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by

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Consistent Voting Systems with a Continuum of Voters

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Abstract

Voting problems with a continuum of voters and finitely many alternatives are considered. The classical Arrow and Gibbard-Satterthwaite theorems are shown to persist in this model, not for single voters but for coalitions of positive size. The emphasis of the study is on strategic considerations, relaxing the nonmanipulability requirement: are there social choice functions such that for every profile of preferences there exists a strong Nash equilibrium resulting in the alternative assigned by the social choice function? Such social choice functions are called exactly and strongly consistent. The study offers an extension of the work of Peleg (1978a) and others. Specifically, a class of anonymous social choice functions with the required property is characterized through blocking coefficients of alternatives, and associated effectivity functions are studied. Finally, representation of effectivity functions by game forms having a strong Nash equilibrium is studied.

Keywords: Nonmanipulability, exact and strong consistency, social choice function, effectivity function, blocking coefficients, game form.
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1 Introduction

We consider a classical voting system with many voters and finitely many alternatives. Such a system is representative of political elections on the local or national level. As an, in our view, best approximation we model voters as elements of a nonatomic measure space. In particular, this approach allows us to accommodate the fact that in such voting systems single voters have negligible influence on the final outcome, and to avoid potential combinatorial peculiarities of a model with a large but finite number of voters.

The purpose of the paper is to present an extensive study of such voting systems with an emphasis on strategic aspects. If we talk about strategic aspects in this model, we necessarily deal with strategic voting by groups of voters (coalitions). This does not have to imply that coalitions actually get together to coordinate their voting behavior. Although single voters are negligible for the final outcome, they may nevertheless derive utility from voting and, thus, may also vote strategically, possibly resulting in strategic behavior of groups of equally-minded voters.

As a prelude, we derive versions of the Arrow (1963) and Gibbard (1973) and Satterthwaite (1975) theorems in this model, based on these results for finitely many voters. This works since the number of different preference relations is finite, and therefore every preference profile induces a finite partition of the set of voters; each element of the partition can then be regarded as one voter. Our result on social welfare functions—functions that assign a preference relation to every profile of preference relations—which are independent of irrelevant alternatives (cf. Arrow, 1963) is closely related to the result of Kirman and Sondermann (1972) in the sense that we establish the existence of "invisible dictators"—arbitrarily small dictator coalitions of positive measure. Our work is, however, independent of theirs since we impose the domain restriction that voter profiles be measurable. Our version of the Gibbard-Satterthwaite theorem for social choice functions—functions that assign an alternative to every profile of preferences—shows that the requirement of nonmanipulability again results in the existence of invisible dictators.

Inspired by this negative Gibbard-Satterthwaite type of result, in the bulk of the paper we concentrate on social choice functions satisfying the weaker requirement of exact and strong consistency (ESC). This means that for every given profile of preferences there is another profile which (i) is a strong Nash equilibrium—no coalition can profitably deviate—in the strategic game in which each voter reports a preference and the outcome is evaluated according to given "true" preferences, and (ii) results in the same alternative as the true preferences. In other words, insincere voting may occur but is likely to result, insofar as a particular strong Nash equilibrium is likely to result, in the same alternative as if the voters had voted sincerely. In view of the Gibbard-Satterthwaite result for our model, this seems the best that can be achieved. ESC social choice functions were first studied in Peleg (1978a,b) for finitely many voters. See also Dutta and Pattanaik (1978); Ishikawa and Nakamura (1980); and Kim and Roush (1981).

In order to investigate ESC social choice functions we make extensive use of effectivity functions. Effectivity functions were first introduced by Moulin and Peleg (1982); see also Abdou and Keiding (1991) for an overview. For a given social choice function the associated effectivity function describes, more abstractly, the power of each coalition in terms of subsets of alternatives into which the coalition can force the final alternative by an appropriate choice of preferences. We show that, for an ESC social choice function, the associated effectivity function is, among other things, stable: its core, consisting of those alternatives that cannot be blocked by any coalition, is nonempty for every profile of preferences.

The main results of the paper concentrate on anonymous ESC social choice functions. These induce so called blocking coefficients on the alternatives, and it turns out that this assignment of blocking coefficients is additive. A coalition can block a subset of alternatives (and thus, is effective for the complement) if the size of the coalition is either larger than or at least as large as the sum of the blocking coefficients of the subset. Depending on these two cases, we call a subset of alternatives either an i-set or an e-set. This distinction is the source of major differences with the finite case as treated in Peleg (1978a,1991), Oren (1981), Polishchuk (1978), and Holzman (1986a,b); see also Peleg (1984, Chapter 5). Conversely, we define effectivity functions by specifying i-sets and e-sets and associated blocking coefficients, and show that these effectivity functions are stable.

We show that for effectivity functions with exactly one alternative that must be strictly blocked, i.e., exactly one i-alternative, every element of the core can be obtained by a so called feasible elimination procedure and conversely, and that any anonymous selection from the core is an anonymous ESC social choice function and conversely. We also show that these results are not true for an arbitrary number of i-alternatives. Specifically, in that case an anonymous selection from the core does not have to lead to an anonymous ESC social choice function.

A social choice function can be viewed as a game form: the strategy set of each voter is the set of potential preferences, and a strategy combination fed into the social choice function leads to an alternative, which is evaluated according to the true preferences. An ESC social choice function is therefore a game form that has a strong Nash equilibrium for each profile of preferences with the special property that this equilibrium leads to the same alternative as the true preferences. In the final part of this paper we broaden our view by starting with an arbitrary effectivity function and looking for a game form which represents this effectivity function—meaning that it preserves the power induced by this effectivity function—and which for every profile of preferences has a strong Nash equilibrium. We find that maximality and stability are necessary and sufficient conditions; the proof of this is an extension of the argument in Peleg and Moulin (1982).

The organization of the paper is as follows. In Section 2 the basic model is presented, and in Section 3 the classical approaches by Arrow and Gibbard-Satterthwaite are considered within our model. The core of the paper is Section 4, where we investigate ESC social choice functions. The broadening to strong Nash consistent representation of effectivity functions is considered in Section 5, and Section 6 concludes.

2 The basic model

Let $(\Omega, \Sigma, \lambda)$ be a nonatomic measure space. Here Ω is the set of *voters* or *players*, Σ is the σ-field of permissible *coalitions*, and λ is a nonatomic measure on Σ. The number $\lambda(S)$ for a coalition S is interpreted as the size of S. By $\Sigma_0 = \Sigma \setminus \{\emptyset\}$ we denote the set of all nonempty coalitions, and by Σ_+ we denote the set of all coalitions S with $\lambda(S) > 0$. Throughout we assume $\Omega \in \Sigma_+$ and $\lambda(\Omega) < \infty$.

Let A be a finite set of *alternatives*. We assume throughout that $|A| \geq 2$. (For a finite set D, $|D|$ denotes the number of elements of D.) A linear ordering of A is a complete, transitive, and antisymmetric binary relation on A. The set of all linear orderings of A is denoted by $L(A)$.

A *profile* (of preferences) is a measurable function $\mathbf{R} : \Omega \to L(A)$ (i.e., for each $R \in L(A)$, $\{t \in \Omega \mid \mathbf{R}(t) = R\}$ is in Σ). Two profiles $\mathbf{R}_1$ and $\mathbf{R}_2$ are *equivalent*, written $\mathbf{R}_1 \sim \mathbf{R}_2$, if $\lambda(\{t \in \Omega \mid \mathbf{R}_1(t) \neq \mathbf{R}_2(t)\} = 0$. Let ρ denote the set of all profiles.

In our model, a *partition* of Ω is a finite collection of pairwise disjoint sets in Σ_+ the union of which has measure equal to $\lambda(\Omega)$. Let $R_1, \ldots, R_{|A|!}$ be an enumeration of the elements of $L(A)$. Each profile $\mathbf{R}$ results in a collection $\mathcal{P} = \{S_1, \ldots, S_{|A|!}\}$ of subsets of Ω with $S_k = \{t \in \Omega \mid \mathbf{R}(t) = R_k\} \in \Sigma$ for each $1 \leq k \leq |A|!$. We denote by $\mathcal{P}(\mathbf{R})$ the collection obtained from $\mathcal{P}$ by omitting the sets of measure 0 and call this the *partition generated by* $\mathbf{R}$.

A *social choice function* (SCF) is a surjective function $F : \rho \to A$ that satisfies

(1) for all $\mathbf{R}_1, \mathbf{R}_2 \in \rho$, if $\mathbf{R}_1 \sim \mathbf{R}_2$, then $F(\mathbf{R}_1) = F(\mathbf{R}_2)$.

A *social welfare function* (SWF) is a function $f : \rho \to L(A)$ that satisfies

(2) for all $\mathbf{R}_1, \mathbf{R}_2 \in \rho$, if $\mathbf{R}_1 \sim \mathbf{R}_2$, then $f(\mathbf{R}_1) = f(\mathbf{R}_2)$.

Conditions (1) and (2) imply that social choice functions and social welfare functions do not depend on the preferences of coalitions of measure 0. In particular, because of nonatomicity, single agents do not have any influence at all.

3 Classical approaches: independence and nonmanipulability

In this section we investigate, in our model, the implications of the approaches by Arrow (1963) to social welfare functions and by Gibbard (1973) and Satterthwaite (1975) to social choice functions. The former approach concentrates on the well known independence of irrelevant alternatives condition, while the latter approach concentrates on nonmanipulability.

3.1 Independence of irrelevant alternatives

For a profile $\mathbf{R}$ and $a, b \in A$, $a \neq b$, we say that a *Pareto dominates* b if $\lambda(\{t \in \Omega \mid b\mathbf{R}(t)a\}) = 0$. A social welfare function f satisfies *Pareto* (P) if for all $\mathbf{R} \in \rho$ and $a, b \in A$ with $a \neq b$, if a Pareto dominates b, then $af(\mathbf{R})b$.

For $B \subseteq A$, $R \in L(A)$, and $\mathbf{R} \in \rho$, we denote by $R|B$ the restriction of R to B, and we define $\mathbf{R}|B$ by $(\mathbf{R}|B)(t) = \mathbf{R}(t)|B$ for all $t \in \Omega$. A social welfare function f satisfies *Independence of Irrelevant Alternatives* (IIA) if for all $\mathbf{R}_1, \mathbf{R}_2 \in \rho$ and $B \subseteq A$ with $|B| = 2$ and $\mathbf{R}_1|B = \mathbf{R}_2|B$ we have $f(\mathbf{R}_1)|B = f(\mathbf{R}_2)|B$. In words, the societal preference between two alternatives should only depend on the individual preferences between these alternatives and not on individual preferences between other alternatives.

We shall characterize all social welfare functions satisfying Pareto and IIA. This characterization is closely related to the results in Kirman and Sondermann (1972). It is independent because of our domain restriction of measurability of preference profiles.

We need the following concept. A collection $\mathcal{D} \subseteq \Sigma_+$ is called an *ultrafilter* if (i) $D \cap D' \in \mathcal{D}$ for all $D, D' \in \mathcal{D}$ and (ii) $D \in \mathcal{D}$ or $\Omega \backslash D \in \mathcal{D}$ for every $D \in \Sigma_+$.

Let $\mathcal{P} = \{D_1, \ldots, D_k\}$ be a partition of Ω. Let $\mathcal{D}$ be an ultrafilter. We claim that there is at least one i for which $D_i \in \mathcal{D}$. If not, then by property (ii) of $\mathcal{D}$, $D^i := \bigcup_{j=1,\ldots,k,\ j \neq i} D_j \in \mathcal{D}$ for every $i = 1, \ldots, k$, so by property (i), $\emptyset = \bigcap_{i=1,\ldots,k} D^i \in \mathcal{D}$, a contradiction since $\emptyset \notin \Sigma_+$. Hence, there is an i with $D_i \in \mathcal{D}$ and by property (i) again there is exactly one such i. Also, if a partition $\mathcal{P}'$ of Ω is coarser than $\mathcal{P}$ (i.e., each element of $\mathcal{P}$ is contained in an element of $\mathcal{P}'$; we also say that $\mathcal{P}$ is finer than $\mathcal{P}'$) then (i) implies $D \subseteq D'$, where D and D' are the elements of $\mathcal{P}$ and $\mathcal{P}'$ that are in $\mathcal{D}$, respectively. Therefore, there is a well defined mapping d that assigns to each partition its element in $\mathcal{D}$, and d satisfies:

(3) If $\mathcal{P}'$ is coarser than $\mathcal{P}$, then $d(\mathcal{P}) \subseteq d(\mathcal{P}')$.

The following lemma shows that also the converse holds.

Lemma 3.1 *Let d be a mapping that assigns to each partition of Ω exactly one element. Suppose d satisfies (3). Then the collection*

$$\mathcal{D} = \{D \in \Sigma_+ \mid \text{there is a partition } \mathcal{P} \text{ of } \Omega \text{ with } D = d(\mathcal{P})\}$$

is an ultrafilter.

Proof. Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be partitions and $D^1 = d(\mathcal{P}^1)$, $D^2 = d(\mathcal{P}^2)$. We show that $D^1 \cap D^2 \in \mathcal{D}$. Consider the join $\mathcal{P}$ of $\mathcal{P}^1$ and $\mathcal{P}^2$, i.e., the partition

$$\mathcal{P} = \{D \cap E \mid D \in \mathcal{P},\ E \in \mathcal{P}',\ D \cap E \in \Sigma_+\}.$$

Obviously, $\mathcal{P}$ is finer than both $\mathcal{P}^1$ and $\mathcal{P}^2$. Suppose $D^* = d(\mathcal{P})$. Then by (3), both $D^* \subseteq D^1$ and $D^* \subseteq D^2$, hence $D^* \subseteq D^1 \cap D^2$. By definition of $\mathcal{P}$ therefore, $D^* = D^1 \cap D^2$, which implies $D^1 \cap D^2 \in \mathcal{D}$.

Finally let $D \in \Sigma_+$. If $\lambda(D) = \lambda(\Omega)$ then $D = d(\{D\})$, so $D \in \mathcal{D}$. Otherwise, either $D = d(\{D, \Omega \backslash D\})$ or $\Omega \backslash D = d(\{D, \Omega \backslash D\})$, hence either $D \in \mathcal{D}$ or $\Omega \backslash D \in \mathcal{D}$.

Thus, $\mathcal{D}$ is an ultrafilter. □

We now associate with an ultrafilter $\mathcal{D}$ an SWF $f^{\mathcal{D}}$, as follows. For a profile $\mathbf{R} \in \rho$ let D be the unique element from $\mathcal{P}(\mathbf{R})$ that is in $\mathcal{D}$. Define $f^{\mathcal{D}}(\mathbf{R}) := R$ where $R = \mathbf{R}(t)$ for (all) $t \in D$.

Obviously, $f^{\mathcal{D}}$ satisfies Pareto. We first show:

Lemma 3.2 *$f^{\mathcal{D}}$ satisfies IIA.*

Proof. Let $a, b \in A$ $(a \neq b)$ and $\mathbf{R}_1, \mathbf{R}_2 \in \rho$ such that $\mathbf{R}_1|\{a,b\} = \mathbf{R}_2|\{a,b\}$. Let, for $i = 1, 2$, $f^{\mathcal{D}}(\mathbf{R}_i) = R_i = \mathbf{R}_i(t_i)$ for some $t_i \in D_i$, $D_i \in \mathcal{D}$ as in the definition of $f^{\mathcal{D}}$. Since $D_1 \cap D_2 \in \mathcal{D}$, we have $D_1 \cap D_2 \neq \emptyset$, so we may take $t_1 = t_2$ and hence $R_1|\{a,b\} = R_2|\{a,b\}$. Consequently,

$$f^{\mathcal{D}}(\mathbf{R}_1)|\{a,b\} = R_1|\{a,b\} = R_2|\{a,b\} = f^{\mathcal{D}}(\mathbf{R}_2)|\{a,b\}.$$

This proves that $f^{\mathcal{D}}$ satisfies IIA. □

Conversely, let f be an SWF satisfying Pareto and IIA. If $|A| = 2$ then f could for instance be majority rule. Assume now $|A| > 2$. We will show that f is of the form $f^{\mathcal{D}}$ for some ultrafilter $\mathcal{D}$. To this end, let $\mathcal{P} \subseteq \Sigma_+$ be a partition of Ω. Regard every element of $\mathcal{P}$ as a separate agent. By the classical theorem of Arrow (see Arrow, 1963) for a finite number of agents there is a fixed element of $\mathcal{P}$, call it $d^f(\mathcal{P})$, such that, for every profile $\mathbf{R} \in \rho$ that is measurable with respect to $\mathcal{P}$ (i.e., $\mathbf{R}(t) = \mathbf{R}(t')$ for all $D \in \mathcal{P}$ and $t, t' \in D$), we have $f(\mathbf{R}) = \mathbf{R}(t)$ for (all) $t \in d^f(\mathcal{P})$. (Note that, if $\mathbf{R}$ is measurable with respect to $\mathcal{P}$, then $\mathcal{P}$ is at least as fine as $\mathcal{P}(\mathbf{R})$, the partition generated by $\mathbf{R}$.) Let

$$\mathcal{D}^f := \{d^f(\mathcal{P}) \mid \mathcal{P} \subseteq \Sigma_+ \text{ is a partition}\}.$$

Lemma 3.3 *$\mathcal{D}^f$ is an ultrafilter.*

Proof. By Lemma 3.1, it is sufficient to prove that d^f satisfies (3). Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions with $\mathcal{P}'$ coarser than $\mathcal{P}$. Let $D' \in \mathcal{P}'$ with $d^f(\mathcal{P}) \subseteq D'$. Let $R, Q \in L(A)$ be different and take a profile $\mathbf{R} \in \rho$ that is measurable with respect to $\mathcal{P}'$, and hence with respect to $\mathcal{P}$, and with $\mathbf{R}(t) = R$ for all $t \in D'$ and with $\mathbf{R}(t) = Q$ otherwise. Then $f(\mathbf{R}) = R$ since $R = \mathbf{R}(t)$ for (all) $t \in d^f(\mathcal{P})$. Hence, $d^f(\mathcal{P}') = D'$, so that $d^f(\mathcal{P}) \subseteq d^f(\mathcal{P}')$. □

Lemmas 3.2 and 3.3 have the following corollary.

Corollary 3.4 *Let $|A| \geq 3$. A social welfare function f satisfies Pareto and IIA if and only if there is an ultrafilter $\mathcal{D}$ with $f = f^{\mathcal{D}}$.*

The next result shows existence.

Theorem 3.5 *There exists a social welfare function satisfying Pareto and IIA.*

Proof. By Corollary 3.4 it is sufficient to show that there exists an ultrafilter of sets in Σ_+.

A *filter* in Σ_+ is a collection $\mathcal{F} \subseteq \Sigma_+$ satisfying

(i) for all $D, D' \in \mathcal{F}$, $D \cap D' \in \mathcal{F}$;

(ii) for all $D \in \mathcal{F}$ and $D' \in \Sigma_+$ with $D \subseteq D'$, $D' \in \mathcal{F}$.

Let $\mathcal{U}$ be the collection of all filters $\mathcal{F}$ that satisfy, additionally,

(iii) for all $D \in \mathcal{F}$ and $D' \in \Sigma_+$ with $D' \subseteq D$ and $\lambda(D) = \lambda(D')$, $D' \in \mathcal{F}$.

Any set of positive measure together with all its subsets of the same measure and all measurable supersets of these form a filter, so $\mathcal{U}$ is non-empty. The inclusion relation is a partial ordering on $\mathcal{U}$ and each chain in $\mathcal{U}$ has an upper bound, namely the union of all filters in the chain. Hence, Zorn's Lemma implies that $\mathcal{U}$ has a maximal element, say $\mathcal{D}$. We claim that $\mathcal{D}$ is an ultrafilter. If not, then there is a $D \in \Sigma_+$ such that $D \notin \mathcal{D}$ and $\Omega \backslash D \notin \mathcal{D}$ ($D \in \mathcal{D}$ and $\Omega \backslash D \in \mathcal{D}$ is not possible by (i)). By (ii), we have $D' \cap D \neq \emptyset$ and $D' \cap (\Omega \backslash D) \neq \emptyset$ for every $D' \in \mathcal{D}$ and by (iii), we have $\lambda(D' \cap D) > 0$ and $\lambda(D' \cap (\Omega \backslash D)) > 0$. Now consider the collection $\mathcal{D}'$ obtained by adding to $\mathcal{D}$ the collection $\{D' \cap D \mid D' \in \mathcal{D}\}$. Then it is easy to check that $\mathcal{D}'$ is a filter in $\mathcal{U}$ that is larger than $\mathcal{D}$, contradicting the maximality of $\mathcal{D}$. Hence, $\mathcal{D}$ is an ultrafilter. □

It is easy to construct an ultrafilter. For instance, let $\Omega = [0,1]$ and let λ be the Lebesgue measure. If $\mathcal{D}$ is an ultrafilter, then for any $t \in [0,1]$ exactly one of the two intervals $[0,t]$ and $[t,1]$ must be in $\mathcal{D}$. Suppose, for the sake of the argument, that this is always the lower one, $[0,t]$. Then 0 is an "invisible dictator" in the sense of Kirman and Sondermann (1972). Of course, the singleton 0 does not have any power at all, but always needs, roughly, a coalition of positive measure in any arbitrarily small neighborhood to exercise its "dictatorship".

3.2 Nonmanipulability

In this subsection and in the remainder of the paper we focus on social choice functions. Clearly, an SCF F cannot be manipulated by a single player because of (1). We will see, however, that if A contains at least three alternatives, then for F to be nonmanipulable it has to exhibit an "invisible dictator" as above. (For $|A| = 2$, we can again take the majority rule, which is not manipulable.) We proceed to a precise formulation of this result.

Let $\mathbf{R} \in \rho$ and $S \in \Sigma$. The social choice function F is *manipulable by S at $\mathbf{R}$* if there exists a $Q \in L(A)$ with the following property: if $\mathbf{R}_1 \in \rho$ is a profile with $\mathbf{R}_1(t) = \mathbf{R}(t)$ for all $t \notin S$ and $\mathbf{R}_1(t) = Q$ for all $t \in S$, then $F(\mathbf{R}) \neq F(\mathbf{R}_1)$ and $F(\mathbf{R}_1)\mathbf{R}(t)F(\mathbf{R})$ for all $t \in S$. (Clearly, if F is manipulable by S at $\mathbf{R}$, then $\lambda(S) > 0$.) We call F *nonmanipulable* if there exist no $\mathbf{R} \in \rho$ and $S \in \Sigma$ such that F is manipulable by S at $\mathbf{R}$. In words, it can never happen that all members of a coalition obtain a preferred alternative if that coalition coordinates on an untruthful preference profile. See also Remark 3.9 below.

We associate with an ultrafilter $\mathcal{D}$ an SCF $F^{\mathcal{D}}$, as follows. For a profile $\mathbf{R} \in \rho$ let D be the unique element of $\mathcal{P}(\mathbf{R})$ that is in $\mathcal{D}$. Define $F^{\mathcal{D}}(\mathbf{R}) := x$ where xRy for all $y \in A$ and $R = \mathbf{R}(t)$ for (all) $t \in D$. We have:

Lemma 3.6 *$F^{\mathcal{D}}$ is nonmanipulable.*

Proof. Let $\mathbf{R} \in \rho$. Clearly, if a coalition S can manipulate at $\mathbf{R}$, then $S \cap D = \emptyset$, where D is the element of $\mathcal{P}(\mathbf{R})$ in $\mathcal{D}$. Hence, a manipulation of

S results in a profile $\mathbf{R}'$ such that $\mathcal{P}(\mathbf{R}')$ shares D with $\mathcal{P}(\mathbf{R})$. But then D is also the element of $\mathcal{P}(\mathbf{R}')$ that is in $\mathcal{D}$ by condition (i) of an ultrafilter. So $F^{\mathcal{D}}(\mathbf{R}') = F^{\mathcal{D}}(\mathbf{R})$. □

Conversely, let F be an SCF satisfying nonmanipulability. Let $|A| > 2$. In order to apply the Gibbard (1973) and Satterthwaite (1975) theorem we need to make sure that the range condition is satisfied. Therefore, we fix profiles $\mathbf{R}_1, \ldots, \mathbf{R}_{|A|}$ in ρ such that $|\{F(\mathbf{R}_i) \mid i = 1, \ldots, |A|\}| = |A|$ (this is possible since F is surjective by assumption). For an arbitrary partition $\mathcal{P} \subseteq \Sigma_+$ of Ω let $\mathcal{P}^*$ be the coarsest common refinement of $\mathcal{P}$ and the generated partitions $\mathcal{P}(\mathbf{R}_i)$, $i = 1, \ldots, |A|$. Regard every element of $\mathcal{P}^*$ as a separate agent. By the Gibbard-Satterthwaite theorem for a finite number of agents there is a fixed element D^* of $\mathcal{P}^*$ such that, for every profile $\mathbf{R} \in \rho$ that is measurable with respect to $\mathcal{P}^*$, we have $f(\mathbf{R}) = x$ where x is the top element of $\mathbf{R}(t)$ for (all) $t \in D^*$. Denote by $d^F(\mathcal{P})$ the element of $\mathcal{P}$ that contains D^* and let

$$\mathcal{D}^F := \{d^F(\mathcal{P}) \mid \mathcal{P} \subseteq \Sigma_+ \text{ is a partition}\}.$$

Lemma 3.7 *$\mathcal{D}^F$ is an ultrafilter.*

Proof. The proof is analogous to the proof of Lemma 3.3 if we assume that R and Q in that proof have different top elements. □

Lemmas 3.6 and 3.7 have the following corollary.

Corollary 3.8 *Let $|A| \geq 3$ and let $F : \rho \to A$ be an SCF. Then F is nonmanipulable if and only if there is an ultrafilter $\mathcal{D}$ with $F = F^{\mathcal{D}}$.*

As a consequence of Corollary 3.8, the only social choice functions that guarantee sincere voting if there are at least three alternatives exhibit again an "invisible dictator". (Existence of such SCFs follows similarly to Theorem 3.5.) In the next section we will therefore relax the nonmanipulability requirement.

Remark 3.9 Our nonmanipulability condition has necessarily the form of coalitional nonmanipulability since single agents have no influence. Nevertheless, it can be weakened to a version that is a closer approximation of individual nonmanipulability. Call F ε-manipulable if for every $\varepsilon > 0$ there is a profile $\mathbf{R} \in \rho$ and a coalition $S \in \Sigma$ with $\lambda(S) < \varepsilon$ such that F is manipulable by S at $\mathbf{R}$. Call F non-ε-manipulable if it is not ε-manipulable. This means that there is an $\varepsilon > 0$ such that at no profile coalitions with size smaller than ε can manipulate. Clearly, non-ε-manipulability is weaker than nonmanipulability, hence for every ultrafilter $\mathcal{D}$ the SCF $F^{\mathcal{D}}$ satisfies it. Conversely, suppose that the SCF F is non-ε-manipulable. Take $\varepsilon > 0$ so small that no coalition of size smaller than ε can ever manipulate, and take an arbitrary partition $\mathcal{P}_{|A|+1}$ of Ω such that each element of $\mathcal{P}_{|A|+1}$ has size smaller than ε. Modify the definition of $\mathcal{P}^*$ preceding Lemma 3.7 such that $\mathcal{P}^*$ is now the coarsest common refinement of $\mathcal{P}_{|A|+1}$ and $\mathcal{P}(\mathbf{R}_i)$, $i = 1, \ldots, |A|$. Then Lemma 3.7 and Corollary 3.8 continue to hold if we replace nonmanipulability by non-ε-manipulability.

4 Exactly and strongly consistent social choice functions

For $|A| \geq 3$, Corollary 3.8 implies that every SCF $F : \rho \to A$ is manipulable unless it exhibits an invisible dictator. Thus, we cannot guarantee sincere voting but we are still interested in optimizing the chance of reaching the sincere outcome $F(\mathbf{R})$ for every profile $\mathbf{R} \in \rho$. To be more precise, observe that for every $\mathbf{R} \in \rho$ the pair $(F, \mathbf{R})$ defines a game in strategic form in a natural way: each player $t \in \Omega$ has strategy set $L(A)$ and preference $\mathbf{R}(t)$ on A for evaluating any outcome $F(\mathbf{R}^*) \in A$, $\mathbf{R}^* \in \rho$. For $S \in \Sigma_0$, denote by ρ^S the set of all measurable functions $\mathbf{R}^S : S \to L(A)$. Let $\mathbf{R} \in \rho$. The profile $\mathbf{Q}$ is a *strong Nash equilibrium* (SNE) of the game $(F, \mathbf{R})$ if for every $S \in \Sigma_+$ and every $\mathbf{V}^S \in \rho^S$, there exists $T \in \Sigma_+$ with $T \subseteq S$ and $F(\mathbf{Q})\mathbf{R}(t)F(\mathbf{Q}^{\Omega \setminus S}, \mathbf{V}^S)$ for every $t \in T$. The SCF F is *exactly and strongly consistent* (ESC) if for every $\mathbf{R} \in \rho$ there exists an SNE $\mathbf{Q}$ of $(F, \mathbf{R})$ such that $F(\mathbf{Q}) = F(\mathbf{R})$. Thus, if F is an ESC SCF, then for every profile there is a strong Nash equilibrium profile that results in the same outcome, and therefore F is not necessarily distorted. Exactly and strongly consistent SCFs were introduced in Peleg (1978a).

Before we proceed to an investigation of ESC SCFs, we first consider a simple example. In our model an SCF $F : \rho \to A$ is *anonymous* if for all

$\mathbf{R}_1, \mathbf{R}_2 \in \rho$ we have:

(4) if $\lambda(\{t \in \Omega \mid \mathbf{R}_1(t) = R\}) = \lambda(\{t \in \Omega \mid \mathbf{R}_2(t) = R\})$ for all $R \in L(A)$ then $F(\mathbf{R}_1) = F(\mathbf{R}_2)$.

For a profile $\mathbf{R}$, call an alternative $a \in A$ *Pareto optimal* with respect to $\mathbf{R}$ if it is not Pareto dominated by some other element of A (see the beginning of Subsection 3.1), and denote by $\text{PAR}(\mathbf{R})$ the set of Pareto optimal alternatives with respect to $\mathbf{R}$.

Example 4.1 Let $s \in A$ be a designated alternative, called the status quo, and let $R_0 \in L(A)$ be fixed. Define an SCF $F : \rho \to A$ by

$$F(\mathbf{R}) = \begin{cases} s & \text{if } s \in \text{PAR}(\mathbf{R}) \\ a & \text{if } s \notin \text{PAR}(\mathbf{R}) \text{ and } a \text{ is the } R_0\text{-maximum} \\ & \quad \text{of } \{b \in \text{PAR}(\mathbf{R}) \mid b \text{ Pareto dominates } s\} \end{cases}$$

for all $\mathbf{R} \in \rho$. We show that F is an anonymous ESC SCF. Obviously, F is surjective. Now let $\mathbf{R} \in \rho$. We distinguish the following possibilities.

(i) $s \in \text{PAR}(\mathbf{R})$.

Let $\mathbf{Q} \in \rho$ satisfy $s\mathbf{Q}(t)a$ for all $t \in \Omega$ and $a \in A\backslash\{s\}$. Then $\mathbf{Q}$ is an SNE of $(F, \mathbf{R})$ and $F(\mathbf{Q}) = F(\mathbf{R})$.

(ii) $s \notin \text{PAR}(\mathbf{R})$.

Let q be the R_0-maximum of

$$B = \{b \in \text{PAR}(\mathbf{R}) \mid b \text{ Pareto dominates } s\}.$$

Define $\mathbf{Q} \in \rho$ by $q\mathbf{Q}(t)s\mathbf{Q}(t)a$ for all $t \in \Omega$ and $a \in A\backslash\{s, q\}$. Then $F(\mathbf{Q}) = q = F(\mathbf{R})$ and $\mathbf{Q}$ is an SNE of $(F, \mathbf{R})$. Indeed, Ω does not have a profitable deviation from $\mathbf{Q}$ since q is Pareto optimal with respect to $\mathbf{R}$. Now let $S \in \Sigma_+$, $\lambda(S) < 1$, and $\mathbf{V}^S \in \rho^S$. Then $F(\mathbf{Q}^{\Omega\backslash S}, \mathbf{V}^S) \in \{s, q\}$. Hence, $\mathbf{V}^S$ cannot be a profitable deviation for S.

In the remainder of this section we first study ESC social choice functions through their associated effectivity functions. Such effectivity functions provide an alternative description of the power that coalitions have as the result of using a specific social choice function in the society. This description is independent of the preference profile. We show that exact and strong consistency implies that the associated effectivity function has a number of interesting properties: maximality, stability, and convexity. Next, we narrow down on anonymous ESC SCFs.

4.1 Effectivity functions of ESC social choice functions

We start with introducing the concept of an effectivity function in our model. Then we derive the effectivity function associated with an ESC social choice function, and some interesting properties of it.

First a notation: for a set D we denote by $P(D)$ the set of all subsets of D and by $P_0(D) = P(D)\backslash\{\emptyset\}$ the set of all non-empty subsets of D.

Definition 4.2 An *effectivity function* (EF) is a function $E : \Sigma \to P(P_0(A))$ that satisfies the following conditions: (i) $E(\Omega) = P_0(A)$; (ii) $E(\emptyset) = \emptyset$; (iii) $A \in E(S)$ for every $S \in \Sigma_0$; and (iv) if $S_1, S_2 \in \Sigma_0$ and $\lambda(S_1\backslash S_2)+\lambda(S_2\backslash S_1) = 0$, then $E(S_1) = E(S_2)$.

If $B \in E(S)$ we sometimes say that S *is effective for* B: the interpretation is that the coalition S can guarantee that the outcome (alternative) is in B. Condition (iv) in Definition 4.2 is specific for our model. It says that the effectivity function does not distinguish between coalitions that differ only in a set of measure 0.

An effectivity function E is *superadditive* if for all $S_1, S_2 \in \Sigma$ with $S_1 \cap S_2 = \emptyset$ and all $B_1 \in E(S_1)$ and $B_2 \in E(S_2)$ we have: $B_1 \cap B_2 \in E(S_1 \cup S_2)$. The EF E is *monotonic* if for all $S, S^* \in \Sigma$ and $B, B^* \in P_0(A)$ with $B \in E(S)$, $S \subseteq S^*$ and $B \subseteq B^*$, we have $B^* \in E(S^*)$. These two conditions are natural and satisfied by many effectivity functions.

An EF E is *maximal* if for all $S \in \Sigma_0$ and $B \in P_0(A)$ we have: if $B \notin E(S)$ then $A\backslash B \in E(\Omega\backslash S)$.

Let $F : \rho \to A$ be a social choice function. We associate with F an effectivity function E^F as follows. Let $S \in \Sigma_0$ and let $B \in P_0(A)$. Call S effective for B if there exists an $\mathbf{R}^S \in \rho^S$ such that $F(\mathbf{R}^S, \mathbf{Q}^{\Omega\backslash S})$ is in B for every $\mathbf{Q}^{\Omega\backslash S} \in \rho^{\Omega\backslash S}$. Formally, $E^F(\emptyset) = \emptyset$ and for $S \in \Sigma\backslash\{\emptyset\}$

$$E^F(S) = \{B \in P_0(A) \mid S \text{ is effective for } B\}.$$

It is easy to see that E^F is superadditive and monotonic. We will prove that, if F is exactly and strongly consistent, then E^F is also maximal. We first prove the following lemma. (For readers familiar with the terminology, this lemma shows that for an ESC SCF α-effectivity and β-effectivity coincide.)

Lemma 4.3 *Let $F : \rho \to A$ be an ESC social choice function. Let $S \in \Sigma_0$ and $B \in P_0(A)$. Suppose that for every $\mathbf{R}^{\Omega\backslash S} \in \rho^{\Omega\backslash S}$ there exists $\mathbf{R}^S \in \rho^S$ such that $F(\mathbf{R}^S, \mathbf{R}^{\Omega\backslash S}) \in B$. Then $B \in E^F(S)$.*

Proof. Let $\mathbf{R}_0^S$ satisfy $b\mathbf{R}_0(t)a$ for all $b \in B$, $a \in A\backslash B$, and $t \in S$. We claim that $F(\mathbf{R}_0^S, \mathbf{R}^{\Omega\backslash S})$ is in B for all $\mathbf{R}^{\Omega\backslash S} \in \rho^{\Omega\backslash S}$. Indeed, assume to the contrary that there exists $\mathbf{R}_0^{\Omega\backslash S} \in \rho^{\Omega\backslash S}$ such that $F(\mathbf{R}_0) = y \notin B$. Let $\mathbf{Q}$ be an SNE of $(F, \mathbf{R}_0)$ such that $F(\mathbf{Q}) = y$. By the assumption in the lemma there exists $\mathbf{V}^S \in \rho^S$ such that $F(\mathbf{V}^S, \mathbf{Q}^{\Omega\backslash S}) = x \in B$. As $x\mathbf{R}_0(t)y$ for all $t \in S$, this contradicts the fact that $\mathbf{Q}$ is an SNE of $(F, \mathbf{R}_0)$. □

Lemma 4.3 has the following corollaries.

Corollary 4.4 *If $F : \rho \to A$ is an ESC social choice funtion, then E^F is maximal.*

Proof. Let $S \in \Sigma_0$ and $B \in P_0(A)$. If $B \notin E^F(S)$ then for every $\mathbf{R}^S \in \rho^S$ there exists an $\mathbf{R}^{\Omega\backslash S} \in \rho^{\Omega\backslash S}$ such that $F(\mathbf{R}) \in A\backslash B$. By Lemma 4.3, $A\backslash B \in E^F(\Omega\backslash S)$. □

Corollary 4.5 *Let $F : \rho \to A$ be ESC. Let $S \in \Sigma_0$ and $B \in E^F(S)$. If $\mathbf{R}^S \in \rho^S$ satisfies $b\mathbf{R}(t)a$ for all $b \in B$, $a \in A\backslash B$, and $t \in S$, then $F(\mathbf{R}^S, \mathbf{Q}^{\Omega\backslash S})$ is in B for all $\mathbf{Q}^{\Omega\backslash S} \in \rho^{\Omega\backslash S}$.*

Corollary 4.5 follows directly from the proof of Lemma 4.3.

We now turn to cores of effectivity functions. Let $E : \Sigma \to P(P_0(A))$ be an EF and let $\mathbf{R} \in \rho$. Let $B \in P_0(A)$, $x \in A\backslash B$, and $S \in \Sigma$. We say that B *dominates* x *via* S *at* $\mathbf{R}$ if $B \in E(S)$ and $b\mathbf{R}(t)x$ for all $b \in B$ and $t \in S$. Also, x is *dominated at* $\mathbf{R}$ if there exists $B \in P_0(A)$ and $S \in \Sigma$ such that B dominates x via S at $\mathbf{R}$. If b is not dominated at $\mathbf{R}$ then b is *undominated* at $\mathbf{R}$.

Definition 4.6 The *core* $C(E, \mathbf{R})$ is the set of all undominated alternatives at $\mathbf{R}$. The EF E is *stable* if $C(E, \mathbf{R}) \neq \emptyset$ for all $\mathbf{R} \in \rho$.

We next prove that the effectivity functions associated with ESC social choice functions are stable.

Theorem 4.7 *Let $F : \rho \to A$ be an ESC SCF. Then E^F is stable, and $F(\mathbf{R}) \in C(E^F, \mathbf{R})$ for all $\mathbf{R} \in \rho$.*

Proof. Let $\mathbf{R} \in \rho$ and $x = F(\mathbf{R})$. We claim that $x \in C(E^F, \mathbf{R})$. Indeed, assume to the contrary that x is dominated by B via S at $\mathbf{R}$. Let $\mathbf{Q}$ be an SNE of $(F, \mathbf{R})$ such that $x = F(\mathbf{Q})$. As $B \in E^F(S)$, there exists $\mathbf{V}^S \in \rho^S$ such that $y = F(\mathbf{V}^S, \mathbf{Q}^{\Omega \setminus S}) \in B$. Since $y\mathbf{R}(t)x$ for all $t \in S$, we arrive at the desired contradiction. □

Thus, we have proved that the effectivity function associated with an ESC social choice function is maximal and stable, and that the alternative assigned by the SCF is always in the core of the associated EF. Next, we show that such effectivity functions have special structure.

We call an effectivity function $E : \Sigma \to P(P_0(A))$ *subadditive* if for all $S_1, S_2 \in \Sigma$ and $B_1, B_2 \in P_0(A)$ with $B_1 \in E(S_1)$, $B_2 \in E(S_2)$ and $B_1 \cap B_2 = \emptyset$, we have $B_1 \cup B_2 \in E(S_1 \cap S_2)$.

For $S \in \Sigma$, $B, B' \subseteq A$ and $\mathbf{R}^S \in \rho^S$ we write $B\mathbf{R}^S B'$ if $b\mathbf{R}^S(t)b'$ for all $b \in B$, $b' \in B'$, and $t \in S$.

Lemma 4.8 *Let the effectivity function $E : \Sigma \to P(P_0(A))$ be maximal and stable. Then E is subadditive.*

Proof. Let $S_1, S_2 \in \Sigma$ and $B_1, B_2 \in P_0(A)$ with $B_1 \in E(S_1)$, $B_2 \in E(S_2)$ and $B_1 \cap B_2 = \emptyset$. Assume, contrary to what we wish to prove, that $B_1 \cup B_2 \notin E(S_1 \cap S_2)$. We distinguish the following possible cases:

(i) $S_1 \cap S_2 = \emptyset$.

Let $\mathbf{R} \in \rho$ satisfy $B_1\mathbf{R}^{S_1}A\backslash B_1$ and $B_2\mathbf{R}^{S_2}A\backslash B_2$. Then $C(E, \mathbf{R}) = \emptyset$, contradicting the stability of E.

(ii) $S_1 \cap S_2 \neq \emptyset$.

In this case $B_1 \cup B_2 \neq A$ (otherwise $B_1 \cup B_2 = A \in E(S_1 \cap S_2)$). By the maximality of E, $A\backslash(B_1 \cup B_2) \in E(\Omega\backslash(S_1 \cap S_2))$. Denote $B_3 = A\backslash(B_1 \cup B_2)$ and $S_3 = \Omega\backslash(S_1 \cap S_2)$. Consider the profile $\mathbf{R}$ given by the following table:

$R^{S_1 \cap S_2}$	$R^{S_1 \backslash S_2}$	$R^{\Omega \backslash S_1}$
B_1	B_3	B_2
B_2	B_1	B_3
B_3	B_2	B_1.

As $B_1\mathbf{R}^{S_1}B_2$, $B_3\mathbf{R}^{S_3}B_1$, and $B_2\mathbf{R}^{S_2}B_3$, it follows that $C(E, \mathbf{R}) = \emptyset$, contradicting the stability of E. □

The proof of the following lemma is similar to that of Lemma 4.8 and therefore omitted.

Lemma 4.9 *Let the effectivity function* $E : \Sigma \to P(P_0(A))$ *be maximal and stable. Then* E *is superadditive.*

We call an effectivity function *convex* if for all $S_1, S_2 \in \Sigma$ and $B_1 \in E(S_1)$, $B_2 \in E(S_2)$ we have $B_1 \cap B_2 \in E(S_1 \cup S_2)$ or $B_1 \cup B_2 \in E(S_1 \cap S_2)$.

We conclude this part with the following result.

Theorem 4.10 *A maximal and stable effectivity function is convex.*

Observe that Theorem 4.10 strengthens Lemmas 4.8 and 4.9 since convexity implies subadditivity and as superadditivity, as is easy to check.

The proof of Theorem 4.10 is completely analogous to the proof of Theorem 6.A.9 in Peleg (1984). Theorem 4.10 implies in particular that the effectivity function associated with an ESC social choice function is convex.

4.2 The blocking coefficients of an anonymous ESC social choice function

In the remainder of Section 4 we concentrate on anonymous ESC social choice functions. Anonymity is a natural requirement for voting procedures. Moreover, imposing this condition will enable us to derive much more detailed results on both social choice functions and effectivity functions.

Let $F : \rho \to A$ be an anonymous ESC social choice function, with associated effectivity function E^F. In Subsection 4.1 we established that E^F is maximal, stable, and therefore convex. Here, we will study the additional implications of anonymity.

For $B \in P_0(A) \backslash \{A\}$, the *blocking coefficient* is the real number

(5) $\beta(B) = \inf\{\lambda(S) \mid A \backslash B \in E^F(S)\}.$

The number $\beta(B)$ is well defined since F is anonymous. It represents the minimum size of a blocking coalition of B (cf. Corollary 4.5). We call B an *e-set* ("e" from "equality") if $S \in \Sigma$ and $\lambda(S) = \beta(B)$ imply that $A \backslash B \in E^F(S)$; otherwise, B is called an *i-set* ("i" from "inequality").

If $B_1, B_2 \in P_0(A)\backslash\{A\}$ and $B_1 \cup B_2 \neq A$, then

(6) $\beta(B_1 \cup B_2) \leq \beta(B_1) + \beta(B_2)$.

To see this, note that we may assume that the right hand side is smaller than $\lambda(\Omega)$. Let $\varepsilon > 0$ be small and let $S_i \in \Sigma$ with $\lambda(S_i) = \beta(B_i) + \varepsilon$ and $A\backslash B_i \in E(S_i)$ for $i = 1, 2$, such that $S_1 \cap S_2 = \emptyset$. By superadditivity, $A\backslash(B_1 \cup B_2) \in E(S_1 \cup S_2)$, hence $\beta(B_1 \cup B_2) \leq \beta(B_1) + \beta(B_2) + 2\varepsilon$. By letting ε approach 0, (6) follows.

For every $B \in P_0(A)\backslash\{A\}$ we have

(7) $\beta(B) + \beta(A\backslash B) \geq \lambda(\Omega)$

because otherwise there would be disjoint coalitions S and T with $B \in E^F(S)$ and $A\backslash B \in E^F(T)$, contradicting the superadditivity of E^F. We shall now show the reverse inequality. Assume $\beta(B) > 0$ otherwise there is nothing left to prove. For every $0 < \delta < \beta(B)$ and $S \in \Sigma$ with $\lambda(S) = \delta$ we have $A\backslash B \notin E^F(S)$. Hence by maximality of E^F, $B \in E^F(\Omega\backslash S)$, so $\beta(A\backslash B) \leq \lambda(\Omega) - \delta$. This implies the reverse inequality of (7), hence

(8) $\beta(B) + \beta(A\backslash B) = \lambda(\Omega)$

for every $B \in P_0(A)\backslash\{A\}$. Superadditivity, maximality and (8) imply

(9) B is an e-set $\Leftrightarrow A\backslash B$ is an i-set

for every $B \in P_0(A)\backslash\{A\}$.

Moreover, monotonicity of E^F clearly implies monotonicity of the function $\beta(\cdot)$:

(10) $B_1 \subseteq B_2 \Rightarrow \beta(B_1) \leq \beta(B_2)$

for all $B_1, B_2 \in P_0(A)\backslash\{A\}$.

We now show that blocking coefficients are actually additive.

Theorem 4.11 *$\beta(\cdot)$ is additive.*

Proof. Let $B_i \in P_0(A)$, $i = 1, 2$, with $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 \neq A$. In view of (6) it is sufficient to prove that $\beta(B_1 \cup B_2) \geq \beta(B_1) + \beta(B_2)$. By (10) we may assume $\beta(B_i) > 0$ for $i = 1, 2$. Let S and T satisfy $\lambda(S) < \beta(B_1)$, $\lambda(T) < \beta(B_2)$, and $S \cap T = \emptyset$. Then by (8), $B_1 \in E^F(\Omega\backslash S)$ and $B_2 \in E^F(\Omega\backslash T)$.

By the subadditivity of E^F (Lemma 4.8), $B_1 \cup B_2 \in E^F(\Omega \backslash (S \cup T))$. Thus, by definition of $\beta(\cdot)$ and superadditivity of E^F, $\beta(B_1 \cup B_2) \geq \lambda(S) + \lambda(T)$. Since, by (8) and (10), $\beta(B_1) + \beta(B_2) \leq \lambda(\Omega)$, we can choose $\lambda(S)$ and $\lambda(T)$ as close to $\beta(B_1)$ and $\beta(B_2)$, respectively, as desired, which completes the proof. □

We note that Theorem 4.11 is a substantial deviation from the case with finitely many voters, see Theorem 5.2.16 in Peleg (1984).

In view of Theorem 4.11 and (8) it is useful to define $\beta(A) = \lambda(\Omega)$ and let A be an i-set.

For e-sets we have the following corollary.

Corollary 4.12 *If B_1 and B_2 are disjoint e-sets, then $B_1 \cup B_2$ is an e-set.*

Proof. Let B_1 and B_2 be disjoint e-sets. Take disjoint coalitions S_1 and S_2 of sizes $\beta(B_1)$ and $\beta(B_2)$, respectively, then $A \backslash B_1 \in E^F(S_1)$ and $A \backslash B_2 \in E^F(S_2)$. By superadditivity, $A \backslash (B_1 \cup B_2) \in E^F(S_1 \cup S_2)$. Since $\lambda(S_1 \cup S_2) = \beta(B_1 \cup B_2)$ by Theorem 4.11, we conclude that $B_1 \cup B_2$ is an e-set. □

A similar result holds for i-sets:

Corollary 4.13 *If B_1 and B_2 are disjoint i-sets, then $B_1 \cup B_2$ is an i-set.*

Proof. Let B_1 and B_2 be disjoint i-sets. We are done if $B_1 \cup B_2 = A$. Otherwise, let $S, T, U \in \Sigma$ be pairwise disjoint sets with $U = \Omega \backslash (S \cup T)$ and with $\lambda(S) = \beta(B_1)$ and $\lambda(T) = \beta(B_2)$. Then $A \backslash B_1 \notin E^F(S)$ and $A \backslash B_2 \notin E^F(T)$. Then by maximality $B_1 \in E^F(U \cup T)$ and $B_2 \in E^F(U \cup S)$, so by subadditivity $B_1 \cup B_2 \in E^F(U)$. Therefore, by superadditivity, $A \backslash (B_1 \cup B_2) \notin E^F(S \cup T)$. This implies that $B_1 \cup B_2$ is an i-set. □

Example 4.14 For the effectivity function associated with the ESC SCF of Example 4.1 we have $\beta(s) = \lambda(\Omega)$ and $\beta(a) = 0$ for $a \in A \backslash \{s\}$: any positive coalition can block any set of alternatives not containing s by putting s on top of its preference profile. The i-sets are A and every $B \subseteq A \backslash \{s\}$. In particular, if $|A| \geq 3$ and $a \in A \backslash \{s\}$ then $\{a, s\}$ is an e-set.

A final observation concerning e-sets is stated in the following lemma.

Lemma 4.15 *If B_1 and B_2 are e-sets, then $B_1 \cap B_2$ or $B_1 \cup B_2$ are e-sets.*

Proof. Let B_1 and B_2 be e-sets. In view of Corollary 4.12 we may assume $B_1 \cap B_2 \neq \emptyset$. Choose pairwise disjoint sets S_1, S_2, and S_3 in Σ_0 such that $\lambda(S_1) = \beta(B_1 \cap B_2)$, $\lambda(S_2) = \beta(B_1) - \beta(B_1 \cap B_2)$, and $\lambda(S_3) = \beta(B_2) - \beta(B_1 \cap B_2)$. Define $T_1 = S_1 \cup S_2$ and $T_2 = S_1 \cup S_3$. Then $\lambda(T_1) = \beta(B_1)$, $\lambda(T_2) = \beta(B_2)$, $\lambda(T_1 \cap T_2) = \beta(B_1 \cap B_2)$, and $\lambda(T_1 \cup T_2) = \beta(B_1 \cup B_2)$. By assumption, $A \backslash B_1 \in E^F(T_1)$ and $A \backslash B_2 \in E^F(T_2)$. Since E^F is convex (Theorem 4.10), $A \backslash (B_1 \cup B_2) \in E^F(T_1 \cup T_2)$ or $A \backslash (B_1 \cap B_2) \in E^F(T_1 \cap T_2)$. Thus, $B_1 \cup B_2$ or $B_1 \cap B_2$ are e-sets. □

In the next section we will consider effectivity functions satisfying all properties on e-sets and i-sets derived above but not necessarily derived from ESC social choice functions.

4.3 Systems of e-sets and i-sets

We start by specifying a system of e-sets and i-sets and associated blocking coefficients, imposing the conditions that were derived above as properties of the effectivity function E^F associated with an anonymous ESC social choice function F. Then we define the effectivity function E associated with this system, and show that E shares the main properties of an effectivity function E^F, namely, maximality, convexity, and stability.

Let $\beta : P_0(A) \to [0, \lambda(\Omega)]$ and let $\{\mathbf{i}, \mathbf{e}\}$ be a partition of $P_0(A)$ satisfying

(11) β is additive, and $\beta(A) = \lambda(\Omega)$.

(12) For all $B \in P_0(A) \backslash \{A\}$, $B \in \mathbf{e} \Leftrightarrow A \backslash B \in \mathbf{i}$, and $A \in \mathbf{i}$.

(13) For all $B_1, B_2 \in \mathbf{e}$, we have $B_1 \cap B_2 \in \mathbf{e}$ or $B_1 \cup B_2 \in \mathbf{e}$.

Properties (11)–(13) are also the properties of the e-sets and i-sets of the EF associated with an anonymous ESC social choice function, as derived in Subsection 4.2 (the property in Corollary 4.13 can be derived from the three conditions above). Next, for a system $(\beta; \mathbf{e}, \mathbf{i})$ satisfying (11)–(13), we define an effectivity function E by $E(\emptyset) = \emptyset$ and

(14) For all $B \in \mathbf{e}$ and $S \in \Sigma_0$, if $\lambda(S) \geq \beta(B)$ then $A \backslash B \in E(S)$.

(15) For all $B \in \mathbf{i}$ and $S \in \Sigma$, if $\lambda(S) > \beta(B)$ then $A \backslash B \in E(S)$.

We now show that the effectivity function E defined in this way shares the following properties of an EF derived from an anonymous ESC SCF: maximality, convexity, and stability.

Theorem 4.16 *Let $(\beta; \mathbf{e}, \mathbf{i})$ be a system satisfying (11)–(13) and let E be the derived effectivity function. Then E is maximal and convex.*

Proof. Maximality of E is straightforward, so we only prove convexity. Let $B_1, B_2 \in P_0(A)$ and $A \backslash B_i \in E(S_i)$ for $i = 1, 2$. Then $\lambda(S_1) \geq \beta(B_1)$ and $\lambda(S_2) \geq \beta(B_2)$, hence

$$
\begin{aligned}
(16) \quad \lambda(S_1 \cap S_2) + \lambda(S_1 \cup S_2) &= \lambda(S_1) + \lambda(S_2) \\
&\geq \beta(B_1) + \beta(B_2) \\
&= \begin{cases} \beta(B_1 \cap B_2) + \beta(B_1 \cup B_2) & \text{if } B_1 \cap B_2 \neq \emptyset \\ \beta(B_1 \cup B_2) & \text{if } B_1 \cap B_2 = \emptyset. \end{cases}
\end{aligned}
$$

If $\lambda(S_1 \cap S_2) > \beta(B_1 \cap B_2)$ then $(A \backslash B_1) \cup (A \backslash B_2) \in E(S_1 \cap S_2)$; and if $\lambda(S_1 \cup S_2) > \beta(B_1 \cup B_2)$ then $(A \backslash B_1) \cap (A \backslash B_2) \in E(S_1 \cup S_2)$; so we are done. Otherwise, there are only equality signs in (16), which implies that both B_1 and B_2 are in $\mathbf{e}$. In that case, the desired result follows from (13). □

Theorem 4.17 *Let $(\beta; \mathbf{e}, \mathbf{i})$ be a system satisfying (11)–(13) and let E be the derived effectivity function. Then E is stable.*

Proof. The theorem follows from Theorem 4.16 and stability of any convex EF in our model, see Theorem 6.1 in the Appendix. □

In the remainder of Section 4 we consider the following question: Given a system $(\beta; \mathbf{e}, \mathbf{i})$ and associated effectivity function E, is there an (anonymous) ESC social choice function F such that $E = E^F$? We tackle this problem through the study of so-called feasible elimination procedures (Subsection 4.4) and their relation with the core (Subsection 4.5). Our affirmative answers to this question are collected in Corollary 4.27. A counterexample, showing that the answer to the question in complete generality is negative, is given in Subsection 4.6.

4.4 Feasible elimination procedures

In this section we describe a procedure that, in the end, will result in an anonymous ESC social choice function. Later, in Subsection 4.5, we will see that in this way we obtain a characterization of a class of ESC social choice functions.

Let $|A| = m \geq 2$. Let $s \in A$ be a designated element, sometimes called the *status quo*, and let real numbers $\beta(a) \geq 0$ be given with $\sum_{a \in A} \beta(a) = \lambda(\Omega)$ and with $\beta(a) > 0$ for all $a \neq s$.

Definition 4.18 Let $\mathbf{R} \in \rho$. A *pseudo feasible elimination procedure* (p.f.e.p.) is a sequence $(x_{i_1}, C_1; \ldots; x_{i_{m-1}}, C_{m-1}; x_{i_m})$ that satisfies the following conditions:

(17) $A = \{x_{i_1}, \ldots, x_{i_m}\}$;

(18) for all $j, k = 1, \ldots, m-1$ with $j \neq k$, we have
$$C_j \subseteq \Omega,\ C_j \cap C_k = \emptyset,\ \text{and } \lambda(C_j) \geq \beta(x_{i_j});$$

(19) for all $j = 1, \ldots, m-1$ and all $t \in C_j$,
$$y\mathbf{R}(t)x_{i_j} \text{ for all } y \in \{x_{i_j}, \ldots, x_{i_m}\}.$$

As $\sum_{a \in A} \beta(a) = \lambda(\Omega)$, it is obvious that for each profile there always exists at least one p.f.e.p. A more demanding procedure is the following.

Definition 4.19 Let $\mathbf{R} \in \rho$. A p.f.e.p. $(x_{i_1}, C_1; \ldots; x_{i_{m-1}}, C_{m-1}; x_{i_m})$ is a *feasible elimination procedure* (f.e.p.) if it satisfies the following condition:

(20) $x_{i_m} = s$ or $[s = x_{i_j}$ for some $j < m$ and $\lambda(C_j) > \beta(s)]$.

We shall now prove the existence of f.e.p.'s in our model and then relate them to ESC social choice functions. We start with the following lemma.

Lemma 4.20 *Let* $\mathbf{R} \in \rho$ *and let* $x \in A$ *satisfy*
$$\lambda(\{t \in \Omega \mid y\mathbf{R}(t)x \textit{ for all } y \in A\}) > \beta(x).$$
Then there exists a p.f.e.p. $(x, C_x; x_{i_1}, C_1; \ldots; x_{i_{m-1}})$ *with* $\lambda(C_x) > \beta(x)$.

Proof. The proof will be by induction on m. The case $m = 2$ is obvious. Let $m \geq 3$. We define

(21) $A^* = \{y \in A \mid \lambda(\{t \in \Omega \mid z\mathbf{R}(t)y \text{ for all } z \in A\}) > \beta(y)\}$.

By assumption, $x \in A^*$. We distinguish the following cases.

(i) $|A^*| \geq 2$.

Let $y \in A^* \backslash \{x\}$ and choose $C_y \subseteq \Omega$ such that $\lambda(C_y) = \beta(y)$ and $C_y \subseteq \{t \in \Omega \mid z\mathbf{R}(t)y \text{ for all } z \in A\}$. Define the profile $\mathbf{Q} \in \rho$ as follows. If $t \in \Omega \backslash C_y$ with $z\mathbf{R}(t)y$ for all $z \in A$, then let $x\mathbf{Q}(t)A\backslash\{x, y\}\mathbf{Q}(t)y$; otherwise, $\mathbf{Q}(t) = \mathbf{R}(t)$. Consider the restricted profile $\mathbf{Q}_1 = \mathbf{Q}^{\Omega \backslash C_y}|A\backslash\{y\}$. By the induction hypothesis and by the construction of $\mathbf{Q}$ there exists a p.f.e.p. $(x, C_x; x_{i_1}, C_1; \ldots; x_{i_{m-2}})$ with respect to $\mathbf{Q}_1$ such that $\lambda(C_x) > \beta(x)$ and $C_x \subseteq \{t \in \Omega \mid z\mathbf{R}(t)x \text{ for all } z \in A\}$. Then the p.f.e.p.
$(x, C_x; y, C_y; x_{i_1}, C_1; \ldots; x_{i_{m-2}})$ is as required.

(ii) $A^* = \{x\}$.

Let $\hat{C}_x$ satisfy $\hat{C}_x \subseteq \{t \in \Omega \mid y\mathbf{R}(t)x \text{ for all } y \in A\}$ and $\lambda(\hat{C}_x) = \beta(x)$. Consider the profile $\mathbf{R}_1 = \mathbf{R}^{\Omega \backslash \hat{C}_x}|A\backslash\{x\}$. For $y \neq x$ let $C_y = \{t \in \Omega \backslash \hat{C}_x \mid z\mathbf{R}(t)y \text{ for all } z \in A\backslash\{x\}\}$. We distinguish two subcases.

(ii.1) $\lambda(C_y) = \beta(y)$ for all $y \neq x$.

Choose $\bar{y} \in A\backslash\{x\}$ such that

$$\lambda(\{t \in \Omega \backslash \hat{C}_x \mid z\mathbf{R}(t)\bar{y}\mathbf{R}(t)x \text{ for all } z \in A\backslash\{x\}\}) > 0.$$

Let $\hat{C} \subseteq \{t \in \Omega \backslash \hat{C}_x \mid z\mathbf{R}(t)\bar{y}\mathbf{R}(t)x \text{ for all } z \in A\backslash\{x\}\}$ satisfy $\lambda(\hat{C}) > 0$. (Observe that $\hat{C} \subseteq C_{\bar{y}}$, hence $\lambda(\hat{C}) \leq \beta(\bar{y})$.) Let $C_x = \hat{C}_x \cup \hat{C}$, and let $A\backslash\{x, \bar{y}\} = \{y_1, \ldots, y_{m-2}\}$. Then $(x, C_x; y_1, C_{y_1}; \ldots; y_{m-2}, C_{y_{m-2}}; \bar{y})$ is a p.f.e.p. as required.

(ii.2) There exists $\bar{y} \neq x$ such that $\lambda(C_{\bar{y}}) > \beta(\bar{y})$.

By the induction hypothesis there exists a p.f.e.p. $(\bar{y}, \hat{C}_{\bar{y}}; x_{i_1}, C_1; \ldots, x_{i_{m-2}})$ with respect to $\mathbf{R}_1$ such that $\lambda(\hat{C}_{\bar{y}}) > \beta(\bar{y})$. Choose $\hat{C} \subseteq \{t \in \hat{C}_{\bar{y}} \mid z\mathbf{R}(t)\bar{y}\mathbf{R}(t)x \text{ for all } z \in A\backslash\{x\}\}$ such that $0 < \lambda(\hat{C}) \leq \lambda(\hat{C}_{\bar{y}}) - \beta(\bar{y})$. Then $(x, \hat{C}_x \cup \hat{C}; \bar{y}, \hat{C}_{\bar{y}}\backslash\hat{C}; x_{i_1}, C_1; \ldots; x_{i_{m-2}})$ is a p.f.e.p. as required. □

Next, we establish the existence of feasible elimination procedures.

Theorem 4.21 *For every $\mathbf{R} \in \rho$ there is an f.e.p. with respect to $\mathbf{R}$.*

Proof. Let $\mathbf{R} \in \rho$. The proof is by induction on m. The case $m = 2$ is obvious. Let $m \geq 3$. Define A^* as in (21). We distinguish the following possibilities.

(i) $A^* = \emptyset$.

For $a \in A$ let $C(a) = \{t \in \Omega \mid y\mathbf{R}(t)a \text{ for all } y \in A\}$. Then $\lambda(C(a)) = \beta(a)$ for all $a \in A$. Let $A \backslash \{s\} = \{a_1, \ldots, a_{m-1}\}$.
Then $(a_1, C(a_1); \ldots; a_{m-1}, C(a_{m-1}); s)$ is an f.e.p.

(ii) $A^* \neq \emptyset$ and $s \notin A^*$.

Let $y \in A^*$ and let $C_y \subseteq \{t \in \Omega \mid z\mathbf{R}(t)y \text{ for all } z \in A\}$ satisfy $\lambda(C_y) = \beta(y)$. By the induction hypothesis for $\mathbf{R}^{\Omega \backslash C_y}|A \backslash \{y\}$ there exists an f.e.p. $(x_{i_1}, C_1; \ldots; x_{i_{m-1}})$ for the restricted profile. Then $(y, C_y; x_{i_1}, C_1; \ldots; x_{i_{m-1}})$ is an f.e.p. for $\mathbf{R}$.

(iii) $s \in A^*$.

This case follows from Lemma 4.20. □

We shall use the existence of feasible elimination procedures established in Theorem 4.21 to derive the existence of an interesting class of ESC social choice functions. Let $\mathbf{R} \in \rho$. Call $x \in A$ $\mathbf{R}$-*maximal* if there exists an f.e.p. $(x_{i_1}, C_1; \ldots; x_{i_m})$ with respect to $\mathbf{R}$ such that $x = x_{i_m}$. Further, denote

$$M(\mathbf{R}) = \{x \in A \mid x \text{ is } \mathbf{R}\text{-maximal}\}.$$

$M(\cdot)$ is an anonymous social choice correspondence that satisfies Pareto. (A social choice *correspondence* is set-valued; the definitions of anonymity and Pareto are analogous to those for SCFs.)

The following remark will be very useful in the sequel.

Remark 4.22 Let $\mathbf{R} \in \rho$ and let $x \in A \backslash \{s\}$ satisfy

$$\lambda(\{t \in \Omega \mid y\mathbf{R}(t)x \text{ for all } y \in A\}) \geq \beta(x).$$

Then $x \notin M(\mathbf{R})$. This is so since $\lambda(\bigcup_{y \in A \backslash \{x\}} \{t \in \Omega \mid A \backslash \{y\}\mathbf{R}(t)y\}) \leq \lambda(\Omega) - \beta(x)$ and s has to be eliminated strictly in an f.e.p.

Theorem 4.23 *Let the SCF $F : \rho \to A$ be a selection from $M(\cdot)$, that is, $F(\mathbf{R}) \in M(\mathbf{R})$ for every $\mathbf{R} \in \rho$. Then F is ESC.*

Proof. Let $\mathbf{R} \in \rho$ and $x = F(\mathbf{R})$. Then there exists an f.e.p. $(x_{i_1}, C_1; \ldots; x_{i_{m-1}}, C_{m-1}; x)$ with respect to $\mathbf{R}$. Choose $\mathbf{Q} \in \rho$ that satisfies $y\mathbf{Q}(t)x_{i_j}$ for all $t \in C_j$, $y \in A$, and $j = 1, \ldots, m-1$. We claim that $F(\mathbf{Q}) = F(\mathbf{R})$ and that $\mathbf{Q}$ is an SNE of the game $(F, \mathbf{R})$. We distinguish the following cases.

(i) $x = s$.

By Remark 4.22 $F(\mathbf{Q}) = s$. Now assume, on the contrary, that $\mathbf{Q}$ is not an SNE of $(F, \mathbf{R})$. Then there exist $S \in \Sigma_+$ and $\mathbf{V}^S \in \rho^S$ such that $F(\mathbf{Q}^{\Omega \setminus S}, \mathbf{V}^S) = y$, $y \neq s$, and $y\mathbf{R}(t)s$ for all $t \in S$. Let $y = x_{i_j}$ for some $1 \leq j \leq m-1$. Then $S \cap C_j = \emptyset$ because $s\mathbf{R}(t)x_{i_j}$ for all $t \in C_j$. Hence, by Remark 4.22, $F(\mathbf{Q}^{\Omega \setminus S}, \mathbf{V}^S) \neq x_{i_j}$, which is the desired contradiction.

(ii) $x \neq s$.

Then $s = x_{i_{j_0}}$ for some $j_0 \leq m-1$. Hence, by definition of an f.e.p., $\lambda(C_{j_0}) > \beta(s)$. Hence, it is not possible to eliminate all $x \neq s$ in an f.e.p. with respect to $\mathbf{Q}$, and therefore $F(\mathbf{Q}) \neq s$. Thus, by Remark 4.22 applied to all $x' \in A \setminus \{x, s\}$, $F(\mathbf{Q}) = x$. The proof that $\mathbf{Q}$ is an SNE of $(F, \mathbf{R})$ is analogous to that in case (i), observing that a profitable deviation from $\mathbf{Q}$ can never result in s since $\lambda(C_{j_0}) > \beta(s)$. □

Let $\hat{F}$ be an anonymous selection from $M(\cdot)$. (E.g., for every $\mathbf{R} \in \rho$ select the maximal element in $M(\mathbf{R})$ according to a fixed order $R_0 \in L(A)$.) By Theorem 4.23 $\hat{F}$ is an anonymous ESC SCF, and therefore its associated effectivity function $E^{\hat{F}}$ is characterized by blocking coefficients (say) $\hat{\beta}(B)$ for $B \in P_0(A)$ (cf. (5)). Since alternatives assigned by $\hat{F}$ result from feasible elimination procedures with weights $\beta(a)$ ($a \in A$), it is easy to check that $\hat{\beta}(a) = \beta(a)$ for every $a \in A$, and that $\{s\}$ is an i-set whereas all other singleton sets are e-sets. By the results established in Subsection 4.2, it follows that a set $B \subseteq A$ is an i-set if and only if it contains s. Also, Theorems 4.7 and 4.23 imply that $M(\mathbf{R}) \subseteq C(E^{\hat{F}}, \mathbf{R})$ for all $\mathbf{R} \in \rho$. More generally, $M(\mathbf{R}) \subseteq C(E, \mathbf{R})$ for all $\mathbf{R} \in \rho$, where E is the effectivity function associated with the system $(\beta, \mathbf{e}, \mathbf{i})$ as above (cf. Subsection 4.3).

In the next section we shall establish a converse of these observations.

4.5 Core and feasible elimination procedures

In this section we prove that for any anonymous ESC social choice function that generates positive blocking coefficients for the e-alternatives and that results in exactly one i-alternative, every element in the core of the associated effectivity function can be obtained by a feasible elimination procedure. The proof basically uses an extension of the "marriage theorem" to our model. We comment on this in Remark 4.25.

Let $(\beta; \mathbf{e}, \mathbf{i})$ be a system satisfying (11)–(13) with $\mathbf{i}$ containing exactly one singleton $\{s\}$ for some designated $s \in A$, and with coefficients $\beta(y)$ for $y \in A \backslash \{s\}$ positive. Let E be the associated effectivity function and let $\mathbf{R} \in \rho$ and $x \in C(E, \mathbf{R})$ (cf. Theorem 4.17). For every $y \in A \backslash \{x\}$ denote

$$S(y) = \{t \in \Omega \mid x\mathbf{R}(t)y\}.$$

The fact that $x \in C(E, \mathbf{R})$ is equivalent to

$$(22) \quad \begin{aligned} &\lambda(\bigcup_{y \in B} S(y)) \geq \beta(B) \quad \text{if } s \notin B \\ &\lambda(\bigcup_{y \in B} S(y)) > \beta(B) \quad \text{if } s \in B. \end{aligned}$$

In this setting we have the following result.

Theorem 4.24 *There exist disjoint measurable sets $C(y)$ ($y \in A \backslash \{x\}$) such that (i) $C(y) \subseteq S(y)$ for every $y \in A \backslash \{x\}$; (ii) $\lambda(C(y)) \geq \beta(y)$ for all $y \neq x$ and $\lambda(C(s)) > \beta(s)$.*

Proof. Before we start with the actual proof we note that, if $x \neq s$, we may increase $\beta(s)$ with a small $\varepsilon > 0$ and decrease $\beta(x)$ with the same amount (note that $\beta(x) > 0$). In this way, all inequalities in (22) still hold as weak inequalities and it is sufficient to prove (ii) in the theorem with only weak inequalities. Moreover, we may regard x as the i-alternative instead of s. For the rest of the proof we assume that this is the case.

We prove the theorem by induction on $|A| = m \geq 2$. The case $m = 2$ is obvious, so we concentrate on the induction step for $m \geq 3$. We first make the following observation.

Remark. Suppose there exists a set $B^* \subseteq A \backslash \{x\}$ with $\emptyset \neq B^* \neq A \backslash \{x\}$, such that $\lambda(\bigcup_{y \in B^*} S(y)) = \beta(B^*)$. Then, for every $B \subseteq A \backslash (\{x\} \cup B^*)$ we have $\lambda(\bigcup_{y\in B} S(y)\backslash \bigcup_{\hat{y}\in B^*} S(\hat{y})) \geq \beta(B)$. Hence, we can decompose our problem into two smaller problems, namely: (i) the problem with set of alternatives $A\backslash B^*$, set of voters $\Omega\backslash \bigcup_{y\in B^*} S(y)$, blocking coefficients unchanged, and preferences $\mathbf{R}(t)$ restricted to $A\backslash B^*$; (ii) the problem with set of alternatives $B^* \cup \{x\}$, set of agents $\bigcup_{y\in B^*} S(y)$, blocking coefficients $\hat{\beta}(x) = 0$ and $\hat{\beta}(y) = \beta(y)$ unchanged for $y \in B^*$, and preferences $\mathbf{R}(t)|B^* \cup \{x\}$ for $t \in \bigcup_{y\in B^*} S(y)$. Then we can apply the induction hypothesis to both smaller problems and find sets $C(y)$ $(y \in A\backslash\{x\})$ as required.

We now proceed to the induction step. Let $m \geq 3$. We are done if there is a decomposition possible as in the Remark, so suppose there is none. Let $b \in A\backslash\{x\}$ and consider the set $S = S(b)\backslash \bigcup_{y\in A\backslash\{x,b\}} S(y)$. We distinguish two cases.

Case 1: $\lambda(S) \geq \beta(b)$. Note that for every $t \in S$, $y\mathbf{R}(t)x\mathbf{R}(t)b$ for all $y \neq x, b$. Hence, since $x \in C(E, \mathbf{R})$, $0 \leq \lambda(S) \leq \beta(x) + \beta(b)$. Now take $C(b)$ equal to S, and apply the induction hypothesis to the problem with set of alternatives $A\backslash\{b\}$, set of voters $\Omega\backslash S$, blocking weights β' unchanged except $\beta'(x) = \beta(x) - (\lambda(S) - \beta(b))$, and preferences equal to the original preferences restricted to $A\backslash\{b\}$.

Case 2: $\lambda(S) < \beta(b)$. We also know $\lambda(S(b)) > \beta(b)$ otherwise $\lambda(S(b)) = \beta(b)$ by (22), and we would have a decomposition as in the Remark with $B^* = \{b\}$. Now choose a measurable set S^* satisfying $S \subseteq S^* \subseteq S(b)$ and $\lambda(S^*) = \beta(b)$ (this is possible by Lyapunov's theorem). Consider the set of vectors

$$\text{(23)} \quad \{\lambda(S^* \cup T \cup \bigcup_{y\in B} S(y))_{B\subseteq A\backslash\{b,x\},\ B\neq A\backslash\{b,x\}} \mid \emptyset \subseteq T \subseteq S(b)\backslash S^*\}.$$

For $T = S(b)\backslash S^*$ and $B = \emptyset$ we have

$$\text{(24)} \quad \lambda(S^* \cup T \cup \bigcup_{y\in B} S(y)) = \lambda(S(b)) > \beta(b)$$

and for $T = S(b)\backslash S^*$ and $B \subseteq A\backslash\{x, b\}$ arbitrary we have

$$\text{(25)} \quad \lambda(S^* \cup T \cup \bigcup_{y\in B} S(y)) \geq \beta(b) + \beta(B)$$

by (22). For $B \subseteq A\backslash\{x, b\}$ with $B \neq A\backslash\{x, b\}$ and $T \subseteq S(b)\backslash S^*$ consider the expression

$$\lambda(S^* \cup T \cup \bigcup_{y\in B} S(y)) \quad = \quad \lambda(T) + \lambda(S^* \cup \bigcup_{y\in B} S(y))$$

$$-\lambda(T \cap (S^* \cup \bigcup_{y \in B} S(y))).$$

This is an affine function of two measures $\lambda(T)$ and $\lambda(T \cap (S^* \cup \bigcup_{y \in B} S(y)))$. As B varies on $\{B' \mid B' \subseteq A \backslash \{b, x\},\ B' \neq A \backslash \{b, x\}\}$ we obtain an affine combination of two vector measures. Hence, its range (23) is compact and convex by Lyapunov's theorem. By (24) and (25) we can choose $T = T_0$ such that all inequalities in (25) are still valid but with at least one equality, say for B_0. Now set $S_0 = S^* \cup T_0$, and set $B^* = B_0 \cup \{b\}$. On $S(b) \backslash S_0$ change the preferences by shifting b over x. The problem with the new profile is decomposable according to the Remark. Applying the Remark, we obtain the desired sets: in particular, the resulting set $C(b)$ is a subset of S_0 and therefore of $S(b)$. This concludes the proof of the theorem. $\square$

Remark 4.25 Consider the following continuous version of the discrete "marriage theorem" (cf. Halmos and Vaughan, 1950). Let $S_1, \ldots, S_k$ be measurable sets and let $\beta_1, \ldots, \beta_k$ be nonnegative real numbers such that for all nonempty subsets $B \subseteq \{1, \ldots, k\}$ we have $\lambda(\bigcup_{i \in B} S_i) \geq \sum_{i \in B} \beta_i$. Then there exist disjoint measurable sets $C_1, \ldots, C_k$ with $C_i \subseteq S_i$ and $\lambda(C_i) \geq \beta_i$ for every $i \in \{1, \ldots, k\}$. As an interpretation of this result, think of $1, \ldots, k$ as the names of different crops, the S_i as the parcels of land suited to grow crop i, and the β_i as the required minimal area to grow crop i in order to reach a certain production level. Alternatively, i may be a computer program, S_i the part of memory suited to store program i, and β_i the minimal storage requirement. The "marriage theorem" provides conditions under which it is possible to reach the required production of crops, or storage of computer programs. Theorem 4.24 gives a version of this theorem suited for our context, in particular also for deriving Theorem 4.26 below. For a proof of a slightly less general version of the continuous "marriage theorem" see Hart and Kohlberg (1974, p. 171).

Theorem 4.24 can be used to prove the following result. Let F be a social choice function that always selects from the core of E: then F is ESC. The proof (cf. Lemma 5.3.6 in Peleg, 1984) is as follows. Consider the sets $C(y)$ as in Theorem 4.24. Construct a profile $\mathbf{Q} \in \rho$ where, for every $y \in A \backslash \{x\}$ and every $t \in C(y)$, $z\mathbf{Q}(t)y$ for all $z \in A \backslash \{y\}$. Then it is not much work to show that $\mathbf{Q}$ is an SNE in the game $(F, \mathbf{R})$ and $F(\mathbf{Q}) = x$. This result is also implied by Theorem 4.26 below combined with Theorem 4.23.

For every $\mathbf{R} \in \rho$ let $M(\mathbf{R})$ be the set of alternatives attainable through a feasible elimination procedure, as in Subsection 4.4. We have already established that these alternatives are in the core $C(E, \mathbf{R})$. We will now show the converse. Call $b \in A$ a *bottom alternative* of $\mathbf{R}$ if the set $\hat{S}(b) = \{t \in \Omega \mid y\mathbf{R}(t)b \text{ for all } y \in A\}$ has measure $\lambda(\hat{S}(b)) \geq \beta(b)$, with strict inequality sign for $b = s$.

Theorem 4.26 *Let $(\beta; \mathbf{e}, \mathbf{i})$ be a system satisfying (11)–(13) with $\mathbf{i}$ containing exactly one singleton $\{s\}$ for some designated $s \in A$, and with all coefficients $\beta(y)$ $(y \in A \backslash \{s\})$ positive. Let E be the associated effectivity function and let $\mathbf{R} \in \rho$ and $x \in C(E, \mathbf{R})$. Then $x \in M(\mathbf{R})$. In particular, if b is a bottom alternative of $\mathbf{R}$, then there is an f.e.p. $(b, C_b; y_{i_1}, C_1; \ldots; y_{i_{m-2}}, C_{m-2}; x)$ such that $C_b \subseteq \hat{S}(b)$.*

Proof. Let b be a bottom alternative. If $b = s$ we slightly increase the blocking coefficient of b (as in the beginning of the proof of Theorem 4.24) so that we still have $\lambda(\hat{S}(b)) \geq \beta(b)$.

The proof is by induction on $m = |A|$. For $m = 2$ the result is obvious again. Let $m \geq 3$.

(i) First suppose that the problem is decomposable into two subproblems with sets of alternatives $\{x\} \cup B^*$ and $A \backslash B^*$ as in the proof of Theorem 4.24, and with $b \in B^*$. Note that all voters in the problem with $A \backslash B^*$ rank B^* above x. By the induction hypothesis, each of the subproblems has an f.e.p. leading to x, with the one in the first subproblem starting with b. Let $|B^*| = k$, let $(b, C_b; y_1, C_1; \ldots; y_{k-1}, C_{k-1}; x)$ be an f.e.p. in the problem with $\{x\} \cup B^*$ and let $(x_1, \hat{C}_1; \ldots; x_{m-k-1}, \hat{C}_{m-k-1}; x)$ be an f.e.p. in the problem with $A \backslash B^*$. Then

$$(b, C_b; x_1, \hat{C}_1; \ldots; x_{m-k-1}, \hat{C}_{m-k-1}; y_1, C_1; \ldots; y_{k-1}, C_{k-1}; x)$$

is an f.e.p. for the orginal problem.

(ii) Next, suppose the problem is not decomposable in this way. As in the proof of Theorem 4.24 let $S = S(b) \backslash \bigcup_{y \in A \backslash \{x,b\}} S(y)$ and distinguish two cases as there. In Case 1, $\lambda(S) \geq \beta(b)$, we take again $C(b) = S$, observing that $S \subseteq \hat{S}(b)$. Applying the induction hypothesis, we let $(y_{i_1}, C_1; \ldots; y_{i_{m-2}}, C_{m-2}; x)$ be an f.e.p. in the problem with set of alternatives $A \backslash \{b\}$, then
$(b, C_b; y_{i_1}, C_1; \ldots; y_{i_{m-2}}, C_{m-2}; x)$ is as desired.

In Case 2, we proceed again as in the proof of Theorem 4.24 but we make sure that S_0 there is chosen in such a way that $\lambda(S_0 \cap \hat{S}(b)) \geq \beta(b)$. This is possible since $S \subseteq \hat{S}(b) \subseteq S(b)$ and so we can choose S^* (which is a subset of S_0 by construction) such that $S^* \subseteq \hat{S}(b)$. We have now again a decomposition as in (i) of this proof: since b is eliminated first, shifting b over x in the original preferences of voters in $S(b)\backslash S_0$ does not change the restriction of these preferences to $A\backslash B^*$. □

We note that Theorem 4.26 may also be adapted to apply to the case with finitely many agents and, thus, provides an alternative and shorter proof of Theorem 5.4.2 in Peleg (1984).

The following corollary is a summary of the main results of Subsections 4.4 and 4.5.

Corollary 4.27 *(i) Let F be an anonymous ESC social choice function. Suppose that the associated effectivity function E has exactly one i-alternative, and the blocking coefficients of the e-alternatives are all positive. Then $C(E, \cdot) = M(\cdot)$ and F is a selection from this set. (ii) Let $(\beta; \mathbf{e}, \mathbf{i})$ be a system satisfying (11)–(13) such that $\mathbf{i}$ contains exactly one singleton and all singletons in $\mathbf{e}$ have positive blocking coefficients. Then, for the associated effectivity function E, $C(E, \cdot) = M(\cdot)$, and any anonymous selection from these sets is an anonymous ESC social choice function.*

4.6 A counterexample

A natural question is whether Corollary 4.27 can be extended to general systems $(\beta; \mathbf{e}, \mathbf{i})$. The following example shows that, in the case of three alternatives and positive blocking coefficients, there must be exactly one i-alternative for an anonymous ESC social choice function to exist.

Example 4.28 Let A consist of three different alternatives, $A = \{a, b, c\}$, let $\Omega = [0, 1]$, and let λ be the Lebesgue measure. Let F be an ESC social choice function and suppose that the associated effectivity function $E = E^F$ is characterized by blocking coefficients $\beta(a)$, $\beta(b)$, and $\beta(c)$, all positive and summing to 1. Suppose that a is an e-alternative and that b and c are i-alternatives. This is the only remaining case if we do not have exactly one i-alternative: we will in fact show that this case is not possible.

Consider a partition $\{B_1, B_2, B_3, B_4\}$ of $[0,1]$ with $\lambda(B_1) = \lambda(B_2) = \frac{1}{2}\beta(a)$, $\lambda(B_3) = \beta(b)$, and $\lambda(B_4) = \beta(c)$. The profile $\mathbf{R}$ is given in the following table.

B_1	B_2	B_3	B_4
b	c	a	a
c	b	c	b
a	a	b	c

Note that $C(E, \mathbf{R}) = \{b, c\}$: a is blocked via $B_1 \cup B_2$ but b and c are not blocked and basically occur symmetrically in this profile. Assume that $\mathbf{Q}$ is an SNE of the game $(F, \mathbf{R})$ and $F(\mathbf{R}) = b$. (The case $F(\mathbf{R}) = c$ is analogous.)

Suppose there is an $S \in \Sigma_+$ with $S \subseteq B_1 \cup B_2$ and $a\mathbf{Q}(t)c$ for all $t \in S$. Consider the deviation $\mathbf{V}^{B_3 \cup B_4}$ with $a\mathbf{V}(t)c\mathbf{V}(t)b$. In the profile $(\mathbf{Q}^{B_1 \cup B_2}, \mathbf{V}^{B_3 \cup B_4})$, b is dominated by $\{a, c\}$ via $B_3 \cup B_4$ and c is dominated by $\{a\}$ via $S \cup B_3 \cup B_4$: observe that the latter coalition has measure strictly larger than $\beta(b) + \beta(c)$. Hence, $C(E, (\mathbf{Q}^{B_1 \cup B_2}, \mathbf{V}^{B_3 \cup B_4})) = \{a\}$, thus $F(\mathbf{Q}^{B_1 \cup B_2}, \mathbf{V}^{B_3 \cup B_4}) = a$. Therefore, $\mathbf{V}^{B_3 \cup B_4}$ is a profitable deviation from $\mathbf{Q}$ for the coalition $B_3 \cup B_4$, a contradiction. We conclude that $c\mathbf{Q}(t)a$ for all $t \in B_1 \cup B_2$.

Next, consider the deviation $\mathbf{W}^{B_2 \cup B_3}$ given by $c\mathbf{W}(t)a\mathbf{W}(t)b$ for all $t \in B_2 \cup B_3$. Since, by the previous argument, $c\mathbf{Q}(t)a$ for all $t \in B_1$, we have that in the profile $(\mathbf{Q}^{B_1 \cup B_4}, \mathbf{W}^{B_2 \cup B_3})$, alternative a is dominated by $\{c\}$ via $B_1 \cup B_2 \cup B_3$: $\lambda(B_1 \cup B_2 \cup B_3) = \beta(a) + \beta(b)$ and $\{a, b\}$ is an e-set. Also, b is dominated by $\{a, c\}$ through $B_2 \cup B_3$ which has measure strictly larger than $\beta(b)$. Hence, $C(E, (\mathbf{Q}^{B_1 \cup B_4}, \mathbf{W}^{B_2 \cup B_3})) = \{c\}$ and therefore $F((\mathbf{Q}^{B_1 \cup B_4}, \mathbf{W}^{B_2 \cup B_3})) = c$. So $\mathbf{W}^{B_2 \cup B_3}$ is a profitable deviation from $\mathbf{Q}$ for the coalition $B_2 \cup B_3$. This is again a contradiction. Hence, the game $(F, \mathbf{R})$ does not have an SNE.

Examples 4.1 and 4.14 describe an anonymous ESC social choice function with exactly one e-alternative and with all i-alternatives having blocking coefficients 0. This shows that it is not possible to extend Example 4.28 to a general counterexample for all systems different from the ones in Corollary 4.27. The question of existence of anonymous ESC social choice functions in case that more than one alternative must be strictly blocked, is therefore still open.

5 Strong Nash consistent representation of effectivity functions

In this section the focus remains on strategic behavior by coalitions, but we relax the requirement that such strategic behavior should result in the same outcome as truthful revelation of preferences. In order to be more precise, we start with the definition of a game form and its associated effectivity function.

For every $t \in \Omega$ let $\mathcal{S}^t$ be a nonempty set, let $\mathcal{S} = \prod_{t \in \Omega} \mathcal{S}^t$ be the Cartesian product, and let $\pi : \mathcal{S} \to A$ be a surjective function. We assume that for all $\sigma, \tau \in \mathcal{S}$ we have (i) $\{t \in \Omega \mid \sigma(t) \neq \tau(t)\} \in \Sigma$ and (ii) $\pi(\sigma) = \pi(\tau)$ if $\lambda(\{t \in \Omega \mid \sigma(t) \neq \tau(t)\}) = 0$. (With some abuse of notation we still use the symbol $\mathcal{S}$.) Then $\Gamma = (\mathcal{S}, \pi)$ is a *game form*, the sets $\mathcal{S}^t$ $(t \in \Omega)$ are the *strategy sets*, $\sigma \in \mathcal{S}$ is a *strategy profile*, and π is the *outcome function*. For $\mathbf{R} \in \rho$, $(\Gamma, \mathbf{R})$ is a game in the obvious way.

A straightforward example of a game form is a social choice function F.

A strategy profile $\sigma \in \mathcal{S}$ is a *strong Nash equilibrium* of $(\Gamma, \mathbf{R})$ if for all $S \in \Sigma_+$ and all $\tau^S \in \mathcal{S}^S$ we have $\lambda(\{t \in S \mid \pi(\sigma)\mathbf{R}(t)\pi(\sigma^{\Omega \setminus S}, \tau^S)\}) > 0$. The game form Γ is *strong Nash consistent* if $(\Gamma, \mathbf{R})$ has a strong Nash equilibrium for every $\mathbf{R} \in \rho$.

As an example, if F is an ESC social choice function, then F is strong Nash consistent as a game form.

The effectivity function E^Γ associated with a game form Γ is defined by $E^\Gamma(\emptyset) = \emptyset$, and for $S \in \Sigma_0$ and $B \in P_0(A)$, $B \in E^\Gamma(S)$ if there is a $\sigma^S \in \mathcal{S}^S$ such that $\pi(\sigma^S, \sigma^{\Omega \setminus S}) \in B$ for every $\sigma^{\Omega \setminus S} \in \mathcal{S}^{\Omega \setminus S}$. It is easy to check that E^Γ indeed satisfies all requirements in Definition 4.2. Also note that this definition coincides with the definition of E^F for an SCF F.

For a game form Γ and an effectivity function E, we say that Γ *represents* E if $E = E^\Gamma$. The purpose of this section is to establish necessary and sufficient conditions under which an effectivity function can be represented by a strong Nash consistent game form. In this way we broaden the perspective taken so far in considering ESC social choice functions. In the case of an ESC SCF, obviously the associated effectivity function E^F is strong Nash represented by F, with the special property that there is always a strong Nash equilibrium that results in the same outcome as truthful revelation of preferences. In the broader perspective taken in this section, we consider general effectivity functions as representing the power embedded within the society of voters, and look for a decentralization of this power by means of a game form that preserves it and for which there is always a "stable" situation in the sense of a strong Nash equilibrium.

The following lemma extends Corollary 4.4 and Theorem 4.7.

Lemma 5.1 *Let* Γ *be a strong Nash consistent game form. Then* $E = E^{\Gamma}$ *is maximal and stable.*

Proof. We first show that E is maximal. Let $S \in \Sigma_0$ with $\lambda(S) < \lambda(\Omega)$ and $B \in P_0(A)$ with $B \notin E(S)$. Suppose, for contradiction, that $A \backslash B \notin E(\Omega \backslash S)$. Take $\mathbf{R} \in \rho$ with $B\mathbf{R}^S(A \backslash B)$ and $(A \backslash B)\mathbf{R}^{\Omega \backslash S}B$. Let σ be an SNE of $(\Gamma, \mathbf{R})$ and $\pi(\sigma) = x$. If $x \in B$ then, since $B \notin E(S)$, there is $\tau^{\Omega \backslash S}$ with $\pi(\sigma^S, \tau^{\Omega \backslash S}) \in A \backslash B$; hence $\Omega \backslash S$ has a profitable deviation. Similarly, one can prove $x \notin A \backslash B$. Since $x \in A$, this is a contradiction.

To show that E is stable, take $\mathbf{R} \in \rho$ and let σ be an SNE of $(\Gamma, \mathbf{R})$. Let $x = \pi(\sigma)$, then it is straightforward to show that $x \in C(E, \mathbf{R})$. □

The main result of this section is the following theorem, which establishes the converse of Lemma 5.1. Its proof is an adaptation of the construction in Moulin and Peleg (1982).

Theorem 5.2 *Let the effectivity function* E *be stable and maximal. Then* E *can be represented by a strong Nash consistent game form.*

Proof. Fix an arbitrary preference $R_0 \in L(A)$. We construct a game form $\Gamma = (\mathcal{S}, \pi)$ as follows.

For every $t \in \Omega$ let $\mathcal{S}^t$ consist of all functions $\sigma^t : \{S \in \Sigma_0 \mid t \in S\} \to P_0(A)$ that satisfy, for all $S, T \in \Sigma$ with $t \in S \subseteq T$:

(26) $\sigma^t(S) \in E(S)$, $\sigma^t(T) \in E(T)$, and $\sigma^t(T) \subseteq \sigma^t(S)$.

Let $\mathcal{S}$ be the subset of $\prod_{t \in \Omega} \mathcal{S}^t$ consisting of all strategy profiles σ that satisfy (i) and (ii) in the general definition of a game form above, and moreover: $\{t \in S \mid \sigma^t(S) = B\}$ is a measurable set for every $S \in \Sigma_0$ and every $B \in E(S)$.

Let $\sigma \in \mathcal{S}$. For every $S \in \Sigma_0$ define an equivalence relation $\sim_\sigma$ on S by

$$s \sim_\sigma t \Leftrightarrow \sigma^s(S) = \sigma^t(S)$$

for all $s, t \in S$. Let $D(S) = D(S, \sigma)$ denote the partition of S generated by $\sim_\sigma$. (Note that every element of such a partition is indeed a measurable set.) Define $H_0(\sigma) = \{\Omega\}$, $H_1(\sigma) = D(\Omega, \sigma)$, and if $H_k(\sigma) = \{S_{k,1}, \ldots, S_{k,l_k}\}$ for $k \geq 0$ define

$$H_{k+1}(\sigma) = \bigcup_{j=1}^{l_k} D(S_{k,j}).$$

Consider the (infinite) tree with root Ω, set of nodes $\bigcup_{k=0}^{\infty} H_k(\sigma)$, and, for each node $S_{k,j}$ a finite set of emanating edges, labelled by $\{\sigma^t(S_{k,j}) \in E(S_{k,j}) \mid t \in S_{k,j}\}$, and ending in the corresponding nodes in $D(S_{k,j}, \sigma) \subseteq H_{k+1}(\sigma)$. For every path in this tree, by the monotonicity requirement in (26), there is a $k \geq 0$ such that the edges on the path have some constant label B after the k-th node. Let $J \subseteq P_0(A)$ be the set of labels that occur in this way, that is, as a constant label on some path; say $J = \{B_1, \ldots, B_l\}$. For every $B_j \in J$ choose a path P_j that has B_j as constant label after some node on the path. In this way we obtain a finite collection of paths $\{P_1, \ldots, P_l\}$ in the tree, and thus there is an r such that (i) all these paths have constant label at each edge on the path following the r-th node, and (ii) all paths are disjoint after the r-th node. Choose an arbitrary $k > r$, and for each $B_j \in J$ let S_{k,i_j} be the k-th node on the path P_j corresponding to B_j. Then $B_j \in E(S_{k,i_j})$ by requirement (i) and the sets S_{k,i_j} $(B_j \in J)$ are disjoint by requirement (ii). Hence, by stability of E, $\bar{B} = \bigcap_{j=1}^{l} B_j \neq \emptyset$.

Define $\pi(\sigma)$ to be the maximal element of $\bar{B}$ according to R_0. This completes the definition of the game form Γ.

Next we show that Γ represents E, i.e., that $E^\Gamma = E$. First, let $B \in E(S)$ for some $S \in \Sigma_0$. For every $t \in S$ let σ^t be the strategy that assigns B to every $T \in \Sigma$ with $T \supseteq S$, and A otherwise. (Note that this is possible since E is monotonic.) Then there is a path from the root Ω that has constant label B. Hence, $B \in J$, so that $\bar{B} \subseteq B$. Therefore, $\pi(\sigma^S, \tau^{\Omega \setminus S}) \in B$ for all $\tau^{\Omega \setminus S} \in \Sigma^{\Omega \setminus S}$. Thus, $B \in E^\Gamma(S)$. For the converse, suppose $B \notin E(S)$. Then by maximality of E, we have $A \setminus B \in E(\Omega \setminus S)$, hence by the previous argument $A \setminus B \in E^\Gamma(\Omega \setminus S)$. Hence by stability of E we have $B \notin E^\Gamma(S)$. Altogether, $E = E^\Gamma$.

Finally, we prove that Γ is strong Nash consistent. Let $\mathbf{R} \in \rho$ and let $a \in C(E, \mathbf{R})$. For every $T \in \Sigma \setminus \{\emptyset\}$ define

$$B^T = \{b \in A \mid b \mathbf{R}^T a,\ b \neq a\}.$$

Since $a \in C(E, \mathbf{R})$ we have $B^T \notin E(T)$ for every $T \in \Sigma_0$. So by maximality of E we have $A \backslash (B^{\Omega \backslash T}) \in E(T)$ for every $T \in \Sigma_0$. Define, for all $t \in \Omega$ and all $T \in \Sigma$ with $t \in T$,

$$\sigma^t(T) := \begin{cases} \{a\} & \text{if } \lambda(T) = \lambda(\Omega) \\ A \backslash (B^{\Omega \backslash T}) & \text{if } \lambda(T) < \lambda(\Omega). \end{cases}$$

Then σ^t satisfies (26) because $a \in A \backslash (B^{\Omega \backslash T})$ for all $T \in \Sigma$ with $t \in T$ and $\lambda(T) < \lambda(\Omega)$, and for $t \in S \subseteq T$ with $\lambda(T) < \lambda(\Omega)$ we have $\sigma^t(T) \subseteq \sigma^t(S)$. Then $\pi(\sigma^\Omega) = a$. It remains to prove that $\sigma = \sigma^\Omega$ is an SNE of $(\Gamma, \mathbf{R})$. Let $T \in \Sigma_0$ and $\mu^T \in \mathcal{S}^T$. It is sufficient to prove that $\pi(\sigma^{\Omega \backslash T}, \mu^T) \notin B^T$. If $\lambda(T) = \lambda(\Omega)$ then this follows from $a \in C(E, \mathbf{R}) \subseteq \mathrm{PAR}(\mathbf{R})$. Otherwise, by construction of π, $\pi(\sigma^{\Omega \backslash T}, \mu^T) \in A \backslash (B^{\Omega \backslash T'})$ for some $T' \supseteq \Omega \backslash T$. Hence, $\pi(\sigma^{\Omega \backslash T}, \mu^T) \notin B^{\Omega \backslash T'}$, so $\pi(\sigma^{\Omega \backslash T}, \mu^T) \notin B^T$, which was to be proved. □

For the case of finitely many voters the partitions that are used to define the outcome function in the game form in the proof of Theorem 5.2 are constant after a finite number of steps. For that case the game form is almost identical to the one used in Moulin and Peleg (1982): the difference is the monotonicity requirement in (26), which is an improvement in the sense that the strategy sets are smaller than in the Moulin-Peleg game form. In our model with infinitely many voters, however, it is possible that these partitions never become constant. The monotonicity requirement on the strategies nevertheless guarantees that there is a collection of subsets of alternatives for which disjoint coalitions are effective and which becomes constant in a finite number of steps.

6 Concluding remarks

In this paper we have carried out a thorough study of a voting system with a measurable space of voters in which single voters are powerless, and a finite number of alternatives. We have derived analogues of the classical theorems of Arrow and Gibbard-Satterthwaite, resulting in the existence of invisible dictators in the sense of Kirman and Sondermann (1972). The emphasis of the paper is on exactly and strongly consistent social choice functions, which we have studied through their effectivity functions and also through feasible elimination procedures. The latter may, in principle, be turned into algorithms for computing (anonymous) ESC social choice functions with given blocking coefficients. The final part of the paper presents a characterization of effectivity functions through game forms that have strong Nash equilibria whatever the preference profile.

It is our hope that the model under consideration provides an improved approximation of large voting systems compared to a model with a finite or discrete number of voters.

Appendix: Stability of convex functions

Let $E : \Sigma \to P(P_0(A))$ be *convex*, that is, for all $S_1, S_2 \in \Sigma$ and $B_1 \in E(S_1)$, $B_2 \in E(S_2)$ we have $B_1 \cap B_2 \in E(S_1 \cup S_2)$ or $B_1 \cup B_2 \in E(S_1 \cap S_2)$. We will show that such a function is stable, i.e., has a nonempty core for each $\mathbf{R} \in \rho$, where the core is defined similarly as for an EF. The proof of this result is an adaptation of the proof of Theorem 6.A.7 in Peleg (1984) to our model and is based, similarly, on a result on the core of a specific cooperative game.

An *n-person cooperative game without sidepayments* is a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of players and $v : P(N) \to \mathbb{R}^N_+$ (the nonnegative orthant of $\mathbb{R}^N$) satisfies: (i) $v(\emptyset) = \emptyset$ and $v(S) \neq \emptyset$ if $S \neq \emptyset$; (ii) $v(S)$ is closed for every $S \in P(N)$; (iii) for all $S \in P(N)$ and $x \in v(S)$, if $y \in \mathbb{R}^N_+$ with $y_i \leq x_i$ for all $i \in S$, then $y \in v(S)$; and (iv) $v_p(S)$, the projection of $v(S)$ on $\mathbb{R}^S_+$, is bounded. The *core* of a game (N, v), denoted $C(N, v)$, consists of all vectors x in $v(N)$ such that, for every $S \in P(N)$ and every $y \in v(S)$, there is an $i \in S$ with $x_i \geq y_i$. A game (N, v) is *ordinally convex* if for all $S, T \in P(N)$, we have $v(S) \cap v(T) \subseteq v(S \cap T) \cup v(S \cup T)$. We will use the result that every ordinally convex game has a nonempty core (cf. Greenberg, 1985, for a short proof).

Theorem 6.1 *Let $E : \Sigma \to P(P_0(A))$ be convex. Then E is stable.*

Proof. Let $\mathbf{R} \in \rho$. Consider the partition $\mathcal{P}(\mathbf{R}) \subseteq \Sigma_+$ generated by $\mathbf{R}$ and write $\mathcal{P}(\mathbf{R}) = \{T_1, \ldots, T_n\}$. Let $N = \{1, \ldots, n\}$, then the function E induces a function $\tilde{E} : P(N) \to P(P_0(A))$ by $\tilde{E}(S) := E(\bigcup_{i \in S} T_i)$ for every $S \in P(N)$. Obviously, $\tilde{E}$ is again convex.

For each $i \in N$ let the function $u_i : A \to I\!R_+$ represent the preference $\mathbf{R}(t)$ for (all) $t \in T_i$. For every $i \in N$ define the function $w_i : P_0(A) \to I\!R_+$ by

(27) $w_i(B) = \min\{u_i(x) \mid x \in B\}$ for all $B \in P_0(A)$.

For every $S \in P_0(N)$ define

(28) $v(S) = \{r \in I\!R^N_+ \mid \text{there is a } B \in \tilde{E}(S) \text{ with } r_i \leq w_i(B) \text{ for all } i \in S\}$.

With $v(\emptyset) := \emptyset$, (N, v) is a well defined cooperative game. Moreover, this game is ordinally convex (see *Claim b* in Peleg, 1984, p. 149, using that $\tilde{E}$ is convex) and therefore has a nonempty core. The proof of the theorem will be complete by the following claim.

Claim: If $C(N, v) \neq \emptyset$, then $C(E, \mathbf{R}) \neq \emptyset$.

Indeed, let $r \in C(N, v)$. Then there exist $B \in P_0(A)$ such that $r_i \leq w_i(B)$ for every $i \in N$. Let $x \in B$. We claim that $x \in C(E, \mathbf{R})$. Assume, for contradiction, that this is not the case. Then there exists $B' \in P_0(A)$ and a coalition $T \in \Sigma_+$ such that $B' \in E(T)$ and $B'\mathbf{R}(t)x$ for every $t \in T$. Without loss of generality we may assume $T = \bigcup_{i \in S} T_i$ for some $S \in P_0(N)$. Hence, $w_i(B') > u_i(x)$ for every $i \in S$. Since $B' \in \tilde{E}(S)$ we can take $q \in v(S)$ with $q_i = w_i(B')$ for every $i \in S$. Hence, $q_i > u_i(x) \geq w_i(B) \geq r_i$ for every $i \in S$, contradicting $r \in C(N, v)$. □

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