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DISSOLVING A COMMON VALUE PARTNERSHIP IN A REPEATED 'QUETO' GAME

by

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Dissolving a common value partnership in a repeated 'queto' game*

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Abstract

We analyse a common value, alternating ascending bid, first price auction as a repeated game of incomplete information where the bidders hold equal property rights to the object auctioned off. Consequently they can accept (by *quitting*) or *veto* any proposed settlement. We characterise the essentially unique, sequentially rational dynamic Bayesian Nash equilibrium of this game under incomplete information on one side and discuss its properties. (JEL *C7,D82,D44,J12.* Keywords: *Repeated games, Incomplete information, Common value auctions, Partnership dissolution.*)

Introduction

We analyse a common value, alternating ascending bid, first price auction as a repeated (dynamic) game of incomplete information. We discuss a setting where two players hold *equal* ownership titles to a single indivisible object—the partnership—which has the same objective value to both. These ownership or property rights give the players veto power over any settlement they oppose. The value of the object is decided initially through a casting move by Nature who draws the common value from a publicly known distribution on some known support. Subsequently both players are sent private signals fully informing player one (P1) and giving no additional information to player two (P2). We always normalise the lower value to zero and start the auction at the reservation price of zero—hence we exclude the fraction of the object's value over which players do not hold conflicting views from our deliberation process. P2 is given initial *possession*—which is defined as ownership conditional on the partner not exercising a veto—and P1 starts bidding. P2 accepts any positive initial offer for the object, and keeps it if no such offer realises—in which latter case the game is over. The same rules apply to any successive period of the game. Any bid strictly higher than the previous bid leads to transferral of possession to the higher bidder. Bids at or below the current price end the

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game. We call this game a ‘queto’ game because, each period, the moving player has the choice between *quitting* the auction and accepting the current offer and *vetoing* the proposed sharing rule and bidding the necessary compensation up.

Apart from the above used auction-interpretation, the model can be interpreted in at least three different ways. The most general is that of a *repeated game of incomplete information* in the general framework developed by Mertens, Sorin, and Zamir (1994). In this context, the stage game is of extensive-form and spans two periods, is repeated indefinitely and has the characteristics of a constant-sum game. The game is *dynamic* because stage actions and payoffs generally dependent on the history of play and on private information. The third interpretation is that of a particular *bargaining* situation where parties bargain over sharing a commonly valued pie and make payments every period. Here, exiting means agreeing to the proposed sharing rule (the current price) while bidding up is to propose a different rule.

We offer two alternative motivations for the study of this setup. The first is a *partnership dissolution* game, where two parties bargain over the dissolution of a jointly owned and commonly valued partnership over the value of which the partners are asymmetrically informed. There is a major cost which players incur if not dissolving the partnership which we assume throughout the paper to be sufficiently high for both players to participate. Players alternatingly offer strictly increasing payments to their opponents in exchange for full ownership of the partnership. If such a bid is accepted, the accepting party exits, gets the final bid and thus ends the game while the winner of the auction gets the partnership. If the offer is not accepted, the declining party has to submit a strictly higher offer to the opponent. This is the motivation we elaborate on in the remainder of the paper. As an alternative interpretation we propose the short-run trade on an idealised *stock market* where traders simultaneously submit bids every period in an one-shot auction. Each period, the highest bidder obtains possession of the asset for payment of his bid to last period’s owner. Only when the share is issued does trading involve the owner of the underlying asset—all subsequent trades of the share are between traders. All traders know the current price and the resell value at is the common value of the object traders are asymmetrically informed about. In this interpretation, the key assumption is that a player can only realise the resell value if there is ongoing trade on this asset—if a player does not offer the object for resale but keeps it, the resell value vanishes.

We will look for a sequentially rational, dynamic Bayesian Nash equilibrium.[1] Existence of equilibria in repeated games of incomplete information on one side with discontinuous payoffs in strategy space—such as ascending first price auctions—is a consequence of the more general arguments made by Mertens, Sorin, and Zamir (1994, chp. IV) and Simon, Spież, and Toruńczyk (1995). The plan of the paper is to present the model and illustrate it with some examples. After a short discussion of the properties of the equilibrium in these examples we present the general results in the last section.

[1] For the definition and details consult Geanakoplos (1994).

Literature

Much of the literature on the sharing of an indivisible object to which all participants have titles comes under the heading of 'dissolving a partnership'. The area of applicability is, however, much wider and includes divorce settlements, out-of-court resolutions of legal disputes, many public good cases, and—more generally—any division of an asset among holders of property rights. The key contributions use a one-shot mechanism design approach and analyse the set of individually rational and incentive compatible mechanisms mainly in terms of efficiency. This approach was pioneered by Cramton, Gibbons, and Klemperer (1987) whose main conclusion is that—with private valuations—efficient trade can be achieved if both parties initially own significant shares of the partnership. Neeman (1999) studies a private value, public good case and Jehiel and Pauzner (2002) allow for interdependent values and conclude that there is a wide class of situations where efficient trade cannot be reached. They characterise a second-best solution and provide comparative statics on the partners' expected welfare with respect to the ex-ante property rights structure. Kittsteiner (2000) bridges the gap to the dynamic auctions literature and the idea of his interdependent value version of the 'k-double auction with veto' is related to our stage game. A recent survey on the partnership dissolution literature is provided by Moldovanu (2001). A discussion of the bargaining setting is provided by Samuelson (1984) who analyses the case where one side is informed on the private value.

The segment of the theoretical literature which we are most closely related to is that of repeated games with incomplete information. The study of these games was initiated and developed by Aumann and Maschler (1966), reprinted in Aumann and Maschler (1994), who also establish the existence of the value. It saw numerous contributions among which several of the most notable are by the authors of Mertens, Sorin, and Zamir (1994) who also provide a comprehensive survey. This literature, however, is typically concerned with the study of uniform or discounted long-run payoffs which do not arise naturally in our case. The structure of our game is similar to the quitting games discussed in Vieille and Solan (2001). Their results, however, apply to complete information stochastic games.[2]

1 The model

We first define the general model for arbitrary settings of informational incompleteness. The set of players—the partners—is denoted by $N = \{1, 2\}$. Players jointly own the object in *equal* shares and are not wealth constrained. Players are *risk-neutral* and final payoffs are given by the *undiscounted sum* of stage payoffs u^t. We denote the *common value* of the indivisible object to be shared—the partnership—by $\theta \in \Theta \equiv \{\underline{\theta}, \bar{\theta}\}$. The players *types* are the private *signals* $s_i \in S_i$ they receive on this value. We define a *state* of the world as a triple $\omega = (\theta, s_1, s_2) \in$

[2] Vieille and Solan (2001) define quitting games as sequential games in which, at any stage, each player has the choice between continuing and quitting. The game ends as soon as at least one player chooses to quit; this player then receives a payoff which depends on the set of players who did choose to quit. If the game never ends, the payoff to each player is zero.

$(\Theta \times S_1 \times S_2) \equiv \Omega$. There is some publicly known prior (joint) probability distribution φ^0 over Ω which is refined into the player's *beliefs* about the state of the world on the basis of the players' observed behaviour.

Stage *actions* are bids $b_i^t : \{b^\tau\}_{\tau=1}^{t-1} \times S_i \mapsto \Delta(B)$ taking values in the set of possible stage actions B. This set defines the constant minimal bidding increase ν. Bids are transfers (and therefore stage payoffs) to the opponent in exchange for possession of the object. Players can monitor their opponents' actions perfectly well and enjoy perfect recall. A player's repeated game *strategy* $\beta_i(s_i)$ is a complete, contingent plan a profile of which is denoted by $\beta(s)$. The discrete time t *price* of the object p^{t-1} is last period's highest bid, or—since there is only one bid—just b^{t-1}. A *stage game* consists of two alternating bids—P1 bids at odd periods and P2 at even periods t. The repeated queto game is denoted by Γ. The rules of the game are that bids have to be strictly increasing and alternating. Initial possession is given to P2, the informed party P1 makes the first move and the game ends if a player chooses to exit in which case the current possessor becomes owner of the object. If neither player ever exits, the game continues forever.

In the special case of incomplete information on one side considered in the remainder of this paper, the signals fully inform P1 and give no additional information to P2. In this case, the state space Ω reduces to just Θ and the uninformed party's beliefs can be conveniently summarised as $\varphi = \text{prob}(\theta = \bar{\theta})$ where φ^0 is the common prior. This extreme case of incomplete information on one side is sufficient to explain all cases where one party has an informational advantage over the other. A desirable relaxation of the common knowledge of priors assumption and the case of incomplete information on both sides remain for future work.

2 An example

We put a discrete grid on the space of possible bids B—which defines minimal bidding increments ν—and restrict the value space Θ to two elements. For concreteness let $\Theta = \{0, 3\}$ and $B = \mathbb{N}$ (hence $\nu = 1$). The players' private information is given by P1's signal $s_1 = \theta$ and P2's signal $s_2 = \varphi^0 = {}^1\!/_2$.

2.1 Perfect information

In cases where both bidders have perfect information we cannot have a pooling equilibrium. We write an equilibrium (path) for the case of $\theta = 0$ by $(\beta_1^*, \beta_2^*) = (e, e)$, meaning that P1 exits by playing $b_1^* = 0$ at t=1 (or everywhere else) and P2 exits whenever she gets to move.[3] This unique symmetric equilibrium leads to the outcome $(0, 0)$. A symmetric equilibrium for the case of $\theta = 3$ is to always increase the current price by the minimal increase $\nu = 1$ starting from $p^0 = 0$; this yields $(1, 2)$ or $({}^{\theta-1}\!/_2, {}^{\theta+1}\!/_2)$. The game tree for this case is shown in figure 1 below. The precise outcome is determined by θ, of course, and by the choice of the bidding grid. The

[3] We denote a player's generic exit choice by 'e' to cover cases where all $b \in \{0, 1, \ldots, p^{t-1}\}$ lead to exit.

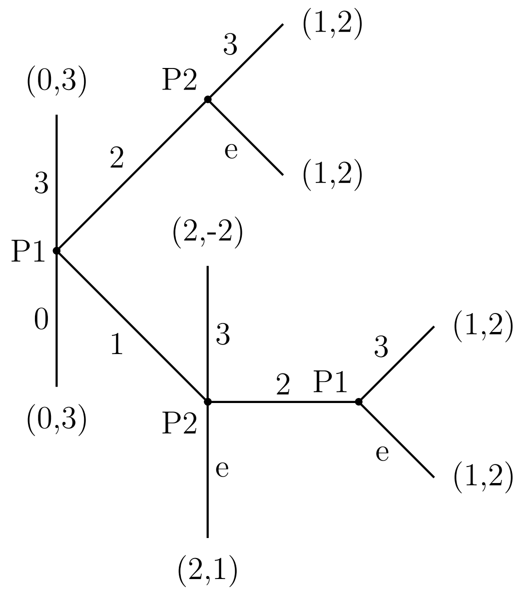

Figure 1: The perfect information game tree for $\theta = 3$.

outcome for a continuous bidding space with $\nu \to 0$ is that both players can achieve close to ${}^{\theta}/_{2}$ and the second-mover advantage vanishes. Since the perfect information case corresponds to the case where P1 fully and immediately reveals the value of the object, it will be the yardstick to measure the success of any partially or gradually revealing strategy below.

2.2 Incomplete information on one side

Now the uninformed party needs to form beliefs about the possible realisations of θ. We can identify a number of fully and immediately revealing (separating) equilibrium candidates corresponding to the above discussion on perfect information. A more interesting, only gradually revealing equilibrium candidate involves P1 using type dependent lotteries at his moves. This equilibrium candidate is

$$\beta_o^* = \left(\left\{ \lambda^t (p^{t-1} + \nu) + (1 - \lambda^t) e \right\}_{t=1}^{\bar{\theta}}, \left\{ \mu^t (p^{t-1} + \nu) + (1 - \mu^t) e \right\}_{t=2}^{\bar{\theta}-1} \right) \tag{2.1}$$

where the increment for t is 2 and λ, μ are probabilities. We will now discuss the sequence of equilibrium play prescribed by β_o^* in our example of $\theta \in \{0, 3\}$.

t=0: Nature decides on θ and sends the signals $s_1 = \theta$ and $s_2 = \varphi^0 = \text{prob}(\theta = \bar{\theta}) = {}^{1}/_{2}$.

t=1: The minimum acceptable bid not ending the game is $p^0 + \nu = 1$. Depending on the true state, P1 uses the type-dependent lottery $b_1^1 = \{\lambda 1 + (1 - \lambda) e\}$ in case of $\theta = 0$ and plays $b_1^1 = 1$ for sure in the case of $\theta = 3$. Therefore, in the case of $\theta = 0$, P1 must be indifferent between his continuation payoff and zero. After observing $b_1^1 = 1$, P2 uses the conditional probability embedded in P1's announced equilibrium lotteries to compute her posterior. The t=1 lottery is

prob(b_1^1 |	$\theta = 0$,	$\theta = 3$)
0	$1-\lambda$	0
≥ 1	λ	1

which induces P2 to use Bayes' rule to revise her prior $\varphi^0 = {}^1\!/_2$ to

$$\text{prob}(\theta = 4|b_1^1 = 1) = \frac{\text{prob}(b_1^1 = 1|\theta = 3)\,\text{prob}(\theta = 3)}{\text{prob}(b_1^1 = 1)} = \frac{1}{1+\lambda} = \varphi. \tag{2.2}$$

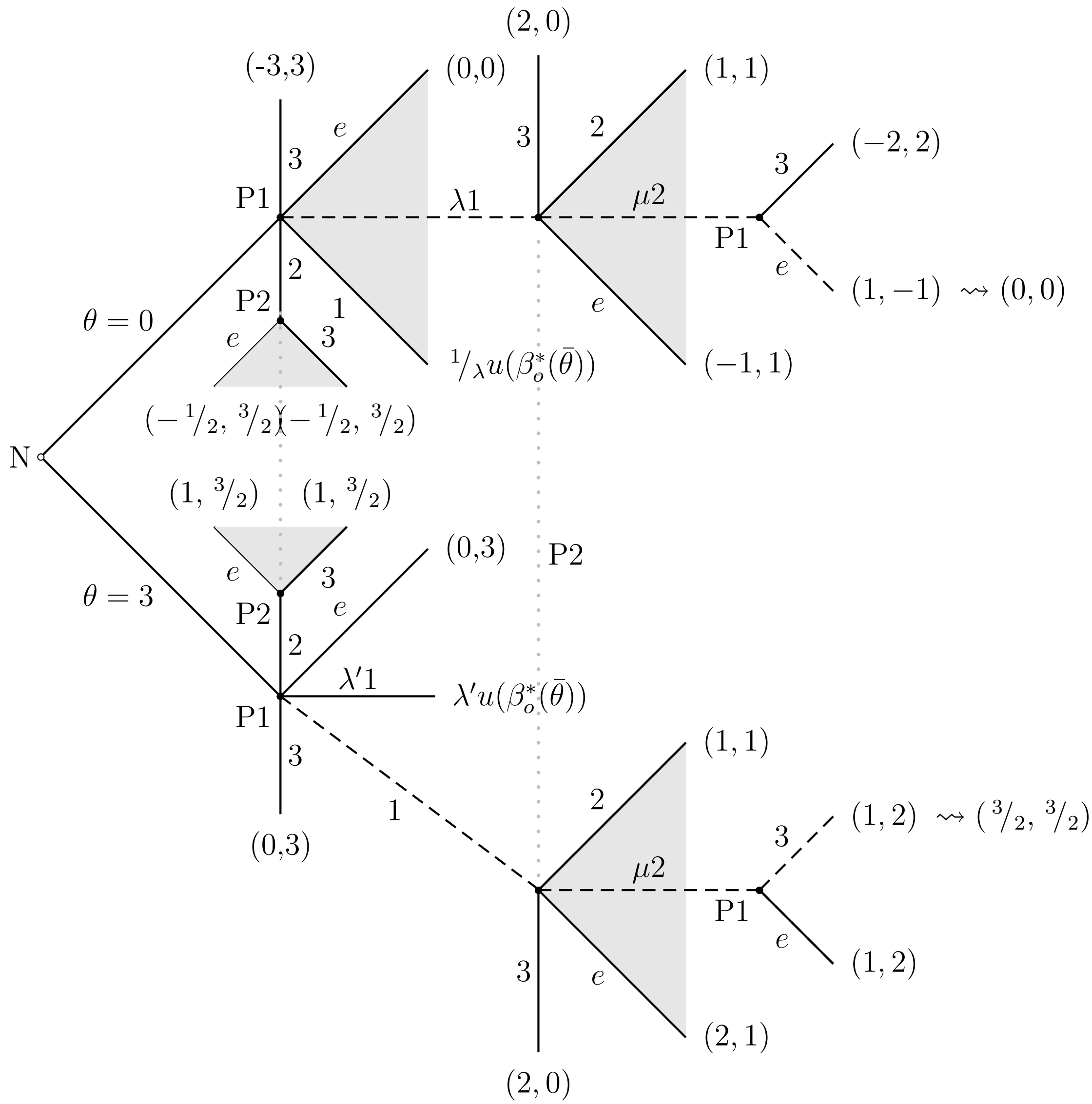

Figure 2: Possible deviations in the one-sided incomplete information example with $\theta \in \{0, 3\}$.

t=2: The minimum acceptable bid not ending the game is $p^1 + \nu = 2$. Given the posterior φ, P2 will play the mixed action

$$b_2^2 = \mu 2 + (1-\mu)e$$

for any $\mu \in [0, 1]$, because—through the appropriately chosen signal λ—she is made indifferent between zero (the outcome in case of exiting) and $\text{E}[\varphi(1) + (1-\varphi)(-2)]$ (the continuation outcome). In particular she is willing to play $\mu = {}^1\!/_2$ which makes P1 indifferent between exiting and bidding 1 in the low-value branch as required. For (λ, ϕ) to be optimal they have to fulfill

$$0 = 2\varphi + (1 - \varphi)(-1) \tag{2.3}$$

in addition to φ being generated from the application of Bayes' rule (2.2).

This implies $(\lambda^*, \varphi^*) = ({}^1\!/_2, {}^2\!/_3)$.

t=3: Given the bid b_2^2, P1 finds it optimal to play $b_1^3 = e$ in case of $\theta = 0$ and $b_1^3 = 3$ in the case of $\theta = 3$. He thus reveals the value of θ at final stage.

We use figure 2 to check for profitable deviations and find none.[4] This completes the argument to show that (2.1) is indeed an equilibrium of the game with one sided incomplete information leading to the outcomes

	P1	P2
$\mathrm{E}[u(\beta^*_{o3}(s\vert\underline{\theta}))]$	0	0
$\mathrm{E}[u(\beta^*_{o3}(s\vert\bar{\theta}))]$	${}^3\!/_2$	${}^3\!/_2$
$\mathrm{E}[u(\beta^*_{o3}(s))]$	${}^3\!/_4$	${}^3\!/_4$

Some reflection shows that there cannot be equilibria involving $\lambda' = \text{prob}(b_1^1|\bar{\theta}) < 1$ because (i) a mixture $\lambda'(b_1^1 = 1) + (1 - \lambda')(b_1^1 = 0)$ would result in P1's payoff of $\lambda'(2 - \mu) + (1 - \lambda')0$ which can only be optimal for $\lambda' = 1$, and (ii) a mixture $\lambda'(b_1^1 = 1) + (1 - \lambda')(b_1^1 = 2)$ would result in a payoff of 1 with certainty which is also suboptimal. Hence there are no partially revealing equilibria not involving the action $b_1^1 = 1$ in the high value case.

We interpret β_o^* such that P1 is using the type dependent lottery $b_1^1 = \lambda 1 + (1 - \lambda)0$ in the low-value case in order to induce P2 not to exit but to continue bidding up with some positive probability in the next stage. This necessitates to balance the mixture probability λ with the posterior φ that probability induces. On the other hand P1 profits from inducing P2 to exit early in the high-value branch since then his payoffs are higher than what he can obtain from just splitting the value.

The example readily extends to more general values and finer bidding grids as long as the equivalent of (2.3) and the condition on φ being derived using Bayes' rule are not violated. For example it is tedious but straightforward to check that the case of $\theta \in \{0, 5\}$ has the equilibrium path

$$\begin{aligned}\beta^*_{o5} \;=\; & ((\{{}^7\!/_8 1 + (1 - {}^7\!/_8)e\}, \{{}^2\!/_7 3 + (1 - {}^2\!/_7)e\}), \\ & (\{{}^1\!/_2 2 + (1 - {}^1\!/_2)e\}, \{{}^3\!/_4 4 + (1 - {}^3\!/_4)e\}))\end{aligned}$$

with

	P1	P2
$\mathrm{E}[u(\beta^*_{o5}(s\vert\underline{\theta})]$	0	0
$\mathrm{E}[u(\beta^*_{o5}(s\vert\bar{\theta})]$	$3\,{}^1\!/_8$	$1\,{}^7\!/_8$
$\mathrm{E}[u(\beta^*_{o5}(s)]$	$1\,{}^9\!/_{16}$	${}^{15}\!/_{16}$

[4] Vertices on the equilibrium path are dashed. The greyed areas symbolise the range of mixed strategies.

3 Results

Assumption 1. *Bids b_i, the minimal constant bidding increment ν, and the possible values $\theta \in \left\{\underline{\theta}, \bar{\theta}\right\}$, $\underline{\theta} \leq \bar{\theta}$, take only finite values in $\mathbb{N}$.*[5]

3.1 Perfect information

Proposition 1. *Under perfect information and assumption 1, the queto game Γ_p with common value $\theta \geq 2$ has the pure strategy subgame perfect equilibrium*

$$\beta_p^*(s) = \begin{cases} \left(\{p^{t-1}+\nu\}_{t=1}^{\theta}, \{p^{t-1}+\nu\}_{t=2}^{\theta-1}\right) & \text{for odd } \theta, \\ \left(\{p^{t-1}+\nu\}_{t=1}^{\theta-1}, \{p^{t-1}+\nu\}_{t=2}^{\theta}\right) & \text{otherwise} \end{cases} \tag{3.1}$$

for t-increment of 2, leading to the outcome (which is Pareto-dominating among all equilibria)

$$u(\beta_p^*) = \begin{cases} \left(\frac{\theta-\nu}{2}, \frac{\theta+\nu}{2}\right) & \text{for odd } \theta, \\ \left(\frac{\theta}{2}, \frac{\theta}{2}\right) & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. It is easy to see that every premature exit incurs a weakly lower payoff for the player who exits—hence such deviations are dominated. Upward deviations ('jump' bids) by i leave the accumulated net payoffs unchanged at stages where i moves while transferring the extent of the relative jump to the opponent in states where $j = 3 - i$ moves. Since the game is constant θ-sum, this cannot increase i's payoffs.

Since we are in a multistage game of perfect information, every information set starts a subgame of Γ_p. Each subgame following a deviation (i.e. all subgames off the equilibrium path) is either already contained in identical form in the equilibrium path or with the deviatior's payoffs uniformly reduced by the gap between equilibrium bid and jump bid and the deviator's opponent's payoffs uniformly enlarged by the same gap. Thus the strategic situation following a deviation is identical to that on some part of the equilibrium path. Hence the equilibrium actions prescribe an equilibrium in any subgame and β_p^* is subgame perfect. □

Corollary 1. *Every fully revealing strategy can—after revelation—achieve at best (3.2).*

3.2 Incomplete information on one side

Assumption 2. *The conditional probabilities* $\text{prob}(b_1^t|\theta)$, $t \geq 1$ *are defined as*

$\text{prob}(\ b_1^t \mid$	$\underline{\theta},$	$\bar{\theta}\ )$
e	$1-\lambda^t$	0
$b_1^t > p^{t-1}$	λ^t	1

[5] Since some results below require $\nu \to 0$, it is occasionally more convenient to normalise the high value to 1 and define $\nu = {}^{1}/_{\bar{\theta}}$.

We refer to the queto game with one sided incomplete information under assumptions 1–2 as Γ_o. For an initial price $p^0 = 0$, the proposed equilibrium of Γ_o is $\beta_o^*(s) =$

$$\begin{cases} \left(\{(p^{t-1}+\nu)\lambda^t + e(1-\lambda^t)\}_{t=1}^{\bar{\theta}}, \{(p^{t-1}+\nu)\mu^t + e(1-\mu^t)\}_{t=2}^{\bar{\theta}-1}\right), \text{ for odd } \bar{\theta} > 2, \\ \left(\{(p^{t-1}+\nu)\lambda^t + e(1-\lambda^t)\}_{t=1}^{\bar{\theta}-1}, \left(\{(p^{t-1}+\nu)\mu^t + e(1-\mu^t)\}_{t=2}^{\bar{\theta}-2}, e\right)\right), \text{ for even } \bar{\theta} \geq 2, \end{cases} \tag{3.3}$$

for t-increment of 2, where both players react to every deviation by playing their equilibrium actions—based on the observed play and P2 updating her beliefs in accordance with assumption 2—and all mixture probabilities fulfill the below equilibrium conditions (1–3).

In the opening game, P1 decides whether to signal in the $\underline{\theta}$-branch or not. He will signal at t provided that P2's signal-less expected payoff before t is not higher than what she is led to believe will be her payoff at t+1 after observing his signal and updating her prior on θ. If this is indeed the case, P2 will not deviate from β_o^* and, moreover, she will be willing to play a mixed action at t+1 provided that her continuation payoff equals what she can secure there. Since β_o^* assures her precisely this payoff, she will mix with probability μ^{t+1} between exiting and continuing by increasing the current price by the minimal bid ν. Both players repeat these mixed actions in turn until the final stage of the game where P1 exits for sure in the low-value branch and bids $\bar{\theta}$ otherwise. This endgame is shown in figure 3.

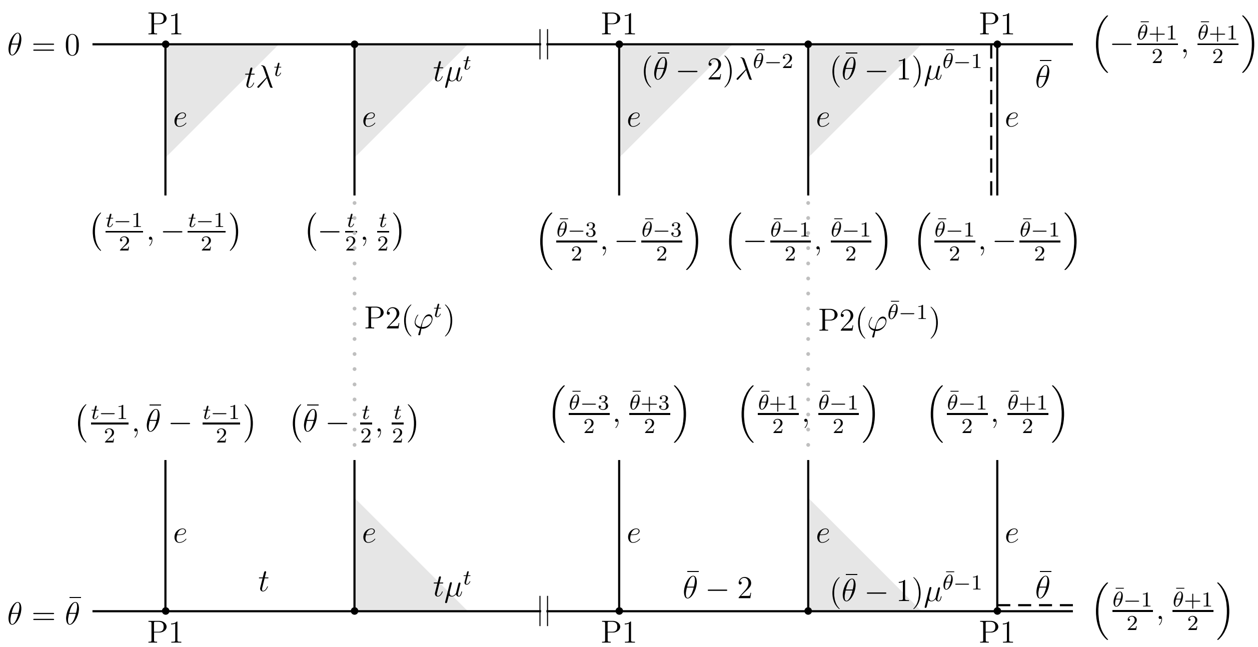

Figure 3: The endgame for odd $\bar{\theta}$ according to $\beta_o^*(\theta)$.

The equilibrium conditions which have to be fulfilled each period are:

1. μ^t (even t): P1 mixes with any λ^{t-1} iff $\mathrm{E}[u_1^t(\beta_o^*(s|\underline{\theta}))] = u_1^{t-1}(e^*)$, or[6]

$$(1-\mu^t)u_1^t(e^*) + \mu^t\, \mathrm{E}[u_1^t(\beta_o^*(s|\underline{\theta}))] = u_1^{t-1}(e^*) \quad \Rightarrow \mu^t = \frac{p^{t-1}}{p^{t-1}+\nu};$$

2. φ^t (even t): P2 mixes with any μ^t iff $\mathrm{E}^t[u_2^{t+1}(\beta_o^*(s))] = u_2^t(e^*)$, or

$$(1-\varphi^t)\left[(1-\lambda^{t+1})u_2^{t+1}(e^*|\underline{\theta}) + \lambda^{t+1}\, \mathrm{E}^t[u_2^{t+2}(\beta_o^*)]\right] + \varphi^t\, \mathrm{E}^t[u_2^{t+2}(\beta_o^*)] = u_2^t(e^*)$$

where φ^t evolves according to Bayes' rule

$$\varphi^t = \frac{1\varphi^{t-2}}{1\varphi^{t-2} + \lambda^{t-1}(1-\varphi^{t-2})} \quad \text{implying } \varphi^{\bar{\theta}-1} = \frac{\bar{\theta}-\nu}{\bar{\theta}} \text{ and } \lambda^{\bar{\theta}} = 0, \text{ in particular}$$

3. and payoffs at any t along the equilibrium path are given (for ν=1) by:

$$\begin{array}{lll} \text{for odd } t: & u_2^t(e^*) = \theta - \frac{t-1}{2}, & u_1^t(e^* \vee b^*) = \frac{t-1}{2}, \\ \text{otherwise:} & u_1^t(e^*) = \theta - \frac{t}{2}, & u_2^t(e^* \vee b^*) = \frac{t}{2}. \end{array}$$

Definition 1. *An equilibrium is called* essentially unique *if it is unique up to the final stage of the quitting game Γ_o, but has an arbitrary final action by P1 for odd $\theta = \bar{\theta}$.*

Theorem *The equilibrium (3.3) is essentially unique and sequentially rational in the quitting game Γ_o. For small ν, its outcome converges to*

$$\mathrm{E}[u(\beta_o^*(s|\bar{\theta}))] = \left({}^{5\bar{\theta}}/_8,\, {}^{3\bar{\theta}}/_8\right), \quad \mathrm{E}[u(\beta_o^*(s|\underline{\theta}))] = \left({}^{\bar{\theta}}/_8,\, -{}^{\bar{\theta}}/_8\right), \quad \mathrm{E}^0[u(\beta_o^*(s))] = \left({}^{3\bar{\theta}}/_8,\, {}^{\bar{\theta}}/_8\right).$$

This summarises our main result which we will establish in the remainder of this section.

Lemma 1. *If (3.3) is an equilibrium of Γ_o, then P1 starts signalling at period $t_s = {}^{\bar{\theta}}/_4 - \nu$.*

Proof. Denote the period where P1 starts signalling by t_s and the associated equilibrium candidate (where P1 signals at t_s) by $\hat{\beta}$. We only consider the case where $\bar{\theta}$ is odd but a similar argument applies for even $\bar{\theta}$. In equilibrium, t_s is the first period where P2 can be made indifferent between her current $t < t_s$ sure payoff and her belief-induced payoffs at $t > t_s$.[7] This amounts to the condition

$$u_2^{t_s+1}(e^*) = \varphi^{t_s+1}\, \mathrm{E}\left[u_2^{t_s}(\hat{\beta}(s|\bar{\theta}))\right] + (1-\varphi^{t_s+1})u_2^{t_s}(e^*) \tag{3.4}$$

where $\varphi^{t_s+1} \geq {}^1/_2$ and P2's expected payoff from $\hat{\beta}$, given $\theta = \bar{\theta}$, is defined recursively as

$$\mathrm{E}[u_2^{t_s}(\hat{\beta}(s|\bar{\theta}))] = (1-\mu^{t_s+1})u_2^{t_s+1}(e^*) + \mu^{t_s+1}\left((1-\mu^{t_s+3})u_2^{t_s+3}(e^*) + \mathrm{E}[u_2^{t_s+3}(\hat{\beta}(s|\bar{\theta}))]\right).$$

[6] To avoid going overboard on the notation we write the payoff following $\beta_o^*(s)$ but exiting at t by $u^t(e^*)$.

[7] Notice that there may be beliefs smaller than ${}^1/_2$ which would sustain the equilibrium with earlier signalling but P1 has no way of inducing such beliefs using his equilibrium strategy.

As it turns out, it is more convenient to write the above as a sum of the form

$$\mathrm{E}[u_2^{t_s}(\hat{\beta}(s|\bar{\theta}))] = \sum_{\tau=1}^{\bar{\theta}-1}\left(\prod_{t=1}^{\tau-1}\mu^{2t}\right)u_2^{2\tau}(e^*)(1-\mu^{2\tau}) + \left(\prod_{t=1}^{\bar{\theta}+1}\mu^{2t}\right)u_2^{\bar{\theta}}(e^*) \tag{3.5}$$

where P2's stage payoffs—following our equilibrium conditions—are defined as $u_2^t(e^*) = \frac{\nu t}{2}$ and $u_2^{\bar{\theta}}(e^*) = \frac{\bar{\theta}+\nu}{2}$. Plugging these into (3.4) for the minimal posterior belief $\varphi^{t_s+1} = {}^1/_2$ and rearranging terms results in a condition where t_s is the earliest period in which

$$t_s \leq \frac{2}{3}\,\mathrm{E}[u_2^{t_s}(\hat{\beta}(s|\bar{\theta}))] - \nu$$

is true. For $\nu \to 0$, (3.5) converges to ${}^{3\theta}/_8$ and hence $t_s = {}^{\theta}/_4$ as required. $\square$

Lemma 2. *Expected payoffs* $\mathrm{E}[u_i^t(\beta_o^*)]$, $i = \{1, 2\}$, *from playing* Γ_o *are monotonic in* t.

Proof. Let $\nu = 1$ and consider P1. P1's payoffs are constant (and equal to $u_1^{t_s}(e^*)$) in case of $\underline{\theta}$. Payoffs along the $\bar{\theta}$-branch are given at t by

$$\mathrm{E}[u_1^t(\beta_o^*(s|\bar{\theta}))] = (1-\mu^{t+1})u_1^{t+1}(e^*) + \mu^{t+1}\,\mathrm{E}[u_1^{t+2}(\beta_o^*(s|\bar{\theta}))]$$

where all involved $u_1^t(e^*) = \bar{\theta} - {}^t/_2$, $t \leq \bar{\theta}+1$ are strictly decreasing in t. Therefore, given $\theta = \bar{\theta}$, we have for all $t \leq \bar{\theta}+1$

$$\mathrm{E}[u_1^t(\beta_o^*(s|\bar{\theta}))] > \mathrm{E}[u_1^{t+2}(\beta_o^*(s|\bar{\theta}))].$$

Since payoffs for P1 are monotonically decreasing and we are in a constant sum game, the opposite is true for P2's payoffs. $\square$

Lemma 3. *No 'jump' deviation from (3.3) can be profitable in* Γ_o.

Proof. Let $\nu = 1$ and consider a jump bid (i.e. upward deviation) at t_j by P1 first. No jump bids are profitable before P1 starts signalling at t_s because—given assumption 2—no posterior $\varphi \geq {}^1/_2$ can be formed from Bayes' rule to make P2 indifferent between her certain pre-signalling payoff and the expected payoff after the jump bid. So $t_j \geq t_s$.

We argue that jump bids by P1 shorten the game and work like certain transfers to P2. To see this, define a deviation strategy $\hat{\beta}$ equaling β_o^* everywhere except for the single jump $\hat{b}_1^{t_j} \leq \bar{\theta}$ at t_j. Denote the gap between the deviation and the equilibrium bid by $g = \hat{b}_1^{t_j} - b_1^{t_j} > 0$. The effect of the jump bid is—while leaving the exit payoffs in states where P1 moves entirely unchanged—to decrease P1's payoffs (increase P2's payoffs) by g in states where P2 moves.

It is easy to see that in the low-value state no deviation of P1 can pay because P2 will simply equilibrate the deviation with his following two payoff expectations and leave P1 indifferent. For $\bar{\theta}$, the expected payoff following a deviation $\hat{\beta}$ at t_j excludes at least one payoff opportunity of the form $(1-\mu^{t_j+1})u_1^{t_j+1}(e^*)$ which is contained in β_o^* but not in the deviation path. We will show that this decreases expected payoffs. (i) If g is odd, the jump at t_j results in an expected payoff which is (minus some uniform payoff gap) already contained in a part of the equilibrium payoff sequence following t_j

$$\mathrm{E}[u_1^{t_j}(\hat{\beta}_g(s|\bar{\theta}))] + {}^g\!/_2 = \mathrm{E}[u_1^{t_j+g}(\beta_o^*(s|\bar{\theta}))].$$

From the monotonicity of payoffs (lemma 2), we know that this cannot induce P1 to jump. (ii) Similarly, for an even-valued jump g, the deviation expectation is directly dominated

$$\mathrm{E}[u_1^{t_j}(\tilde{\beta}_g(s|\bar{\theta}))] + \frac{g+1}{2} < \mathrm{E}[u_1^{t_j}(\beta_o^*(s|\bar{\theta}))]$$

where $\tilde{\beta}$ is identical to β_o^* with all probabilities μ *higher* than in equilibrium.
The argument for P2 is symmetric. □

Proposition 2. *The queto game Γ_o with common value $\theta > 1$ and signals $s = (\theta, \varphi^0)$ has the dynamic Bayesian Nash equilibrium (3.3).*

Proof. We look at one-shot deviations first and consider only odd values of θ.

- P1: Clearly, no exiting before t_s—defined in lemma 1—pays for P1 because all payoffs at odd $t < t_s$ are strictly lower than $u_1^{t_s}(e^*)$. Given $\underline{\theta}$, exiting after and at t_s cannot be profitable because P1 is playing a mixed action including 'e' in its support, so P1 is indifferent by construction of β_o^*. Given the true state is $\bar{\theta}$, P1 will not exit prematurely because $u_1^t(e^*) < u_1^{\bar{\theta}}(e^*)$ for any odd $t < \bar{\theta}$. Since lemma 3 excludes jump bids, P1 has no reason to deviate.

- P2: At any $t < t_s$, P2's payoffs from exiting are strictly lower than $u_2^{t_s}(e^*)$. P2 mixes at every stage after t_s and is therefore indifferent between exiting and the action prescribed by β_o^* for all $t > t_s$. Again jump bids are excluded by lemma 3. Hence P2 will not deviate.

Since no one-shot deviation pays, there are no profitable more complicated multi-stage deviations either. The same argument holds for even values of $\bar{\theta}$. □

Proposition 3. *The equilibrium (3.3) is sequentially rational.*

Proof. Sequential rationality is the requirement that (i) players cannot gain by deviating from β_o^* at *any* information set, and that (ii) off-equilibrium path beliefs are consistent. Consider any such deviation; since the relative payoff consequences of the players' actions are unchanged for the deviator and increased uniformly by the jump for the opponent, all monotonicity properties of the payoffs are preserved and the argument from lemma 3 goes through unchanged along the deviation path. Therefore the choice situation on the deviation path is not strategically different from the situation on the β_o^*-path with the qualification that for even jumps the even-$\bar{\theta}$-equilibrium changes to the odd-$\bar{\theta}$ version and vice versa. Hence β_o^* is an equilibrium following any deviation and requirement (i) is fulfilled. (ii) is fulfilled because P2's beliefs follow from the priors, assumption 2 and (repeated) application of Bayes' rule everywhere. □

Proposition 4. *There is no separating equilibrium in the queto game* Γ_o *for* $t < \bar{\theta}$.

Proof. Let $\bar{\theta}$ be odd, the initial price $p^0 = 0$ and $\hat{\beta}_1$ be a separating strategy for the informed P1 with full revelation at $t = 1$ as part of the equilibrium candidate

$$\hat{\beta} = \begin{cases} \left(\{p^{t-1} + \nu\}_{t=1}^{\bar{\theta}}, \{p^{t-1} + \nu\}_{t=2}^{\bar{\theta}-1}\right) & \text{in case of } \theta = \bar{\theta}, \text{ and} \\ (e, e) & \text{otherwise.} \end{cases}$$

Then P1 has incentives to deviate immediately to $b_1^1(\hat{\beta}(s|\bar{\theta}))$ in the case of $\theta = \underline{\theta}$. Because of payoff monotonicity in t, the same argument holds true for all but the last period of the game where P1 is to move. The same argument can be made for even $\bar{\theta}$. $\square$

Proposition 5. *The equilibrium (3.3) is essentially unique in the queto game* Γ_o.

Proof. Consider first an odd value $\bar{\theta}$. In his final move as part of any equilibrium, P1 will reveal all information by exiting for certain in the low-value branch. For $t < \bar{\theta}$, this is a separating action because in the high-value branch all future payoffs for P1 are strictly higher than $u_1^t(e^*)$ and therefore $b_1^t(\beta(s|\bar{\theta})) = e$ is strictly dominated for all $t < \bar{\theta}$. Hence we are looking at a separating strategy where P1 fully reveals the object's value before $t = \bar{\theta}$. But from proposition 4 we know that no separating equilibrium can exist before $t = \bar{\theta}$. Hence (3.3) is unique up to the final stage. An equivalent argument holds for even values of $\bar{\theta}$ for all $t < \bar{\theta} - 1$. $\square$

Proposition 6. *For* $\nu \to 0$*, the payoffs from playing (3.3) in* Γ_o *converge to*

$$\mathrm{E}[u(\beta_o^*(s|\bar{\theta}))] = \left({}^{5\bar{\theta}}\!/_8, {}^{3\bar{\theta}}\!/_8\right), \quad \mathrm{E}[u(\beta_o^*(s|\underline{\theta}))] = \left({}^{\bar{\theta}}\!/_8, -{}^{\bar{\theta}}\!/_8\right), \quad \mathrm{E}^0[u(\beta_o^*(s))] = \left({}^{3\bar{\theta}}\!/_8, {}^{\bar{\theta}}\!/_8\right).$$

Proof. Lemma 1 establishes that, for $\nu \to 0$, P1 starts signalling at $t_s = {}^{\bar{\theta}}\!/_4$. P1's payoff from exiting there is $u_1^{t_s}(e^*) = {}^{\bar{\theta}}\!/_8$. Since he plays a mixed action, this equals $\mathrm{E}[u_1(\beta_o^*(s|\underline{\theta}))]$. From the zero-sum nature of the $\underline{\theta}$-game follows that $\mathrm{E}[u_2(\beta_o^*(s|\underline{\theta}))] = -{}^{\bar{\theta}}\!/_8$.

Since P2 responds by mixing to P1's signal irrespective of the true state of the world, lemma 1 also pins down the payoffs in the high-value case as

$$\mathrm{E}[u_1(\beta_o^*(s|\bar{\theta}))] = \sum_{\tau=1}^{\bar{\theta}-1} \left(\prod_{t=1}^{\tau-1} \mu^{2t}\right) u_1^{2\tau}(e^*)(1 - \mu^{2\tau}) + \left(\prod_{t=1}^{\bar{\theta}+1} \mu^{2t}\right) u_1^{\bar{\theta}}(\beta_o^*(s|\bar{\theta}))$$

where P1's stage payoffs are given as $u_1^t(e^*) = \bar{\theta} - \frac{\nu t}{2}$ and $u_1^{\bar{\theta}}(\beta_o^*(s|\bar{\theta})) = \frac{\bar{\theta}-\nu}{2}$. Given $\nu \to 0$, the above sums to $\mathrm{E}[u_1(\beta_o^*(s|\bar{\theta}))] = {}^{5\bar{\theta}}\!/_8$ and the $\bar{\theta}$-sum properties of this branch of the game ensure that P2's equilibrium payoff is given by $\mathrm{E}[u_2(\beta_o^*(s|\bar{\theta}))] = {}^{3\bar{\theta}}\!/_8$. $\square$

Remark: Since we only consider the pure common value case, (3.3) is fully ex-post efficient. Moreover, the player with the higher signal *always* wins the object.

Conclusion

We present the essentially unique way of dividing a jointly owned, common value object between two players in a repeated queto game. The intuition behind the equilibrium β_o^* is that P1 will start signalling a high value as soon as P2 is receptive to such a signal. This entails P1 foregoing considerable payoff-opportunities in the opening game. P1's signalling fixes the payoffs for the players in the low value branch because P2 must play a mixed action in order to make P1 indifferent between continuing and exiting (i.e. signalling). This mixture increases P1's payoffs in the high value case beyond what he can get from the simple splitting of the perfect information case. The informed P1 realises positive profits even when the object is worthless while the uninformed P2 is still given incentives to participate. Notice that β_o^* requires perfection in order to survive.

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