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NOTE ON THE OPTIMUM PRICING OF ANNUITIES

by

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Note on the Optimum Pricing of Annuities

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Abstract

In a perfectly competitive market for annuities with full information, the price of annuities is equal to individuals (discounted) survival probabilities. That is, prices are *actuarially fair*. In contrast, the pricing implicit in social security systems invariably allows for cross subsidization between different risk groups (males/females). We examine the utilitarian approach to the optimum pricing of annuities and show how the solution depends on the joint distribution of survival probailities and incomes in the population.

1 Introduction

In a perfectly competitive market for annuities with full information, the price of annuities is equal to individuals (discounted) survival probabilities. That is, prices are *actuarially fair* . In contrast, the pricing implicit in social security systems invariably allows for cross subsidization between different risk groups, implying transfers from high to low risk individuals. For example, most social security systems provide the same bene ts to males and females of equal age with equal income and retirement histories in spite of the higher life expectancy of females.[1]

We want to examine the utilitarian approach to this issue using the theory of optimum commodity taxation. Consider a population that consists of H

[1] Further subsidization is provided when females are allowed to retire earlier.

individuals. Denote the expected utility of individual h by $U_h, h = 1, 2, ..., H$. Utilitarianism attempts to maximize a social welfare function, W, which depends on the U_h s:

$$W = W\left(U_1, U_{2,...}, U_H\right). \qquad (1)$$

W depends positively on, and is assumed to be symmetric in, the U_h s.

Each individual lives for either one or two periods, and individuals differ in their survival probabilities. Let p_h be the probability that individual h lives for two periods; c_{1h} be consumption of individual h in period 1 and c_{2h} be consumption of individual h in period 2, if he or she is living then. Utility derived from consumption, $c > 0$, by any individual in any period during life is $u(c)(> 0)$. It is the same in either period so there is no *time preference.* When not alive, utility is 0. Expected utility of individual h is thus

$$U_h = u\left(c_{1h}\right) + p_h u(c_{2h}). \qquad (2)$$

The economy has a given amount of resources, R, which can be used in either period and they can be carried forwards without any gain or loss. With a large number of individuals, expected consumption in the two periods must therefore equal the given resources:

$$\sum_{h=1}^{H} c_{1h} + \sum_{h=1}^{H} p_h c_{2h} = R \qquad (3)$$

Maximization of (1) s.t. (3) yields the condition that consumption is equal in both periods, $c_{1h} = c_{2h} = c_h$, for all $h = 1, 2, ..., H$. Consequently, expected utility, (2), becomes $U_h = u\left(c_h\right)\left(1 + p_h\right)$ and the resource constraint, (3), is $\sum_{h=1}^{H} c_h(1 + p_h) = R$. The *First-Best* optimum allocation of consumption, c_h, among individuals depends on the welfare function as given by the F.O.C.

$$\frac{\partial W}{\partial U_h} u'(c_h) = \text{ constant, } \textit{for all } h = 1, 2, ..., H. \qquad (4)$$

For the case of an additive W, i.e. the sum of expected utilities, condition (4) implies equal consumption for all: $c_h = c; h = 1, 2, ..., H$.

Optimum consumption, satisfying (3) and (4), can be supported by a competitive annuity market accompanied by an optimum income distribution. In a competitive (full information) market with a zero rate of interest, annuities, i.e. second period consumption, are priced by survival probabilities. Individuals maximizing expected utility subject to a budget constraint

$$c_{1h} + p_h c_{2h} = y_h \qquad h = 1, 2, ..., H, \tag{5}$$

where y_h is individual h s income, choose $c_{1h} = c_{2h} = c_h = \frac{y_h}{1+p_h}$. There is typically a unique allocation of incomes that supports the *First-Best*, condition (4). In particular, with additive W, the optimum y_h are proportional to p_h:

$$y_h = \frac{1 + p_h}{\sum_{h=1}^{H} (1 + p_h)} \cdot R \tag{6}$$

Accordingly, optimum consumption is equal in all periods and for all individuals: $c_{1h} = c_{2h} = \frac{R}{\sum_{h=1}^{H}(1+p_h)}$; $h = 1, 2, ..., H$. Optimum expected utility, on the other hand, increases with p_h: $U_h = u\left(\frac{R}{\sum_{h=1}^{H}(1+p_h)}\right)(1 + p_h)$. Hence, the utilitarian *First-Best* optimum has inequality in expected utilities and equality in consumption levels, as pointed out by Arrow (1992). This result is similar to Mirrlees optimum income tax model (1971) where individuals differ in their productivity.[2] Maximization of the sum of utilities leads to a First Best allocation that provides higher (expected) utility to those with a higher capacity to produce utility. This result carries over for all general concave (*egalitarian*) welfare functions.[3]

2 Optimum Pricing of Annuities

Governments cannot engage in unconstrained lump-sum redistributions of incomes. In contrast, most annuities are supplied directly by government-run

[2] In Mirrlees model with additive utilities, the *First-Best* has all individuals with equal consumption and those with higher productivity, having a lower disutility for generating *income,* are assigned to produce a higher income and hence have a lower utility.

[3] In the extreme case, with $W = Min\,[U_1, U_2, ..., U_H]$, expected utilities are equalized, i.e. $U_h = u\left(\frac{y_h}{1+p_h}\right)(1 + p_h) =$ constant for all $h = 1, 2, ..., H$. Since utility is concave, this implies that $c_h = \frac{y_h}{1+p_h}$ and y_h decrease with p_h.

social security systems and taxes/subsidies can be applied to annuity prices offered by private pension funds. Prices of annuities can thus be used by governments to improve social welfare. Although deviations from actuarially fair prices entail distortions (i.e., efficiency losses), distributional improvements may outweigh the costs.

Suppose individual h purchases annuities at a price of q_h. With an income y_h, his or her budget constraint is

$$c_{1h} + q_h c_{2h} = y_h, \qquad h = 1, 2, ..., H \tag{7}$$

Maximization of (2) subject to (7) yields demands $\hat{c}_{ih} = \hat{c}_{ih}(q_h, p_h, y_h)$, $i = 1, 2$, and $h = 1, 2, ..., H$. Maximized expected utility, $\hat{U}_h$, is $\hat{U}_h\ (q_h, p_h, y_h) = u(\hat{c}_{1h}) + p_h u(\hat{c}_{2h})$.

Assume that total subsidies/taxes on annuities must equal zero,

$$\sum_{h=1}^{H} (q_h - p_h)\, \hat{c}_{2h} = 0 \tag{8}$$

Maximization w.r.t prices $(q_1, ..., q_H)$of $W\left(\hat{U}_1, \hat{U}_{2,}..., \hat{U}_H\right)$ subject to (8) yields F.O.C.

$$\frac{\partial W}{\partial \hat{U}_h}\frac{\partial \hat{U}_h}{\partial q_h} + \lambda\left[\hat{c}_{2h} + (q_h - p_h)\frac{\partial \hat{c}_{2h}}{d q_h}\right] = 0, \qquad h = 1, 2, ..., H, \tag{9}$$

where $\lambda > 0$ is the shadow price of constraint (8). In elasticity form, using Roy s identity $\left(\frac{\partial \hat{U}_h}{\partial q_h} = -\frac{\partial \hat{U}_h}{\partial y_h}\hat{c}_{2h}\right)$, (9) can be written

$$\frac{q_h - p_h}{q_h} = \frac{\theta_h}{\varepsilon_h} \tag{10}$$

where $\varepsilon_h = -\frac{q_h}{\hat{c}_{2h}}\frac{\partial \hat{c}_{2h}}{\partial q_h}$ is the *price elasticity* of second period consumption of individual h, and $\theta_h = 1 - \frac{1}{\lambda}\frac{\partial W}{\partial \hat{U}_h}\frac{\partial \hat{U}_h}{\partial y_h}$ is the *net* social value of a marginal transfer to individual h through the optimum pricing scheme. Equation (10) is a variant of the well-known *inverse-elasticity* optimum tax formula which combines equity (θ_h) and efficiency $\left(\frac{1}{\varepsilon_h}\right)$considerations.

The implication of (10) for the optimum pricing of annuities depends on the welfare function, W, and on the joint distribution of incomes, $(y_1, ..., y_H)$, and probabilities $(p_1, ..., p_H)$.

For concreteness, let W be the sum of expected utilities. Then $\frac{\partial W}{\partial \hat{U}_h} = 1, h = 1, 2, ..., H$. Assume further that $U_h = \ln c_{1h} + p_h \ln c_{2h}$. Then,

$$\hat{c}_{1h} = \frac{y_h}{1+p_h} \quad ; \quad \hat{c}_{2h} = \frac{y_h}{1+p_h}\frac{p_h}{q_h} \tag{11}$$

and

$$\hat{U}_h = (1+p_h)\ln\left(\frac{y_h}{1+p_h}\right) + p_h \ln\left(\frac{p_h}{q_h}\right) \tag{12}$$

Conditions (10) and (8) now yield the solution

$$q_h = \phi\left(\frac{\beta_h}{\sum_{h=1}^{H}\beta_h}\right),$$

$$\textit{where } \phi = \sum_{h=1}^{H} p_h > 0 \textit{ and } \beta_h = \frac{p_h y_h}{1+p_h} > 0 \tag{13}$$

Consider two special cases of (13):

(a) _Equal Incomes_: $(y_h = y = \frac{R}{H}; h = 1, ..., H.)$

Condition (13) becomes $q_h = \overline{\phi}\left(\frac{p_h}{1+p_h}\right)$, where
$\overline{\phi} = \frac{\sum_{h=1}^{H} p_h}{\sum_{h=1}^{H}\left(\frac{p_h}{1+p_h}\right)} \; (> 1)$.

It is seen (Figure 1) that optimum pricing involves subsidization (taxation) of individuals with high (low) survival probabilities.[4]

(b) $\underline{y_h = y\,(1+p_h)}$

This, one recalls, is the First-Best utilitarian income distribution and since all price elasticities are equal to unity, we see from (13), as expected, that $q_h = p_h$, i.e., efficiency prices.

[4]For Figure 1, it can be shown that $\frac{\overline{\phi}}{2} < 1$.

More generally, it is seen from (13) that a higher correlation between incomes, y_h, and survival probabilities, p_h, decreases - and possibly eliminates - the subsidization of high survival individuals. In contrast, a negative correlation between incomes and survival probabilities (as, presumably, in the female/male case) leads to subsidies for high survival individuals, possibly to the commonly observed uniform pricing rule.

References

[1] Arrow, K.J., (1992), Sex Differentiation in Annuities: Re ections on Utilitarianism and Inequality , in R. Selten (ed.) *Rational Interactions (*Springer), pp.333-336

[2] Mirrlees, J.A. (1971), An Exploration in the Theory of Optimal Income Taxation , *Review of Economic Studies*, 38 (114), April, pp. 175-208.

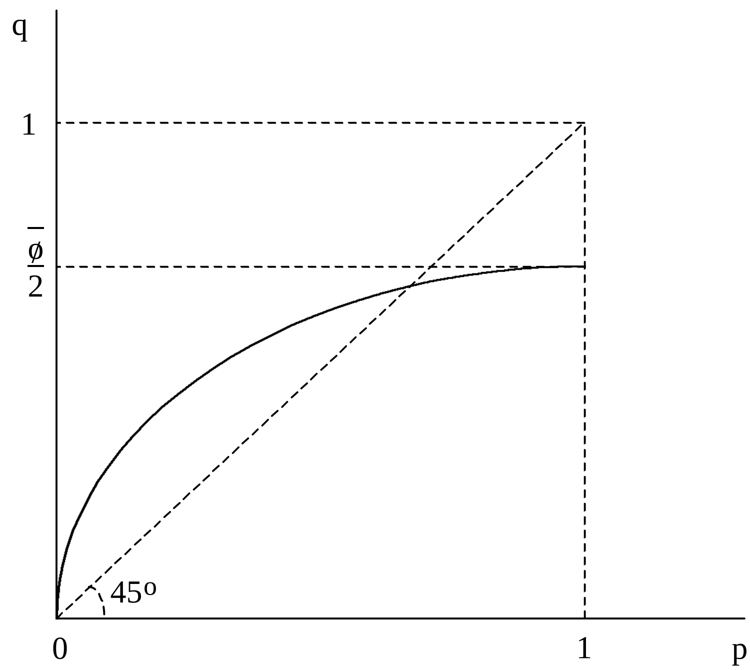

Figure 1