker is

$$EU^D_{\text{uninformed}}(q^*, x) = -\psi\sigma^2.$$

In this context we can interpret ψ as the decision-maker's degree of risk aversion: lower ψ makes the informational asymmetry less costly for her.

As an alternative to an uninformed decision, the decision-maker may hire an expert who has the skills to identify the true state of nature. The decision-maker gets a report $x^R \in \mathbb{R}$ from the expert, forms posterior beliefs based on the report, and takes an action $q = Q(x^R)$.

We assume that the utility of the expert who observes the true state of nature, x, and reports x^R is given by:[17]

$$U^E(x, x^R) = \alpha Q(x^R) - \beta(x - x^R)^2, \tag{15}$$

where $\alpha, \beta > 0$. The utility function (15) of the expert captures two conflicting interests. On the one hand, we assume that the misrepresentation is costly for the expert. This cost may come from many sources: there is a possibility of a legal or regulatory action; the act of misrepresentation may involve real costs (more on this below); or the expert's reputation (self-esteem) may suffer as a result. This assumption clearly differentiates this model from the models of experts based on the "cheap talk" assumption (see Crawford and Sobel (1982), and recent papers by Krishna and Morgan (2001) and Morgan and Stocken (2003)), which in this case is tantamount to $\beta = 0$. The specific functional form we use implies that the marginal cost increases in the degree of misrepresentation, and β represents the severity of the penalty. This cost provides an incentive for the expert to tell the truth. On the other hand, the expert may be biased: in our specification the bias takes the form of his preference for a "larger" action. This may come directly form the expert's preferences, i.e., a "green minded" expert trying to influence environmental regulation, or be motivated by his compensation. This bias causes the expert to misrepresent his report in such a way as to increase the action chosen by the decision maker. Parameter α represents the degree of the bias. The relative importance of the two incentives is represented by the ratio $\frac{\alpha}{\beta}$.

[17] Notice that his utility does not include the compensation from the decision-maker, which is assumed to be constant. Derivation of the optimal expert contract in this case is outside of the scope of this paper, and is left for future research.

Both agents maximize their expected utility. Since, the decision maker minimizes a quadratic function conditional on the report, his optimal decision is

$$Q(x^R) = E(\tilde{x}|x^R). \tag{16}$$

Plugging (16) in (15) we obtain that the expert maximizes:

$$U^E(x, x^R) = \alpha E(\tilde{x}|x^R) - \beta(x - x^R)^2. \tag{17}$$

Notice that we are back to the optimization of the manager in Equation (2), thus the remaining derivations are identical (assuming $c = 1$). The two types of equilibrium exist. The only significant difference is that the partially pooling equilibrium does not always Pareto dominate the perfectly revealing one. In fact the efficiency depends on the value of parameter ψ. To formalize this statement we analyze the total surplus to both agents in the separating and partially pooling equilibria, given by

$$\begin{aligned}
\Gamma_s &\equiv \int_{-\infty}^{\infty} \left[U^E(x, \rho_s^*(x)) + U^D(x, x)\right] f(x)dx \\
\Gamma_p &\equiv \int_{-\infty}^{\infty} \left[U^E(x, \rho_p^*(x)) + U^D(x, Q_p^*(\rho_p^*(x))\right] f(x)dx.
\end{aligned}$$

We prove the following proposition:[18]

Proposition 6 *There exists a* $\psi^* > 0$*, such that for all* $0 < \psi < \psi^*$, $\Gamma_s < \Gamma_p$.

The intuition behind this result is clear: in the partially pooling equilibrium the expert gains by introducing noise, while the decision-maker suffers from making mistakes in the pooling region. If the penalty for mistakes (also her risk aversion), ψ, is sufficiently low, then the imposed additional variance is not very costly for the decision maker. An adjustment of the expert's fee is sufficient in this case to convince her to play the partially pooling equilibrium.

The above framework applies to many examples in all areas of life. Many experts suffer a cost for misrepresentation. At the same time these experts frequently have a direct interest in the action their client will take, which may cause some bias in their report. A litigant in a tort case needs a legal opinion on his chances of winning before deciding on whether to settle. A lawyer on a contingent contract will tend

[18] The proof is straightforward. It is omitted for brevity, but is available from the authors upon request.

to understate his chances of winning a larger award, while a lawyer charging an hourly fee will tend to overstate it. A real estate agent has an incentive to bias information to buyers and sellers alike to facilitate a deal; and any salesman tries to exaggerate his product advantages over the competition. Auto mechanics are notorious for their pessimism about the state of our cars, while money managers are equally famous for their overoptimism about the returns they promise. Various interest groups lobby regulators and politicians to make decisions favorable to their cause; in the process they are not averse to manipulating the information. Sell-side stock analysts who want future business for their investment banks from firms they cover may inflate their reports. Line workers may misreport to middle managers, who in turn may misreport to top management. Military commanders in the battlefield may overstate the severity of an attack to obtain air support for their unit. The higher-up military staff and intelligence officials routinely modify information they supply to politicians to secure funding and influence policy (the recent inquiries in the US, UK and Australia about the intelligence on weapons of mass destruction in Iraq are alleged to be examples of such behavior). The list goes on.

The cost for misrepresenting agents may be quite substantial. The Bible tells us about the spies that Moses sends to the land of Canaan (Numbers, 13-38). They come back with two reports: a few, apparently not eager to fight, grossly exaggerate the danger, claiming they had seen giants, while others paint a more accurate picture. The former are consequently put to death by the Higher Authority, while the latter live. While such severity of punishment is unusual in our times, it is reasonable to assume that misrepresentation of information is potentially costly to experts in all cases presented above. In the legal and medical professions malpractice suits, and sanctions by professional associations impose significant costs on false advice. Salesmen (or their employers) have to worry about repeat customers, and word-of-mouth reputation. Sell-side analysts may lose credibility with the investors, which would make them useless to their employers. Managers can be fired, or prosecuted for significant misreporting, and military commanders can be discharged or even court-martialed for misconduct of this sort. Even politicians, who never seem to bear the full cost of making false statements to their constituents, have to worry about reelection. In all these cases an increase in misrepresentation is likely to increase its marginal cost.

The message is quite clear: our model suggests that when an expert is somewhat biased, yet suffers negative consequences if the truth is distorted, she may prefer the partially pooling equilibrium, since it reduces the expected penalty for misrepresentation. The expert will intentionally obfuscate the signal in the intermediate range (home-made noise), while making it precise, if biased, for extreme realizations. The client, as long as he is not too risk averse, may play along in choosing to play this equilibrium, since it reduces the expert's fees.

7 Robustness

In this section we test the robustness of the partially pooling equilibrium. First, we refine the out-of-equilibrium pricing, by making it monotone. Then, we demonstrate the existence of other partially pooling equilibria, and show that they all possess the same attributes. In particular, they all ex-ante Pareto dominate the perfectly separating equilibrium. This family of equilibria nests our suggested partially pooling equilibrium as a special case.

7.1 Out-of-Equilibrium Beliefs

The literature on equilibrium refinements offers a multitude of concepts to limit the freedom of the modeler in choosing "reasonable" out of equilibrium beliefs in signalling games. The most prevalent criterion for refinement is the "intuitive criterion" of Cho and Kreps (1987). It is straightforward to verify that our partially pooling equilibrium, as specified in Proposition 2 survives this criterion. Actually, the intuitive criterion works best in models with just two types of informed parties, therefore it doesn't impose much restriction on out of equilibrium beliefs in our continuous type framework. Other criteria such as the "divinity criterion" (Banks and Sobel (1987)) were developed for finite type games, and we find them hard to interpret in our framework. Consequently, we follow a direction introduced by Harrington (1987), requiring that the pricing function be monotonically increasing in reports (on and off the equilibrium path).

In Section 3 we assumed that if investors observe an out-of-equilibrium report $x^R \in (a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta})$ then they believe that the manager is "mistakenly" playing the benchmark linear equilibrium. This out-of-equilibrium pricing function has the undesirable property of non-monotonicity (see Figure 2). For instance, if a manager increases his report from b to $b + \varepsilon$ where $\varepsilon > 0$ is sufficiently small the price drops. Although this price reaction cannot happen in equilibrium, this property of the pricing function is highly counter-intuitive. In this section we show that our equilibrium does not hinge on this choice, and that the out-of-equilibrium pricing function can be made monotone, preserving all the other properties obtained so far. To show this, we show in the following lemma that a sufficient condition for an out-of-equilibrium pricing function to support our partially pooling equilibrium, is that the types 'a' and 'b' are indifferent between following the equilibrium strategy and deviating from it.

Lemma 2 *Consider any out-of-equilibrium report $x^R \in (a + \frac{\alpha c}{2\beta}, b) \cup (b, b + \frac{\alpha c}{2\beta})$ combined with an out-of-equilibrium pricing function $P(x^R)$. The following holds:*

1. *If $x^R \in (a + \frac{\alpha c}{2\beta}, b)$, and if type '$a$' is indifferent between the equilibrium report of b, and the out-of-equilibrium report of x^R, then all other types $x' \neq a$ strictly prefer the equilibrium report b over the out-of-equilibrium report x^R.*

2. *If $x^R \in (b, b + \frac{\alpha c}{2\beta})$, and if type '$b$' is indifferent between the equilibrium report of b, and the out-of-equilibrium report of x^R, then all other types $x' \neq b$ strictly prefer the equilibrium report b over the out-of-equilibrium report x^R.*

Proof: In the Appendix.

Based on Lemma 2, the partially pooling equilibrium strategy $\rho_p^*(\cdot)$ is said to be supported by a *tight* pricing function $P(x^R)$, if for all $x^R \in (a + \frac{\alpha c}{2\beta}, b)$, type '$a$' is indifferent between the equilibrium strategy and deviating to x^R, and for all $x^R \in (b, b + \frac{\alpha c}{2\beta})$, type '$b$' is indifferent between following the equilibrium strategy and deviating to x^R. We can now prove the following:

Proposition 7 *There exists a unique tight pricing function that supports the partially pooling strategy $\rho_p^*(\cdot)$. This pricing function is given by*

$$P_t^*(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a + \frac{\alpha c}{2\beta} \text{ or } x^R > b + \frac{\alpha c}{2\beta} \\ c \cdot d(a,b) & x^R = b \\ c\left(a - \frac{\alpha c}{4\beta} + \frac{\beta}{\alpha c}(x^R - a)^2\right) & x^R \in [a + \frac{\alpha c}{2\beta}, b) \\ c\left(b - \frac{\alpha c}{4\beta} + \frac{\beta}{\alpha c}(x^R - b)^2\right) & x^R \in (b, b + \frac{\alpha c}{2\beta}] \end{cases}.$$

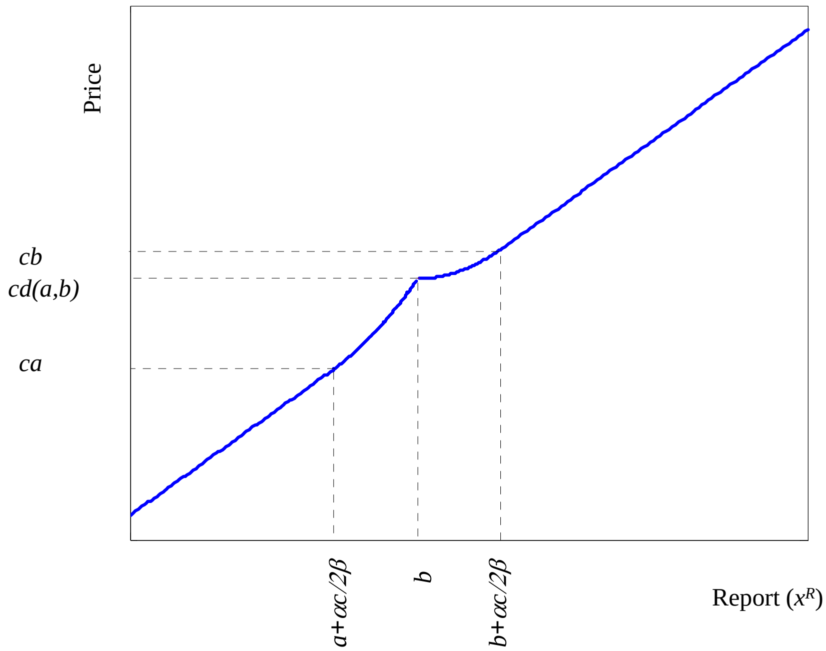

Figure 6: The Monotone Pricing Function

Proof: The cases $x^R < a + \frac{\alpha c}{2\beta}$, $x^R > b + \frac{\alpha c}{2\beta}$, and $x^R = b$ are identical to these cases in Proposition 2, and are determined uniquely using Bayes rule. As for the pooling region: for all $x^R \in [a + \frac{\alpha c}{2\beta}, b)$, we look for a pricing function $P_t^*(x^R)$ that makes type 'a' indifferent between deviating to x^R and sticking to the equilibrium. This indifference implies that this pricing function must satisfy

$$\alpha ca - \beta(\frac{\alpha c}{2\beta})^2 = \alpha P_t^*(x^R) - \beta(x^R - a)^2.$$

Solving for $P_t^*(x^R)$ yields the required result. A similar calculation applies for the case $x^R \in (b, b+\frac{\alpha c}{2\beta})$. Lemma 2 implies that this out-of-equilibrium pricing guarantees that no type will be willing to deviate from the partially pooling strategy $\rho_p^*(\cdot)$. ■

It is easy to verify that the unique tight pricing function given in Proposition 7 is strictly increasing and continuous. Figure 6 presents the pricing function using the tight out-of-equilibrium beliefs.

7.2 Other Partially Pooling Equilibria

In Section 3 we have shown the existence and uniqueness of a partially pooling equilibrium, in which the manager reports the upper bound of the pooling interval. That equilibrium, however, is not the only partially pooling equilibrium in this model. Actually, there exists a continuum of similar partially pooling equilibria. Equilibria in this family differ by the pooling report the manager makes when the realized earnings are in the interval $[a, b]$. While in our original equilibrium all managers with earnings in $[a, b]$ report the upper bound b, we can create other equilibria in which managers report $b + \eta$, for all $\eta \in (-\frac{\alpha c}{2\beta}, \frac{\alpha c}{2\beta})$. Our original equilibrium corresponds to the case $\eta = 0$. All the equilibria in this family have similar attributes; moreover, they all ex-ante Pareto dominate the benchmark separating equilibrium. This is demonstrated in the next proposition.

Proposition 8 *For all $\eta \in (-\frac{\alpha c}{2\beta}, \frac{\alpha c}{2\beta})$, there exists a unique interval $[a_\eta, b_\eta]$ such that the reporting strategy*

$$\rho_p^\eta(x) \equiv \begin{cases} b_\eta + \eta & x \in [a_\eta, b_\eta] \\ x + \frac{\alpha c}{2\beta} & otherwise \end{cases},$$

joint with the pricing function

$$P^\eta(x^R) = \begin{cases} c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R < a_\eta + \frac{\alpha c}{2\beta} \;\; or \;\; x^R > b_\eta + \frac{\alpha c}{2\beta} \\ c\left(b_\eta - \frac{\alpha c}{4\beta} + \frac{\beta}{\alpha c}\eta^2\right) & x^R = b_\eta + \eta \\ c\left(x^R - \frac{\alpha c}{2\beta}\right) & x^R \in [a_\eta + \frac{\alpha c}{2\beta}, b_\eta + \eta) \cup (b_\eta + \eta, b_\eta + \frac{\alpha c}{2\beta}] \end{cases},$$

constitute an equilibrium. This equilibrium ex-ante Pareto dominates the perfectly separating equilibrium.

Proof: Similar to the proofs of Propositions 2 and 3. ■

The family of equilibria described above nests our original partially pooling equilibrium as a special case. We have chosen to focus on this equilibrium because it is the simplest one analytically, and completely represents the family. All the welfare implications, comparative statics, and empirical predictions carry on to the entire family of equilibria. The choice of η just changes the length, and position of the pooling interval.

8 Conclusions

Managers have some leeway in earnings reporting but these distortions are not costless for them. Managers' compensation is based on the stock price, which depends on investors' beliefs about the true earnings. The result is that managers can and do manipulate earnings in their favor. To infer the true earnings from the report, investors try to undo the manipulation. The conventional equilibrium in the literature is perfectly revealing: the investors are not fooled, yet the manager pays the cost of manipulation. We show an alternative equilibrium in which managers add homemade noise to their reports, which prevents investors from inferring the truth, and reduces the cost of manipulation for the managers. This equilibrium ex-ante Pareto dominates the fully revealing equilibrium, and endogenously generates discontinuity in the earnings reports. Such discontinuity is well documented in the literature, yet there has been no theory to generate such discontinuity endogenously. Instead, the literature has interpreted this behavior as "meeting and beating" targets, which arises from behavioral arguments on investors' information processing. While not mutually exclusive with the extant explanations, the advantage of this approach is that the behavior arises endogenously in equilibrium, and thus we can generate new empirical predictions with respect to several measurable parameters. In particular, we can make comparisons across various levels of reliance on stock-based compensation, accounting standards, growth options, and degree of information asymmetry.

The model is not specific to managers and investors. In fact we conjecture that many situations in which a decision-maker hires a better-informed, yet biased expert, will tend to exhibit similar pooling behavior. We show that the pooling equilibrium can Pareto dominate the fully revealing one in this setting as well, but not for all parameter values.

References

[1] Abarbanell, J., and R. Lehavy, 2000, Biased Forecasts or Biased Earnings? The Role of Reported Earnings in Explaining Apparent Bias and Over/Underreaction in Analysts' Earnings Forecasts, working paper.

[2] Banks J. S. and J. Sobel, 1987, Equilibrium Selection in Signaling Games, Econometrica 55, 647-661.

[3] Bergstresser D., and T. Philippon, 2002, CEO Incentives and Earnings Management: Evidence from the 1990s, working paper.

[4] Bhattacharya, U., H. Daouk, and M. Welker, 2003, The World Price of Earnings Opacity, The Accounting Review 78, 641-678.

[5] Burgstahler D. and I. Dichev, 1997, Earnings Management to Avoid Earnings Decreases and Losses, Journal of Accounting and Economics 24, 99-126.

[6] Cho I. and D. M. Kreps, 1987, Signaling Games and Stable Equilibria, Quarterly Journal of Economics 102, 179-221.

[7] Crawford, V., and J. Sobel, 1982, Strategic Information Transmission, Econometrica 50, 1431-1451.

[8] Degeorge, F., J. Patel, and R. Zeckhauser, 1999, Earnings Management to Exceed Thresholds, Journal of Business 72, 1-33.

[9] Dye R. A., 1988, Earnings Management in an Overlapping Generations Model, Journal of Accounting Research 26, 195-235.

[10] Fischer, P. E., and R. E. Verrecchia, 2000, Reporting Bias, The Accounting Review 75, 229-245.

[11] Fundenberg D., and J. Tirole, 1995, A Theory of Income and Dividend Smoothing Based on Incumbency Rents, Journal of Political Economy 103, 75-93.

[12] Goel A. M., and A. V. Thakor, 2003, Why Do Firms Smooth Earnings?, Journal of Business 76, 151-192.

[13] Grossman, S. J., and J. Stiglitz, 1980, On the Impossibility of Informationally Efficient Markets, American Economic Review,70, 393-408.

[14] Guttman, I., Kadan, O., and E. Kandel, 2003, Analysts: Target Setters or Information Providers, working paper.

[15] Hall B. J., and J. B. Liebman, 1998, Are CEOs Really Paid Like Bureaucrats?, Quarterly Journal of Economics 113, 653-691.

[16] Harrington, J. E., 1987, Oligopolistic Entry Deterrence Under Incomplete Information, Rand Journal of Economics 18, 211-231.

[17] Healy P. M., 1985, The Effect of Bonus Schemes on Accounting Decisions, Journal of Accounting and Economics 7, 85-107.

[18] Healy, P. M., and J. Wahlen, 1999, A Review of the Earnings Management Literature and Its Implications for Standard Setting, Accounting Horizons 13, 365–383.

[19] Jensen M. C., and K. J. Murphy, 1990, CEO Incentives - It's Not How Much You Pay, But How, Harvard Business Review 90, 138-153.

[20] Johnson, N. L., S. Kotz, and N. Balakrishnan, 1994, Continuous Univariate Distributions, 2^{nd} Edition, New York : Wiley & Sons.

[21] Kedia S., 2003, Do Executive Stock Options Generate Incentives for Earnings Management? Evidence from Accounting Restatements. working paper.

[22] Krishna V., and J. Morgan, 2001, A Model of Expertise, Quarterly Journal of Economics 116, 747-75.

[23] Leuz C., Nanda D., and P. D. Wysocki, 2003, Earnings Management and Investor Protection: an International Comparison, Journal of Financial Economics 69, 505–527.

[24] Milgrom, P, and J. Roberts, 1986, Relying on the Information of Interested Parties, Rand Journal of Economics 17, 18-32.

[25] Morgan, J., and P. Stocken, 2003, An Analysis of Stock Recommendations, Rand Journal of Economics 34, 183-203.

[26] Rangan S., 1998, Earnings Management and the Performance of Seasoned Equity Offerings, Journal of Financial Economics 50, 101-122.

[27] Riley, J. G., 1979, Informational Equilibrium, Econometrica 47, 331-359.

[28] Shivakumar L., (2000), Do Firms Mislead Investors by Overstating Earnings before Seasoned Equity Offerings? Journal of Accounting and Economics 29, 339-371.

[29] Stein, J. C., 1989, Efficient Capital Markets, Inefficient Firms: A Model of Myopic Corporate Behavior, Quarterly Journal of Economics 104, 655-669.

[30] Teoh S. H., Welch I., and T. J. Wong, 1998, Earnings Management and the Long-Run Market Performance of Initial Public Offerings, Journal of Finance 53, 1935-1974.

[31] Trueman B., and S. Titman, 1988, An Explanation of Accounting Income Smoothing, Journal of Accounting Research 26, suppl.: S127-S139.

[32] Verrecchia, R. E., 2003, Essays on Reporting, Journal of Accounting and Economics, forthcoming.

9 Appendix

Proof of Proposition 1

Let $\rho_s(\cdot)$ be a perfectly separating, continuously differentiable reporting strategy. Since $\rho_s(\cdot)$ is perfectly separating it can be inverted; thus, let $\varphi_s = \rho_s^{-1}$. It follows that the only pricing function consistent with $\rho_s(\cdot)$ is given by $P_s(\cdot) = c\varphi_s(\cdot)$. The utility of the manager given earnings of x and a report of x^R is given by

$$U^M(x, x^R) = \alpha c\varphi_s(x^R) - \beta(x^R - x)^2. \qquad (18)$$

The first order condition of (18) with respect to x^R is

$$\frac{d}{dx^R}\varphi_s(x^R) - \frac{2\beta}{\alpha c}x^R + \frac{2\beta}{\alpha c}x = 0.$$

Since in equilibrium $x = \varphi_s(x^R)$, we obtain the following linear, first-order differential equation for an equilibrium

$$\frac{d}{dx^R}\varphi_s(x^R) = -\frac{2\beta}{\alpha c}\varphi_s(x^R) + \frac{2\beta}{\alpha c}x^R.$$

All potential solutions of this equation are given by

$$\varphi_s(x^R) = x^R - \frac{\alpha c}{2\beta} + Ke^{-\frac{2x^R\beta}{c\alpha}},$$

where K is a constant. We claim that $K = 0$. Indeed, suppose on the contrary that $K > 0$, then a simple calculation shows that $\varphi_s(x^R)$ is strictly convex and has a unique minimum at $x^R = -\frac{c\alpha}{2\beta} \ln \frac{c\alpha}{2k\beta}$. This implies that $\varphi_s(x^R)$ is bounded from below, contrary to the fact that x can take any value in $\mathbb{R}$ (it is drawn from a normal distribution). Similarly, we can rule out the case $K < 0$. Therefore, we obtain $\varphi_s(x^R) = x^R - \frac{\alpha c}{2\beta}$, $P_s(x^R) = c\left(x^R - \frac{\alpha c}{2\beta}\right)$ and $\rho_s(x) = x + \frac{\alpha c}{2\beta}$, as required. ■

Proof of Proposition 2

It is sufficient to show that there exists a unique $a \in \mathbb{R}$, such that $E(\tilde{x}|\tilde{x} \in [a, a + \frac{\alpha c}{\beta}]) = a + \frac{3\alpha c}{4\beta}$. As $b = a + \frac{\alpha c}{\beta}$ we shall denote the conditional expectation by $d(a) = E(\tilde{x}|\tilde{x} \in [a, a + \frac{\alpha c}{\beta}])$ instead of $d(a, b)$. Thus, we will show that there exists a unique $a \in \mathbb{R}$, such that $d(a) = a + \frac{3\alpha c}{4\beta}$. The conditional expectation $d(a)$ is the expectation of a truncated normal random variable over the interval $[a, a + \frac{\alpha c}{\beta}]$. It is well known (see Johnson, Kotz and Balakrishnan (1994)) that $d(a)$ may be expressed using the following formula:

$$d(a) = x_0 - \sigma^2 \frac{f(a + \frac{\alpha c}{\beta}) - f(a)}{F(a + \frac{\alpha c}{\beta}) - F(a)} \qquad a \in \mathbb{R}. \tag{19}$$

Also, notice that the first derivative of the normal density satisfies:

$$f'(x) = -\frac{x - x_0}{\sigma^2} f(x). \tag{20}$$

The following two lemmas are needed in order to establish the existence of the required a.

Lemma 3 *The following holds for any $s > 0$:*

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x+s)}{f(x)} &= 0 \\ \lim_{x\to-\infty} \frac{f(x+s)}{f(x)} &= \infty. \end{aligned}$$

Proof. For any $s > 0$, and $x \in \mathbb{R}$ we have

$$\frac{f(x+s)}{f(x)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{(x+s-x_0)^2}{2\sigma^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{(x-x_0)^2}{2\sigma^2}} = \exp -\frac{s(2x - 2x_0 + s)}{2\sigma^2}.$$

The result follows by taking the appropriate limits. ■

Lemma 4 *The following holds:*

$$\begin{aligned} \lim_{a\to\infty} [d(a) - a] &= 0 \\ \lim_{a\to-\infty} [d(a) - a] &= \frac{\alpha c}{\beta}. \end{aligned}$$

Proof. By (19), and using L'Hopital's law we have

$$\begin{aligned}
\lim_{a\to\infty} d(a) &= x_0 - \sigma^2 \lim_{a\to\infty} \frac{f(a+\frac{\alpha c}{\beta}) - f(a)}{F(a+\frac{\alpha c}{\beta}) - F(a)} = x_0 - \sigma^2 \lim_{a\to\infty} \frac{f'(a+\frac{\alpha c}{\beta}) - f'(a)}{f(a+\frac{\alpha c}{\beta}) - f(a)} \\
&= x_0 + \sigma^2 \lim_{a\to\infty} \frac{\frac{a+\frac{\alpha c}{\beta}-x_0}{\sigma^2} f(a+\frac{\alpha c}{\beta}) - \frac{a-x_0}{\sigma^2} f(a)}{f(a+\frac{\alpha c}{\beta}) - f(a)} \\
&= x_0 + \lim_{a\to\infty} \frac{(a+\frac{\alpha c}{\beta} - x_0) f(a+\frac{\alpha c}{\beta}) - (a-x_0) f(a)}{f(a+\frac{\alpha c}{\beta}) - f(a)} \\
&= \lim_{a\to\infty} \left[a + \frac{\alpha c}{\beta} \frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta}) - f(a)} \right]
\end{aligned}$$

Now, by plugging $s = \frac{\alpha c}{\beta}$ in Lemma 3 it follows that

$$\lim_{a\to\infty} d(a) - a = \frac{\alpha c}{\beta} \lim_{a\to\infty} \frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta}) - f(a)} = \frac{\alpha c}{\beta} \lim_{a\to\infty} \frac{1}{1 - \frac{f(a)}{f(a+\frac{\alpha c}{\beta})}} = 0,$$

as required.

As for the second part, repeating the previous analysis we obtain

$$\lim_{a\to-\infty} d(a) = \lim_{a\to-\infty} \left[a + \frac{\alpha c}{\beta} \frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta}) - f(a)} \right].$$

Using Lemma 3 it follows that

$$\lim_{a\to-\infty} [d(a) - a] = \frac{\alpha c}{\beta} \lim_{a\to-\infty} \frac{f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta}) - f(a)} = \frac{\alpha c}{\beta} \lim_{a\to-\infty} \frac{1}{1 - \frac{f(a)}{f(a+\frac{\alpha c}{\beta})}} = \frac{\alpha c}{\beta},$$

as required. ■

It is now easy to prove the existence of a required a. Indeed, define $H(a) \equiv d(a) - a - \frac{3\alpha c}{4\beta}$. From Lemma 4 it follows that $\lim_{a\to-\infty} H(a) = \frac{\alpha c}{4\beta} > 0$, and $\lim_{a\to\infty} H(a) = -\frac{3\alpha c}{4\beta} < 0$. Thus, from the continuity of $H(a)$ we conclude that there exists an $a \in \mathbb{R}$ such that $H(a) = 0$.

Our next step is to prove the uniqueness of the chosen a. We shall accomplish this by showing that $H(a)$ is strictly decreasing, namely, that $d'(a) < 1$. For brevity we shall assume that $x_0 = 0$. This shortens the presentation and has no effect on the results.

From Lemma 4: $\lim_{a\to\infty} d'(a) = \lim_{a\to-\infty} d'(a) = 1$. Also, denote $k(a) \equiv d(a) - a$. Notice that, for all $a \in \mathbb{R}$: $0 \le d(a) \le \frac{\alpha c}{\beta}$, and $k'(a) = d'(a) - 1$. Also from Lemma 4: $\lim_{a\to\infty} d(a) = 0$, $\lim_{a\to-\infty} d(a) = \frac{\alpha c}{\beta}$.

Differentiating $d(a)$ and using the fact that $f'(x) = -\frac{x}{\sigma^2}f(x)$ we obtain

$$d'(a) = \frac{(a+\frac{\alpha c}{\beta})f(a+\frac{\alpha c}{\beta}) - af(a)}{F(a+\frac{\alpha c}{\beta}) - F(a)} + \sigma^2\left(\frac{f(a+\frac{\alpha c}{\beta}) - f(a)}{F(a+\frac{\alpha c}{\beta}) - F(a)}\right)^2 \tag{21}$$
$$= -\frac{a}{\sigma^2}d(a) + \frac{d(a)^2}{\sigma^2} + \frac{\frac{\alpha c}{\beta}f(a+\frac{\alpha c}{\beta})}{F(a+\frac{\alpha c}{\beta}) - F(a)} = \frac{1}{\sigma^2}d(a)k(a) + \frac{\frac{\alpha c}{\beta}f(a+\frac{\alpha c}{\beta})}{F(a+\frac{\alpha c}{\beta}) - F(a)}.$$

Using this equation we can evaluate $d'(\cdot)$ at $a = -\frac{\alpha c}{2\beta}$. We accomplish this in the next lemma.

Lemma 5 $d'(-\frac{\alpha c}{2\beta}) = \frac{\frac{\alpha c}{\beta}f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta})-F(-\frac{\alpha c}{2\beta})} < 1.$

Proof. From (21) we have:

$$d'(-\frac{\alpha c}{2\beta}) = \frac{1}{\sigma^2}d(-\frac{\alpha c}{2\beta})k(-\frac{\alpha c}{2\beta}) + \frac{\frac{\alpha c}{\beta}f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})}.$$

Since $f(\frac{\alpha c}{2\beta}) = f(-\frac{\alpha c}{2\beta})$ we have: $d(-\frac{\alpha c}{2\beta}) = 0$; therefore $d'(-\frac{\alpha c}{2\beta}) = \frac{\frac{\alpha c}{\beta}f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta})-F(-\frac{\alpha c}{2\beta})}$. Now, suppose on the contrary that $\frac{\frac{\alpha c}{\beta}f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta})-F(-\frac{\alpha c}{2\beta})} \geq 1$. Since $\frac{\alpha c}{\beta} > 0$, this implies $\frac{F(\frac{\alpha c}{2\beta})-F(-\frac{\alpha c}{2\beta})}{\frac{\alpha c}{\beta}} \leq f(\frac{\alpha c}{2\beta})$. However, by the mean value theorem we have $\frac{F(\frac{\alpha c}{2\beta})-F(-\frac{\alpha c}{2\beta})}{\frac{\alpha c}{\beta}} = \frac{F(\frac{\alpha c}{2\beta})-F(-\frac{\alpha c}{2\beta})}{\frac{\alpha c}{2\beta}-(-\frac{\alpha c}{2\beta})} = f(\xi)$ for some $\xi \in (-\frac{\alpha c}{2\beta}, \frac{\alpha c}{2\beta})$. But since f is normal with mean zero, it follows that $f(\xi) > f(\frac{\alpha c}{2\beta})$ for all $\xi \in (-\frac{\alpha c}{2\beta}, \frac{\alpha c}{2\beta})$. This constitutes a contradiction. ■

In order to proceed we need the following lemma, which uses the symmetry of the normal distribution.

Lemma 6 $k(a) = \frac{\alpha c}{2\beta}$ *if and only if* $a = -\frac{\alpha c}{2\beta}$.

Proof. If $a = -\frac{\alpha c}{2\beta}$ then by the symmetry of f around 0: $f(a) = f(a+\frac{\alpha c}{\beta})$, and thus $d(a) = 0$, and $k(a) = \frac{\alpha c}{2\beta}$.

To prove the "only if" part of the lemma recall that $d(a) = \frac{\int_a^{a+\frac{\alpha c}{\beta}} f(x)dx}{F(a+\frac{\alpha c}{\beta})-F(a)}$. Denote $\Delta(a) \equiv d(a) - (a+\frac{\alpha c}{2\beta})$. We may write

$$\Delta(a) = \frac{1}{F(a+\frac{\alpha c}{\beta}) - F(a)}\int_a^{a+\frac{\alpha c}{\beta}}(x-(a+\frac{\alpha c}{2\beta}))f(x)dx$$
$$= \frac{\int_a^{a+\frac{\alpha c}{2\beta}}(x-(a+\frac{\alpha c}{2\beta}))f(x)dx + \int_{a+\frac{\alpha c}{2\beta}}^{a+\frac{\alpha c}{\beta}}(x-(a+\frac{\alpha c}{2\beta}))f(x)dx}{F(a+\frac{\alpha c}{\beta}) - F(a)}.$$

Changing variables in the right hand integral to $\eta = 2a + \frac{\alpha c}{\beta} - x$ we obtain:

$$\Delta(a) = \frac{\int_a^{a+\frac{\alpha c}{2\beta}}(x-(a+\frac{\alpha c}{2\beta}))f(x)dx - \int_a^{a+\frac{\alpha c}{2\beta}}((a+\frac{\alpha c}{2\beta})-\eta)f(2a+\frac{\alpha c}{\beta}-\eta)d\eta}{F(a+\frac{\alpha c}{\beta})-F(a)} \tag{22}$$

$$= \frac{\int_a^{a+\frac{\alpha c}{2\beta}}(x-(a+\frac{\alpha c}{2\beta}))[f(x)-f(2a+\frac{\alpha c}{\beta}-x)]dx}{F(a+\frac{\alpha c}{\beta})-F(a)}.$$

Now, suppose that $k(a) = \frac{\alpha c}{2\beta}$, namely, $\Delta(a) = 0$, and suppose on the contrary that $a \neq -\frac{\alpha c}{2\beta}$. Consider first the case of $a < -\frac{\alpha c}{2\beta}$. In this case, the symmetry of f around 0 implies that for all $x \in (a, a+\frac{\alpha c}{2\beta}) : f(x) < f(2a+\frac{\alpha c}{\beta}-x)$. But from (22) it follows that $\Delta(a) > 0$ - a contradiction. A similar argument shows that it cannot be the case that $a > -\frac{\alpha c}{2\beta}$. We conclude that $a = \frac{\alpha c}{2\beta}$. ■

The condition $H(a) = 0$ implies that $k(a) = \frac{3\alpha c}{4\beta}$. Thus, Lemma 6 implies that we can assume $a \neq \frac{\alpha c}{2\beta}$. Since $d(a)$ is increasing in a, and $d(\frac{\alpha c}{2\beta}) = 0$, it follows that $d(a) \neq 0$, and $f(a+\frac{\alpha c}{\beta}) \neq f(a)$. Using this observation we can solve (21) for $k(a)$ and obtain

$$k(a) = \frac{\sigma^2 d'(a)}{d(a)} - \frac{\alpha c}{\beta}\sigma^2 \frac{\frac{f(a+\frac{\alpha c}{\beta})}{F(a+\frac{\alpha c}{\beta})-F(a)}}{d(a)} = \frac{\sigma^2 d'(a)}{d(a)} + \frac{\frac{\alpha c}{\beta}f(a+\frac{\alpha c}{\beta})}{f(a+\frac{\alpha c}{\beta})-f(a)} \tag{23}$$

Differentiating (23) yields

$$k'(a) = \frac{\sigma^2 d''(a)}{d(a)} - \sigma^2\left(\frac{d'(a)}{d(a)}\right)^2 + \frac{\alpha c}{\beta}\frac{-\frac{a+\frac{\alpha c}{\beta}}{\sigma^2}f(a+\frac{\alpha c}{\beta})[f(a+\frac{\alpha c}{\beta})-f(a)]}{[f(a+\frac{\alpha c}{\beta})-f(a)]^2} \tag{24}$$

$$+\frac{\alpha c}{\beta}\frac{[\frac{a+\frac{\alpha c}{\beta}}{\sigma^2}f(a+\frac{\alpha c}{\beta}) - \frac{a}{\sigma^2}f(a)]f(a+\frac{\alpha c}{\beta})}{[f(a+\frac{\alpha c}{\beta})-f(a)]^2}$$

$$= \frac{\sigma^2 d''(a)}{d(a)} - \sigma^2\left(\frac{d'(a)}{d(a)}\right)^2 + \frac{(\frac{\alpha c}{\beta})^2}{\sigma^2}\frac{f(a)f(a+\frac{\alpha c}{\beta})}{[f(a+\frac{\alpha c}{\beta})-f(a)]^2}$$

The following notation is useful. For all $a \in \mathbb{R}$ denote $Q(a) \equiv \frac{(\frac{\alpha c}{\beta})^2 f(a)f(a+\frac{\alpha c}{\beta})}{[F(a+\frac{\alpha c}{\beta})-F(a)]^2}$. Notice that $Q(a) > 0$ for all a. The next lemma shows that $Q(\cdot)$ is bounded from above by 1.

Lemma 7 *For all $a \in \mathbb{R}$, $Q(a) < 1$.*

Proof. Differentiating Q we obtain for all $a \in \mathbb{R}$:

$$\begin{aligned} Q'(a) &= (\frac{\alpha c}{\beta})^2 \frac{-\frac{a}{\sigma^2} f(a) f(a+\frac{\alpha c}{\beta}) - \frac{a+\frac{\alpha c}{\beta}}{\sigma^2} f(a) f(a+\frac{\alpha c}{\beta})}{[F(a+\frac{\alpha c}{\beta}) - F(a)]^2} \\ &\quad -(\frac{\alpha c}{\beta})^2 \frac{2(f(a+\frac{\alpha c}{\beta}) - f(a)) f(a) f(a+\frac{\alpha c}{\beta})}{[F(a+\frac{\alpha c}{\beta}) - F(a)]^3} \\ &= -\frac{a}{\sigma^2} Q(a) - \frac{a+\frac{\alpha c}{\beta}}{\sigma^2} Q(a) + \frac{2}{\sigma^2} d(a) Q(a) \\ &= \frac{Q(a)}{\sigma^2} \left[2(d(a) - a) - \frac{\alpha c}{\beta} \right]. \end{aligned}$$

Since $Q(a) > 0$ for all a, it follows that $Q'(a) = 0$ if and only if $d(a) - a = \frac{\alpha c}{2\beta}$. And from Lemma 6 we conclude that $Q'(a) = 0$ if and only if $a = -\frac{\alpha c}{2\beta}$. We shall now show that $a = -\frac{\alpha c}{2\beta}$ is a global maximum for Q. Indeed, differentiating Q once again we obtain

$$Q''(a) = \frac{Q'(a)}{\sigma^2} \left[2(d(a) - a) - \frac{\alpha c}{\beta} \right] + \frac{Q(a)}{\sigma^2} [2(d'(a) - 1)].$$

It follows that

$$Q''(-\frac{\alpha c}{2\beta}) = \frac{Q(\frac{\alpha c}{2\beta})}{\sigma^2} [2(d'(-\frac{\alpha c}{2\beta}) - 1)].$$

Thus, from Lemma 5 we conclude that $Q''(-\frac{\alpha c}{2\beta}) < 0$, and $a = -\frac{\alpha c}{2\beta}$ is a global maximum. Given this, in order to show that $Q(a) < 1$ for all a, it is sufficient to show that $Q(-\frac{\alpha c}{2\beta}) < 1$. Indeed:

$$Q(-\frac{\alpha c}{2\beta}) = \frac{(\frac{\alpha c}{\beta})^2 f(\frac{\alpha c}{2\beta})^2}{[F(a+\frac{\alpha c}{\beta}) - F(a)]^2}.$$

However, from Lemma 5 we know that $d'(-\frac{\alpha c}{2\beta}) = \frac{\frac{\alpha c}{\beta} f(\frac{\alpha c}{2\beta})}{F(\frac{\alpha c}{2\beta}) - F(-\frac{\alpha c}{2\beta})} < 1$; therefore $Q(-\frac{\alpha c}{2\beta}) = (d'(-\frac{\alpha c}{2\beta}))^2 < 1$. ■

We are now ready to show that $H(a) = d(a) - a - \frac{3\alpha c}{4\beta}$ is strictly decreasing in a, namely, that $d'(a) < 1$ for all $a \in \mathbb{R}$. We will show that this is true for all $a \in (-\infty, -\frac{\alpha c}{2\beta}]$. A parallel argument shows that this assertion is true also for all $a \in (-\frac{\alpha c}{2\beta}, \infty)$.

Suppose on the contrary that $d'(a) \geq 1$ for some a values in $(-\infty, -\frac{\alpha c}{2\beta}]$. Note that $\lim_{a \to -\infty} d'(a) = 1$, and from Lemma 5, $d'(-\frac{\alpha c}{2\beta}) < 1$. It follows that there exists an $\hat{a} \in (-\infty, -\frac{\alpha c}{2\beta})$ such that $d'(\hat{a}) \geq 1$ and $d''(\hat{a}) = 0$. Substituting $\hat{a}$ in (24) we obtain

$$\begin{aligned}
k'(\hat{a}) &\leq \frac{-\sigma^2}{d(\hat{a})^2} + \frac{(\frac{\alpha c}{\beta})^2}{\sigma^2} \frac{f(\hat{a})f(\hat{a}+\frac{\alpha c}{\beta})}{[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} \\
&= -\frac{1}{\sigma^2} \frac{[F(\hat{a}+\frac{\alpha c}{\beta}) - F(\hat{a})]^2}{[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} + \frac{(\frac{\alpha c}{\beta})^2}{\sigma^2} \frac{f(\hat{a})f(\hat{a}+\frac{\alpha c}{\beta})}{[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} \\
&= \frac{(F(\hat{a}+\frac{\alpha c}{\beta}) - F(\hat{a}))^2}{\sigma^2[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} \left[(\frac{\alpha c}{\beta})^2 \frac{f(\hat{a})f(\hat{a}+\frac{\alpha c}{\beta})}{(F(\hat{a}+\frac{\alpha c}{\beta}) - F(\hat{a}))^2} - 1 \right] \\
&= \frac{(F(\hat{a}+\frac{\alpha c}{\beta}) - F(\hat{a}))^2}{\sigma^2[f(\hat{a}+\frac{\alpha c}{\beta}) - f(\hat{a})]^2} (Q(\hat{a}) - 1).
\end{aligned}$$

But from Lemma 7 it follows that $Q(\hat{a}) - 1 < 0$, and therefore: $k'(\hat{a}) < 0$, or equivalently $d'(\hat{a}) < 1$ - a contradiction. This shows that there is a unique a that satisfies $H(a) = 0$, as required. ■

Proof of Proposition 5

For brevity we assume $x_0 = 0$.[19] Since we are interested in the impact of σ, we view a and d as functions of σ. Define

$$H(a, \sigma) \equiv d(a, \sigma) - a - \frac{3\alpha c}{4\beta}.$$

The relation between a and σ is given by the implicit equation $H(a, \sigma) = 0$. In the proof of Proposition 2 we have shown that $\frac{\partial H(a,\sigma)}{\partial a} < 0$ for all $a, \sigma \in \mathbb{R}$. By the implicit function theorem we have

$$\frac{\partial a(\sigma)}{\partial \sigma} = -\frac{\frac{\partial H(a,\sigma)}{\partial \sigma}}{\frac{\partial H(a,\sigma)}{\partial a}}.$$

Thus, to show that $\frac{\partial a(\sigma)}{\partial \sigma} < 0$ it is sufficient to show that $\frac{\partial H(a,\sigma)}{\partial \sigma} < 0$. We have

[19] A different choice of x_0 would shift $a(\sigma)$ by a constant, and thus it has no effect on $\frac{\partial a}{\partial \sigma}$.

$$\begin{aligned}
\frac{\partial H(a,\sigma)}{\partial \sigma} &= \frac{\partial d(a,\sigma)}{\partial \sigma} = \frac{\partial}{\partial \sigma}\frac{\int_a^{a+\frac{\alpha c}{\beta}} xf(x)dx}{F(a+\frac{\alpha c}{\beta})-F(a)} = \frac{\partial}{\partial \sigma}\frac{\frac{1}{\sqrt{2\pi\sigma^2}}\int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\frac{1}{\sqrt{2\pi\sigma^2}}\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx} \\
&= \frac{\int_a^{a+\frac{\alpha c}{\beta}} \frac{x^3}{\sigma^3}e^{-\frac{x^2}{2\sigma^2}}dx \cdot \int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx - \int_a^{a+\frac{\alpha c}{\beta}} \frac{x^2}{\sigma^3}e^{-\frac{x^2}{2\sigma^2}}dx \cdot \int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\left[\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx\right]^2} \\
&= \frac{1}{\sigma^3}\left[\frac{\int_a^{a+\frac{\alpha c}{\beta}} x^3e^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx} - \frac{\int_a^{a+\frac{\alpha c}{\beta}} x^2e^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{x^2}{2\sigma^2}}dx}\cdot\frac{\int_a^{a+\frac{\alpha c}{\beta}} xe^{-\frac{x^2}{2\sigma^2}}dx}{\int_a^{a+\frac{\alpha c}{\beta}} e^{-\frac{\pi^2}{2\sigma^2}}dx}\right] \\
&= \frac{1}{\sigma^3}\left[\frac{\int_a^{a+\frac{\alpha c}{\beta}} x^3f(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)d\pi} - \frac{\int_a^{a+\frac{\alpha c}{\beta}} x^2f(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)dx}\cdot\frac{\int_a^{a+\frac{\alpha c}{\beta}} xf(x)dx}{\int_a^{a+\frac{\alpha c}{\beta}} f(x)dx}\right] \\
&= \frac{1}{\sigma^3}\left[E(\tilde{x}^3|a\le\tilde{x}\le a+\frac{\alpha c}{\beta}) - E(\tilde{x}^2|a\le\tilde{x}\le a+\frac{\alpha c}{\beta})\cdot E(\tilde{x}|a\le\tilde{x}\le a+\frac{\alpha c}{\beta})\right] \\
&= \frac{1}{\sigma^3}Cov(\tilde{x}^2,\tilde{x}|a\le\tilde{x}\le a+\frac{\alpha c}{\beta}).
\end{aligned}$$

Thus, the sign of $\frac{\partial H(a,\sigma)}{\partial \sigma}$ is equal to the sign of the $Cov(\tilde{y},\, \tilde{y}^2)$, where $\tilde{y}$ is a random variable obtained from a truncation of $\tilde{x}$ between a and $a+\frac{\alpha c}{\beta}$. It can be shown that this covariance is strictly negative, as required.[20]

Given that $a(\sigma)$ is decreasing in σ, we know that $a_0 \equiv \lim_{\sigma\to 0} a(\sigma)$ exists. It is easy to see that for any fixed $a < x_0$ and $b > a$ we have

$$\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a,b]) = \begin{cases} b & x_0 \notin [a,b] \\ x_0 & x_0 \in [a,b] \end{cases}. \tag{25}$$

We claim first that there exists an $a_0 > 0$ such that $a_0 + \frac{\alpha c}{\beta} > x_0$. Indeed, suppose on the contrary that $a_0 + \frac{\alpha c}{\beta} \le x_0$. This implies by (25) that $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a(\sigma), a(\sigma)+\frac{\alpha c}{\beta}]) = a_0 + \frac{\alpha c}{\beta}$, contradicting the fact that for all σ, $E(\tilde{x}|\tilde{x}\in[a(\sigma), a(\sigma)+\frac{\alpha c}{\beta}]) = a(\sigma) + \frac{3\alpha c}{4\beta}$. Now, for all $\varepsilon > 0$ sufficiently small we have: $a_0 - \varepsilon + \frac{\alpha c}{\beta} > x_0$. Thus, by (25) we have: $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a_0-\varepsilon, a_0-\varepsilon+\frac{\alpha c}{\beta}]) = x_0$. From the continuity of the conditional expectation and since ε is arbitrary we conclude that $\lim_{\sigma\to 0} E(\tilde{x}|\tilde{x}\in[a(\sigma), a(\sigma)+\frac{\alpha c}{\beta}]) = x_0$. And, hence $\lim_{\sigma\to 0} a(\sigma) = x_0 - \frac{3\alpha c}{4\beta}$, as required.

[20] The proof is technical. It applies to any symmetric and continuous distribution, and not only to the normal distribution. It relies on the fact that the truncation interval $[a, a+\frac{\alpha}{\beta}]$ is tilted to the left-hand side of the distribution. We omit the proof here for brevity, but it is available upon request.

As for the case of $\sigma \to \infty$. For all fixed a and b we have: $E(\tilde{x}|\tilde{x} \in [a,b] \to \frac{a+b}{2}$. Indeed, by applying L'Hopital's law we obtain

$$\begin{aligned}
\lim_{\sigma\to\infty}\left[x_0 - \sigma^2 \frac{f(b)-f(a)}{F(b)-F(a)}\right] &= x_0 - \lim_{\sigma\to\infty} \sigma^2 \frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}} - e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\int_a^b e^{\frac{-(x-x_0)^2}{2\sigma^2}}dx} \\
&= x_0 - \frac{1}{b-a}\lim_{\sigma\to\infty} \frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}} - e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\frac{1}{\sigma^2}} \\
&= x_0 - \frac{1}{b-a}\lim_{\sigma\to\infty} \frac{e^{\frac{-(b-x_0)^2}{2\sigma^2}} - e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{\frac{1}{\sigma^2}} \\
&= x_0 - \frac{1}{b-a}\lim_{\sigma\to\infty} \frac{\frac{(b-x_0)^2}{\sigma^3}e^{\frac{-(b-x_0)^2}{2\sigma^2}} - \frac{(a-x_0)^2}{\sigma^3}e^{\frac{-(a-x_0)^2}{2\sigma^2}}}{-\frac{2}{\sigma^3}} \\
&= x_0 + \frac{(b-x_0)^2-(a-x_0)^2}{2(b-a)} = \frac{a+b}{2}
\end{aligned}$$

This calculation implies that if $a_\infty \equiv \lim_{\sigma\to\infty} a(\sigma)$ were finite, we would have that $d(a(\sigma)) \to a_\infty + \frac{\alpha c}{2\beta}$ - a contradiction to the fact $d(a(\sigma)) = a(\sigma) + \frac{3\alpha c}{4\beta}$ for all σ. ■

Proof of Lemma 2

We shall prove Part 1 of the lemma. The proof of Part 2 is symmetric.

Suppose $x^R \in (a + \frac{\alpha c}{2\beta}, b)$ is an out-of-equilibrium report, and let $P(x^R)$ be the price in case a report of x^R is observed. We claim that in this case, if the 'a' type is indifferent between submitting a report of b (equilibrium report) or x^R (deviating), then all other types $x' \neq a$ strictly prefer to stick to their equilibrium report. We shall consider three cases.

Case 1: $x' \in (a,b]$. Since the 'a' type is indifferent between submitting b, and deviating to x^R, we obtain

$$\alpha cd - \beta(b-a)^2 = \alpha P(x^R) - \beta(x^R - a)^2. \tag{26}$$

The payoff to type $x' \in (a,b]$ from reporting x^R is: $\alpha P - \beta(x^R - x')^2$. It follows that the largest benefit from deviating to a report of x^R is incurred when the type is equal to the report, namely: $x' = x^R$. In this case, the payoff in case of deviation is αP, while the payoff on the equilibrium path is: $\alpha cd - \beta(x^R - b)^2$. By (26), the difference between the payoff on the equilibrium path, and the payoff in case of deviation is

$$\begin{aligned}\alpha cd - \beta(x^R - b)^2 - \alpha P &= -\beta(x^R - b)^2 + \beta(b-a)^2 - \beta(x^R - a)^2 \\ &= 2\beta(x^R - a)(b - x^R) > 0.\end{aligned}$$

Thus, type x' strictly prefers to stick to his equilibrium report.

Case 2: $x' < a$. Since the 'a' type is indifferent between submitting $a + \frac{\alpha c}{2\beta}$, and deviating to x^R we obtain

$$\alpha ca - \beta(\frac{\alpha c}{2\beta})^2 = \alpha P(x^R) - \beta(x^R - a)^2. \tag{27}$$

Now, if type x' follows the equilibrium he obtains: $\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2$. If on the other hand he deviates to x^R he obtains: $\alpha P(x^R) - \beta(x^R - x')^2$. Using (27) we obtain that the difference is

$$\begin{aligned}\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2 - \alpha P(x^R) + \beta(x^R - x')^2 &= \alpha cx' - \alpha ca - \beta(x^R - a)^2 + \beta(x^R - x')^2 \\ &= \beta(a - x')(2x^R - x' - a - \frac{\alpha c}{\beta}) \\ &> \beta(a - x')(2(a + \frac{\alpha c}{2\beta}) - x' - a - \frac{\alpha c}{\beta}) \\ &= \beta(a - x')^2 > 0,\end{aligned}$$

where the penultimate inequality follows since $x^R > a + \frac{\alpha c}{2\beta}$. Thus, type x' is better off sticking to the equilibrium strategy.

Case 3: $x' > b$. In Case 1, we have shown that if type 'a' is indifferent between the two alternatives, then type 'b' strictly prefers to stick to the equilibrium. Thus

$$\alpha cb - \beta(\frac{\alpha c}{2\beta})^2 > \alpha P(x^R) - \beta(x^R - b)^2.$$

Therefore

$$\alpha P(x^R) + \beta(\frac{\alpha c}{2\beta})^2 < \alpha cb + \beta(x^R - b)^2.$$

We conclude that

$$\begin{aligned}\alpha cx' - \beta(\frac{\alpha c}{2\beta})^2 - \alpha P(x^R) + \beta(x^R - x')^2 &> \alpha cx' - \alpha cb - \beta(x^R - b)^2 + \beta(x^R - x')^2 \\ &= \beta(x' - b)(x' + b + \frac{\alpha c}{\beta} - 2x^R) \\ &> \beta(x' - b)(x' - b + \frac{\alpha c}{\beta}) > 0,\end{aligned}$$

where the penultimate inequality follows since $x^R < b$. Thus, the deviation is not profitable. This concludes the proof. ■