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OPTIMUM AND RISK-CLASS PRICING OF ANNUITIES

by

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OPTIMUM AND RISK-CLASS PRICING OF ANNUITIES*

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Abstract

When information on longevity (survival functions) is unknown early in life, individuals have an interest to insure themselves against future 'risk-class' classification. Accordingly, the *First-Best* typically involves transfers across states of nature. Competitive equilibrium cannot provide such transfers if insurance firms are unable to *precommit* their customers. On the other hand, public insurance plans that do not distinguish between 'risk-class' realizations are also inefficient. It is impossible, *a-priori*, to rank these alternatives from a welfare point of view.

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1 Demand for Annuities

Consider an individual who wants to decide on his or her optimum consumption at different ages and choose an age for retirement in the presence of uncertainty about the length of life. Assume at first that this uncertainty is represented by a known *survival distribution function*, $F(z)$, which is the probability to survive to age z. Subsequently we shall analyze the more realistic case of uncertain survival probabilities early in life.

Let T be maximum lifetime.[1] Then, $F(0) = 1$, $F(T) = 0$ and $F(z)$ is non-increasing in z.

Assume that $f(z)$, the density of $1 - F(z)$ (the probability of dying at age z), exists for all z, $0 \leq z \leq T$. Consumption at age z is denoted by $c(z)$. Utility of consumption, $u(c)$, is independent of age, increasing in c, and displays risk aversion ($u'(c) > 0$ and $u''(c) < 0$). When working, the individual provides one unit of labor. Disutility of work, $a(z)$, is independent of consumption and increasing with age ($a'(z) > 0$). Contingent on survival, individuals work between age zero and R, i.e. retirement occurs at R.

The individual's objective is to maximize *expected lifetime utility*, V. With no subjective discount rate,

$$V = \int_0^T F(z)u(c(z))dz - \int_0^R F(z)a(z)dz. \tag{1}$$

Let wages at age z be $w(z)$. Savings, $w(z) - c(z)$, whether positive or negative, are assumed to incur a *zero* rate of interest.[2]

With no bequest motive, individuals save only in order to finance consumption, particularly during retirement. This is achieved efficiently by continuously annuitizing savings through the purchase of *'deferred-annuities'* that will start payments upon retirement (Sheshinski (1999)).

Expected lifetime consumption is equal to expected lifetime income:

$$\int_0^T F(z)c(z)dz - \int_0^R F(z)w(z)dz = 0. \tag{2}$$

[1] Formally, it is possible to allow $T = \infty$.

[2] It is well-known how to modify the results for positive rates of interest and subjective discount rates.

Maximization of (1) subject to (2) yields an optimum constant consumption flow, $c(z) = c^*$, $0 \leq z \leq T$ which depends on the age of retirement By (2),

$$c^* = c^*(R) = \frac{W(R)}{\bar{z}} \tag{3}$$

where $\bar{z} = \int_0^T F(z)dz$ is *life-expectancy*[3] and $W(R) = \int_0^R F(z)w(z)dz$ is *expected wages* until retirement.

The condition for optimum retirement is written:

$$\phi(R) - a(R) = 0 \tag{4}$$

where $\phi(R) = u'(c^*(R))w(R)$ is the additional utility from a small postponement of retirement. Condition (4) determines optimum retirement so as to balance these benefits against instantaneous labor disutility, $a(R)$. We assume that $w'(R) \leq 0$. This ensures that (4) has a unique solution, denoted R^*, which satisfies second-order conditions.[4]

Optimum expected utility, V^*, is

$$V^* = u(c^*(R^*))\ \bar{z} - \int_0^{R^*} F(z)a(z)dz. \tag{5}$$

Fully insuring against lifetime uncertainty, c^*, R^* and V^* are the *First-Best* allocation.

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[3]Integrating by parts: $\int_0^T F(z)dz = \int_0^T zf(z)dz$.

[4]The sufficient condition is that at any solution, R^*, to (4), ϕ strictly decreases:

$$\frac{\phi'(R^*)}{\phi(R^*)} = \frac{w'(R^*)}{w(R^*)} - \sigma(R^*)\frac{w(R^*)}{W(R^*)}F(R^*) < 0,$$

where $\sigma = -\dfrac{u''(c)c}{u'(c)} > 0$. This condition clearly holds when $w'(R) \leq 0$.

2 'Risk-Classes': Ranking of Survival Functions

Individuals who share a common survival function are called a *'risk-class'*. We want to consider a population that consists of a number of risk-classes and analyze the implications of alternative annuity pricing schemes in the presence of such heterogeneity. It will be useful first to formalize the notion that one survival function has a shorter life-span or is more 'risky' than another. Our approach is a direct application of the theory of *Stochastic-Dominance*.

2.1 Ranking of Survival Function

Consider two survival functions, $F_1(z)$ and $F_2(z)$, $0 \leq z \leq T$, both satisfying $F_i(0) = 1$, $F_i(T) = 0$ and $F_i(z)$ non-increasing in z, $i = 1, 2$. The conditional probability of dying at age z, $\frac{f_i(z)}{F_i(z)}$, is termed the *'Hazard-Rate'* of $F_i(z)$.

Definition 1. (*'Single Crossing'* or *'Stochastic-Dominance'*): The function $F_1(z)$ is said to (strictly) stochastically dominate $F_2(z)$ if the *'Hazard-Rates'* satisfy

$$\frac{f_2(z)}{F_2(z)} > \frac{f_1(z)}{F_1(z)}, \qquad 0 \leq z \leq T. \tag{6}$$

In words, the rate of decrease of survival probabilities, $\frac{d \ln F(z)}{dz} = -\frac{f(z)}{F(z)}$, is smaller at all ages with distribution 1 than with 2.

Two implications of this definition are important. First, consider the functions $\frac{F_i(z)}{\int_0^T F_i(z)dz}$, $0 \leq z \leq T$, $i = 1, 2$. Being positive and their integral over $(0, T)$ equal to one, they must intersect (cross) at least once over this range. At any such crossing, when $\frac{F_1(z)}{\int_0^T F_1(z)dz} = \frac{F_2(z)}{\int_0^T F_2(z)dz}$, condition (6) implies that $\frac{d}{dz}\left(\frac{F_1(z)}{\int_0^T F_1(z)dz}\right) > \frac{d}{dz}\left(\frac{F_2(z)}{\int_0^T F_2(z)dz}\right)$. Hence, there can be only a *single crossing*. That is, there exists an age z_c, $0 < z_c < T$, such that (Figure 1),

$$\frac{F_1(z)}{\int_0^T F_1(z)dz} \lesseqgtr \frac{F_2(z)}{\int_0^T F_2(z)dz} \text{ as } z \lesseqgtr z_c. \tag{7}$$

Intuitively, (7) means that the dominant (dominated) distribution has higher (lower) survival rates, relative to life expectancy, at older (younger) ages.

Second, since $F_i(0) = 1$, $i = 1, 2$, it follows from (7) that

$$\bar{z}_1 = \int_0^T F_1(z)dz > \int_0^T F_2(z)dz = \bar{z}_2, \tag{8}$$

i.e., stochastic dominance implies *higher life expectancy.*

2.2 Risk-Class Pricing of Annuities

Suppose that the population consist of two risk classes represented by survival functions $F_i(z)$, $i = 1, 2$. Otherwise individuals are identical (i.e., same preferences and incomes). Assume that group 1's survival function stochastically dominates, according to (6), that of group 2. In particular, group 1 has a higher life expectancy. In a perfectly competitive market, when firms can identify annuity purchasers according to the risk-class to which they belong, the analysis in section 2 applies to each group separately. This leads us to the following:

Proposition 1. *When $F_1(z)$ stochastically dominates $F_2(z)$, then $c_1^*(R_1^*) < c_2^*(R_2^*)$ and $R_1^* > R_2^*$.*

Proof. Applying (3) to each group, $c_i^* = c_i^*(R) = \frac{\int_0^R F_i(z)w(z)dz}{\int_0^T F_i(z)dz}$, $i = 1, 2$.

By (7),

$$c_i^*(R) = \frac{\int_0^R F_1(z)w(z)dz}{\int_0^T F_1(z)dz} < \frac{\int_0^R F_2(z)w(z)dz}{\int_0^T F_2(z)dz} = c_2^*(R), \tag{9}$$

for all R, $0 < R < T$. It follows from (4) that $\phi_1(R) > \phi_2(R)$ and hence, since $a'(R) > 0$, that $R_1^* > R_2^*$ and, further more, since $w'(R) \leq 0$ and $a'(R) > 0$, $c_1^*(R_1^*) < c_2^*(R_2^*)$ ॥.

When wages are the same for all individuals, those with higher life expectancy partially compensate for higher longevity by retiring later, but their optimum consumption remains lower throughout.

3 Uncertain Future Survival Functions

The assumption that uncertainly in lifetime duration is represented by a known survival function is not realistic. Survival probabilities are difficult to predict, particularly early in life, since they depend on health and other circumstances which only unfold overtime. Accordingly, we shall now assume that early in life individuals do not know to what risk class they will belong later on. Consequently, they have an interest in insurance against alternative risk classifications later in life. Such insurance typically involves transfers across different risk classes ('states of nature') and is actuarially fair on average.

Risk classes with higher than average life expectancy face unfavorably priced annuities while the others face favorably priced annuities. It is desirable to have ex-ante insurance that allows consumption levels and retirement ages to deviate from those that would be chosen when annuity prices are actuarially fair for each risk class separately.[5]

We model the uncertainty about future risk classification as follows. All individuals have the same known survival function, $F(z)$, between ages zero to M, well before retirement. At that age, there is a probability p, $0 < p < 1$, that the survival function becomes $F_1(z)$ and $1 - p$ that it becomes $F_2(z)$.

Assuming that preferences do not vary with the realized risk class, expected lifetime utility is

[5] More generally, this applies not only to retirement but to other labor supply attributes (e.g., hours of work and effort) not modelled here.

$$
\begin{aligned}
V =& \int_0^M F(z)u(c(z))dz + p\int_M^T F_1(z)u(c_1(z))dz + (1-p)\int_M^T F_2(z)u(c_2(z))dz - \\
& - \int_0^M F(z)a(z)dz - p\int_0^{R_1} F_1(z)a(z)dz - (1-p)\int_M^{R_2} F_2(z)a(z)dz
\end{aligned} \tag{10}
$$

where $c_i(z)$, $M \leq z \leq T$, and R_i are consumption and retirement age under realization of survival function i, $i = 1, 2$. Choices obey a zero expected profits constraint:

$$
\begin{aligned}
& \int_0^M F(z)c(z)dz + p\int_M^T F_1(z)c_1(z)dz + (1-p)\int_M^T F_2(z)c_2(z)dz - \\
& - \int_0^M F(z)w(z)dz - p\int_0^{R_1} F_1(z)w(z)dz - (1-p)\int_M^{R_2} F_2(z)w(z)dz = 0
\end{aligned} \tag{11}
$$

Maximization of (10) subject to (11) yields optimum consumption, which is constant at all ages and across states: $c(z) = c_1(z) = c_2(z) = c^*$. Similarly, optimum retirement ages are equal for both risk classes: $R_1 = R_2 = R$. By (11), c^* is given, in analogy to (3), by

$$
c^* = c^*(R) = \beta\frac{W_1(R)}{\bar{z}_1} + (1-\beta)\frac{W_2(R)}{\bar{z}_2}, \tag{12}
$$

where $\bar{z}_i = \int_0^M F(z)dz + \int_M^T F_i(z)dz$ is life expectancy and $W_i(R) = \int_0^M F(z)w(z)dz + \int_M^R F_i(z)w(z)dz$ is expected wages until retirement in state i, and $\beta = \dfrac{p\bar{z}_1}{p\bar{z}_1 + (1-p)\bar{z}_2}$, $0 < \beta < 1$. Optimum retirement age, R^*, is determined, as before, by condition (4).

We state this result in the following:

Proposition 2. *When preferences are independent of survival function realizations, optimum consumption is uniform and retirement ages are identical for all risk classes.*

The optimum described above entails transfers across risk classes.

Let T_i^* be the optimum transfer to risk-class i, defined as the excess of expected consumption over expected wages from age M to T less expected total savings, S^*, during ages zero to M, $S^* = \int_0^M F(z)(w(z) - c^*)dz$:

$$T_i^* = c^* \int_M^T F_i(z)dz - \int_M^{R^*} F_i(z)w(z)dz - S^* = c^*\overline{z}_i - W_i(R^*). \tag{13}$$

By (12),

$$\begin{aligned} T_1^* &= \overline{z}_1(1-\beta)\left[\frac{W_2(R^*)}{\overline{z}_2} - \frac{W_1(R^*)}{\overline{z}_1}\right] \\ T_2^* &= \overline{z}_2\beta\left[\frac{W_1(R^*)}{\overline{z}_1} - \frac{W_2(R^*)}{\overline{z}_2}\right] \end{aligned} \tag{14}$$

By (7), transfers to the stochastically dominant (dominated) group are positive (negative), $T_1^* > 0 (T_2^* < 0)$. The break-even constraint (11) entails that total expected transfers are zero: $pT_1^* + (1-p)T_2^* = 0$.

4 Competitive Markets: Risk-Class Pricing without Transfers

We have seen that the *First-Best* allocation entails transfers across risk-classes. Competitive insurance markets can implement such a scheme provided that insurance firms can *precommit* their customers, prior to 'risk-class' realization, to stay-on until retirement. This is not plausible. Firms will successfully lure individuals with a short life expectancy, offering them actuarially fair annuities with no transfers to other groups. Consequently, in the absence of transfers between risk-classes, individuals at early ages find themselves not being insured against alternative risk classifications at later ages. We want to study the implications of this market failure.

At age M, expected utility of an individual who belongs to risk class i, denoted V_i, is

$$V_i = \int_M^T F_i(z)u(c_i(z))dz - \int_M^{R_i} F_i(z)a(z)dz, \qquad i = 1, 2. \tag{15}$$

When annuity prices are actuarially fair for each risk class, the zero expected profits constraint is

$$\int_M^T F_i(z)c_i(z)dz - \int_M^{R_i} F_i(z)w(z)dz - S = 0, \qquad i = 1, 2, \tag{16}$$

where S, savings during ages zero to M, are the same for both risk classes. As before, maximization of (15) subject to (16) yields optimum consumption, which is constant at all ages: $c_i(z) = \widehat{c}_i$, $M \leq z \leq T$. By (16),

$$\widehat{c}_i = \widehat{c}_i(R_i, S) = \frac{\int_M^{R_i} F_i(z)w(z)dz + S}{\int_M^T F_i(z)dz}, \qquad i = 1, 2. \tag{17}$$

Optimal retirement age, denoted $\widehat{R}_i(S)$, is determined by the condition:

$$\phi_i(R_i, S) - a(R_i) = 0, \qquad i = 1, 2, \tag{18}$$

where $\phi_i(R_i, S) = u'(\widehat{c}_i(R_i, S))w(R_i)$. At the optimum, with $\widehat{c}_i(\widehat{R}_i(S), S)$ and $\widehat{R}_i(S)$, expected utility, (15), is written $\widehat{V}_i(S)$. By the envelope theorem, $\dfrac{d\widehat{V}_i}{dS} = u'(\widehat{c}_i)$.

Expected utility at age zero, V, is

$$V = \int_0^M F(z)u(c(z))dz + p\widehat{V}_1(S) + (1-p)\widehat{V}_2(S). \tag{19}$$

Maximization of (19) subject to the constraint:

$$S - \int_0^M F(z)w(z)dz + \int_0^M F(z)c(z)dz = 0 \tag{20}$$

yields constant optimum consumption $c(z) = \widehat{c}$, $0 \leq z \leq M$,where, by (20)

$$\widehat{c} = \widehat{c}(S) = \frac{\int_0^M F(z)w(z)dz - S}{\int_0^M F(z)dz}. \tag{21}$$

Optimum savings has to satisfy the condition:

$$u'(\widehat{c}) = pu'(\widehat{c}_1) + (1-p)u'(\widehat{c}_2) \tag{22}$$

At the optimum, marginal utility of consumption between age zero and M is a weighted average of optimum marginal utility of consumption of the two risk classes.

Since $F_1(z)$ stochastically dominates $F_2(z)$, it is seen from (17) that for any R and S, $\widehat{c}_1(R,S) < \widehat{c}_2(R,S)$. This implies, in turn, that $\phi_1(R,S) > \phi_2(R,S)$. It now follows from condition (18) that $\widehat{R}_1 > \widehat{R}_2$ and (since $w'(R) \leq 0$) that $\widehat{c}_1(\widehat{R}_1, S) < \widehat{c}_2(\widehat{R}, S)$.

We summarize:

Proposition 3: *Risk class pricing without transfers implies that at the optimum,* $\widehat{c}_1 < \widehat{c} < \widehat{c}_2$ *and* $\widehat{R}_1 > \widehat{R}_2$.

Comparing (12) with (17) - (18), it can be inferred that *First-Best* consumption and retirement, (c^*, R^*), relate to optimum consumption and retirement under risk class pricing without transfers, $(\widehat{c}_i, \widehat{R}_i)$, $i = 1, 2$, as follows: $\widehat{c}_1 < c^* < \widehat{c}_2$ and $\widehat{R}_1 > R^* > \widehat{R}_2$ (Figure 3). In the *First-Best* allocation, individuals fully insure themselves against risk classification via transfers across risk classes. In the absence of such transfers, individuals with high life expectancy choose to postpone retirement, thereby partially compensating for their lower lifetime consumption. The opposite holds for those with low life expectancy.

Propositions 2 and 3 imply that when preferences are independent of survival function realizations, optimum risk class pricing without transfers is inferior to optimum uniform annuity prices. Importantly, this suggests that social security systems which provide uniform benefits to retirees with the same earnings history and the same retirement age are preferred to private annuity markets which do not provide transfers across risk-classes. We now want to explore whether this conclusion changes when preferences depend on survival function realizations.

5 Welfare Ranking of Uniform vs. Risk-Class Annuity Systems

The inefficiency of risk-class pricing without transfers depends crucially on individuals' desirability to maintain, after the arrival of information about their 'risk-class' classification an optimum level of consumption and retirement age independent of risk-class. Such invariance entails positive and negative transfers across risk-classes. This result does not carry-over to the case when *utility functions are state dependent.*

Consider, for example, the case when disutility from work depends on the state of nature. Thus, let $a_i(z)$ be this disutility in state i, $i = 1, 2$. When $F_1(z)$ stochastically dominates $F_2(z)$, it is natural to assume that $a_1(z) < a_2(z)$, for all $z \geq M$.[6]

Assuming that utility from consumption is independent of the state of nature, it can be shown that the *First-Best* entails constant consumption, c^*, given by

$$c^* = c^*(R_1,\ R_2) = \beta\,\frac{W_1(R_1^*)}{\bar{z}_1} + (1-\beta)\,\frac{W_2(R_2^*)}{\bar{z}_2}, \qquad (23)$$

while optimum retirement ages are determined by the conditions[7]

$$u'(c^*(R_1^*, R_2^*))w(R_i^*) - a_i(R_i^*) = 0, \qquad i = 1, 2. \qquad (24)$$

It is easy to see from (23) - (24) that $R_1^* > R_2^*$.

Whether the optimum entails transfers can be seen (from (14)) to depend on the difference between $\dfrac{W_1(R_1^*)}{\bar{z}_1}$ and $\dfrac{W_2(R_2^*)}{\bar{z}_2}$, that is, on the difference in expected total wages until retirement relative to expected lifetime. This difference can have any sign. For example, let $F_i(z) = e^{-\alpha_i}$, and $w(z) = w$. Then the *First-Best* has *no* transfers iff $\alpha_1 R_1^* = \alpha_2 R_2^*$, i.e. if *the elasticity of optimum retirement relative to expected lifetime is unity.* More generally, transfers to any group can be positive or negative depending on the level of this elasticity.

[6] The relation between $a_i(z)$, for $M \leq z$, and $a(z)$, for $0 \leq z \leq M$ is immaterial for our discussion.

[7] Second-order conditions can be shown to be satisfied.

In principle, therefore, when *First-Best* transfers are zero or negligible, a private market for annuities without transfers will be efficient. On the other hand, if a public social security system does not allow the flexibility in retirement ages implied by the optimum, then the private market will be superior.

When utility of consumption is also state dependent then, in the *First-Best*, both optimum consumption levels and optimum retirement ages depend on risk-class realization. Hence, a social security system which provides uniform consumption (and/or) imposes equal retirement ages is, in general, not efficient.

We are led to the following conclusion:

Proposition 4. *It is impossible to rank, from a welfare point of view, an annuitization system that provides a uniform plan (of consumption and retirement) to all risk classes and a competitive system based on risk-class pricing but without precommitment and hence no transfers.*

References

[1] Sheshinski, E. (1999) "Annuities and Retirement", Department of Economics, The Hebrew University of Jerusalem.

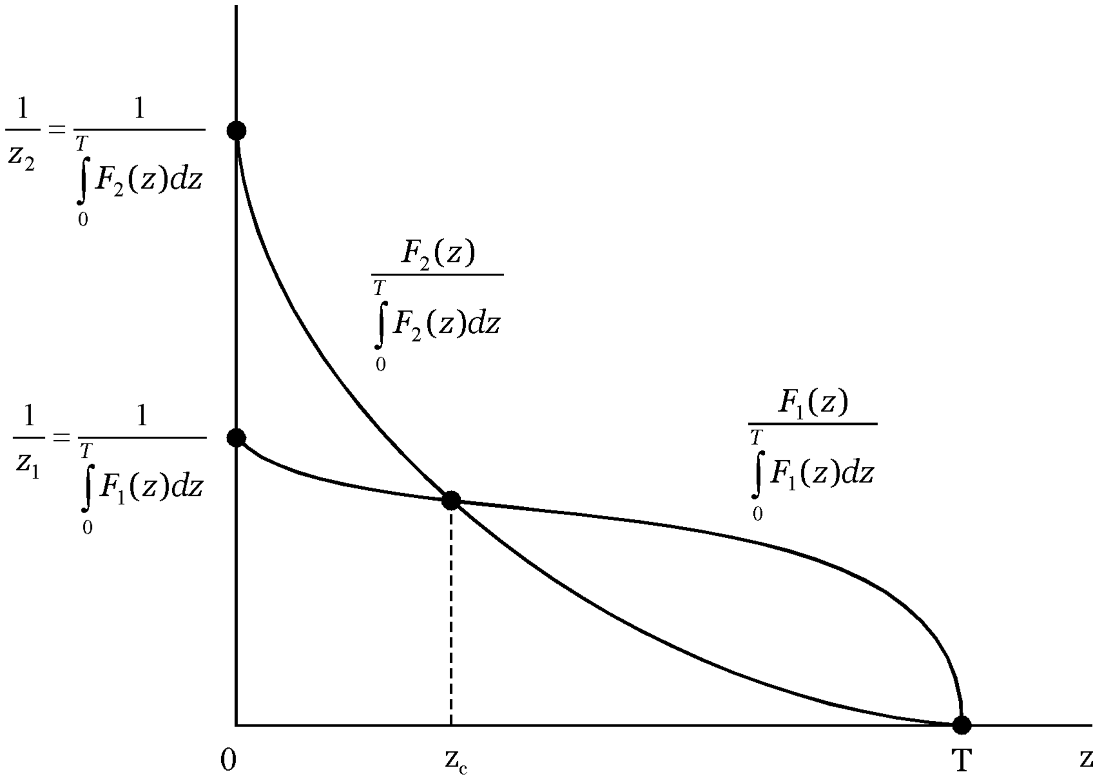
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Figure 1:

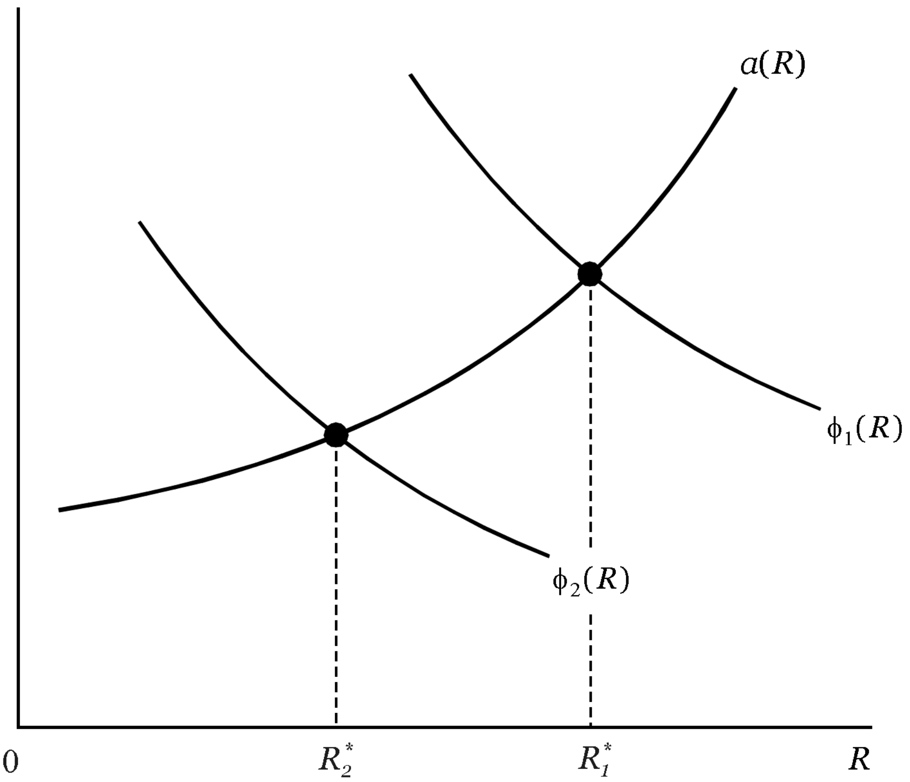

Figure 2:

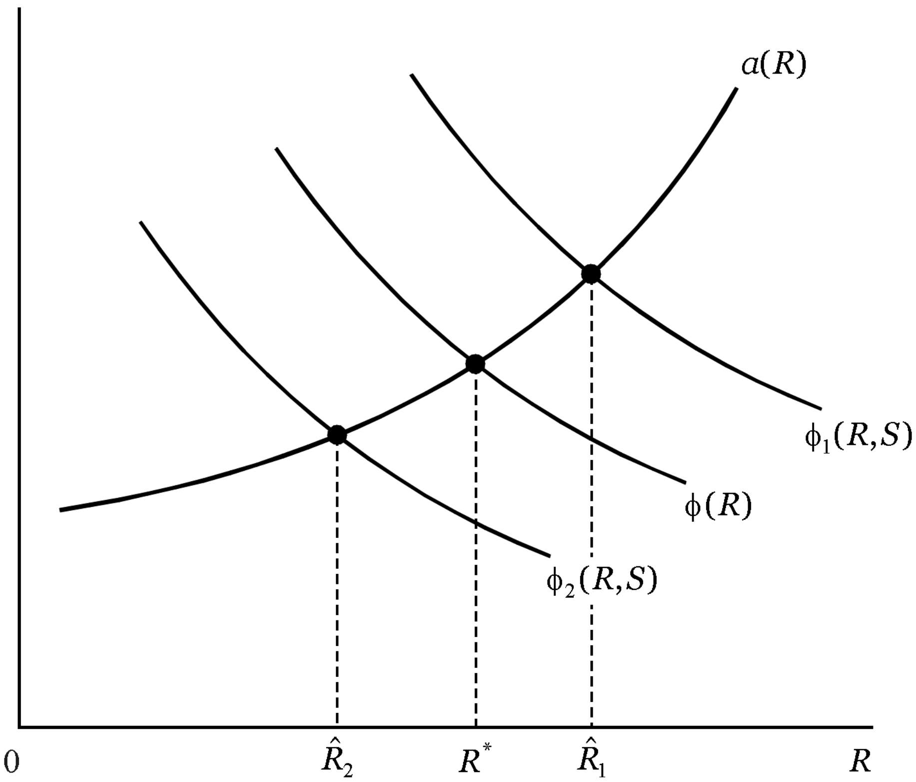

Figure 3: